Stability of Oscillatory Rotating Disk Boundary Layers

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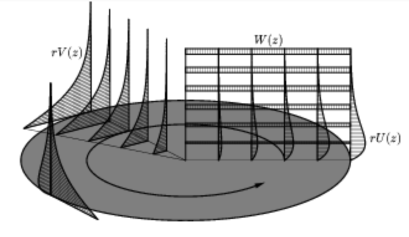
Structure

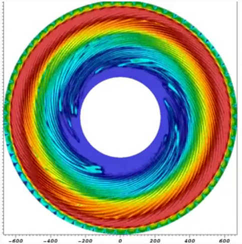
- **Part 1:** What?
- **Part 2:** Why?
- **Part 3:** How?

What?

What?

~~Stability of Oscillatory~~ **Rotating Disk Boundary Layers**

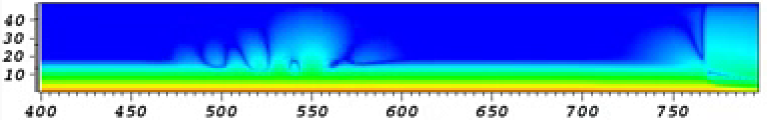




Simulations from E. Appelquist (KTH, Stockholm)

What?

Stability of ~~Oscillatory~~ Rotating Disk Boundary Layers

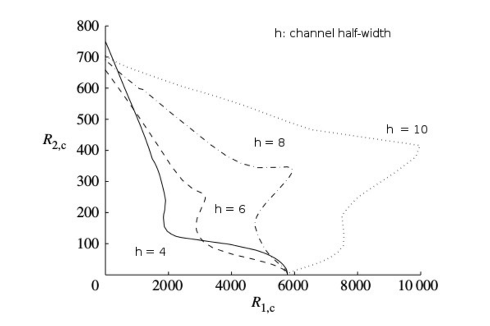


Simulations from E. Appelquist (KTH, Stockholm)

What?

Stability of Oscillatory ~~Rotating Disk~~ Boundary Layers

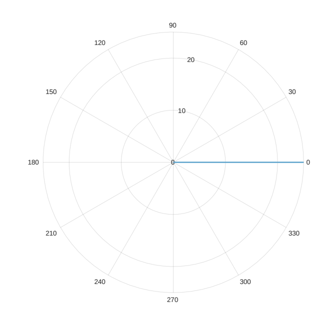
Adding oscillation to *channel* flow can be stabilising

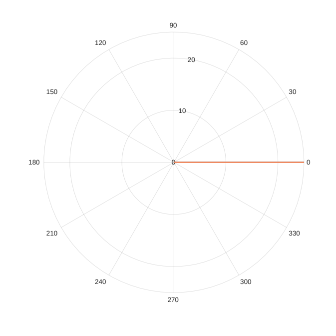


Thomas et. al. (2011)

What?

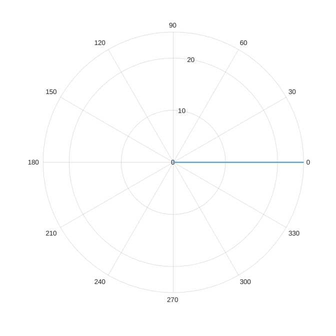
~~Stability of~~ **Oscillatory Rotating Disk Boundary Layers**

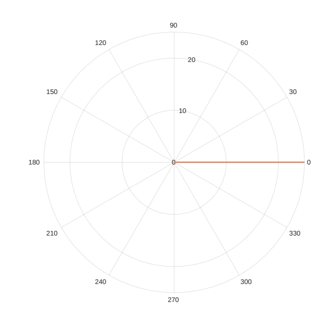




What?

~~Stability of~~ **Oscillatory Rotating Disk Boundary Layers**





Dominant behaviour is Stokes layer for high-frequency, low amplitude oscillations.

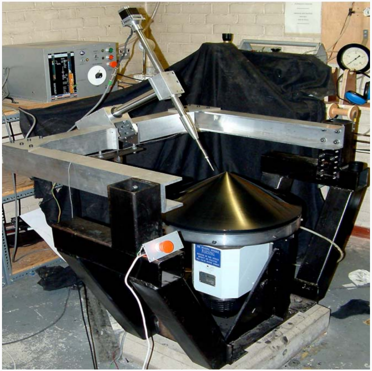
Why?

Why Rotating Disks?

Rotating disks $\approx$ Swept wings

Why Not Just Study Swept Wings?

Reason 1



vs.

Why Not Just Study Swept Wings?

Reason 2

$$
\begin{aligned}
F^2 - G^2 + F'H - F'' &= 0 \\
2FG + G'H - G'' &= 0 \\
2F + H' &= 0
\end{aligned}
$$

vs.

$$
\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla^*)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0
\end{gathered}
$$

How?

How?

- Measure the response of the flow to some *disturbance*.

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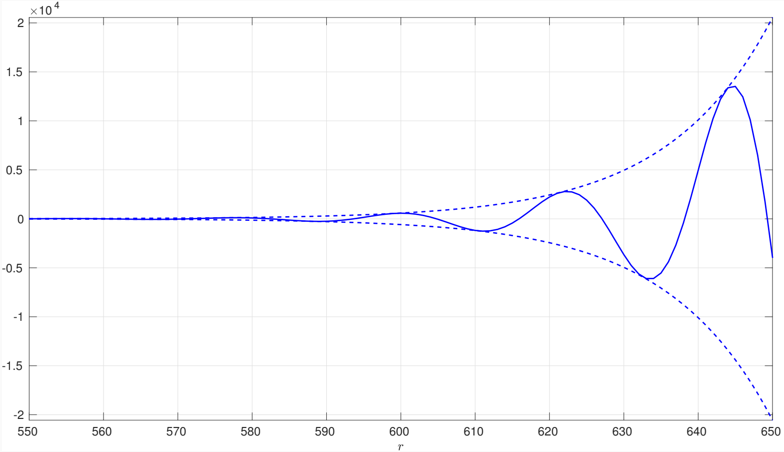
- Exponential evolution $\sim \exp(i\alpha r)$:



$u(r, z = 0, \tau > T_c)$ - *steady:* $R_k = 500$, $n = 32$

How?

- Three-dimensional base flow

$$\mathbf{U}_B = (U, V, W)$$

How?

- Three-dimensional base flow

$$\mathbf{U}_B = (U, V, W)$$

- Boundary conditions:

$$\begin{aligned} V(r, z = 0, t) &= r\Omega(t) \\ &= r\left(\Omega_0 + \epsilon\phi\cos(\phi t)\right) \end{aligned}$$

Ω_0 - constant rotation rate
ϵ - angular displacement
ϕ - oscillation frequency

Approaches

- Three approaches:

Approaches

- Three approaches:
    1. Floquet Theory

Approaches

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    1. Floquet Theory
    2. Linear DNS

Approaches

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  1. Floquet Theory
  2. Linear DNS
  3. *Frozen Flow Analysis*

Approaches

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Approaches

- Three approaches:
    1. Floquet Theory
    2. Linear DNS
    3. ~~*Frozen Flow Analysis*~~
- Solve Navier-Stokes equations using velocity-vorticity formulation.

Approaches

1. **Floquet Theory**

Floquet Theory

- *Floquet* mode approximation:

$$u(r, \theta, z, \tau) \sim \hat{u}(z, \tau) e^{i\alpha r} e^{\mu\tau} e^{in\theta}$$

  - $\hat{u}(z, \tau)$ periodic

Floquet Theory

- *Floquet* mode approximation:

$$u(r,\theta,z,\tau) \sim \hat{u}(z,\tau)e^{i\alpha r}e^{\mu\tau}e^{in\theta}$$

  - $\hat{u}(z,\tau)$ periodic
- Harmonic decomposition gives eigenvalue problem:

$$\sum_{k=-K}^{K} \mathcal{L}_k\{\mu,\alpha; n, R_k, R_s, \varphi\}e^{ik\tau} = 0$$

Floquet Theory

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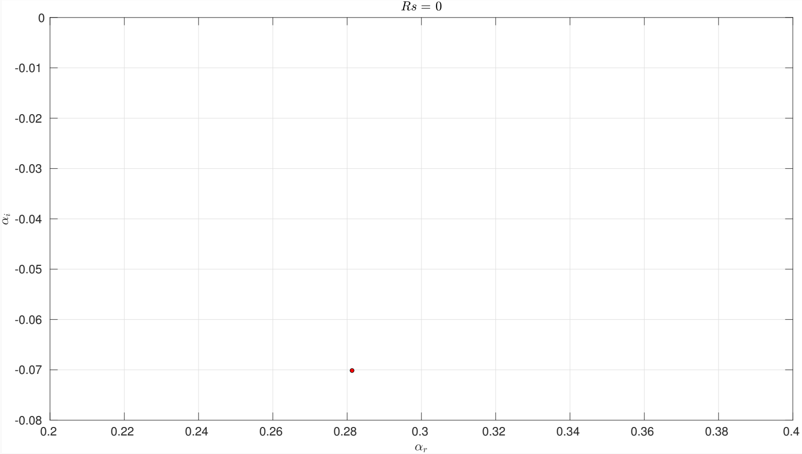
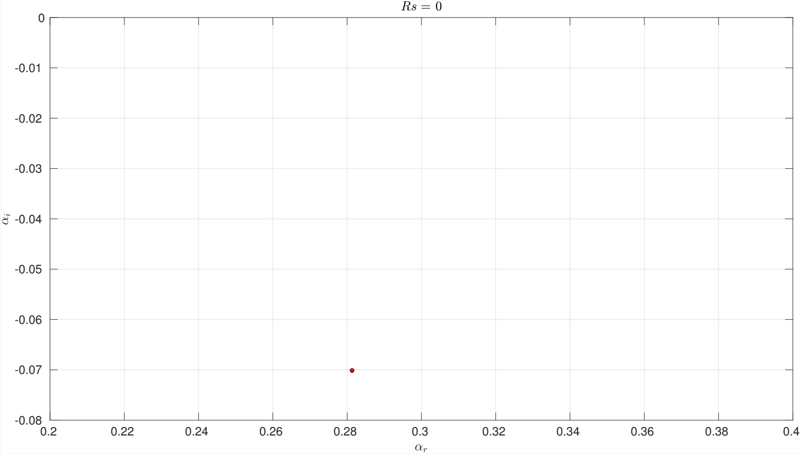
- Specify μ or α as real and solve for the other.

Floquet Results



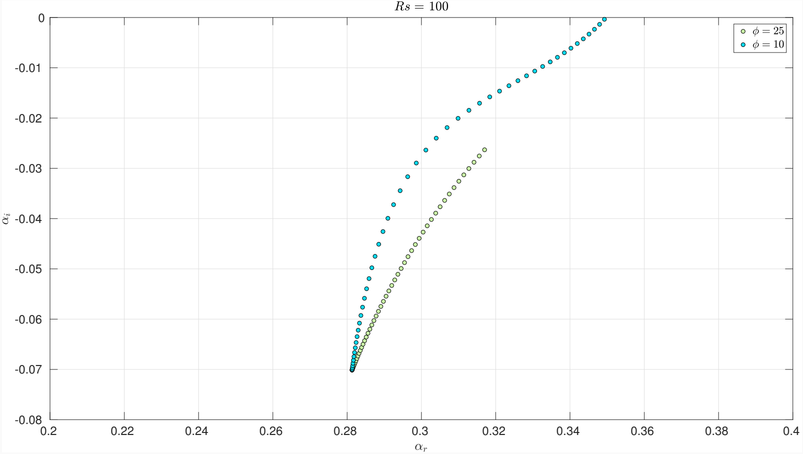
Steady case: $R_k = 500$, $n = 32$, $\alpha_i \approx -0.07$

Floquet Results



$R_k = 500,\ n = 32,\ \varphi = 25,\ R_s \in [0, 100]$

Floquet Results



$R_k = 500,\ n = 32,\ \varphi \in \{10, 25\},\ R_s \in [0, 100]$

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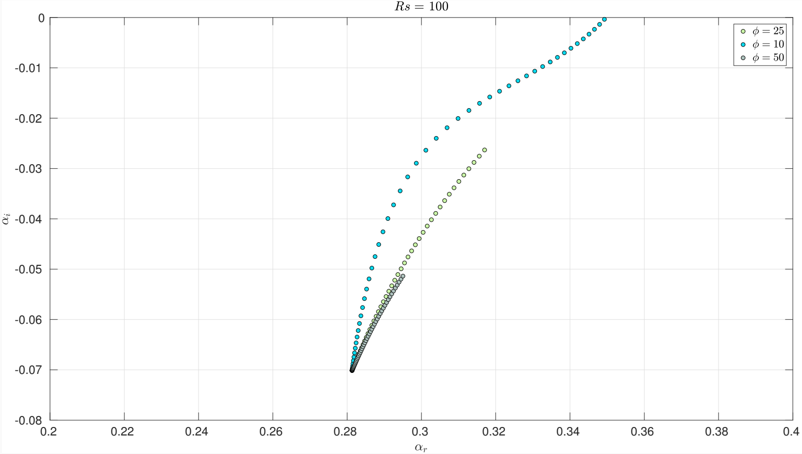
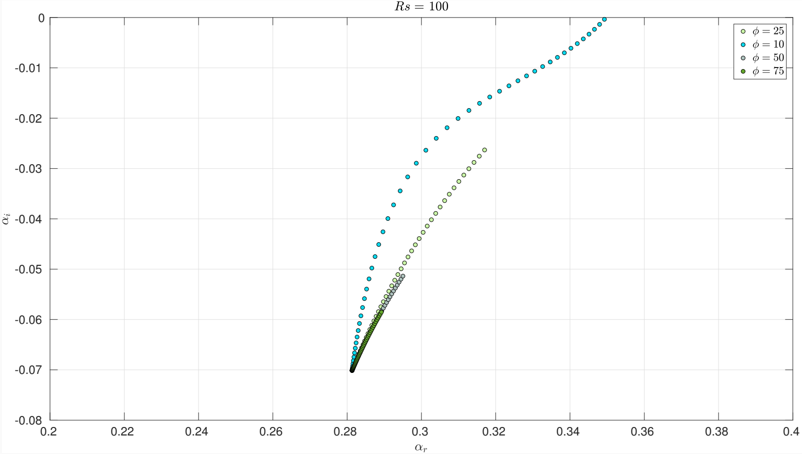


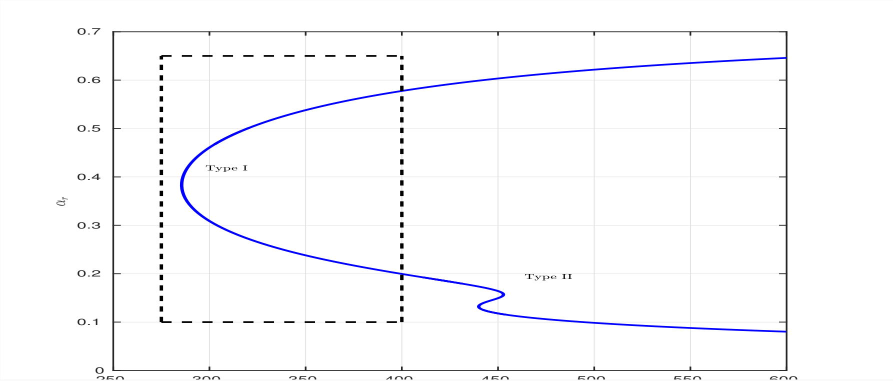
$R_k = 500,\ n = 32,\ \varphi \in \{10, 25, 50\},\ R_s \in [0, 100]$

Floquet Results



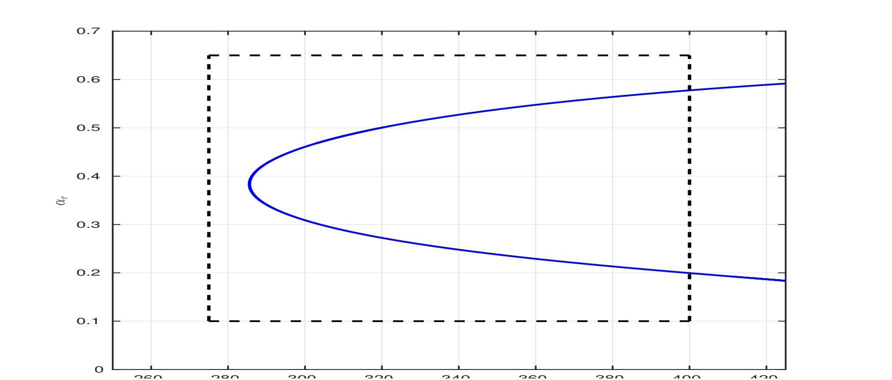
$R_k = 500,\ n = 32,\ \varphi \in \{10, 25, 50, 75\},\ R_s \in [0, 100]$

Neutral Curves



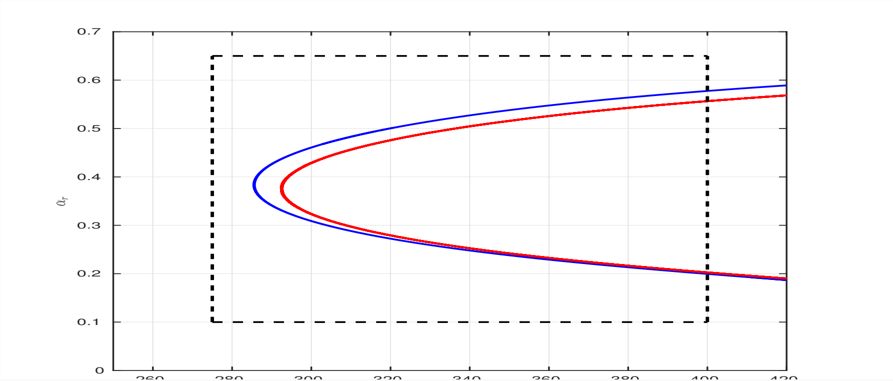
Steady

Neutral Curves



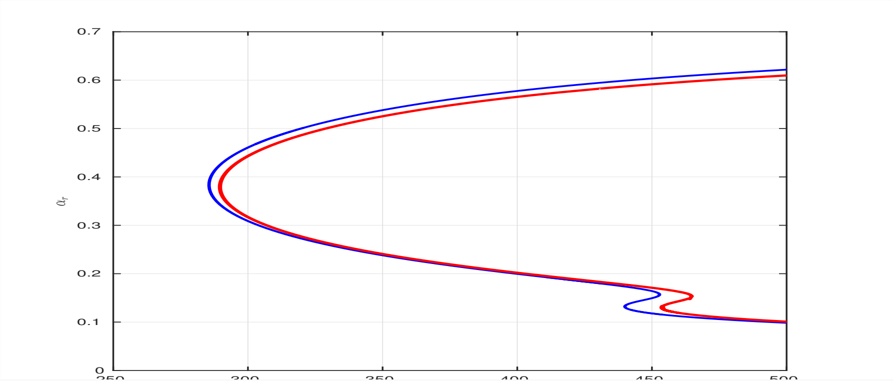
Steady Type I

Neutral Curves



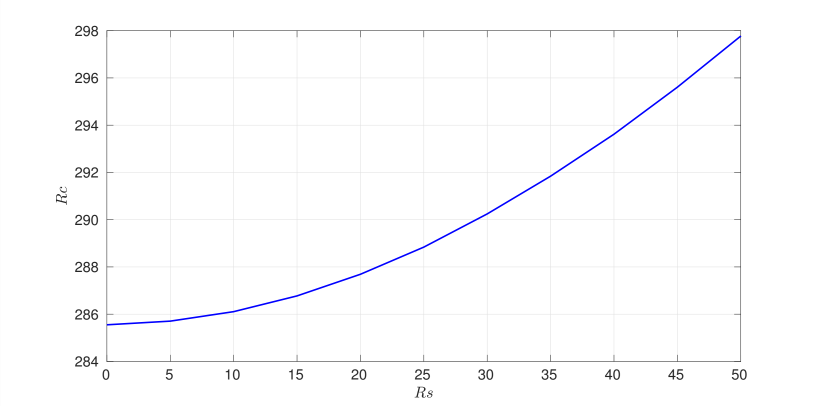
Type I: $Rs = 20$, $\varphi = 25$

Neutral Curves



Full curve: $Rs = 20$, $\varphi = 25$

Neutral Curves



Critical Type I R_c for $\varphi = 25$ and $Rs \in [0, 50]$

Floquet Results

- Clear reduction in spatial growth rates for *small Rs*.

Floquet Results

- Clear reduction in spatial growth rates for *small Rs*.
- *Larger Rs being explored currently - preliminary results indicate intricate behaviour in neutral curves when* $U_w > 1$ *if*

$$G(0,\tau) = 1 + U_w \cos\left(\frac{\varphi}{R_k}\right)$$

Approaches

2. **Linear DNS**

Linear DNS

- Measure the response of the flow to a *stationary* disturbance.

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- Wall motion of the form

$$\zeta(r, \theta, \tau) = e^{-\lambda r^2} e^{in\theta}$$

Linear DNS

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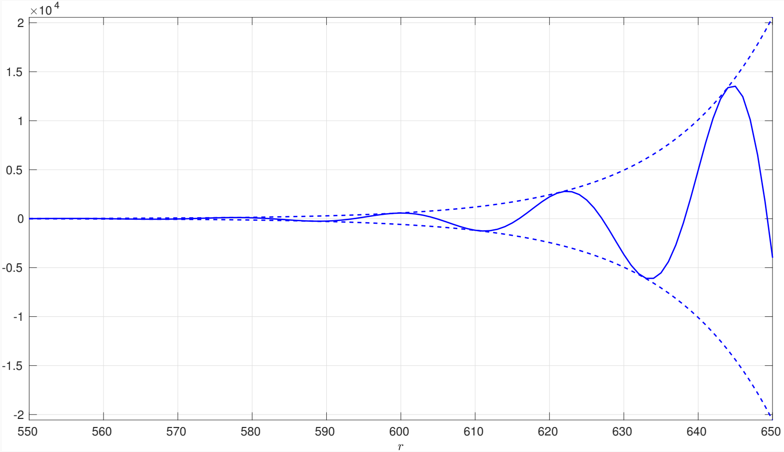

$u(r, z = 0, \tau > T_c)$ - *steady:* $R_k = 500$, $n = 32$

Linear DNS

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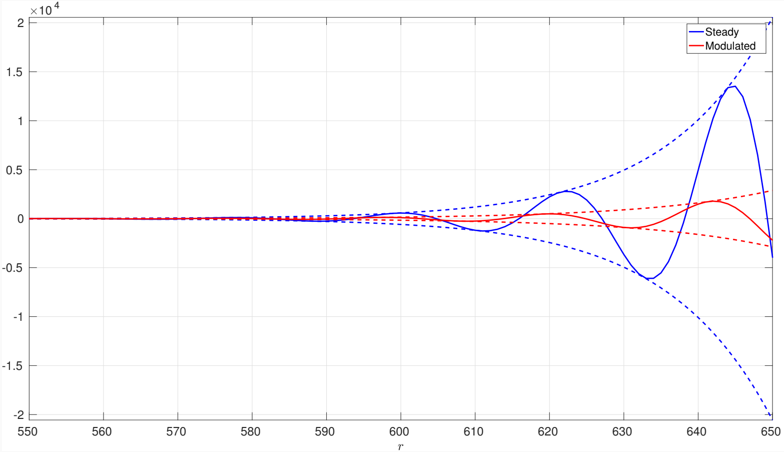

$u(r, z = 0, \tau > T_c)$ - *modulated:* $R_k = 500,\ n = 32,\ Rs = 50,\ \varphi = 25$

DNS

- Normal mode approximation:

$$\mathbf{u}(r, \theta, z, \tau) = u(z, \tau)e^{i\alpha r}e^{in\theta}$$

DNS

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- Given $A = u(r, z=0, \tau > T_c)$, we can calculate:

$$\alpha_i \simeq \frac{-i}{A}\frac{\partial A}{\partial r}$$

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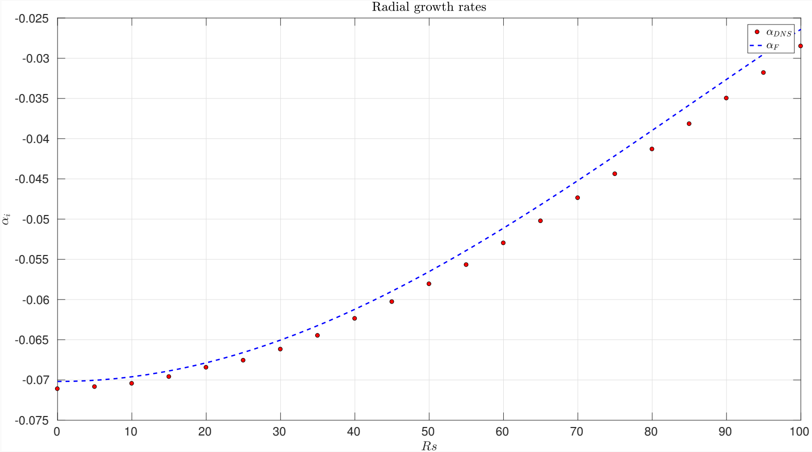
- $-\alpha_i$ gives radial growth rate.

DNS vs. Floquet



Radial Growth Rates: $R_k = 500$, $n = 32$, $\varphi = 25$, $Rs \in \{0, 100\}$

DNS vs. Floquet

DNS

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Floquet

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$$\sum_{k=-K}^{K} \mathcal{L}_k \{\mu, \alpha\} e^{ik\tau} = 0$$

- Specify μ or α as real and solve for the other.

Current & Future Work

- Look at parallels between oscillation and surface roughness.

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- Experimental confirmation.

Current & Future Work

- Look at parallels between oscillation and surface roughness.
- Experimental confirmation.
- Explore torsional oscillations.

Thank You

Approaches

3 **Frozen Flow Analysis**

Frozen Flow Analysis

- Freeze flow, treat τ as parameter:

$$p(r, \theta, z, \tau) = \hat{p}(z; \tau) e^{i\alpha r} e^{in\theta} e^{-i \int^{\tau} \omega}$$

where $\hat{\phi}$ is slowly varying.

Frozen Flow Analysis

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- Dispersion relation:

$$\mathcal{D}(\alpha, \omega; n, R_k, R_s, \varphi, \tau) = 0 \tag{1}$$

Frozen Flow Analysis

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- Specify α and solve (1) for $\omega(\tau)$ at each time-step.

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- Specify ω and solve (1) for $\alpha(\tau)$ at each time-step.

Frozen Flow Analysis

- $\frac{1}{T}\int_0^T \alpha(\tau) \approx \alpha_F$

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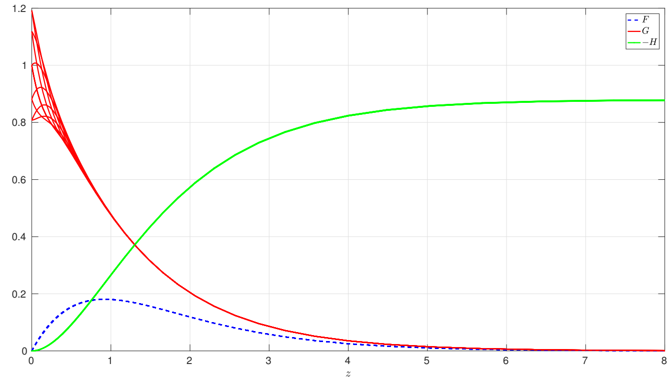
- $\frac{1}{T}\int_0^T \alpha(\tau) \approx \alpha_F$
- $\frac{1}{T}\int_0^T \omega(\tau) \approx \omega_F$

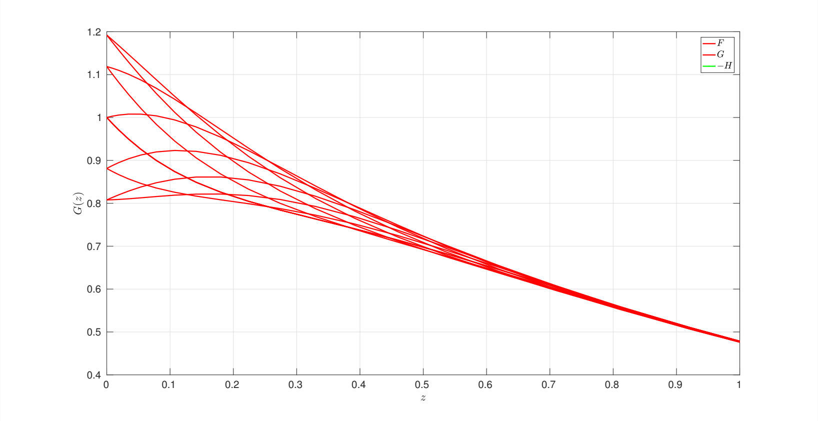
Typical Mean Flow Variation



Base flow variation for $Rs = 10$, $\varphi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

Typical Mean Flow Variation



Azimuthal variation near wall for $Rs = 10$, $\varphi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

Connections with the Stokes Layer

- Write

$$\begin{aligned} \mathbf{U}^T &= \mathbf{U}^S + \mathbf{U}^M \\ &= \mathbf{U}^S + \epsilon \mathbf{U}_1 + \mathcal{O}(\epsilon^2) \end{aligned}$$

Connections with the Stokes Layer

- Write

$$\mathbf{U}^T = \mathbf{U}^S + \mathbf{U}^M$$
$$= \mathbf{U}^S + \epsilon \mathbf{U}_1 + \mathcal{O}(\epsilon^2)$$

- High frequency modulation suggests primary lengthscale:

$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

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- High frequency modulation suggests primary lengthscale:

$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

- Rescale

$$\tilde{z} = \frac{z}{\delta_s}, \qquad \tilde{\tau} = \frac{\varphi}{R}\tau$$

Connections with the Stokes Layer

- Near the wall we have

$$\left(\frac{1}{\delta^2}\right)\frac{\partial F}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)F'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$\left(\frac{1}{R\delta^2}\right)\frac{\partial G}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)G'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$H \sim \delta F$$

with

$$F(0,\tilde{\tau}) = H(0,\tilde{\tau}) = 0, \qquad G(0,\tilde{\tau}) = \cos(\tilde{\tau})$$
$$F \to 0 \quad G \to 0 \quad \text{as} \quad \tilde{z} \to \infty$$

Connections with the Stokes Layer

- To dominant balance:

$$\begin{aligned} \frac{\partial F}{\partial \tilde{\tau}} &= F'', \quad \text{with} \quad F(0) = F(\tilde{z} \to \infty) = 0 \\ \frac{1}{R}\frac{\partial G}{\partial \tilde{\tau}} &= G'', \quad \text{with} \quad G(0) = cos(\tilde{\tau}), \;\; G(\tilde{z} \to \infty) = 0 \end{aligned}$$

Connections with the Stokes Layer

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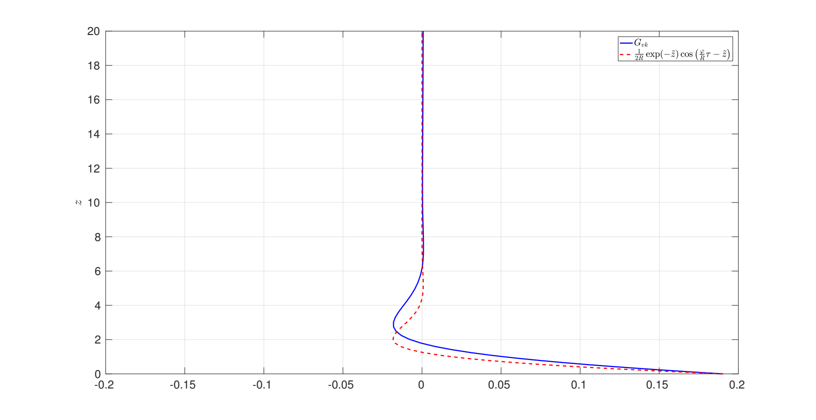
$$\frac{1}{R}\frac{\partial G}{\partial \tilde{\tau}} = G'', \qquad \text{with} \qquad G(0) = cos(\tilde{\tau}), \;\; G(\tilde{z} \to \infty) = 0$$

- Which gives Stokes layer:

$$F = 0$$
$$G = \frac{1}{R}\exp(-\tilde{z})\cos(\tilde{\tau} - \tilde{z})$$

Connections with the Stokes Layer



Comparison between Stokes layer profile and base flow variation

DNS - Wall Motion

- Wall displacement for stationary forcing (steady, rotating frame):

$$\zeta(r, \theta, \tau) = e^{\lambda r^2} e^{in\theta}$$

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$$\theta \to \theta_0 + \int^{\tau} \Omega(\tilde{\tau})$$

- Forcing stationary with respect to modulated disk:

$$\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$$

DNS - Stationary Forcing

- Prescribe wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$

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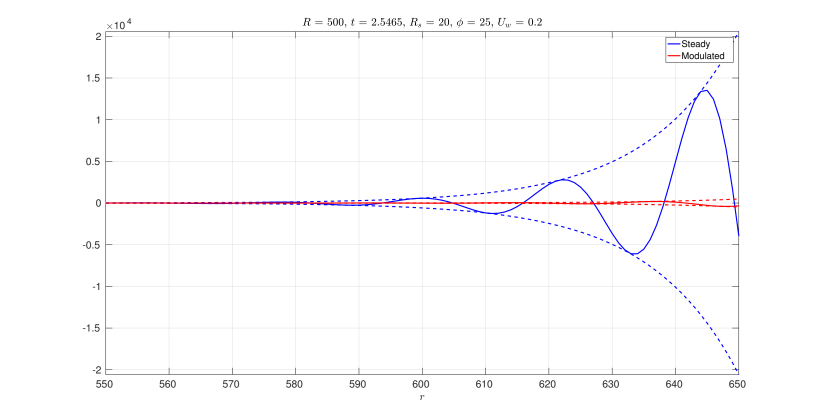
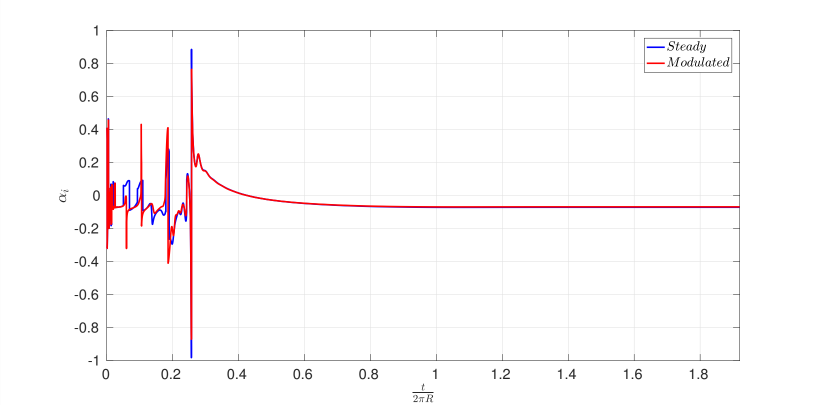


Radial evolution: $R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

- Receptivity issues.

DNS - Stationary Forcing

- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ at fixed r.



$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

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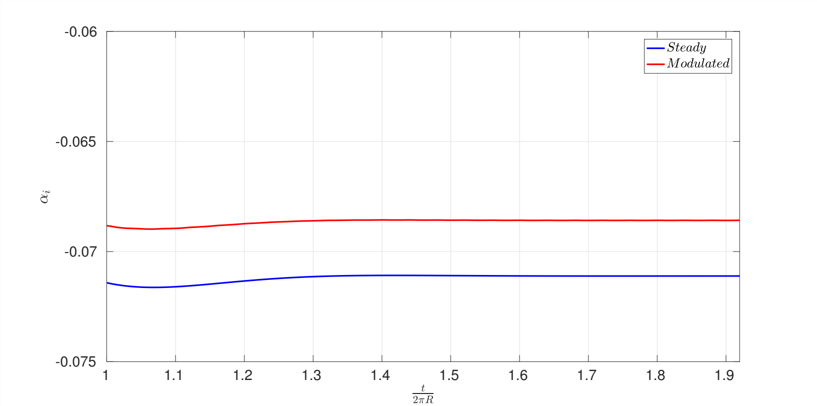
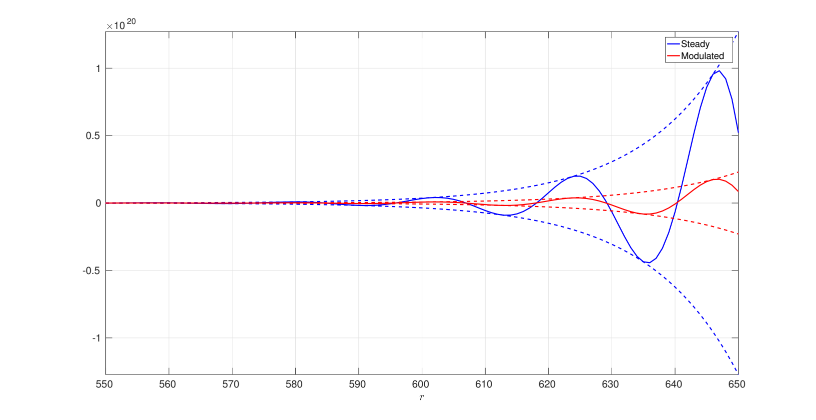


$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

DNS - Stationary Forcing

- Exponential growth reconstructed from $e^{i\alpha r}$



$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

DNS - Results

- Exactly prescribe (α, μ, ψ) from Floquet theory at inflow.

$$\psi(r, \theta, z, \tau) = \hat{\psi}(z, \tau) e^{\mu\tau} e^{i\alpha r} e^{in\theta}$$

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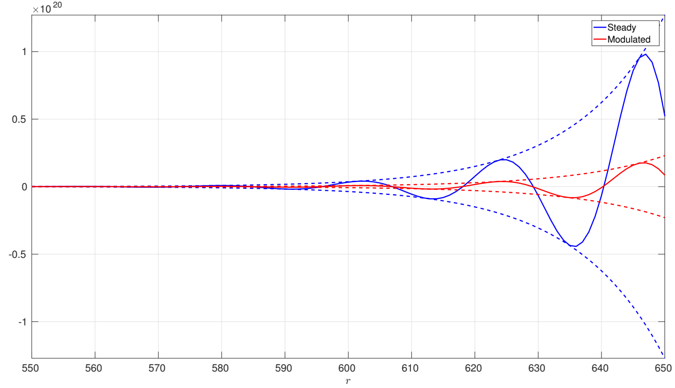

Radial evolution: $R = 500$, $n = 32$, $Rs = 20$, $\varphi = 25$

DNS - Results

- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ for fixed r.

Stationary Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0686i
	$\varphi = 50$	0.2817 - 0.0702i

Inflow Prescribed Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0685i
	$\varphi = 50$	0.2818 - 0.0702i

DNS - Impulsive Forcing

- Prescribe impulsive wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{-\sigma t^2}$

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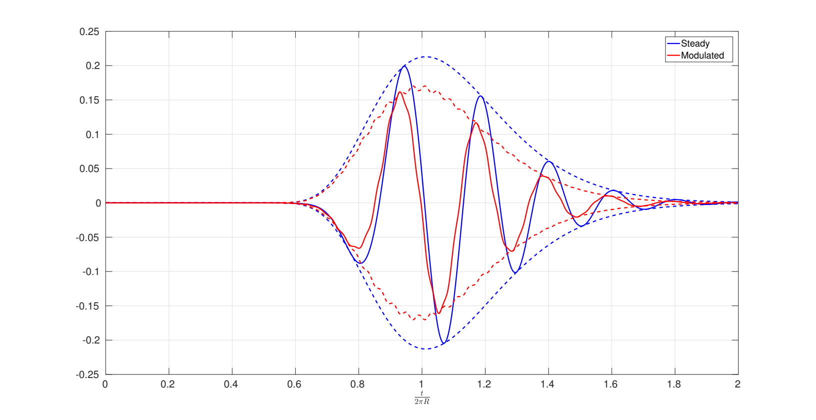

Temporal evolution: $R = 350$, $n = 32$, $Rs = 20$, $\varphi = 25$