Balanced Incomplete Block Design

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12/15/2022

Introduction

There are a treatments and b blocks with each block containing k different treatments where $k \leq a$. Each treatment appears in r blocks and each pair of treatments appears together in λ blocks. a, b, k, r, and λ are not independent. $N = ar = bk$ is the total number of runs.

$\lambda(a-1) = r(k-1)$ for any fixed treatment i_0. There are two different ways to count the total numbers of pairs including treatment i_0:

i) $a-1$ possible pairs, each appears in λ blocks so $\lambda(a-1)$

ii) treatment i_0 appears in r blocks. Within each block, there are $k-1$ pairs including i_0, so $r(k-1)$

$b \geq a$ and has a nonorthogonal design.

Statistical Model

$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$, $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$

Assumptions

Additive model (without interaction), not all y_{ij} exist because of incompleteness, sum of treatment τ_i and sum of block β_j are both equal to 0, and nonorthogonality of treatments and blocks. We use Type III Sums of Squares and lsmeans.

Model Estimates (Least squares estimates)

$\hat{\mu} = \frac{y_{..}}{N}$, $\hat{\tau}_i = \frac{kQ_i}{\lambda a}$, $\hat{\beta}_j = \frac{rQ'_j}{\lambda b}$

Where: $Q_i = y_{i.} - \frac{1}{k}\sum n_{ij}y_{.j}$, where $n_{ij} = 1$ if trt i in block j and 0 otherwise. Represents treatment i's total minus block average, and $\sum Q_i = 0$.

$Q'_j = y_{.j} - \frac{1}{k}\sum n_{ij}y_{i.}$

$Var(Q_i) = \frac{(k-1)r}{k}\sigma^2$

$Var(\hat{\tau}_i) = (\frac{k}{\lambda a})^2 Var(Q_i) = \frac{k(a-1)}{\lambda a^2}\sigma^2$

$Var(\hat{\tau}_i - \hat{\tau}_j) = \frac{2k\sigma^2}{\lambda a}$

Sum of Squares

$SS_T = \sum \sum y_{ij}^2 - y_{..}^2/N$

$SS_{Block} = \frac{1}{k} \sum y_{.j}^2 - y_{..}^2/N$

$SS_{Treatment(adjusted)} = k \sum Q_i^2/\lambda a = \frac{\lambda a}{k} \sum \hat{\tau}_i^2$

TABLE 4.23
Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\frac{k \sum Q_i^2}{\lambda a}$	$a - 1$	$\frac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\frac{1}{k} \sum y_{.j}^2 - \frac{y_{..}^2}{N}$	$b - 1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N - a - b + 1$	$\frac{SS_E}{N-a-b+1}$	
Total	$\sum \sum y_{ij}^2 - \frac{y_{..}^2}{N}$	$N - 1$		

Figure 1: Balanced Incomplete Block Design Table

Reject if $F_0 > F_{\alpha, a-1, N-a-b+1}$

Mean Tests and Contrasts

Must compute adjusted means (lsmeans) $\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i$.

Standard Error of adjusted mean: $\sqrt{MS_E(\frac{k(a-1)}{\lambda a^2} + \frac{1}{N})}$

Contrasts based on adjusted treatment totals

Contrast: $\sum c_i \mu_i$, $\sum c_i = 0$

Estimate: $\sum c_i \hat{\tau}_i = \frac{k}{\lambda a} \sum c_i Q_i$

Sum of Squares: $SS_C = \frac{k(\sum_{i=1}^a c_i Q_i)^2}{\lambda a \sum_{i=1}^a c_i^2}$

Pairwise Comparison

$\tau_1 - \tau_j$

1) Bonferroni: $CD = t_{\alpha/2m, ar-a-b+1} \sqrt{MS_E \frac{2k}{\lambda a}}$

2) Tukey: $CD = \frac{q_\alpha(a, ar-a-b+1)}{\sqrt{2}} \sqrt{MS_E \frac{2k}{\lambda a}}$