Split-Plot Design

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12/29/2022

Introduction

A multifactor factorial design in which the order of the runs can't be completely randomized. Often times there are blocks or replicates that are subdivided into whole plots and are further subdivided into subplots or split-plots. Generally the hard-to-change factor is assigned to the whole plots.

Factors are crossed, randomization restrictions are different that completely randomized, information on factor effects from two levels (or strata), more ppower for main subplot effect and interaction, and randomized factorial design is more powerful if feasible.

Split-plot can be considered as two superimposed blocked designs:

a) A: whole-plot factor(a); B: sub-plot factor (b); r replicates
b) RCBD_A: number of trt: a, number of blk: r
c) RCBD_B: number of trt: b, number of blk: ra

For whole-plots, subdivision to smaller sub-plots are ignored. For sub-plots, whole-plots are considered blocks.

TABLE 14.16
The Experiment on the Tensile Strength of Paper

	Replicate 1			Replicate 2			Replicate 3		
Pulp Preparation Method	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**
Temperature (°F)									
200	30	34	29	28	31	31	31	35	32
225	35	41	26	32	36	30	37	40	34
250	37	38	33	40	42	32	41	39	39
275	36	42	36	41	40	40	40	44	45

Figure 1: Split-Plot Layout

Statistical Model I: All terms included

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijk},$

$i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, a;\ k = 1, 2, \ldots, b$

Assumptions

r_i: block effects (random) $\sim N(0, \sigma_\tau^2)$

β_j: whole-plot factor (A) main effects (fixed)

$(\tau\beta)_{ij}$: whole-plot error (random) $\sim$ normal with $\sigma^2_{\tau\beta}$

γ_k: sub-plot factor (B) main effects (fixed)

$(\tau\gamma)_{ik}$: block-B interaction (random) $\sim$ normal with $\sigma^2_{\tau\gamma}$

$(\beta\gamma)_{jk}$: interaction between A and B (fixed)

$(\tau\beta\gamma)_{ijk}$: sub-plot error (random) $\sim$ normal with $\sigma^2_{\tau\beta\gamma}$

ϵ_{ijk}: random error $\sim N(0, \sigma^2)$

Expected Mean Square

■ **TABLE 14.17**
Expected Mean Squares for Split-Plot Design

	Model Term	Expected Mean Square
	τ_i	$\sigma^2 + ab\sigma_\tau^2$
Whole plot	β_j	$\sigma^2 + b\sigma^2_{\tau\beta} + \frac{rb\sum \beta_j^2}{a-1}$
	$(\tau\beta)_{ij}$	$\sigma^2 + b\sigma^2_{\tau\beta}$
	γ_k	$\sigma^2 + a\sigma^2_{\tau\gamma} + \frac{ra\sum \gamma_k^2}{(b-1)}$
	$(\tau\gamma)_{ik}$	$\sigma^2 + a\sigma^2_{\tau\gamma}$
Subplot	$(\beta\gamma)_{jk}$	$\sigma^2 + \sigma^2_{\tau\beta\gamma} + \frac{r\sum\sum(\beta\gamma)^2_{jk}}{(a-1)(b-1)}$
	$(\tau\beta\gamma)_{ijk}$	$\sigma^2 + \sigma^2_{\tau\beta\gamma}$
	$\epsilon_{(ijk)h}$	σ^2 (not estimable)

Figure 2: Split-Plot Expected Mean Squares Restricted

Estimates and Tests of Fixed Effects

$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{...}$ for $j = 1, 2, \ldots, a$ has test $\beta_j = 0$, $F_0 = MS_A/MS_{rA}$

$\hat{\gamma}_k = \bar{y}_{..k} - \bar{y}_{...}$ for $k = 1, 2, \ldots, b$ has test $\gamma_k = 0$, $F_0 = MS_B/MS_{rB}$

$(\hat{\beta\gamma})_{jk} = \bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}$ has test $(\beta\gamma)_{jk} = 0$, $F_0 = MS_{AB}/MS_{rAB}$

Statistical Model II: Not all terms included

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\beta\gamma)_{jk} + \epsilon_{ijk}$,

$i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, a$; $k = 1, 2, \ldots, b$

Assumptions

r_i: block effects (random) $\sim N(0, \sigma_\tau^2)$

β_j: whole-plot factor (A) main effects (fixed)

$(\tau\beta)_{ij}$: whole-plot error (random) $\sim$ normal with $\sigma_{\tau\beta}^2$

γ_k: sub-plot factor (B) main effects (fixed)

$(\beta\gamma)_{jk}$: interaction between A and B (fixed)

ϵ_{ijk}: random error $\sim N(0, \sigma^2)$

Expected Mean Square

Factor	***E(MS)***
τ_i (Replicates)	$\sigma_\epsilon^2 + ab\sigma_\tau^2$
β_j (A)	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2 + \dfrac{rb\sum \beta_j^2}{a-1}$
$(\tau\beta)_{ij}$	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2$ (whole-plot error)
γ_k (B)	$\sigma_\epsilon^2 + \dfrac{ra\sum \gamma_k^2}{ab-1}$
$(\beta\gamma)_{jk}$ (AB)	$\sigma_\epsilon^2 + \dfrac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
ϵ_{ijk}	σ_ϵ^2 (subplot error)

Figure 3: Split-Plot Expected Mean Squares for Reduced Model