Nested Design

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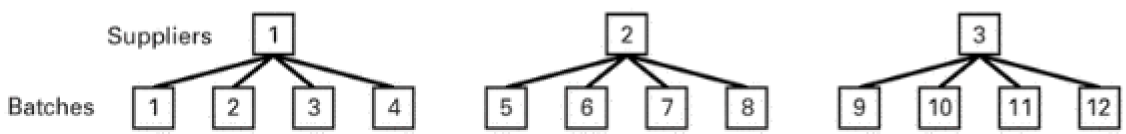
Introduction

It is also known as a hierarchical design. It is a multifactor experiment design that often has one or more random factors. Is similar to crossed design except:

1) Under each fixed level (i) of A, B has b_i levels
2) The levels of B under the same level of A are comparable
3) Under a level of A, the levels of B can be arbitrarily numbered



■ **FIGURE 14.2 Alternate layout for the two-stage nested design**

Figure 1: Nested Layout

Statistical Model

$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{k(ij)}$, $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b$, $k = 1, 2, \ldots, n$

Bracket notation represents nested factor. Cannot include interaction. Factors may be random or fixed. Can use EMS algorithm to derive test.

Sum of Squares

$SS_A = bn\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2 = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$

$SS_{B(A)} = n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..})^2 = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2$

$SS_E = \sum\sum\sum(y_{ijk} - \bar{y}_{ij.})^2 = \sum\sum\sum y_{ijk}^2 - \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2$

$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{\bar{y}_{...}^2}{abn}$

■ **TABLE 14.2**

Analysis of Variance Table for the Two-Stage Nested Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bn \sum (\bar{y}_{i..} - \bar{y}_{...})^2$	$a - 1$	MS_A
B within A	$n \sum \sum (\bar{y}_{ij.} - \bar{y}_{i..})^2$	$a(b - 1)$	$MS_{B(A)}$
Error	$\sum \sum \sum (y_{ijk} - \bar{y}_{ij.})^2$	$ab(n - 1)$	MS_E
Total	$\sum \sum \sum (y_{ijk} - \bar{y}_{...})^2$	$abn - 1$	

Figure 2: Nested ANOVA

Estimates/Assumptions/Tests

Two-Factor Nested Model with Fixed Effects

Assumptions: $\sum_{i=1}^{a} \tau_i = 0$; $\sum_{j=1}^{b} \beta_{j(i)} = 0$ for each i

Estimates: $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$; $\hat{\beta}_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..}$

Tests: MS_A/MS_E for $\tau_i = 0$; $MS_{B(A)}/MS_E$ for $\beta_{j(i)} = 0$

Two-Factor Nested Model with Random Effects

Assumptions: $\tau_i \sim N(0, \sigma_\tau^2)$; $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$

Estimates: $\hat{\sigma}_\tau^2 = (MS_A - MS_{B(A)})/nb$; $\hat{\sigma}_\beta^2 = (MS_{B(A)} - MS_E)/n$

Tests: $MS_A/MS_{B(A)}$ for $\sigma_\tau^2 = 0$; $MS_{B(A)}/MS_E$ for $\sigma_\beta^2 = 0$

Two-Factor Nested Model with Mixed Effects

Assumptions: $\sum_{i=1}^{a} \tau_i = 0$; $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$

Estimates: $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$; $\hat{\sigma}_\beta^2 = (MS_{B(A)} - MS_E)/n$

Tests: MS_A/MS_E for $\tau_i = 0$; $MS_{B(A)}/MS_E$ for $\sigma_\beta^2 = 0$

■ **TABLE 14.1**

Expected Mean Squares in the Two-Stage Nested Design

$E(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$E(MS_A)$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$E(MS_{B(A)})$	$\sigma^2 + \frac{n\sum\sum \beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$E(MS_E)$	σ^2	σ^2	σ^2

Figure 3: Two-Stage Nested EMS

$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_{k(ij)} + \epsilon_{(ijk)l} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

Figure 4: Three-Stage Nested Model

■ TABLE 14.8

Analysis of Variance for the Three-Stage Nested Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bcn \sum_i (\bar{y}_{i...} - \bar{y}_{....})^2$	$a - 1$	MS_A
B (within A)	$cn \sum_i \sum_j (\bar{y}_{ij..} - \bar{y}_{i...})^2$	$a(b - 1)$	$MS_{B(A)}$
C (within B)	$n \sum_i \sum_j \sum_k (\bar{y}_{ijk.} - \bar{y}_{ij..})^2$	$ab(c - 1)$	$MS_{C(B)}$
Error	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{ijk.})^2$	$abc(n - 1)$	MS_E
Total	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{....})^2$	$abcn - 1$	

■ TABLE 14.9

Expected Mean Squares for a Three-Stage Nested Design with A and B Fixed and C Random

Model Term	Expected Mean Square
τ_i	$\sigma^2 + n\sigma_\gamma^2 + \dfrac{bcn \sum \tau_i^2}{a - 1}$
$\beta_{j(i)}$	$\sigma^2 + n\sigma_\gamma^2 + \dfrac{cn \sum \sum \beta_{j(i)}^2}{a(b - 1)}$
$\gamma_{k(ij)}$	$\sigma^2 + n\sigma_\gamma^2$
$\epsilon_{l(ijk)}$	σ^2

Figure 5: Three-Stage Nested Tables