Experiments with Random Factors

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Introduction

Instead of specific set of factor levels chosen for an experiment, the factor levels are chosen at random from a larger population of potential levels. The inference is about the entire population of levels instead of the levels themselves. Often used in measurement system studies.

One Factor Random Effect Model

$y_{ij} = \mu + \tau_i + \epsilon_{ij}$, $i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, n_i$

$\mu =$ grand mean, $\tau_i = i$th treatment effect(random), $\epsilon_{ij} \sim N(0, \sigma^2)$

Because random, $\tau_i \sim N(0, \sigma_\tau^2)$

τ_i and ϵ_{ij} are independent

$\text{Var}(y_{ij}) = \sigma_\tau^2 + \sigma^2$ where σ_τ^2 and σ^2 are called the variance components

Hypotheses in Random Effects (Components of Variance) Model

In fixed effects model, the equality of treatment means is tested. In random effects model, the variance is checked for significance.

$H_0 : \sigma_\tau^2 = 0$ vs. $H_1 : \sigma_\tau^2 > 0$

The ANOVA table is the same as the one found the single factor fixed ANOVA in the ANOVA Introduction.

$E(MS_E) = \sigma^2$, $E(MS_{tr}) = \sigma^2 + n\sigma_\tau^2$

Estimation

$\hat{\sigma}^2 = MS_E$

$\hat{\sigma}_\tau^2 = (MS_{tr} - MS_E)/n$

If unbalanced, use $n_0 = \frac{1}{a-1}(\sum_{i=1}^a n_i - \frac{\sum_{i=1}^a n_i^2}{\sum_{i=1}^a n_i})$

Estimates of σ_τ^2 can be negtive

Statistical Model with Two Random Factors

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$, $i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, b$; $k = 1, 2, \ldots, n$

$\tau_i \sim N(0, \sigma_\tau^2)$, $\beta_j \sim N(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0, \sigma_{\tau\beta}^2)$,

$\mathrm{Var}(y_{ijk}) = \sigma^2 + \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2$

$\mathrm{E}(\mathrm{MS}_A) = \sigma^2 + bn\sigma_\tau^2 + n\sigma_{\tau\beta}^2$,

$\mathrm{E}(\mathrm{MS}_B) = \sigma^2 + an\sigma_\beta^2 + n\sigma_{\tau\beta}^2$,

$\mathrm{E}(\mathrm{MS}_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2$

$\mathrm{E}(\mathrm{MS}_E) = \sigma^2$

Hypothesis Test

$H_0 : \sigma_\tau^2 = 0$ uses $F_0 = MS_A/MS_{AB}$

$H_0 : \sigma_\beta^2 = 0$ uses $F_0 = MS_B/MS_{AB}$

$H_0 : \sigma_{\tau\beta}^2 = 0$ uses $F_0 = MS_{AB}/MS_E$

Variance Components Estimation

$\hat{\sigma}^2 = \mathrm{MS}_E$

$\hat{\sigma}_\tau^2 = (\mathrm{MS}_A$ - $\mathrm{MS}_{AB})/bn$

$\hat{\sigma}_\beta^2 = (\mathrm{MS}_B$ - $\mathrm{MS}_{AB})/an$

$\hat{\sigma}_{\tau\beta}^2 = (\mathrm{MS}_{AB}$ - $\mathrm{MS}_E)/n$

Note: Uses type 1 method which can be negative. RMLE method gives nonnegative results.

Two-Factor Restricted Mixed Effects Model (slightly more general)

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$, $i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, b$; $k = 1, 2, \ldots, n$

$\sum \tau_1 = 0$, $\beta_j \sim N(0, \sigma_\beta^2)$, $(\tau\beta)_{ij} \sim N(0, (a-1)\sigma_{\tau\beta}^2/a)$,

Restriction

$\sum(\tau\beta)_{ij} = 0$ for β level j is an added restriction.

Due to the added restriction,

Not all $(\tau\beta)_{ij}$ are independent. $\mathrm{Cov}((\tau\beta)_{ij},(\tau\beta)_{i'j}) = -\frac{1}{a}\sigma_{\tau\beta}^2$

$\mathrm{Cov}(y_{ijk}, y_{i'j'k'}) = \sigma_\beta^2 - \frac{1}{a}\sigma_{\tau\beta}^2$, $i \neq i'$

E(MS)

$\mathrm{E}(\mathrm{MS}_A) = \sigma^2 + bn \sum \tau_i^2/(a-1) + n\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_B) = \sigma^2 + an\sigma^2_\beta$

$\mathrm{E}(\mathrm{MS}_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_E) = \sigma^2$

Hypotheses Testing

$H_0 : \tau_1 = \tau_2 = ... = 0$ follows $F_0 = MS_A/MS_{AB}$

$H_0 : \sigma^2_\beta = 0$ follows $F_0 = MS_B/MS_E$

$H_0 : \sigma^2_{\tau\beta} = 0$ follows $F_0 = MS_{AB}/MS_E$

Note: F and p-values from SAS output is wrong for fixed variables

Variance Estimates

$\hat{\sigma}^2 = MS_E$

$\hat{\sigma}^2_\beta = (MS_B - MS_E)/an$

$\hat{\sigma}^2_{\tau\beta} = (MS_{AB} - MS_E)/n$

Unrestricted Mixed Model (SAS output)

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$

$\sum \tau_i = 0,\ \beta_j \sim N(0, \sigma^2_\beta),\ (\tau\beta)_{ij} \sim N(0, \sigma^2_{\tau\beta})$

E(MS)

$\mathrm{E}(\mathrm{MS}_A) = \sigma^2 + bn \sum \tau_i^2/(a-1) + n\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_B) = \sigma^2 + an\sigma^2_\beta + n\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_E) = \sigma^2$

Differences Between Restricted and Unrestricted

$\mathrm{E}(\mathrm{MS}_B)$

$\hat{\sigma}^2_\beta = (MS_B - MS_{AB})/an$

Test $H_0 : \sigma^2_\beta = 0$ uses MS_{AB} in the denominator

$\mathrm{Cov}(y_{ijk}, y_{i'j'k'}) = \sigma^2_\beta,\ i \neq i'$

Connection Between Restricted and Unrestricted

$(\bar{\tau\beta})_{.j} = (\sum_i (\tau\beta)_{ij})/a$

$y_{ijk} = \mu + \tau + (\beta_j + (\bar{\tau\beta})_{.j}) + ((\tau\beta)_{ij} - (\bar{\tau\beta})_{.j}) + \epsilon_{ijk}$

Three-Factor Mixed Model (A Fixed)

$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$

$\mathrm{E}(\mathrm{MS}_A) = \sigma^2 + cn\sigma^2_{\tau\beta} + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma} + \frac{bcn\sum \tau_i^2}{a-1}$

$\mathrm{E}(\mathrm{MS}_B) = \sigma^2 + an\sigma^2_{\beta\gamma} + acn\sigma^2_{\beta}$

$\mathrm{E}(\mathrm{MS}_C) = \sigma^2 + an\sigma^2_{\beta\gamma} + abn\sigma^2_{\gamma}$

$\mathrm{E}(\mathrm{MS}_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$

$\mathrm{E}(\mathrm{MS}_{AC}) = \sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$

$\mathrm{E}(\mathrm{MS}_{BC}) = \sigma^2 + an\sigma^2_{\beta\gamma}$

$\mathrm{E}(\mathrm{MS}_{ABC}) = \sigma^2 + n\sigma^2_{\tau\beta\gamma}$

$\mathrm{E}(\mathrm{MS}_E) = \sigma^2$