1 White Noise

WN(σ^2):

$$X_t = \varepsilon_t \sim WN(\sigma^2)$$

A. WN is second-order stationary TS:

1. $E[X_t] = 0$
2. $R_v = Cov(X_t, X_t + v) = \sigma^2 \delta_v = \begin{cases} \sigma^2 & v = 0 \\ 0 & otherwise \end{cases}$

In another words mean is a constant and autocovariance is the function of the lag alone.

B. Autocorrelation function (ACF):

Since we define white noise to be independently distributed $Corr(\varepsilon_s, \varepsilon_t) = 0$, if $s \neq t$, we have that $Cov(\varepsilon_s, \varepsilon_t) = 0$, if $s \neq t$. In other words $R_v = \sigma^2 \delta_v$. Then ACF is given by:

$$\rho_v = \frac{R_v}{R_0} = \frac{\sigma^2 \delta_v}{\sigma^2} = \begin{cases} 1 & v = 0 \\ 0 & otherwise \end{cases}$$

C. Partial autocorrelation function (PACF):
Since the ACF is zero everywhere except the $lag = 0$, the PACF simply copies the behavior of PACF.

D. Spectral Density function:

$$\begin{aligned} f(w) &= \sum_{v=-\infty}^{\infty} R_v cos(2\pi v \omega) \\ &= 0 + R_0 cos(2\pi v \omega) + 0 = R_0 \\ &= \sigma^2 \end{aligned}$$

2 Random Walk

RW:

$$X_t = \varepsilon_t + X_{t-1}$$

Properties: X_0 is initial value.

1. $E[X_0] = \mu_x$
2. $Var(X_0) = \sigma^2 < \infty$
3. $Cov(X_0, \varepsilon_t) = 0$ (X_0 is independent of ε_t).

A. RW is first-order (mean) stationary TS:

$$\begin{aligned}
E[X_t] &= E[\varepsilon_t + X_{t-1}] \\
&= E[\varepsilon_t + (\varepsilon_{t-1} + X_{t-2})] \\
&= E[\sum_{j=1}^{t} \varepsilon_j + X_0] \\
&= E[X_0] = \mu_x
\end{aligned}$$

B. RW **is not** second-order (covariance) stationary TS:

$$\begin{aligned}
Var(X_t) &= Var(X_0 + \sum_{j=1}^{t} \varepsilon_j) \\
&= Var(X_0) + Var(\sum_{j=1}^{t} \varepsilon_j) - Cov(X_0, \sum_{j=1}^{t} \varepsilon_j) \\
&= \sigma_x^2 + \sum_{j=1}^{t} \sigma^2 \\
&= \sigma_x^2 + t\sigma^2 \\
Cov(X_t, X_{t+v}) &= E[X_t X_{t+v}] - E[X_t]E[X_{t+v}] \\
&= E[(X_t)(X_t + \sum_{j=1}^{v} \varepsilon_j)] - \mu_x^2 \\
&= E[X_t^2] + E[X_t \sum_{j=1}^{v} \varepsilon_j] - \mu_x^2 \\
&= (Var(X_t) + \mu_x^2) + E[(X_0 + \sum_{j=1}^{t} \varepsilon_j) \sum_{j=t+1}^{v} \varepsilon_j] - \mu_x^2 \\
&= Var(X_t) + E[X_0 \sum_{j=t+1}^{v} \varepsilon_j] + E[\sum_{j=1}^{t} \varepsilon_j \sum_{j=t+1}^{v} \varepsilon_j] \\
&= \sigma_x^2 + t\sigma^2
\end{aligned}$$

Since covariance depends on time, RW is not covariance stationary TS.

3 Harmonic

$$X_t = \sum_{j=1}^{M} \{A_j cos(2\pi t\omega_j) + B_j sin(2\pi t\omega_j)\}, t \in \mathbb{Z}$$

Properties: $j = 1..M$, $\omega_j \in [0, 0.5]$

1. $E[A_j] = E[B_j] = 0$
2. $Corr(A_j, B_j) = 0$
3. $Var(A_j) = Var(B_j) = \sigma_j^2$

A. Harmonic is second-order stationary TS:

1. The first fact directly follows from (1) $E[A_j] = E[B_j] = 0$:

$$\begin{aligned} E[X_t] &= E[Acos(2\pi t\omega) + Bsin(2\pi t\omega)] && (1)\\ &= 0 && (2) \end{aligned}$$

   Since 0 is a constant.

2. Let $s = t + v$ and let $\phi = 2\pi\omega_j$, then for $M = 1$:

$$\begin{aligned} R_v &= Cov(X_t, X_{t+v})\\ &= E[(X_t - \mu_{X_t})(X_{t+v} - \mu_{X_{t+v}})]\\ &= E[X_t X_{t+v}] - \mu_{X_t}\mu_{X_{t+v}}\\ &= E[X_t X_{t+v}]\\ &= E[(Acos(\phi t) + Bsin(\phi t))(Acos(\phi t + \phi v) + Bsin(\phi t + \phi v))]\\ &= E[A^2 cos(\phi t)cos(\phi t + \phi v) + B^2 sin(\phi t)sin(\phi t + \phi v)\\ &+ AB(cos(\phi t)sin(\phi t + \phi v) + sin(\phi t)cos(\phi t + \phi v))]\\ &= E[A^2 cos(\phi t)cos(\phi t + \phi v) + B^2 sin(\phi t)sin(\phi t + \phi v)]\\ &= \sigma^2 E[cos(\phi t)cos(\phi t + \phi v) + sin(\phi t)sin(\phi t + \phi v)]\\ &= \sigma^2 E[cos(\phi t - \phi t - \phi v)]\\ &= \sigma^2 E[cos(-\phi v)] = \sigma^2 E[cos(\phi v)] = \sigma^2 E[cos(2\pi v\omega_1)]\\ &= \sigma^2 cos(2\pi v\omega_1) \end{aligned}$$

Since mean is a constant and autocovariance is the function of the lag alone, harmonic is the covariance stationary TS.

B. Autocorrelation function (ACF):

$$\begin{aligned} \rho_v &= \frac{R_v}{R_0} = \frac{\sigma^2 cos(2\pi v\omega_j)}{\sigma^2}\\ &= cos(2\pi v\omega_j) \end{aligned}$$

Thus the ACF of the harmonic is the harmonic itself.

C. Partial autocorrelation function (PACF):
PACF is 1 at $v = 0$; then it goes to almost -1 at $v = 1$; then PACF approaches the zero rapidly and stays close to zero thereafter. We believe, that PACF of the harmonic process is itself a harmonic which decays to zero.

D. Spectral Density function:

From the two-spectral representations theorem we know, that the following must be true in order for $f(\omega)$ to be absolutely continuous:

$$\sum_{v=-\infty}^{\infty} |R_v| < \infty$$

Letting M = 1 again and applying the condition:

$$\begin{aligned}\sum_{v=-\infty}^{\infty} |R_v| &= \sum_{v=-\infty}^{\infty} |\sigma^2 cos(2\pi v\omega)| \\ &= \sigma^2 \sum_{v=-\infty}^{\infty} |cos(2\pi v\omega)| \\ &= \infty\end{aligned}$$

The sum above is infinite since we sum up non-negative terms, some of them guaranteed to be positive, and we do it infinite number of times, knowing that $\sigma^2 > 0$. Thus the spectral density function $f(\omega)$ of the harmonic process **is not** an absolutely continuous function of ω.

More generally, for any M we have:

$$\sum_{v=-\infty}^{\infty} |R_v| = \sum_{v=-\infty}^{\infty} \left|\sum_{j=1}^{M} \sigma^2 cos(2\pi v\omega)\right|$$

Which could never converge since the variance is always positive and cosine is oscillating back and forth.

D. Cumulative Spectral Distribution function:

Letting $M = 1$ (for simplicity), the spectral distribution function $F(\omega)$ has a jump discontinuity at ω_1 and $1 - \omega_1$:

$$F(\omega) = \begin{cases} 0, & \omega \in [0, \omega_1) \\ 0.5\sigma^2, & \omega \in [\omega_1, 1 - \omega_1) \\ \sigma^2, & \omega \in [1 - \omega_1, 1] \end{cases}$$

In general, break in continuity occurs at frequencies $\omega_1, \omega_2, ..., \omega_M$ and F has jumps of sizes $\sigma_1^2/2, \sigma_2^2/2, ..., \sigma_M^2/2$.

4 Moving Average

MA(q, β, σ^2) is moving average of order q with coefficients $\beta_0, \beta_1, .., \beta_q$, and error variance σ^2:

$$X_t = \varepsilon_t + \beta_1\varepsilon_{t-1} + \beta_2\varepsilon_{t-2} + ... + \beta_q\varepsilon_{t-q}$$
$$X_t = \sum_{j=0}^{q} \beta_j\varepsilon_{t-j}$$

where $\beta_0 = 1$ and $\varepsilon_t \sim WN(\sigma^2)$.

A. MA is second-order stationary TS.

Consider X_t as filtered version of $Y_t = \varepsilon_t \sim WN(\sigma^2)$:

$$X_t = \sum_{j=-\infty}^{\infty} \beta_j Y_{t-j},\, t \in \mathbb{Z},\, R_{Y,v} = \sigma^2\delta_v$$

Since Y_t is covariance stationary, then, according to Univariate Filter Theorem (UFT), its filtered version X_t is covariance stationary as well.

B. Autocovariance function and ACF:

According to UFT, we can express the autocovariance function of X_t as:

$$\begin{aligned}
R_{X,v} &= \sum_{k=-\infty}^{\infty} R_\beta(k)R_{Y,v-k} \\
&= \sum_{k=-\infty}^{\infty} R_\beta(k)\sigma^2\delta_{v-k} \\
&= \sigma^2 R_\beta(v) \\
&= \sigma^2 \sum_{j=-\infty}^{\infty} \beta_j\beta_{j+|v|} \\
&= \sigma^2 \sum_{j=0}^{q} \beta_j\beta_{j+|v|} \\
&= \sigma^2 \sum_{j=0}^{q-|v|} \beta_j\beta_{j+|v|} \\
&= \begin{cases} \sigma^2 \sum_{j=0}^{q-|v|} \beta_j\beta_{j+|v|} & |v| = 0, 1, 2, ..., q \\ 0 & |v| = q+1, q+2, ... \end{cases}
\end{aligned}$$

ACF is zero for all lags after q:

$$\rho_v = 0,\, \forall |v| = q+1, q+2...$$

C. Partial Autocorrelation function (PACF): PACF is zero for all lags after q:

$$\rho_v = 0, \text{ if } v = q+1, q+2...$$

D. Spectral Density Function:

By UFT for $\omega \in [0, 1]$:

$$
\begin{aligned}
f(\omega) &= |h(e^{2\pi i\omega})|^2 f_{\varepsilon_t}(w) \\
&= \sigma^2 |h(e^{2\pi i\omega})|^2 \\
&= \sigma^2 |\sum_{k=0}^{q} \beta_k e^{2\pi i\omega}|^2
\end{aligned}
$$

Using Two-Spectral Representation Theorem:

$$
\begin{aligned}
f(\omega) &= \sum_{v=-\infty}^{\infty} R_v e^{2\pi i v\omega} \\
&= \sum_{v=-q}^{q} R_v cos(2\pi v\omega) \\
&= R_0 + 2\sum_{v=1}^{q} R_v cos(2\pi v\omega)
\end{aligned}
$$

Let $\Theta = 2\pi\omega$:

$$
\begin{aligned}
cos(v\Theta) &= 2cos(\Theta)cos((v-1)\Theta) - cos((v-2)\Theta) \\
&= 2cos(\Theta)\{2cos(\Theta)cos((v-2)\Theta) - cos((v-3)\Theta)\} - \{2cos(\Theta)cos((v-3)\Theta) - cos((v-4)\Theta)\} \\
&= 2^2 cos^2 cos((v-2)\Theta) - 2cos(\Theta)cos((v-3)\Theta) - 2cos(\Theta)cos((v-3)\Theta) + cos((v-4)\Theta) \\
&= 2^2 cos^2 cos((v-2)\Theta) - 4cos(\Theta)cos((v-3)\Theta) + cos((v-4)\Theta)
\end{aligned}
$$

Thus $f(\omega)$ is q^{th}-degree polynomial of $cos(\Theta)$.

5 Auto Regressive

AR(p, α, σ^2) is auto regressive process of order p with coefficients $\alpha_0, \alpha_1, .., \alpha_p$ and error variance σ^2:

$$X_t + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ... + \alpha_p X_{t-p} = \varepsilon_t$$
$$\sum_{j=0}^{p} \alpha_j X_{t-j} = \varepsilon_t$$

where $\alpha_0 = 1$ and $\varepsilon_t \sim WN(\sigma^2)$, $t \in \mathbb{Z}$.

A. Conditions for second-order stationarity.

If **none of the zeros** of $g(z)$ are equal to 1 in modulus, then X_t is covariance stationary and, as a limit in mean squares, we can write a doubly-infinite MA representation of the AR TS:

$$X_t = \sum_{j=-\infty}^{\infty} \beta_j \varepsilon_{t-j}$$

where β are the coefficients of $h(z) = \frac{1}{g(z)}$.

B. Autocovariance (for order $p = 1$).

Our definition for the TimeSeries class makes the condition stricter: **all zeros** of $g(z)$ are outside of the unit circle (more than 1 in modulus), which allows to have an $MA(\infty)$ representation:

$$X_t = \sum_{j=0}^{\infty} \beta_j \varepsilon_{t-j}$$

And yields the Yuler-Walker equations (YWE):

Consider autocovariance R_v for $AR(1)$:

$$\sum_{j=0}^{1} \alpha_j R_{j-v} = \delta_v \sigma^2$$

Which yields the system of equations:

$$\begin{cases} \sum_{j=0}^{1} \alpha_j R_j = R_0 + \alpha R_1 = \sigma^2 & v = 0 \\ \sum_{j=0}^{1} \alpha_j R_{j-1} = R_0 + \alpha R_1 = 0 & v = 1 \end{cases}$$

Then we can express:

$$R_1 = -\alpha R_0$$
$$R_0 = \frac{\sigma^2}{1 - \alpha^2}$$

Now generalize:

$$\begin{aligned} R_v &= -\alpha R_{v-1} \\ &= -\alpha(-\alpha R_{v-2}) \\ &= (-\alpha)^v R_0 \\ R_v &= \frac{\sigma^2 (-\alpha)^{|v|}}{1 - \alpha^2} \end{aligned}$$

Which allows to express the ACF, which exponentially decays to zero, but never reaches it:

$$\begin{aligned}\rho_v &= \frac{R_v}{R_0}\\ &= \frac{\frac{\sigma^2(-\alpha)^{|v|}}{1-\alpha^2}}{\frac{\sigma^2}{1-\alpha^2}}\\ &= \frac{\sigma^2(-\alpha)^{|v|}}{1-\alpha^2}\frac{1-\alpha^2}{\sigma^2}\\ \rho_v &= (-\alpha)^{|v|}\end{aligned}$$

C. Partial Autocorrelation function (PACF):

Let us consider AR of order p:

$$X_t + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ... + \alpha_p X_{t-p} = \varepsilon_t$$

This is a stochastic difference equation. Depending on the roots of the indicial polynomial and the initial conditions the solution would have the parts of the following forms (for simplicity of representation let's use p = 2):

$$R_v = \begin{cases}(\tau_1 + \tau_2 v)(z^*)^v & \text{one root two times}\\ \tau_1(z_1^*)^v + \tau_2(z_2^*)^v & \text{two distinct roots}\\ \gamma cos(v\theta + \delta)|z^*|^v & \text{conjurate pair } a \pm bi\end{cases}$$

For each of this cases, since $|z_{1,2}^*| < 1$ we have that $R_{v\to\infty} \to 0$, which have different shape however.

Thus PACF of AR process of any order exponentially decays to zero: $\theta_{v\to\infty} \to 0$

D. Spectral Density.

Spectral density is given by UFT:

$$f(w) = \sigma^2 |\frac{1}{g(e^{2\pi i\omega})}|^2$$

which could be written as a rational trigonometric polynomial.

6 Auto Regressive Moving Average

ARMA$(p, q, \alpha, \beta, \sigma^2)$ is auto regressive moving average process of order p, q with coefficients $\alpha_0, \alpha_1, .., \alpha_p$ and $\beta_0, \beta_1, .., \beta_q$, and error variance σ^2:

$$X_t + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ... + \alpha_p X_{t-p} = \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2} + ... + \beta_q \varepsilon_{t-q}$$
$$\sum_{j=0}^{p} \alpha_j X_{t-j} = \sum_{j=0}^{q} \beta_j \varepsilon_{t-j}$$

where $\alpha_0 = 1, \beta_0 = 1$ and $\varepsilon_t \sim WN(\sigma^2)$, $t \in \mathbb{Z}$.

ARMA process could be written using lag operator L (defined as $L^k X_t := X_{t-k}$) as follows:

$$X_t + \alpha_1 L X_t + \alpha_2 L^2 X_t + ... + \alpha_p L^p X_t = \varepsilon_t + \beta_1 L \varepsilon_t + \beta_2 L^2 \varepsilon_t + ... + \beta_q L^q \varepsilon_t$$
$$g(L) X_t = h(L) \varepsilon_t$$
$$X_t = \frac{h(L)}{g(L)} \varepsilon_t$$

where $g(z) = \sum_{j=0}^{p} \alpha_j z^j$ and $h(z) = \sum_{j=0}^{p} \alpha_j z^p$ are the characteristic polynomials of AR and MA parts, respectively.

A. Conditions for second-order stationarity.
Roots of the characteristic polynomial $g(z) = \sum_{j=0}^{p} \alpha_j z^j$ are outside of unit circle. That condition also ensures we can convert ARMA into MA process by converting AR part into MA part.

Optionally, we want ARMA process to be *invertible* - that is the roots of $h(z) = \sum_{j=0}^{p} \alpha_j z^p$ are outside of unit circle. That allows us to convert ARMA into AR process which has lots of nice theoretical properties that we want to use.

B. Autocovariance and ACF.
ACF satisfies:

$$\begin{cases} \sum_{j=0}^{p} \alpha_j R_{j-v} = \sum_{k=0}^{q} \beta_k R_{v-k}, & |v| = 0, 1, 2, ..., q \\ 0, & |v| = q+1, q+2, ... \end{cases}$$

Where

$$R_{v-k} = Cov(X_t, \varepsilon_{t+v}) = \begin{cases} 0, & v = 1, 2, 3... \\ \sigma^2 \gamma_{-v}^2, & v = 0, -1, -2, ... \end{cases}$$

Which yields:

$$\begin{cases} \sum_{j=0}^{p} \alpha_j R_{j-v} = \sigma^2 \gamma_{-v}^2, & v = 0, -1, -2, \dots, -q \\ \sum_{j=0}^{p} \alpha_j R_{j-v} = 0, & v = 1, 2, 3, \dots, q \\ 0, & |v| = q+1, q+2, ... \end{cases}$$

Here γ are the coefficients of the polynomial $\frac{h(z)}{g(z)}$ since $MA(\infty)$ representation of ARMA is $X_t = \frac{h(L)}{g(L)} \varepsilon_t$

C. PACF.
Since ARMA consists of AR and MA processes, it contains the respective traits of their PACF:

1. PACF of AR exponentially decays to zero: $\theta_{v\to\infty} \to 0$
2. PACF of MA is zero for all lags after q: $\rho_v = 0$, if $v = q+1, q+2...$

Thus the PACF of ARMA is too exponentially decays to zero: $\theta_{v\to\infty} \to 0$

D. Spectral Density.
If ARMA process is invertible, then spectral density is given by:

$$f(w) = \sigma^2 \left| \frac{h(e^{2\pi i\omega})}{g(e^{2\pi i\omega})} \right|^2$$

which is a rational trigonometric polynomial since exp could be written using sin and cos and g, h are the the characteristic and indicial polynomials. :

7 Summary of the Models

Process	Autocovariance R_v	PACF θ_v	Spectral Density $f(\omega), \omega \in [0,1]$
WN(σ^2)	$\sigma^2\delta_v$	$0, \forall v \neq 0$	σ^2
RandomWalk	$\sigma_x^2 + t\sigma^2$	$0, \forall v > 1$	Not covariance-stationary
Harmonic	$\sigma^2 cos(2\pi v\omega_j)$	Harmonic, decays to zero fast	$f(\omega)$ is not continuous. $F(\omega) = \begin{cases} 0, & \omega \in [0, \omega_1) \\ 0.5\sigma^2, & \omega \in [\omega_1, 1-\omega_1) \\ \sigma^2, & \omega \in [1-\omega_1, 1] \end{cases}$
MA(q, β, σ^2)	$\begin{cases} \sigma^2 \sum_{j=0}^{q-\lvert v\rvert} \beta_j\beta_{j+\lvert v\rvert} & \lvert v\rvert = 1, .., q \\ 0 & \lvert v\rvert = q+1, ... \end{cases}$	0 after q (at $\lvert v\rvert = q+1, ...$)	$\sigma^2 \lvert h(e^{2\pi i\omega})\rvert^2$
AR(p, α, σ^2)	$\sum_{j=0}^{p} \alpha_j R_{j-v} = \delta_v\sigma^2$	exponential decay to zero: $\theta_{v\to\infty} \to 0$	$\sigma^2 \lvert \frac{1}{g(e^{2\pi i\omega})}\rvert^2$
ARMA(p, q)	$\begin{cases} \sum_{j=0}^{p} \alpha_j R_{j-v} = \sigma^2\gamma_{-v}^2 & v = 0, .., -q \\ \sum_{j=0}^{p} \alpha_j R_{j-v} = 0 & v = 1, 2, ..., q \\ 0 & \lvert v\rvert = q+1, ... \end{cases}$	exponential decay to zero: $\theta_{v\to\infty} \to 0$	$f(w) = \sigma^2 \left\lvert \frac{h(e^{2\pi i\omega})}{g(e^{2\pi i\omega})} \right\rvert^2$