Factorial Design Introduction II

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Post ANOVA Analysis

Multiple Comparisons With No Interactions

Factor level means can be compared to draw conclusions on their effects on response.

$Var(\bar{y}_{i..}) = \frac{\sigma^2}{nb} Var(\bar{y}_{.j.}) = \frac{\sigma^2}{na}$

For A/rows: $Var(\bar{y}_{i..} - \bar{y}_{i'..}) = \frac{2\sigma^2}{nb}$

For B/columns: $Var(\bar{y}_{.j.} - \bar{y}_{.j'.}) = \frac{2\sigma^2}{na}$

Tukey's Method

For Rows: $CD = \frac{q_\alpha(a, ab(n-1))}{\sqrt{2}} \sqrt{MSE \frac{2}{nb}}$

For Columns: $CD = \frac{q_\alpha(b, ab(n-1))}{\sqrt{2}} \sqrt{MSE \frac{2}{na}}$

Bonferroni Method

$CD = t_{\alpha/2m, ab(n-1)} S.E.$ where S.E. depends on whether for rows/columns

Multiple Comparisons With Interactions

$\mu_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij}$ vs. $\mu_{i'j'} = \mu + \tau_{i'} + \beta_{j'} + (\tau\beta)_{i'j'}$

$\hat{\mu}_{ij} = \bar{y}_{ij.}$ and $\hat{\mu}_{i'j'} = \bar{y}_{i'j'.}$

$Var(\bar{y}_{ij.} - \bar{y}_{i'j'.}) = \frac{2\sigma^2}{n}$

There are ab treatment means and $m_0 = \frac{ab(ab-1)}{2}$ pairs

Tukey's method: $CD = \frac{q_\alpha(ab, ab(n-1))}{\sqrt{2}} \sqrt{MSE \frac{2}{n}}$

Bonferroni method: $CD = t_{\alpha/2m, ab(n-1)} \sqrt{MSE \frac{2}{n}}$

Quantitative vs. Qualitative

Fit response curves if need to interpret quantitatively

General Factorial Design and Model

Introduction

Includes all possible level combinations. Has a levels for Factor A, b levels for Factor B, etc. Is a straightforward ANOVA if all fixed effects. Need $n > 1$ replications to test for all possible interactions.

Three Factor Model

Has $nabc$ observations

Statistical Model

$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + \tau\gamma_{ik} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl},$
$i = 1, 2, \ldots, a,\ j = 1, 2, \ldots, b,\ k = 1, 2, \ldots, c,\ l = 1, 2, \ldots, n,$

Sum of Squares

$SS_A = \frac{1}{bcn} \sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn}$
$SS_B = \frac{1}{acn} \sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn}$
$SS_C = \frac{1}{abn} \sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn}$
$SS_{AB} = \frac{1}{cn} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij..}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B = SS_{Subtotals(AB)} - SS_A - SS_B$
$SS_{AC} = \frac{1}{bn} \sum_{i=1}^{a} \sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_C = SS_{Subtotals(AC)} - SS_A - SS_C$
$SS_{BC} = \frac{1}{an} \sum_{j=1}^{b} \sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn} - SS_B - SS_C = SS_{Subtotals(BC)} - SS_B - SS_C$
$SS_{ABC} = \frac{1}{n} \sum \sum \sum y_{ijk.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$
$= SS_{Subtotals(ABC)} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$
$SS_E = SS_T - SS_{Subtotals(ABC)}$
$SS_T = \sum \sum \sum \sum y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$

Additional Notes

Check usual assumptions and diagnostics.

- Normality, homogeneous variance, additivity

For multiple comparisons, extends off of the two-factor case.

Higher order interactions are negligible.

Pooled together with error (increase df_E).

■ TABLE 5.12

The Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn \sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn \sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn \sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn \sum \sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn \sum \sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an \sum \sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n \sum \sum \sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

Figure 1: Three-Factor Table

ANOVA Table

Blocked Factorial Experiment

Also known as complete block factorial design. Interactions between blocks and treatment effects are assumed to be negligible.

Model

$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \epsilon_{ijk}$, $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b$, and $k = 1, 2, \ldots, n$ where δ_k is the effect of the kth block.

■ **TABLE 5.20**

Analysis of Variance for a Two-Factor Factorial in a Randomized Complete Block

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	F_0
Blocks	$\frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}$	$n-1$	$\sigma^2 + ab\sigma_\delta^2$	
A	$\frac{1}{bn}\sum_i y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	$\frac{1}{an}\sum_j y_{.j.}^2 - \frac{y_{...}^2}{abn}$	$b-1$	$\sigma^2 + \frac{an\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$\frac{1}{n}\sum_i\sum_j y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	Subtraction	$(ab-1)(n-1)$	σ^2	
Total	$\sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$		

Figure 2: Two-Factor Block Table

Notes:

(new)SS_E = (old)SS_E - SS_{Blocks}

ANOVA estimate for the variance of the blocks: $\sigma_\delta^2 = \frac{MS_{Blocks} - MS_E}{ab}$