Assignment 4

Aleksandr Salo

Due October 6, 2014

1 Textbook exercises

3.16 Show that the collection of Turing-recognizable languages is closed under the operation of **star.**

Proof (by construction).

1. Let L_{TR} be the collection of Turing-recognizable languages.
2. Let us first prove that L_{TR} is closed under concatenation.
   (a) Let A, B be T.R. languages in L_{TR}. The concatenation of A, B is the language $AB = \{ab|a \in A, b \in B\}$.
   (b) Since A, B are T.R. languages there exist machines M_A, M_B that recognize them. Let us now construct the machine M_{AB} that recognizes AB.
   (c) M_{AB} = "on input w,
      i. Partition w into strings ab.
      ii. Run M_A on a and M_B on b.
      iii. If both M_A and M_B accept, ACCEPT.
      iv. Otherwise, REJECT.
   (d) Such a TM accepts if finds a suitable partition ab to feed into machines that recognize them. Note that there is only a finite number of ways to partition the string hence we can try all the possibilities in finite time. Thus, L_{TR} is closed under concatenation.
3. Now we can proceed with the star operation, which is a unary operation that works by attaching (concatenating) any number of strings in L (including ϵ) together to get a string in the new language. *(Sipser, p. 45).* Yet we just proved, that L_{TR} is closed under concatenation. Hence L_{TR} must be also closed under star operation.

4.7 Let β be the set of all infinite sequences over {0,1}. Show that β is uncountable using a proof by diagonalization.

Proof (by contradiction)

1. The set β is countable if either it is finite or it has the same size as $\mathbb{N}$ *(Sipser 5.14)*. That implies, that in order to proof that set β is **uncountable** we have to show that there is no way to set up a correspondence between β and $\mathbb{N}$.
2. Let assume (for the sake of the following contradiction), that there exists a correspondence f between β and $\mathbb{N}$.
3. Now we can show that f fails to work as it should. Let $f(1) = 010101...$, $f(2) = 000111...$, $f(3) = 110110...$, $f(4) = ...$, and so on, just to make up some values for f. The following table shows a few values of a hypothetical correspondence f between β and $\mathbb{N}$.:

n	$f(n)$
1	**0**10101...
2	0**0**0111...
3	11**0**110...
4	000**1**00...
5	0101**0**0...
$\vdots$	$\vdots$

4. Now let us construct such an **x**, which this imaginable table does not contain. Let the i^{th} digit of **x** be the different from i^{th} digit of **f(n)**. Given sequences in the table, we make x = 11101...
5. Using this construction we ensure that x would not be contained in the table, because we know that x is not $f(n)$ for any n because it differs from $f(n)$ in the n^{th} fractional digit.
6. Thus the imaginable correspondence f fails to work and that is a contradiction that allows us to conclude that β, the set of all infinite sequences over {0,1}, is **uncountable** infinite set.

4.13 Let $A = \{< R, S > \mid$ R and S are regular expressions and $L(R) \subseteq L(S)\}$. Show that A is decidable.

Proof (by construction):

1. The language is decidable if there exists a Turing Machine that decides it.
2. Let us construct such a TM that will decide the language A:
3. M = "on input $< R, S >$,
   (a) if $\overline{L(S)} \cap L(R) = \emptyset$, ACCEPT
   (b) otherwise, REJECT
4. This algorithm will accept if and only if L(R) does **not** contains anything that is not in L(S).
5. To check whether the language is equal to empty set or not we can apply the fact, that regular expressions are equivalent with finite automata in their descriptive power *(Sipser, Thm 1.65)*. Yet we know, that E_{DFA} is decidable (Sipser Thm. 4.4).
6. Note, that if a language is described by a regular expression, then it is regular *(Sipser Thm. 1.55)*. Yet class of the R.L. is closed under intersection and complementation. *(Sipser Thm. 1.45)*.
7. That proves that constructed machine would work and hence the language A is decidable.