Stability of Oscillatory Rotating Disk Boundary Layers

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Wales Mathematics Colloquium
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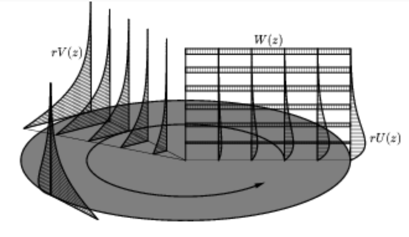
Structure

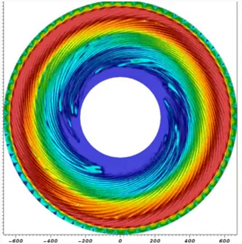
- **Part 1:** What?
- **Part 2:** Why?
- **Part 3:** How?

What?

What?

~~Stability of Oscillatory~~ **Rotating Disk Boundary Layers**

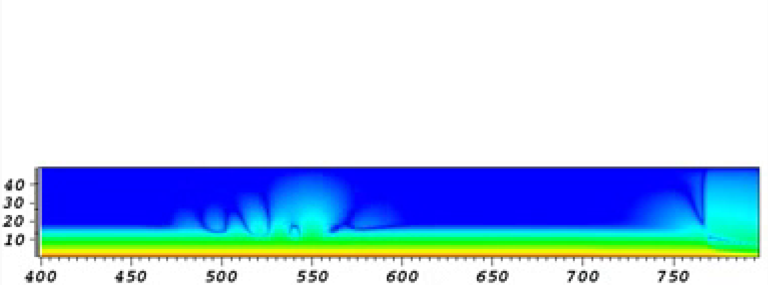




Simulations from E. Appelquist (KTH, Stockholm)

What?

Stability of ~~Oscillatory~~ Rotating Disk Boundary Layers



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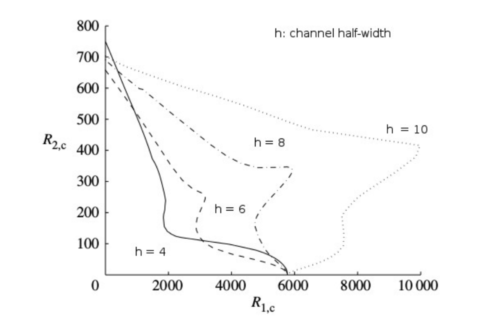
What?

Stability of Oscillatory ~~Rotating Disk~~ Boundary Layers

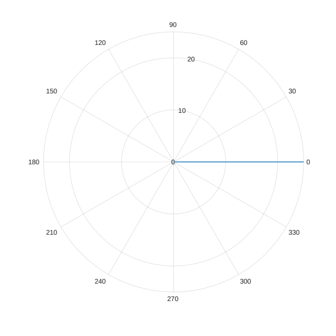
Adding oscillation to *channel* flow can be stabilising

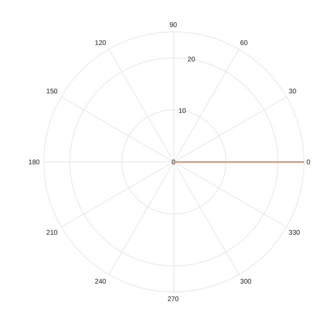


Thomas et. al. (2011)

What?

~~Stability of~~ **Oscillatory Rotating Disk Boundary Layers**



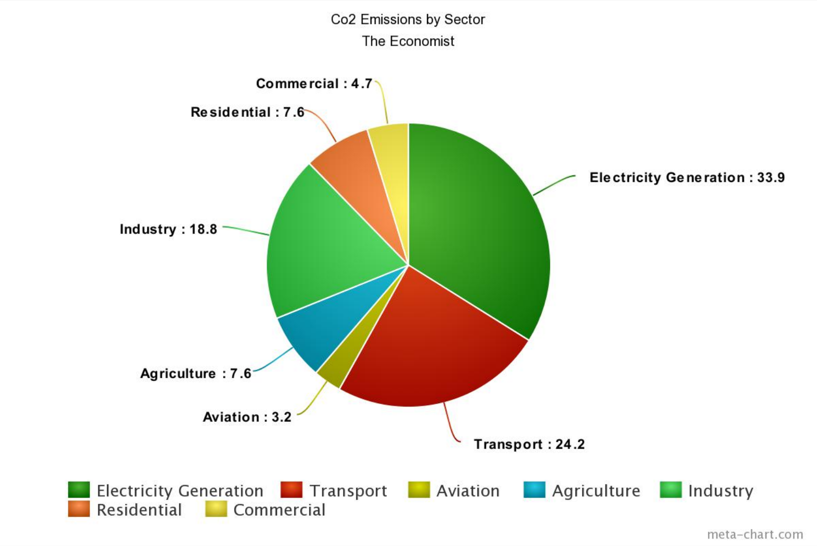


Why?

Why Rotating Disks?

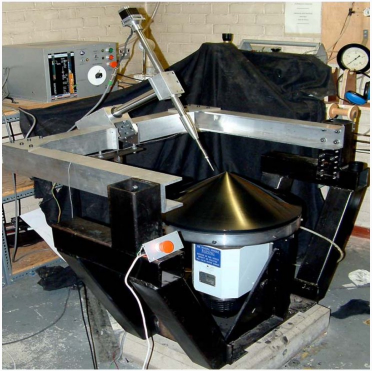
Rotating disks $\approx$ Swept wings

Why Swept Wings?



Why Not Just Study Swept Wings?

Reason 1



vs.

Why Not Just Study Swept Wings?

Reason 2

$$
\begin{aligned}
F^2 - G^2 + F'H - F'' &= 0 \\
2FG + G'H - G'' &= 0 \\
2F + H' &= 0
\end{aligned}
\qquad \text{vs.} \qquad
\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla^*)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0
\end{gathered}
$$

How?

How?

- Measure the response of the flow to some *disturbance*.

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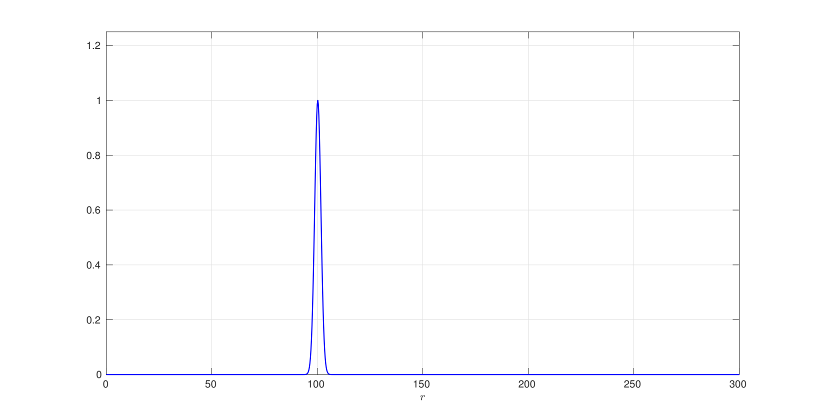
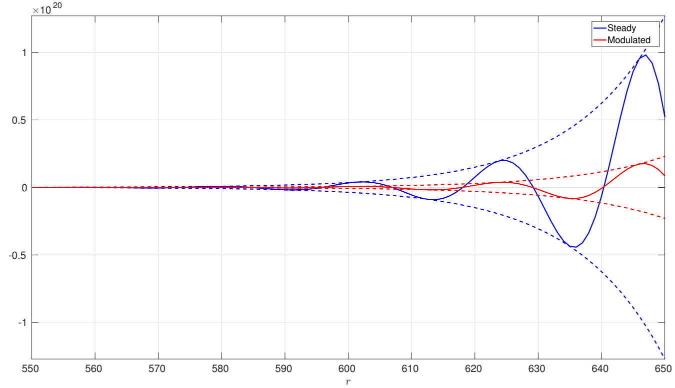
- Force disturbance at some point:



How?

- Measure the response of the flow to some *disturbance*.
- Compute evolution:



Radial evolution of $u(r, z = 0)$

Something a Little More Technical

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- Flow can be expressed as vector in three dimensions:

$$\mathbf{U}(r, \theta, z) = (F, G, H)$$

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$$\begin{aligned} \text{G at disk surface} &= r\Omega(t) \\ &= r(\Omega_0 + \epsilon \cos(\phi t)) \end{aligned}$$

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- Solve the Navier-Stokes equations.

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- Which parameters can we vary?
  - Reynolds number *(actually radial position)*
  - Oscillation amplitude
  - Oscillation frequency

Something a Little More Technical

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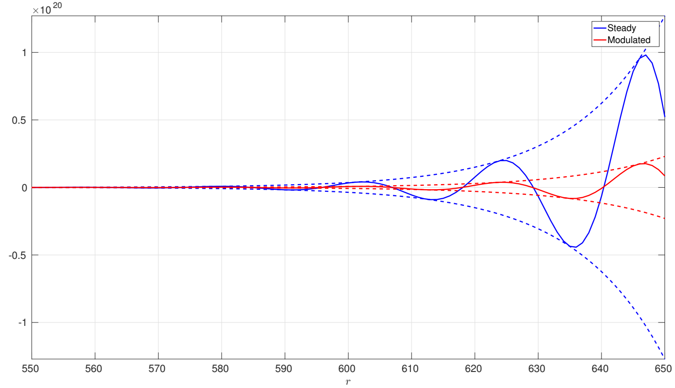
- Exponential growth?



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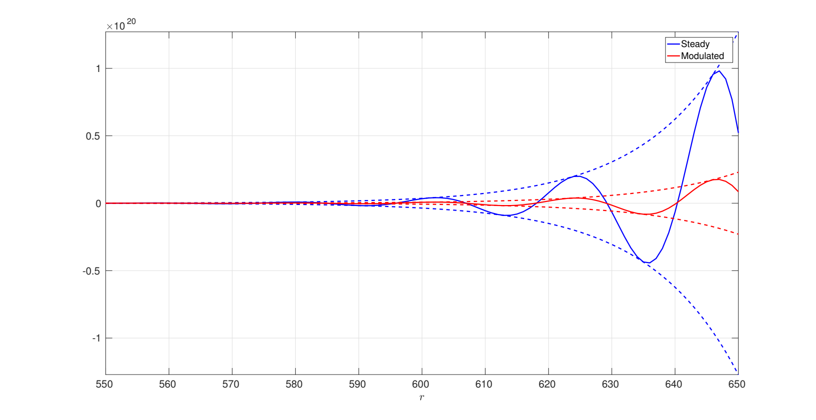
- Exponential growth?



- YES!

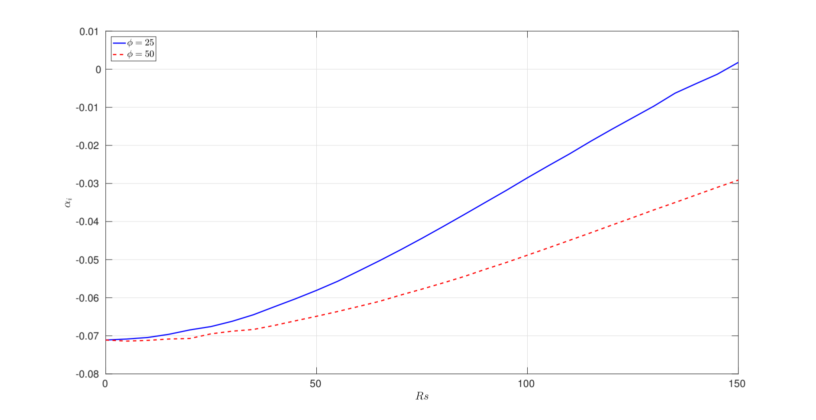
$$\mathbf{u}(r, z, t) \sim u(z, t)e^{i\alpha r}$$

Something a Little More Technical

- α_i gives us the growth rate.
- Give some flow variable A, (we use $A = u(r, z = 0, t)$), we can calculate:

$$\alpha_i \simeq \frac{-i}{A}\frac{\partial A}{\partial r}$$

Something a Little More Technical



Fixed Reynolds number & frequency, varying wall amplitude

Current & Future Work

- Use local techniques to confirm simulation results.
- Look at parallels between oscillation and surface roughness.
- Experimental confirmation.

Thank You

Periodic Modulation - Setup

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Ω_0^* - constant rotation rate
ϵ - angular displacement
ϕ^* - oscillation frequency

Scalings

- Two length scales:

$$\delta_k^* = \sqrt{\frac{\nu^*}{\Omega_0^*}}, \qquad \delta_s^* = \sqrt{\frac{\nu^*}{\phi^*}} \tag{1}$$

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$$R_s = \epsilon\sqrt{\varphi} R_k$$

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- Boundary conditions:

$$G(0, \tau) = 1 + \frac{R_s \sqrt{\varphi}}{R_k} \cos\left(\frac{\varphi}{R_k} \tau\right) \tag{5}$$

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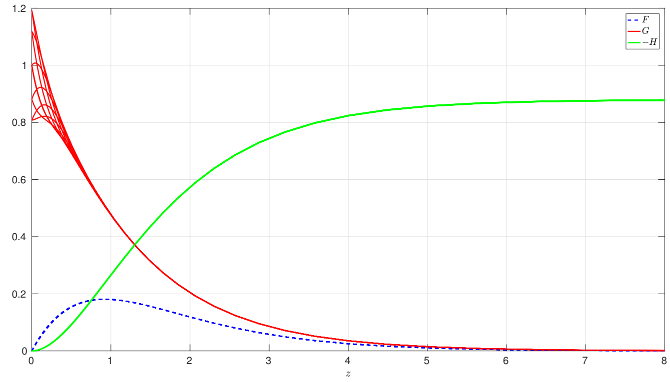
- Three parameters:

$$(R_k, R_s, \varphi) \tag{6}$$

($Rs \to 0$ recovers steady case)

Typical Mean Flow Variation



Base flow variation for $Rs = 10$, $\phi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

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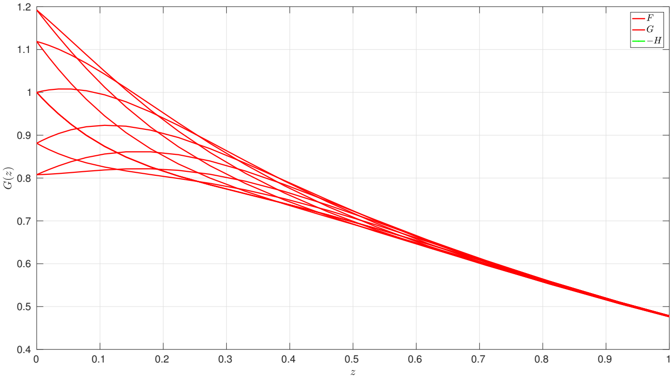

Azimuthal variation near wall for $Rs = 10$, $\phi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

Three Approaches to Stability Analysis

- Floquet theory
- DNS
- Instantaneous (frozen-flow) approximation

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- Focus on *stationary* disturbances

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- Perturbation

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$$\phi(r, \theta, z, \tau) = \hat{\phi}(z, \tau) e^{\mu \tau} e^{i(\alpha r + \beta R \theta)}$$

where $\hat{\phi}$ is periodic.

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  where $\hat{\phi}$ is periodic.

- Gives eigenvalue problem:

$$\sum_{n=-K}^{K} \mathcal{L}\{u_n^r, u_n^\theta, w_n; \mu\} e^{in\tau} = 0$$

  to be solved for (ψ, μ) where (α, β) are prescribed.

DNS - Wall Motion

- Wall displacement for stationary forcing (steady, rotating frame):

$$\zeta(r, \theta, \tau) = e^{\lambda r^2} e^{in\theta}$$

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$$\theta \rightarrow \theta_0 + \int^{\tau} \Omega(\tilde{\tau})$$

- Forcing stationary with respect to modulated disk:

$$\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$$

DNS - Stationary Forcing

- Prescribe wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$

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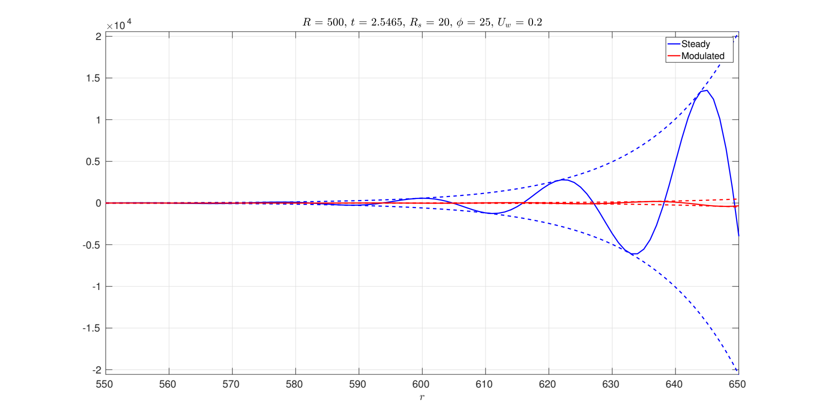
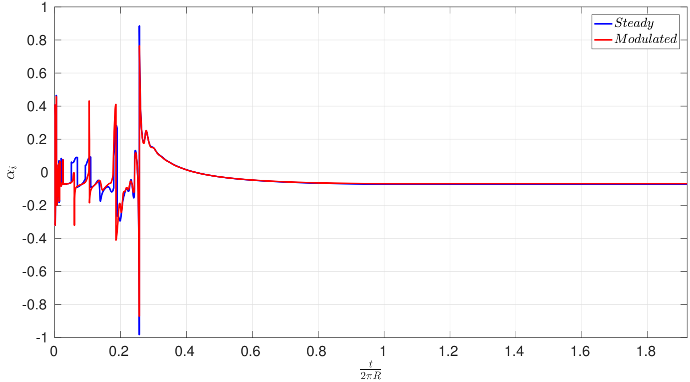


Radial evolution: $R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

- Receptivity issues.

DNS - Stationary Forcing

- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ at fixed r.



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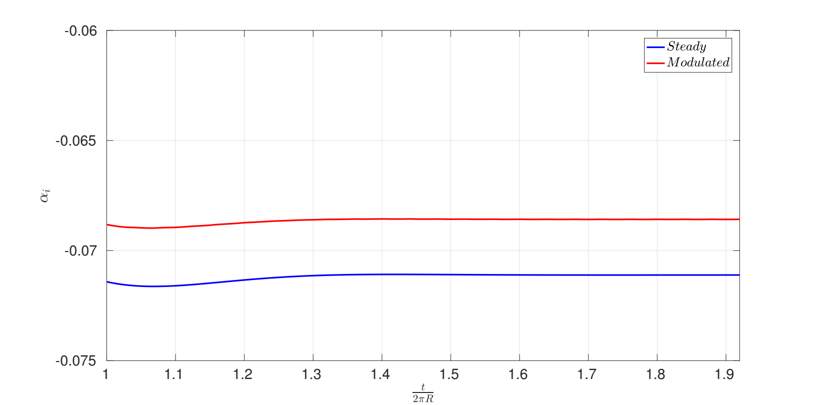

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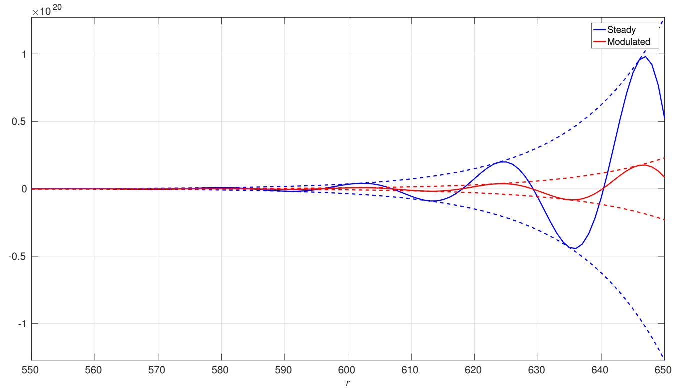
DNS - Stationary Forcing

- Exponential growth reconstructed from $e^{i\alpha r}$



$$R = 500, \; n = 32, \; Rs = 20, \; \varphi = 25$$

DNS - Results

- Exactly prescribe (α, μ, ψ) from Floquet theory at inflow.

$$\psi(r, \theta, z, \tau) = \hat{\psi}(z, \tau) e^{\mu \tau} e^{i \alpha r} e^{i n \theta}$$

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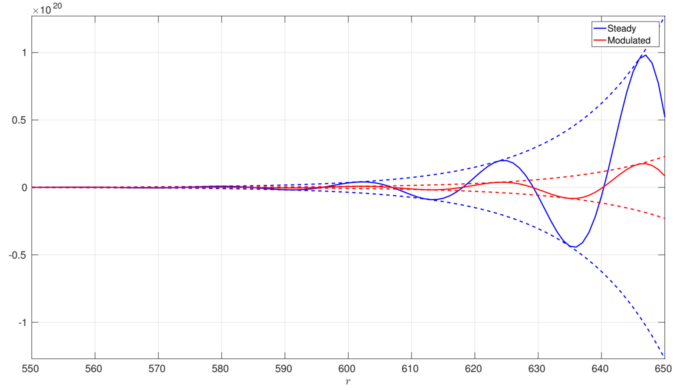

Radial evolution: $R = 500$, $n = 32$, $Rs = 20$, $\varphi = 25$

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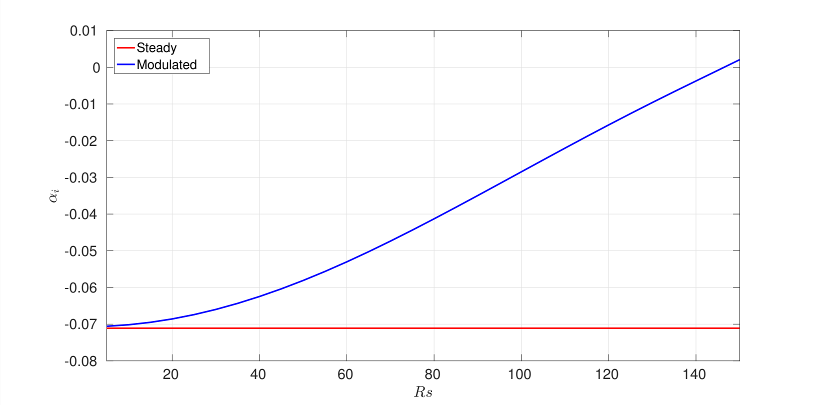
Stationary Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0686i
	$\varphi = 50$	0.2817 - 0.0702i

Inflow Prescribed Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0685i
	$\varphi = 50$	0.2818 - 0.0702i

DNS - Results



Variation of α_i with increasing R_s

DNS - Impulsive Forcing

- Prescribe impulsive wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{-\sigma t^2}$

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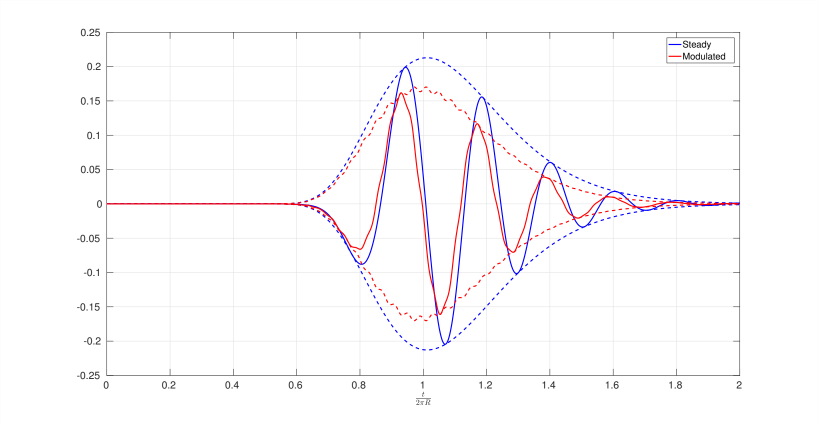

Temporal evolution: $R = 350$, $n = 32$, $Rs = 20$, $\varphi = 25$

Frozen flow approximation

- Freeze flow, treat τ as parameter:

$$\phi(r, \theta, z, \tau) = \hat{\phi}(z; \tau) e^{i(\alpha r + \beta R\theta - \int^{\tau} \omega(\tilde{\tau}))} \tag{7}$$

where $\hat{\phi}$ is slowly varying.

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- Specify α and solve OS equation for $\omega(\tau)$ at each time-step.

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Connections with the Stokes Layer

- Write

$$\begin{aligned}\mathbf{U}^T &= \mathbf{U}^S + \mathbf{U}^M \\ &= \mathbf{U}^S + \epsilon \mathbf{U}_1 + \mathcal{O}(\epsilon^2)\end{aligned}$$

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$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

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- High frequency modulation suggests primary lengthscale:

$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

- Rescale

$$\tilde{z} = \frac{z}{\delta_s}, \qquad \tilde{\tau} = \frac{\varphi}{R}\tau$$

Connections with the Stokes Layer

- Near the wall we have

$$\left(\frac{1}{\delta^2}\right)\frac{\partial F}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)F'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$\left(\frac{1}{R\delta^2}\right)\frac{\partial G}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)G'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$H \sim \delta F$$

with

$$F(0,\tilde{\tau}) = H(0,\tilde{\tau}) = 0, \qquad G(0,\tilde{\tau}) = \cos(\tilde{\tau})$$
$$F \to 0 \quad G \to 0 \quad \text{as} \quad \tilde{z} \to \infty$$

Connections with the Stokes Layer

- To dominant balance:

$$\frac{\partial F}{\partial \tilde{\tau}} = F'', \quad \text{with} \quad F(0) = F(\tilde{z} \to \infty) = 0$$

$$\frac{1}{R}\frac{\partial G}{\partial \tilde{\tau}} = G'', \quad \text{with} \quad G(0) = cos(\tilde{\tau}), \;\; G(\tilde{z} \to \infty) = 0$$

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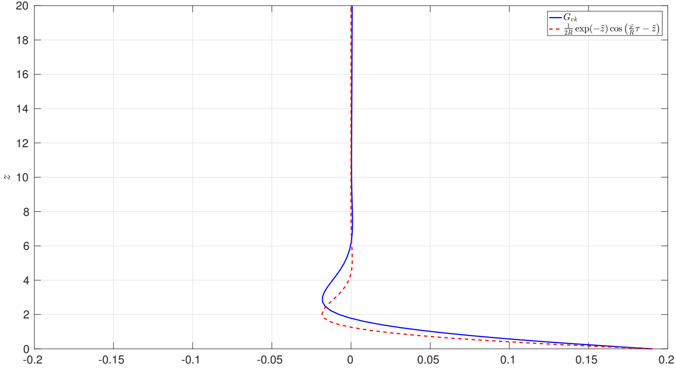
- Which gives Stokes layer:

$$F = 0$$
$$G = \frac{1}{R}\exp(-\tilde{z})\cos(\tilde{\tau} - \tilde{z})$$

Connections with the Stokes Layer



Comparison between Stokes layer profile and base flow variation