Assignment 3

Aleksandr Salo

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1 Textbook exercises and problems

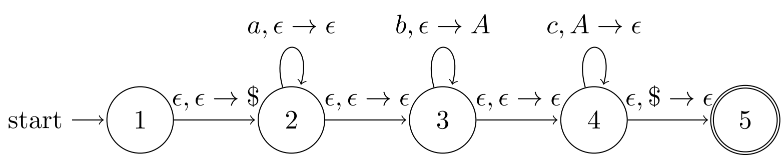
2.2 Do exercise 2.2 from your textbook. You may assume the result from example 2.36, and you may also assume that the class of CFLs is closed under union.

a Let $A = \{a^m, b^n, c^n | m, n \geq 0\}$ and $B = \{a^n, b^n, c^m | m, n \geq 0\}$ together with Example 2.36 to show that the class of context-free languages is not closed under intersection.

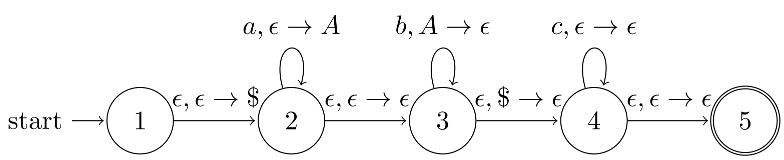
Proof: (by construction and contradiction)

1. The language is CFL if there exists a PDA that recognizes it. Let us show that languages A and B are CFL.

   (a) PDA_A:

   (b) PDA_A:

2. Let C be the intersection $A \cap B$. C must contain equal number of b's and c's as well as equal number of a's and b's. This could happen only if C has equal number of a's, b's and c's. Thus $C = \{a^n, b^n, c^n | n \geq 0\}$

3. By example 2.36 we know that C is not a CFL.

b Use part (a) and DeMorgans law (Theorem 0.20) to show that the class of context-free languages is not closed under complementation.

Proof: (by contradiction)

1. Assume that the class of CFLs is closed under complementation (for the sake of the following contradiction).
2. Let A and B be CFLs. Then we can conclude that $A \cup B$ is a CFL, because the class of CFLs is closed under union.
3. As proved in the theorem 0.20 in the textbook:
   for any two sets A and B: $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (DeMorgan Law).
4. By the assumption at (1) we derive that languages $\overline{A}$ and $\overline{B}$ are CFLs.
5. As proved in part **a**, the class of CFLs is not closed under intersection. Hence $\overline{A} \cap \overline{B}$ not necessarily a CFL.
6. That is a contradiction: $\overline{A} \cap \overline{B}$ must be a CFL for the class of CFLs be closed under complementation.
7. Thus assumption in (1), which led us to contradiction, is not valid. Hence the class of CFLs is **NOT** closed under complementation

2.30 (a): Let $L = \{0^n 1^n 0^n 1^n | n \geq 0\}$. Using the P.L. show that L is not CFL.

Proof: (by contradiction)

1. Assume L is a regular language (for the sake of the following contradiction).
2. Let p be the pumping length for L.
3. Let choose $s = 0^p 1^p 0^p 1^p$ $(|s| \geq p, s \in L)$.
4. Let us show that there is no way to divide s into 5 parts (assume $s = uvxyz$) in a way that fulfills the pumping lemma conditions.
   (a) By third pumping lemma condition $|vxy| < p, vxy$ must not contain 0's, 1's or both. Importantly, by the same condition, vxy can not contain more than one change of symbol (0 to 1 or 1 to 0) and must be withing one of the following boundaries:

$$\overbrace{\underbrace{0^k}_{\text{u}}\underbrace{0^{p-k}1^m}_{\text{vxy}}\underbrace{1^{p-m}0^p1^p}_{\text{z}}}^{\text{uvxyz}} or \overbrace{\underbrace{0^p1^k}_{\text{u}}\underbrace{1^{p-k}0^m}_{\text{vxy}}\underbrace{0^{p-m}1^p}_{\text{z}}}^{\text{uvxyz}} or \overbrace{\underbrace{0^p1^p0^k}_{\text{u}}\underbrace{0^{p-k}1^m}_{\text{vxy}}\underbrace{1^{p-m}}_{\text{z}}}^{\text{uvxyz}} \quad 0 \leq k = m \leq p$$

   Note, that case of xvy containing only 0's or 1's is trivial and specifies any of the other cases mentioned above.

   (b) By second pumping lemma condition $|vy| > 0$, vy must contain at least one 1 or 0.
   (c) By first pumping lemma condition $\forall i \geq 0, uv^ixy^iz \in L$.
   Consider uv^2xy^2z which changed the number of 0's or 1's in one part of the string without changing the number of 0's and 1's in other part of the string. Illustratively:

$$\overbrace{\underbrace{0^k}_{\text{u}}\underbrace{0^{p-k+a}1^{m+b}}_{\text{vxy}}\underbrace{1^{p-m}0^p1^p}_{\text{z}}}^{\text{uvxyz}} or \overbrace{\underbrace{0^p1^k}_{\text{u}}\underbrace{1^{p-k+a}0^{m+b}}_{\text{vxy}}\underbrace{0^{p-m}1^p}_{\text{z}}}^{\text{uvxyz}} or \overbrace{\underbrace{0^p1^p0^k}_{\text{u}}\underbrace{0^{p-k+a}1^{m+b}}_{\text{vxy}}\underbrace{1^{p-m}}_{\text{z}}}^{\text{uvxyz}} \quad a, b \geq 0, a+b > 0$$

   Applying conditions on a, b, k, m we see, that either:
   i. number of 0's in the first line not equal the number of 0's in the second line:

$$k + p - k + a = p + a \neq p$$

   ii. number of 1's in the first line not equal the number of 1's in the second line:

$$m + b + p - m = p + b \neq p$$

   One of these cases must occur. Thereby, string uv^2xy^2z is not in L.
5. Thus, there is no way to divide $s = 0^p 1^p 0^p 1^p$ into 5 parts that satisfy the pumping lemma conditions. This is contradiction. Hence L must not be regular.