Assignment 0

Aleksandr Salo

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Simple exercises

1. Find the value x that maximizes $f(x) = -3x^2 + 24x - 30$
   $f'(x) = -6x + 24$
   $f_{max} : -6x + 24 = 0$
   $x = 4$
   $f''(x) = -6 < 0$
   hence $x = 4$ is a value that maximize f

2. Find the partial derivatives of $g(x)$ with respect to x_0 and x_1: $g(x) = 3x_0^3 - 2x_0x_1^2 + 4x_1 - 8$
   $\frac{\partial g}{\partial x_0} = 9x_0^2 - 2x_1^2$
   $\frac{\partial g}{\partial x_1} = -4x_0x_1 + 4$

3. What is the value of $AB^T + C^{-1}$, if the following define A, B, and C? Use Matlab to check your answer

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

   Transpose B; invert C using Cramer's rule:

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} C^{-1} = \frac{1}{1*2 - 0*0} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB^T + C^{-1} = \begin{bmatrix} 3*1 + 4*2 + 5*3 & 3*4 + 4*5 + 5*6 \\ 6*1 + 7*2 + 8*3 & 6*4 + 7*5 + 8*6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

   Simplifying:

$$\begin{bmatrix} 26 & 62 \\ 44 & 107 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 62 \\ 44 & 107.5 \end{bmatrix}$$

4. Write down the mathematical definitions of the simple Gaussian, multivariate Gaussian, Bernoulli, binomial and exponential distributions.

   Gaussian (Normal) distribution:

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Multivariate Gaussian distribution:

$$f(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

Bernoulli distribution:

$$f(n) = \begin{cases} 1-p & : n = 0 \\ p & : n = 1 \end{cases}$$

Binomial distribution:

$$P_p(n|N) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

Exponential distribution:

$$f(x) = \lambda \exp^{-\lambda x}$$

5. What is the relationship between the Bernoulli and binomial distributions?
   The binomial distribution gives the discrete probability distribution $P_p(n|N)$ of obtaining exactly n successes out of N Bernoulli trials (each with p of success).
   In other words, given sequence of N independent events what is the probability that n would turn to be success?

6. Suppose that random variable $X \sim N(1, 3)$. What is its expected value?
   Answer: 1. Since $N(1, 3)$ denotes normal distr with $\mu = 1$.

7. Suppose that random variable Y has distribution:

$$p(Y = y) = \begin{cases} \exp(-y) & : y \geq 0 \\ 0 & : otherwise \end{cases}$$

- Verify that $\int_{y=-\infty}^{\infty} p(Y = y) = 1$

$$\int_0^\infty exp^{-x} = \lim_{a \to \infty} -\exp^{-x} \binom{a}{0}$$

$$\lim_{a \to \infty} -1(\exp^{-a} - \exp^0)$$

$$\lim_{a \to \infty} -1(\frac{1}{\exp^\infty} - 1) = 1$$

  Thus $\int_{y=-\infty}^{\infty} p(y)$ equals 1 for y ≥ 0, and trivially 0 otherwise; thus infinite integral equal 1.

- Mean:

$$\mu_Y = \int_{y=-\infty}^{\infty} p(y) y dy = \int_0^\infty \frac{y}{\exp^y} = 1$$

- Var:

$$\sigma^2 = Var[Y] = \int_{y=-\infty}^{\infty} p(y)(y - \mu_Y)^2 dy = \int_0^\infty \frac{(y - \mu_Y)^2}{\exp^y} = 1$$

8. What is $E[Y|Y \geq 10]$?

$$E[Y|Y \geq 10] = \int_{y=10}^{\infty} p(y) dy = -(y+1)e^{-y} \binom{\infty}{10} = 0 - (-11e^{-10}) = 0.000499399$$

1 Formatting example

Verify that $\int_{y=-\infty}^{\infty} p(Y = y) = 1$

$$\begin{aligned}
\int_{y=-\infty}^{\infty} p(Y = y) &= \int_{-\infty}^{0} 0 \, dy + \int_{0}^{\infty} e^{-y} \, dy \\
&= -e^{-y} \Big|_{0}^{\infty} \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$