

Time Series Homework 5

Alex Salo

December 1, 2015

Theoretical Part

T5.1 A time series is a harmonic process if:

$$X_t = \sum_{j=1}^M \{A_j \cos(2\pi t\omega_j) + B_j \sin(2\pi t\omega_j)\}, t \in \mathbb{Z},$$

where, for all $j = 1, 2, \dots, M$:

1. $E[A_j] = E[B_j] = 0$
2. $Var(A_j) = Var(B_j) = \sigma_j^2$
3. $Cov(A_j, B_j) = 0$
4. $\omega_j \in [0, 0.5]$

Problems:

(a) For simplicity, let $M = 1$. Prove that a harmonic process is second-order stationary.

To prove that the process is second-order stationary one needs to show two facts about the process to be true:

1. $E[X_t] = \mu$, that is mean function is a constant
2. $Cov(X_s, X_t) = R_{t-s}$, that is the covariance function of any pair of X s that are the same distance apart in time is the same - in other words that the covariance is the function of lag alone.

The first fact directly follows from (1) $E[A_j] = E[B_j] = 0$:

$$E[X_t] = E[A \cos(2\pi t\omega) + B \sin(2\pi t\omega)] \tag{1}$$

$$= 0 \tag{2}$$

Since 0 is a constant.

To show the second fact true, let $s = t + v$ and let $\phi = 2\pi\omega_j$; let us show that the autocovariance function is the function of the lag only:

$$R_v = Cov(X_t, X_{t+v}) \quad (3)$$

$$= E[(X_t - \mu_{X_t})(X_{t+v} - \mu_{X_{t+v}})] \quad (4)$$

$$= E[X_t X_{t+v}] - \mu_{X_t} \mu_{X_{t+v}} \quad (5)$$

$$= E[X_t X_{t+v}] \quad (6)$$

$$= E\left[\left(\sum_{j=1}^M \{A_j \cos(\phi t) + B_j \sin(\phi t)\}\right)\left(\sum_{j=1}^M \{A_j \cos(\phi t + \phi v) + B_j \sin(\phi t + \phi v)\}\right)\right] \quad (7)$$

$$= E[(A \cos(\phi t) + B \sin(\phi t))(A \cos(\phi t + \phi v) + B \sin(\phi t + \phi v))] \quad (8)$$

$$= E[A^2 \cos(\phi t) \cos(\phi t + \phi v) + B^2 \sin(\phi t) \sin(\phi t + \phi v)] \quad (9)$$

$$+ AB(\cos(\phi t) \sin(\phi t + \phi v) + \sin(\phi t) \cos(\phi t + \phi v))] \quad (10)$$

$$= E[A^2 \cos(\phi t) \cos(\phi t + \phi v) + B^2 \sin(\phi t) \sin(\phi t + \phi v)] \quad (11)$$

$$= \sigma^2 E[\cos(\phi t) \cos(\phi t + \phi v) + \sin(\phi t) \sin(\phi t + \phi v)] \quad (12)$$

$$= \sigma^2 E[\cos(\phi t - \phi t - \phi v)] \quad (13)$$

$$= \sigma^2 E[\cos(-\phi v)] \quad (14)$$

$$= \sigma^2 E[\cos(\phi v)] \quad (15)$$

$$= \sigma^2 E[\cos(2\pi v \omega_1)] \quad (16)$$

$$= \sigma^2 \cos(2\pi v \omega_1) \quad (17)$$

$$(18)$$

To get (9) we used the definition of covariance again:

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[XY], \text{ if } E[X] = 0, E[Y] = 0 \end{aligned}$$

Thus giving:

$$E[A_j B_j] = Cov(A_j, B_j) = 0$$

To get (10) we used the definition of variance:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= E[X^2], \text{ if } E[X] = 0 \end{aligned}$$

Thus giving:

$$E[A^2] = E[B^2] = \sigma^2$$

To get (11) we used the trigonometric identity:

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

To get (13) we used the fact that:

$$\cos(-a) = \cos(a)$$

- (b) Letting $M = 1$ again, is the spectral density function $f(\omega)$ of the harmonic process an absolutely continuous function of ω ? Hint: What is the condition on R_v for $f(\omega)$ to be absolutely continuous.

From the two-spectral representations theorem we know, that the following must be true in order for $f(\omega)$ to be absolutely continuous:

$$\sum_{v=-\infty}^{\infty} |R_v| < \infty$$

For $M = 1$:

$$\begin{aligned} \sum_{v=-\infty}^{\infty} |R_v| &= \sum_{v=-\infty}^{\infty} |\sigma^2 \cos(2\pi v\omega)| \\ &= \sigma^2 \sum_{v=-\infty}^{\infty} |\cos(2\pi v\omega)| \\ &= \infty \end{aligned}$$

The sum above is infinite since we sum up non-negative terms, some of them guaranteed to be positive, and we do it infinite number of times, knowing that $\sigma^2 > 0$. Thus the spectral density function $f(\omega)$ of the harmonic process **is not** an absolutely continuous function of ω .

T5.2 A time series X is a moving average process of order q , coefficients $\beta = (\beta_1, \dots, \beta_q)^T$, and error variance σ^2 if

$$X_t = \sum_{j=0}^q \beta_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},$$

where $\beta_0 = 1$ and $\varepsilon_t \sim WN(\sigma^2)$. Prove that the autocovariance function R_v of X is

$$\begin{cases} \sigma^2 \sum_{j=0}^{q-|v|} \beta_j \beta_{j+|v|} & |v| = 0, 1, 2, \dots, q \\ 0 & |v| = q+1, q+2, \dots \end{cases}$$

1. Consider the definition of the filtered time series: X_t is a filtered version of Y_t if we can write it (as a limit in mean square) as a linear function of Y_t :

$$X_t = \sum_{j=-\infty}^{\infty} \beta_j Y_{t-j}, \quad t \in \mathbb{Z},$$

2. Thus, given $X_t = \sum_{j=0}^q \beta_j \varepsilon_{t-j}$, we recover that the original time series has a form:

$$Y_t = \varepsilon_t \sim WN(\sigma^2)$$

From earlier exercise, we know that the autocovariance of white noise is zero everywhere, except when lag equals zero:

$$R_{\varepsilon_t, v} = \sigma^2 \delta_v$$

3. Since Y_t is covariance stationary, then, according to Univariate Filter Theorem (UFT), its filtered version X_t is covariance stationary as well. Furthermore we can express the autocovariance function of X_t as:

$$R_{X, v} = \sum_{k=-\infty}^{\infty} R_{\beta}(k) R_{\varepsilon_t, v-k} \tag{1}$$

$$= \sum_{k=-\infty}^{\infty} R_{\beta}(k) \sigma^2 \delta_{v-k} \tag{2}$$

$$= \sigma^2 R_{\beta}(v) \tag{3}$$

$$= \sigma^2 \sum_{j=-\infty}^{\infty} \beta_j \beta_{j+|v|} \tag{4}$$

$$= \sigma^2 \sum_{j=0}^q \beta_j \beta_{j+|v|} \tag{5}$$

$$= \sigma^2 \sum_{j=0}^{q-|v|} \beta_j \beta_{j+|v|} \tag{6}$$

$$= \begin{cases} \sigma^2 \sum_{j=0}^{q-|v|} \beta_j \beta_{j+|v|} & |v| = 0, 1, 2, \dots, q \\ 0 & |v| = q+1, q+2, \dots \end{cases} \tag{7}$$

To get (3) we used the fact that:

$$\begin{aligned} R_{\varepsilon_t, v} &= \sigma^2 \delta_v \\ R_{\varepsilon_t, v-k} &= \sigma^2 \delta_{v-k} \\ &= \begin{cases} \sigma^2 & v-k=0 \Rightarrow v=k \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which nullifies the sum everywhere, except when $v = k$ thus leaving only one member.

- To get (5)** we noticed, that the product will be zero when $j < 0$ or $j > q$, since we don't have those coefficients.
- To get (6)** we noticed, that as soon as $j + |v| > q$ coefficients $\beta_j = 0$, thus we can further reduce the upper border of the summation.
- To get (7)** we noticed, that the entire sum becomes zero if when lag is larger then the order of MA process ($|v| > q$), since the larger border of the summation becomes less than the lower border $q - |v| < 0$.

T5.3 Use long division to find the coefficients of an MA(1) representation of an ARMA(1, 1) process having AR coefficient α and MA coefficient β ; that is, divide by $1 + z$ by $1 + z$. Do the division in such a way that only nonnegative powers of z are in the answer. What happens for large powers of z if $|\alpha| > 1$?

Long polynomial division yields the following:

$$\begin{array}{r}
 1 + (\beta - \alpha)z - \alpha(\beta - \alpha)z^2 + \alpha^2(\beta - \alpha)z^3 - \alpha^3(\beta - \alpha)z^4 + \dots \\
 1 + \alpha z \overline{) 1 + \beta z} \\
 \underline{-1 - \alpha z} \\
 (\beta - \alpha)z \\
 \underline{-(\beta - \alpha)z - \alpha(\beta - \alpha)z^2} \\
 -\alpha(\beta - \alpha)z^2 \\
 \underline{+\alpha(\beta - \alpha)z^2 + \alpha^2(\beta - \alpha)z^3} \\
 \alpha^2(\beta - \alpha)z^3 \\
 (\text{cont.}) \dots
 \end{array}$$

Generalizing the pattern in the result of the division:

$$\begin{aligned}
 \text{Coef}(p) &= 1 + (\beta - \alpha)z - \alpha(\beta - \alpha)z^2 + \alpha^2(\beta - \alpha)z^3 - \alpha^3(\beta - \alpha)z^4 + \dots \\
 &= 1 + \beta z - \alpha z - \alpha\beta z^2 + \alpha^2 z^2 + \alpha^2\beta z^3 - \alpha^3 z^3 + \dots \\
 &= (1 + \beta z) - (\alpha z + \alpha\beta z^2) + (\alpha^2 z^2 + \alpha^2\beta z^3) - (\alpha^3 z^3 + \alpha^4\beta z^4) + \dots \\
 &= (1 + \beta z) - \alpha z(1 + \beta z) + \alpha^2 z^2(1 + \beta z) - \alpha^3 z^3(1 + \beta z) + \dots \\
 &= \sum_{j=0}^{\infty} (-\alpha)^j z^j (1 + \beta z)
 \end{aligned}$$

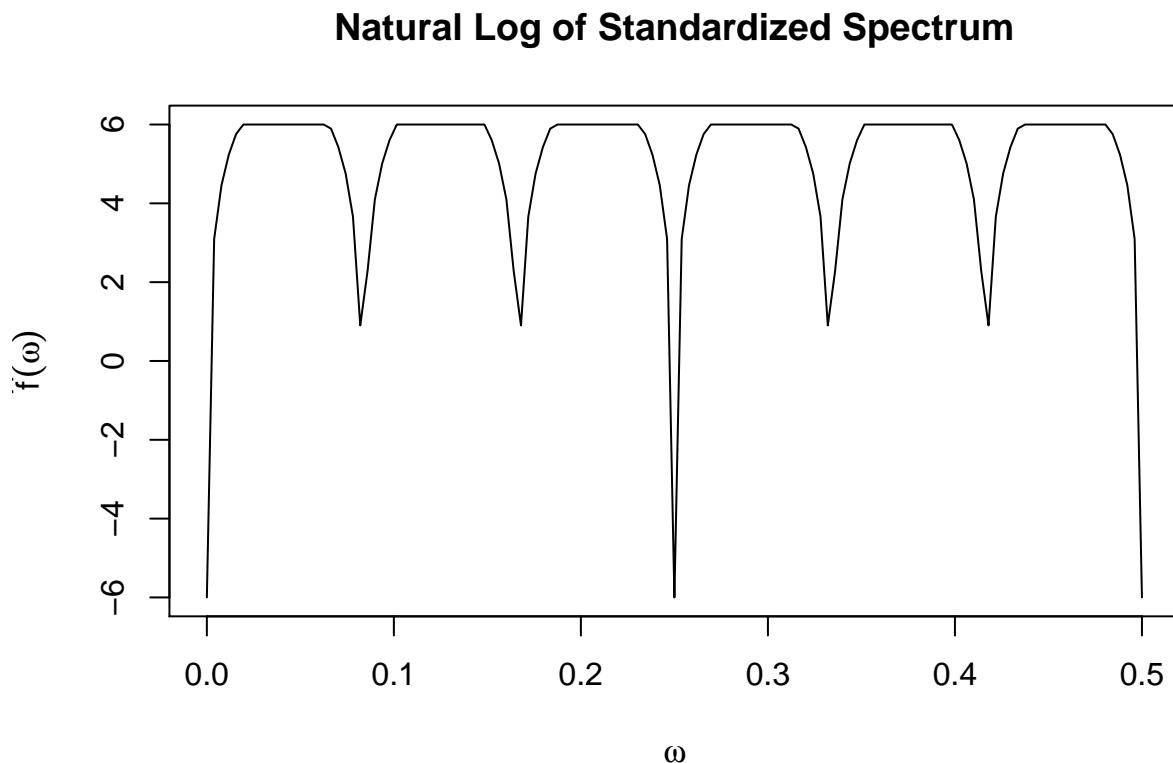
In the infinite sum derived above all powers of z are nonnegative; if $|\alpha| > 1$ then this series diverges.

For large powers of z and we would have a coefficients that are infinitely large (or small), which would mean invalid MA process. In other words, for $ARMA(1, 1, \alpha, \beta)$ we see that if $|\alpha| > 1$ then we would not have an $MA(\infty)$ representation of such a process.

Computational Part

C5.1: Use the `masp` function with $Q = 256$ frequencies to find the square modulus of the frequency transfer function of the 12-th difference filter. The input to `masp` will be that of an `MA(12)` with all coefficients equal to zero, except $\beta_{12} = 1$. Use the `plotsp` function to plot the natural logarithm of the square modulus (don't forget to compute the variance of the filter). Does the result correspond to what we learned about the effect of differencing?

```
Q = 256
D = 12
beta = rep(0, D)
beta[D] = -1
sp = masp(beta, Q)
var = macorr(beta, 1)$var
var = sum(beta^2)
plotsp(sp, Q, var)
```



In the class we learnt that taking the d^{th} difference removes all frequencies of form $\frac{l}{d}, l = 0, 1 \dots d$. Since in the example above we have taken 12th difference, we expect the following frequencies (consider left side $f \in [0, 0.5]$ only since it's symmetrical): $\{0, 0.08(3), 0.1(6), 0.25, 0.(3), 0.41(6), 0.5\}$ to be removed from the spectrum. Exactly these seven inverted spikes we see on `plotsp`. Interestingly, the rational frequencies that happened to have an exact (terminated) value $\{0, 0.25, 0.5\}$ are going all the way down on the `plotsp`, while others (with repeated decimals) $\{0.08(3), 0.1(6), 0.(3), 0.41(6)\}$ go only half way through.

C5.2: What four sets of MA(2) parameters (coefficients and error variances) can lead to the autocovariances $R_0 = 1.3125, R_1 = 0.625, R_2 = 0.25$. Which of these sets leads to a characteristic polynomial having all of its zeros outside the unit circle? Which of these sets does function `corrma` return? Hint: See your notes on the nonidentifiability of the moving average process.

1. Find σ^2 to satisfy autocovariances. According to Univariate Filter Theorem (UFT):

$$\begin{cases} R_v = \sigma^2 \sum_{k=0}^{q-|v|} \beta_k \beta_{k+|v|} & |v| = 0, 1, 2, \dots, q \\ 0 & |v| = q+1, q+2, \dots \end{cases}$$

Which, for MA(2) process, yields the following value for autocovariance function:

$$\begin{aligned} R_0 &= \sigma^2(\beta_0^2 + \beta_1^2 + \beta_2^2) = \sigma^2(1 + \beta_1^2 + \beta_2^2) \\ R_1 &= \sigma^2(\beta_0\beta_1 + \beta_1\beta_2) = \sigma^2\beta_1^2(1 + \beta_2) \\ R_2 &= \sigma^2\beta_0\beta_2 = \sigma^2\beta_2 \end{aligned}$$

To find parametrs then, we need to solve a system of three equations with three unknowns:

$$\begin{aligned} \sigma^2(1 + \beta_1^2 + \beta_2^2) &= 1.3125 \\ \sigma^2\beta_1^2(1 + \beta_2) &= 0.625 \\ \sigma^2\beta_2 &= 0.25 \end{aligned}$$

Simplifying, we get one equation:

$$\begin{aligned} \beta_2 &= \frac{1}{4\sigma^2} \\ \sigma^2\beta_1(1 + \frac{1}{4\sigma^2}) &= 0.625 \\ \beta_1 &= \frac{0.625}{\sigma^2 + 0.25} \\ 1.3125 &= \sigma^2(1 + (\frac{0.625}{\sigma^2 + 0.25})^2 + (\frac{0.25}{\sigma^2})^2) \end{aligned}$$

Solving this (cubic) equation for σ^2 in Mathematica:

$$\text{Solve}[x(1 + (0.625/(x + 0.25))^2 + (0.25/x)^2) == 1.3125, x]$$

Yields the following roots:

$$\begin{aligned} \sigma_1^2 &= 1 \\ \sigma_2^2 &= \frac{1}{16} \\ \sigma_3^2 &= -\frac{1}{8 + i\sqrt{3}} \\ \sigma_4^2 &= -\frac{1}{8 - i\sqrt{3}} \end{aligned}$$

2. Construct the characteristic polynomial and find its zeros. Using the finding that $\sigma_1^2 = 1$, we find the first set of coefficients:

$$\begin{aligned} \beta_1 &= 0.5 \\ \beta_2 &= 0.25 \end{aligned}$$

For MA(2), characteristic polynomial has the following form:

$$\begin{aligned} h(z) &= \sum_{k=0}^2 \beta_k z^k \\ &= 1 + \beta_1 z + \beta_2 z^2 \\ &= 1 + 0.5z + 0.25z^2 \end{aligned}$$

Solving this quadratic equation yields:

$$\begin{aligned} z_1^* &= -1 + i\sqrt{3} \\ z_2^* &= -1 - i\sqrt{3} \end{aligned}$$

Thus the roots of the three other characteristic polynomials are:

$$\begin{aligned} \{z_1^*, \frac{1}{z_2^*}\} &= \{-1 + i\sqrt{3}, \frac{1}{-1 - i\sqrt{3}}\} \\ \{\frac{1}{z_1^*}, z_2^*\} &= \{\frac{1}{-1 + i\sqrt{3}}, -1 - i\sqrt{3}\} \\ \{\frac{1}{z_1^*}, \frac{1}{z_2^*}\} &= \{\frac{1}{-1 + i\sqrt{3}}, \frac{1}{-1 - i\sqrt{3}}\} \end{aligned}$$

Now we check each pair of roots to find the pair, in which both roots are outside of the unit circle.

```
z1 = complex(real=-1, imaginary=sqrt(3))
z2 = complex(real=-1, imaginary=-sqrt(3))
```

```
Mod(z1)
```

```
## [1] 2
```

```
Mod(z2)
```

```
## [1] 2
```

```
Mod(1/z1)
```

```
## [1] 0.5
```

```
Mod(1/z2)
```

```
## [1] 0.5
```

Thus we choose the pair $\{-1 + i\sqrt{3}, -1 - i\sqrt{3}\}$, which, as we already seen, relates to the coefficients $\beta_1 = 0.5, \beta_2 = 0.25$ and variance $\sigma^2 = 1$.

3. Using *corrma* function:

```
#corrma() is under construction
```

C5.3: If X is an AR(1) process with $\alpha = 0.05$, write X_t as a function of X_{t-2} and some ε 's and of X_{t+2} and some ε 's.

1. AR(p) has the form:

$$X_t + \alpha_1 X_{t-1} + \dots + \alpha_p X_{t-p} = \varepsilon_t$$

Thus AR(1) has the form:

$$\begin{aligned} X_t + \alpha X_{t-1} &= \varepsilon_t \\ X_t &= -\alpha X_{t-1} + \varepsilon_t \\ X_t &= -\alpha(-\alpha X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ X_t &= -0.05(-0.05 X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ X_t &= 0.0025 X_{t-2} - 0.05 \varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

Another approach is as follows:

$$\begin{aligned} X_t + \alpha X_{t-1} &= \varepsilon_t \\ X_{t+1} + \alpha X_t &= \varepsilon_{t+1} \\ \frac{1}{\alpha} X_{t+1} + X_t &= \frac{1}{\alpha} \varepsilon_{t+1} \\ X_t &= -\frac{1}{\alpha} (X_{t+1} - \varepsilon_{t+1}) \\ X_t &= -\frac{1}{\alpha} \left(-\frac{1}{\alpha} (X_{t+2} - \varepsilon_{t+2}) - \varepsilon_{t+1} \right) \\ X_t &= \left(\frac{1}{\alpha} \right)^2 X_{t+2} - \left(\frac{1}{\alpha} \right)^2 \varepsilon_{t+2} + \frac{1}{\alpha} \varepsilon_{t+1} \\ X_t &= 400 X_{t+2} - 400 \varepsilon_{t+2} + 20 \varepsilon_{t+1} \end{aligned}$$

C5.4: Which of the following sets of coefficients can be regarded as the coefficients of an AR process?

In order to use Yule-Walker equations (and satisfy the theorem about the relationship between MA and AR processes), we require the roots of the characteristic polynomial (constructed using coefficients α) to be outside of the unit circle.

1. $\alpha = (-0.9, 0.8)$ The process has the following form:

$$X_t - 0.9X_{t-1} + 0.8X_{t-2} = \varepsilon_t$$

Then the characteristic polynomial is:

$$\sum_{j=0}^2 \alpha_j z^j = 1 - 0.9z + 0.8z^2$$

Finding roots of the quadratic equation $8z^2 - 9z + 10 = 0$:

$$\begin{aligned} D &= b^2 - 4ac = 81 - 320 = -239 \\ x_{1,2} &= \frac{-b \pm \sqrt{D}}{2a} = \frac{9 \pm \sqrt{-239}}{16} \\ x_1 &= \frac{9}{16} + i \frac{\sqrt{239}}{16} \\ x_2 &= \frac{9}{16} - i \frac{\sqrt{239}}{16} \end{aligned}$$

Now check whether or not they are outside of the unit circle:

```
z1 = complex(real=9/16, imaginary=sqrt(239)/16)
z2 = complex(real=9/16, imaginary=-sqrt(239)/16)

Mod(z1); Mod(z2)
```

```
## [1] 1.118034
```

```
## [1] 1.118034
```

We see that they are outside of the unit circle, meaning that the set of coefficients can be regarded as the coefficients of an AR process. Alternatively, we can avoid doing the work by hand and use R function to answer the question:

```
polyrt <- function(coef){return(polyroot(c(1, coef)))}

isValidAR <- function(alpha) {
  roots = Mod(polyrt(c(alpha)))
  print(roots)
  if ((roots[1] > 1) && (roots[2] > 1))
    print("Yes, valid set of coefficients for AR process")
  else
    print("No, one or more roots are not outside of the unit circle")
}

alpha = c(-0.9, 0.8)
isValidAR(alpha)
```

```
## [1] 1.118034 1.118034
## [1] "Yes, valid set of coefficients for AR process"
```

Which produces the same result, confirming that this set of coefficient can be regarded as the coefficients of an AR process.

2. $\alpha = (-1.54, 0.4)$

```
alpha=c(-1.54, 0.4)
isValidAR(alpha)
```

```
## [1] 0.8269904 3.0230096
## [1] "No, one or more roots are not outside of the unit circle"
```

3. $\alpha = (1, -0.904, -0.7)$

```
alpha=c(1, -0.904, -0.7)
isValidAR(alpha)
```

```
## [1] 1.1207027 0.7819385 1.6301927
## [1] "No, one or more roots are not outside of the unit circle"
```

4. $\alpha = (-1.22, 1.16, -0.7)$

```
alpha=c(-1.22, 1.16, -0.7)
isValidAR(alpha)
```

```
## [1] 1.092089 1.092089 1.197804
## [1] "Yes, valid set of coefficients for AR process"
```

5. $\alpha = (-0.09, -1.5884, -0.024, 0.9)$

```
alpha=c(-0.09, -1.5884, -0.024, 0.9)
isValidAR(alpha)
```

```
## [1] 1.017835 1.035622 1.035622 1.017835
## [1] "Yes, valid set of coefficients for AR process"
```