Assignment #5

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T5.1 A time series X_t is a **harmonic process** if

$$X_t = \sum_{j=1}^{M} \{A_j \cos(2\pi t\omega_j) + B_j \sin(2\pi t\omega_j)\}, \quad t \in \mathcal{Z},$$

where, for all $j = 1, 2, \ldots, M$, $\mathrm{E}[A_j] = \mathrm{E}[B_j] = 0, \mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2, \mathrm{Cov}(A_j, B_j) = 0$, and $\omega_j \in [0, 0.5]$.

(a) For simplicity, let $M = 1$. Prove that a harmonic process is second-order stationary.

(b) Letting $M = 1$ again, is the spectral density function $f(\omega)$ of the harmonic process an absolutely continuous function of ω? *Hint:* What is the condition on R_ν for $f(\omega)$ to be absolutely continuous. *Warning:* Do not use the spectral distribution function $F(\omega)$ in your argument.

FYI: Letting $M = 1$ (for simplicity), the spectral distribution function $F(\omega)$ has a jump discontinuity at ω_1 and $1 - \omega_1$. In particular,

$$F(\omega) = \begin{cases} 0 & \omega \in [0, \omega_1) \\ 0.5\sigma^2 & \omega \in [\omega_1, 1-\omega_1) \\ \sigma^2 & \omega \in [1-\omega_1, 1], \end{cases}$$

If the harmonic process is of sinusoids with frequencies $\omega_1, \omega_2, \ldots, \omega_M$, then F will have jumps of sizes $\sigma_1^2/2, \sigma_2^2/2, \ldots, \sigma_M^2/2$ at frequencies $\omega_1, \omega_2, \ldots, \omega_M$. If $\omega \neq \omega_j$, then F is differentiable (with respect to ω) and has derivative zero; while at $\omega = \omega_j$, the left derivative is infinite, and the right derivative is zero.

T5.2 A time series X is a moving average process of order q, coefficients $\beta = (\beta_1, \ldots, \beta_q)^T$, and error variance σ^2 if

$$X_t = \sum_{j=0}^{q} \beta_j \varepsilon_{t-j}, \qquad t \in \mathcal{Z},$$

where $\beta_0 = 1$ and $\varepsilon_t \sim \mathrm{WN}(\sigma^2)$. The autocovariance function R_ν of X is

$$R_\nu = \begin{cases} \sigma^2 \sum_{k=0}^{q-|\nu|} \beta_k \beta_{k+|\nu|} & |\nu| = 0, 1, \ldots, q \\ 0 & |\nu| = q+1, q+2, \ldots. \end{cases} \tag{1}$$

Use the Univariate Filter Theorem to prove R_ν is as given in (1).

T5.3 Use long division to find the coefficients of an MA(∞) representation of an ARMA(1, 1) process having AR coefficient α and MA coefficient β; that is, divide by $1 + \beta z$ by $1 + \alpha z$. Do the division in such a way that only nonnegative powers of z are in the answer. What happens for large powers of z if $|\alpha| > 1$?

C5.1 Use the `masp` function with `Q = 256` frequencies to find the square modulus of the frequency transfer function of the 12-th difference filter. The input to `masp` will be that of an MA(12) with all coefficients equal to zero, except $\beta_{12} = -1$. Use the `plotsp` function to plot the natural logarithm of the square modulus (don't forget to compute the variance of the filter). Does the result correspond to what we learned about the effect of differencing?

C5.2 What four sets of MA(2) parameters (coefficients and error variances) can lead to the autocovariances $R_0 = 1.3125, R_1 = 0.625$, and $R_2 = 0.25$. Which of these sets leads to a characteristic polynomial having all of its zeros outside the unit circle? Which of these sets does function `corrma` return? *Hint:* See your notes on the nonidentifiability of the moving average process.

C5.3 If X is an AR(1) process with $\alpha = 0.05$, write X_t as a function of X_{t-2} and some ε's and of X_{t+2} and some εs.

C5.4 Which of the following sets of coefficients can be regarded as the coefficients of an AR process?

(a) $\alpha = (-0.90, 0.80)$.

(b) $\alpha = (-1.54, 0.40)$.

(c) $\alpha = (1.0, -0.904, -0.70)$.

(d) $\alpha = (-1.22, 1.16, -0.70)$.

(e) $\alpha = (-0.09, -1.5884, -0.024, 0.90)$.

C5.5* Use the `idarma` function to find an AR, an MA, and an ARMA process of each of the following kinds.

(a) An excess of high frequency

(b) An excess of low frequency

(c) A single peak in the spectral density function

Once you see a process with the desired characteristic, print the plots, and write the model.

* - The `idarma` function is still being debugged. If it it not ready by Wednesday, November 18th, this part of the assignment will be suitably modified.

Models for "Time Series Models" Section in Notebook

I. White noise

   A. Prove second-order stationarity
   B. Derive autocovariance function
   C. Describe partial autocorrelation function
   D. Derive spectral density function

II. Random walk

   A. Disprove second-order stationarity

III. Harmonic

   A. Prove second-order stationarity
   B. Derive autocovariance function
   C. Describe partial autocorrelation function
   D. Show spectral density function is not absolutely continuous
   E. Give expression for cumulative spectral distribution function.

IV. Moving average of order q with coefficients $\beta_0, \beta_1, \ldots, \beta_q$, and error variance σ^2

   A. Prove second-order stationarity
   B. Derive autocovariance function
   C. Describe partial autocorrelation function
   D. Derive spectral density function; show $f(\omega)$ can be written as a trigonometric polynomial of degree q

V. Autoregressive of order p with coefficients $\alpha_0, \alpha_1, \ldots, \alpha_p$, and error variance σ^2

   A. State conditions for second-order stationarity
   B. Derive autocovariance function for $p = 1$
   C. Describe partial autocorrelation function for any p (not just $p = 1$)
   D. Give expression for spectral density function; show $f(\omega)$ can be written as the reciprocal of a trigonometric polynomial of degree p.

VI. Autoregressive moving average of orders p and q with coefficients $\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_q$, and error variance σ^2

   A. State conditions for second-order stationarity
   B. Derive autocovariance function for $p = 1$
   C. Describe partial autocorrelation function
   D. Give expression for spectral density function; show $f(\omega)$ can be written as a rational trigonometric polynomial

As the last page of this section, include a table that summarizes the behavior of the models. Below are is a start of the table, with the columns, and a few examples of rows.

Process	R_ν or ρ_ν	θ_ν	$f(\omega)$
White noise	0 for all ν	0 for all ν	0 for all ω
$\vdots$	$\vdots$	$\vdots$	$\vdots$
MA(q)	0 for ν	exponential decay for $\nu > 0$	q-th degree trigonometric polynomial
$\vdots$	$\vdots$	$\vdots$	$\vdots$