PSYCHOMETRIKA—VOL. 76, NO. 2, 257–284
APRIL 2011
DOI: 10.1007/S11336-011-9206-8

REGULARIZED GENERALIZED CANONICAL CORRELATION ANALYSIS

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Regularized generalized canonical correlation analysis (RGCCA) is a generalization of regularized canonical correlation analysis to three or more sets of variables. It constitutes a general framework for many multi-block data analysis methods. It combines the power of multi-block data analysis methods (maximization of well identified criteria) and the flexibility of PLS path modeling (the researcher decides which blocks are connected and which are not). Searching for a fixed point of the stationary equations related to RGCCA, a new monotonically convergent algorithm, very similar to the PLS algorithm proposed by Herman Wold, is obtained. Finally, a practical example is discussed.

Key words: generalized canonical correlation analysis, multi-block data analysis, PLS path modeling, regularized canonical correlation analysis.

1. Introduction

The main purpose of this article is to combine the power of multi-block data analysis methods and the flexibility of the Partial Least Squares (PLS) path modeling algorithms proposed by Wold (1985), Lohmöller (1989) and Krämer (2007). These methods deal with the same kind of data and share the same objectives: how to relate several blocks of variables observed on the same set of individuals. The power of multi-block data analysis lies in the fact that it includes a great variety of methods with well identified criteria to be optimized. The great flexibility of PLS path modeling lies in the possibility of taking into account certain hypotheses on connections between blocks: the researcher decides which blocks are connected and which are not. Unfortunately, the criteria optimized by the various options of PLS path modeling algorithms are often unclear. In this paper, we propose a new method called Regularized Generalized Canonical Correlation Analysis (RGCCA). This is a generalization of regularized canonical correlation analysis (Vinod, 1976; Leurgans, Moyeed & Silverman, 1993) to three or more sets of variables. RGCCA combines the power of multi-block data analysis methods and the flexibility of PLS path modeling.

RGCCA is a framework for modeling linear relationships between several blocks of variables observed on the same set of individuals. Considering a network of connections between these blocks, the objective of RGCCA is to find linear combinations of block variables (block components) such that (i) block components explain their own block well and/or (ii) block components that are assumed to be connected are highly correlated. In the literature, many methods exist with the objective of finding block components having these properties. Some of them are based on the maximization of a function of correlations: SUMCOR (sum of correlations method),

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SSQCOR (sum of squared correlations method), SABSCOR (sum of absolute values of the correlations method). Others are based on the maximization of a function of covariances: SUMCOV (sum of covariances method), SSQCOV (sum of squared covariances method), SABSCOV (sum of absolute values of the covariances method). Others are based on the maximization of a function of both correlations and covariances. It appears that RGCCA constitutes a general framework for multi-block data analysis and includes all the previously listed methods as particular cases. Moreover, one of the important properties of RGCCA relies on the fact that blocks are not necessarily fully connected. This flexibility allows a large variety of hierarchical models such as Carroll's (1968a, 1968b) generalized canonical correlation analysis, Chessel and Hanafi's (1996) Multiple Co-Inertia Analysis and those given in Westerhuis, Kourti, and MacGregor (1998) and in Vivien and Sabatier (2003) to be included in the RGCCA framework. RGCCA also contains the PLS approach of Wold–Lohmöller–Krämer as a special case, but only when the option "mode B for all blocks" is selected. The full comparison between PLS path modeling and RGCCA is, however, beyond the scope of this article and will be discussed in a separate paper.

Moreover, in a high-dimensional block setting or in the presence of multicollinearity within blocks (ill-conditioned data blocks), correlation-based methods lead to spurious relationships between blocks. This gives an impression of links between blocks that are invalid when objectively examined. RGCCA constitutes a regularized version of various correlation-based methods and makes a stable analysis of ill-conditioned data blocks possible. It allows for the introduction of a continuum between correlation-based criteria and covariance-based criteria.

There is no analytical solution to RGCCA, so we propose a monotonically convergent algorithm based on a modification of Wold's (1985) PLS algorithm. RGCCA is obtained by following several steps. Firstly, a new generalized canonical correlation analysis (GCCA) method, which takes into account the hypotheses on the connections between sets of variables, is defined at the population level. Secondly, using shrinkage estimates for block covariance matrices, stationary equations related to population GCCA are written at the sample level. Thirdly, these stationary equations are related to a new optimization problem called RGCCA. We have obtained a new monotonically convergent algorithm when searching for a fixed point of these stationary equations.

Finally, we will conclude this paper with a detailed analysis of a practical example where many of the RGCCA possibilities are explored. We will also show how the PLS path modeling drawing conventions can be extended to multi-block data analysis and RGCCA.

This paper is organized as follows:

(1) Definition of population generalized canonical correlation analysis. Construction of the stationary equations at the population level. Searching for a fixed point of the population level stationary equations: construction of a PLS algorithm for population GCCA. Convergence properties of the PLS algorithm for population GCCA.
(2) Construction of the stationary equations at the sample level, using shrinkage estimates of the block covariance matrices.
(3) Definition of regularized generalized canonical correlation analysis (RGCCA).
(4) Searching for a fixed point of the sample level stationary equations: construction of a PLS algorithm for RGCCA. Convergence properties of the PLS algorithm for RGCCA.
(5) The Russett example.

2. Population Generalized Canonical Correlation Analysis

Canonical correlation analysis of two sets of random variables can be extended to three or more sets of variables in many ways. For instance, Kettenring (1971) studied five specific methods: (i) the sum of correlations method (SUMCOR), (ii) the maximum variance method

(MAXVAR), (iii) the sum of squared correlations method (SSQCOR), (iv) the minimum variance method (MINVAR), and (v) the generalized variance method (GENVAR). A sixth method, the sum of absolute value correlations method (SABSCOR), is also considered in this article. This last criterion plays a central role in the PLS approach of Wold (1985).

We propose a modification of the SUMCOR, SSQCOR and SABSCOR methods which takes into account some hypotheses on the connections between the sets of variables. Let us consider J random p_j-dimensional zero mean column vectors $\boldsymbol{x}_j = (x_{j1}, \ldots, x_{jp_j})^t$ defined on the same population and J non-random p_j-dimensional column vectors $\boldsymbol{\alpha}_j = (\alpha_{j1}, \ldots, \alpha_{jp_j})^t$. We also consider a network of connections between the random vectors by defining a design matrix $\mathbf{C} = \{c_{jk}\} : c_{jk} = 1$ if $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ are connected and 0 otherwise.

Consider two linear components $\eta_j = \sum_h \alpha_{jh} x_{jh} = \boldsymbol{\alpha}_j^t \boldsymbol{x}_j$ and $\eta_k = \sum_h \alpha_{kh} x_{kh} = \boldsymbol{\alpha}_k^t \boldsymbol{x}_k$. The correlation between the two random variables η_j and η_k is

$$\rho\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right) = \frac{\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k}{(\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jj} \boldsymbol{\alpha}_j)^{1/2} (\boldsymbol{\alpha}_k^t \boldsymbol{\Sigma}_{kk} \boldsymbol{\alpha}_k)^{1/2}}, \tag{1}$$

where $\boldsymbol{\Sigma}_{jj} = E(\boldsymbol{x}_j \boldsymbol{x}_j^t)$, $\boldsymbol{\Sigma}_{kk} = E(\boldsymbol{x}_k \boldsymbol{x}_k^t)$ and $\boldsymbol{\Sigma}_{jk} = E(\boldsymbol{x}_j \boldsymbol{x}_k^t)$. All $\boldsymbol{\Sigma}_{jj}$ are supposed to be of full rank.

2.1. *Definition of Population Generalized Canonical Correlation Analysis*

We define "population generalized canonical correlation analysis" as the following optimization problem:

$$\begin{aligned} &\underset{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_J}{\text{Maximize}} \quad \sum_{j,k=1, j \neq k}^{J} c_{jk} g\left(\rho\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right) \\ &\text{subject to the constraints} \quad \operatorname{Var}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j\right) = 1, \quad j = 1, \ldots, J, \end{aligned} \tag{2}$$

where g is the identity, the absolute value, or the square function. As the algorithm proposed in this paper for solving optimization problem (2) is close to Wold's (1985) PLS algorithm, we feel that it would be useful and necessary to introduce the PLS terminology here. The centroid scheme introduced by Wold (1985) is related to g being equal to the absolute value. The factorial scheme proposed by Lohmöller (1989) is related to g being equal to the square. The Horst scheme proposed by Krämer (2007) is related to g being equal to the identity. We have limited the optimization problem (2) to these three schemes because they are the most used in the multiblock and PLS path modeling literature. The centroid and factorial schemes usually give close results in practical applications (Noonan & Wold, 1982). The Horst scheme can be very useful when the researcher is looking for positively correlated components. Due to this constraint, the Horst scheme can yield quite different solutions from the two other schemes.

Problem (2) is equivalent to the following optimization problem:

$$\begin{aligned} &\underset{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_J}{\text{Maximize}} \quad \sum_{j,k=1, j \neq k}^{J} c_{jk} g\left(\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k\right) \\ &\text{subject to the constraints} \quad \boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jj} \boldsymbol{\alpha}_j = 1, \quad j = 1, \ldots, J. \end{aligned} \tag{3}$$

The following Lagrangian function of optimization problem (3) is then considered:

$$F(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_J, \lambda_1, \ldots, \lambda_J) = \sum_{j,k=1, j \neq k}^{J} c_{jk} g\left(\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k\right) - \varphi \sum_{j=1}^{J} \frac{\lambda_j}{2} \left(\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jj} \boldsymbol{\alpha}_j - 1\right), \tag{4}$$

where $\lambda_1, \ldots, \lambda_J$ are the Lagrange multipliers and where $\varphi = 1$ when g is the identity or the absolute value and $\varphi = 2$ when g is the square function. We may suppose that $\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k$ is different from 0, because if it were not the case, we would just set the design coefficient c_{jk} to zero. Therefore, we may also consider the derivative g' when g is the absolute value. Canceling the derivatives of the Lagrangian function with respect to $\boldsymbol{\alpha}_j$ and λ_j yields the following stationary equations for population GCCA:

$$\frac{1}{\varphi} \boldsymbol{\Sigma}_{jj}^{-1} \sum_{k=1, k \neq j}^{J} c_{jk} g'\big(\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k\big) \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k = \lambda_j \boldsymbol{\alpha}_j, \quad j = 1, \ldots, J \tag{5}$$

with the normalization constraints

$$\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jj} \boldsymbol{\alpha}_j = 1, \quad j = 1, \ldots, J. \tag{6}$$

These stationary equations have no analytical solution, but they can be used to build a monotonically convergent algorithm for optimization problem (3). This new algorithm, to be described in the next section, appears to be very similar to the Wold's (1985) PLS algorithm.

2.2. A PLS Algorithm for Population GCCA

In the sequel to this paper, we use the expression $\mathrm{Cov}(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k)$ instead of $\boldsymbol{\alpha}_j^t \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k$ for easier mathematical treatments and the function $w(x) = \frac{1}{\varphi} g'(x)$ for easier readability. The function $w(x)$ is equal to 1 for the Horst scheme, to x for the factorial scheme and to $\mathrm{sign}(x)$ for the centroid scheme. In PLS terminology, $\boldsymbol{\alpha}_j$ is a vector of outer weights, $\eta_j = \boldsymbol{\alpha}_j^t \boldsymbol{x}_j$ is called an outer component, and an inner component v_j is defined as follows:

$$v_j = \sum_{k=1, k \neq j}^{J} c_{jk} w\big(\mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big)\big) \boldsymbol{\alpha}_k^t \boldsymbol{x}_k. \tag{7}$$

For the various schemes, we get the following inner components:

$$\text{Horst } (g = \text{identity}): \quad v_j = \sum_{k=1, k \neq j}^{J} c_{jk} \boldsymbol{\alpha}_k^t \boldsymbol{x}_k.$$

$$\text{Factorial } (g = \text{square}): \quad v_j = \sum_{k=1, k \neq j}^{J} c_{jk} \mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big) \boldsymbol{\alpha}_k^t \boldsymbol{x}_k.$$

$$\text{Centroid } (g = \text{absolute value}): \quad v_j = \sum_{k=1, k \neq j}^{J} c_{jk} \mathrm{sign}\big[\mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big)\big] \boldsymbol{\alpha}_k^t \boldsymbol{x}_k.$$

The inner components v_j are useful to simplify the stationary equations (5) for population GCCA. Using the following expression:

$$\begin{aligned} \mathrm{Cov}(\boldsymbol{x}_j, v_j) = E(\boldsymbol{x}_j v_j) &= E\left(\boldsymbol{x}_j \sum_{k=1, k \neq j}^{J} c_{jk} w\big(\mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big)\big) \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right) \\ &= \sum_{k=1, k \neq j}^{J} c_{jk} w\big(\mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big)\big) \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k \\ &= \frac{1}{\varphi} \sum_{k=1, k \neq j}^{J} c_{jk} g'\big(\mathrm{Cov}\big(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\big)\big) \boldsymbol{\Sigma}_{jk} \boldsymbol{\alpha}_k \end{aligned} \tag{8}$$

and the normalization constraints (6), the stationary equations (5) become

$$\boldsymbol{\alpha}_j = \left[\mathrm{Cov}(\boldsymbol{x}_j, v_j)^t \boldsymbol{\Sigma}_{jj}^{-1} \mathrm{Cov}(\boldsymbol{x}_j, v_j)\right]^{-1/2} \boldsymbol{\Sigma}_{jj}^{-1} \mathrm{Cov}(\boldsymbol{x}_j, v_j), \quad j = 1, \ldots, J. \tag{9}$$

It is worth noting that the stationary equations (9) are also obtained by considering the minimum of $E[(v_j - \boldsymbol{\beta}_j^t \boldsymbol{x}_j)^2]$ with respect to $\boldsymbol{\beta}_j$, subject to the normalization constraint $\boldsymbol{\beta}_j^t \boldsymbol{\Sigma}_{jj} \boldsymbol{\beta}_j = 1$ (or equivalently, the maximum of $\mathrm{Cov}(\boldsymbol{\beta}_j^t \boldsymbol{x}_j, v_j)$, subject to the same normalisation constraint). This result is central to the proof of Proposition 2 below.

While the notion of inner component is not usually found in the multi-block literature, it plays a central role in the PLS approach and allows us to design a very efficient algorithm for population GCCA. The following proposition specifies the role of the inner components in the criterion to be maximized.

Proposition 1. *For g equal to the identity, to the square, or to the absolute value, we obtain the following result*:

$$\sum_{j,k=1,k\neq j}^{J} c_{jk} g\left[\mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right] = \sum_{j=1}^{J} \mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, v_j\right). \tag{10}$$

Proof: Equality (10) is obtained from the identity $w(x)x = g(x)$ for g equal to the identity, to the square or to the absolute value:

$$\begin{aligned}
\sum_{j=1}^{J} \mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, v_j\right) &= \sum_{j=1}^{J} \mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \sum_{k=1,k\neq j}^{J} c_{jk} w\left(\mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right) \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right) \\
&= \sum_{j,k=1,k\neq j}^{J} c_{jk} w\left(\mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right) \mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right) \\
&= \sum_{j,k=1,k\neq j}^{J} c_{jk} g\left(\mathrm{Cov}\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right).
\end{aligned}$$

□

It is possible to construct a monotonically convergent algorithm related to optimization problem (3). That is to say, the bounded criterion to be maximized in (3) is increasing at each step of the proposed iterative procedure. Stationary equations (9) and Proposition 1 suggest an iterative algorithm for optimization problem (3):

- Begin with arbitrary normalized outer weights $\boldsymbol{\alpha}_j$, $j = 1, \ldots, J$.
- Compute the inner components v_j, $j = 1, \ldots, J$, according to formula (7).
- Compute new normalized outer weights using formula (9).
- Iterate this procedure.

To obtain a monotonically convergent algorithm, it is necessary to use a sequence of operations similar to the ones used by Wold (1985) and Hanafi (2007) for PLS path modeling. This PLS algorithm for population GCCA is described in Figure 1.

The procedure begins by an arbitrary choice of initial normalized outer weight vectors $\boldsymbol{\alpha}_1^0, \boldsymbol{\alpha}_2^0, \ldots, \boldsymbol{\alpha}_J^0$ (Step A in Figure 1). Suppose outer weight vectors $\boldsymbol{\alpha}_1^{s+1}, \boldsymbol{\alpha}_2^{s+1}, \ldots, \boldsymbol{\alpha}_{j-1}^{s+1}$ have been constructed. The outer weight vector $\boldsymbol{\alpha}_j^{s+1}$ is computed by considering the inner component v_j^s given in Step B in Figure 1, and the formula given in Step C in Figure 1. The procedure is iterated until convergence of the bounded criterion, which is due to the following proposition:

A. Initialisation

A1. Choose J arbitrary vectors $\tilde{\boldsymbol{\alpha}}_1^0, \tilde{\boldsymbol{\alpha}}_2^0, \ldots, \tilde{\boldsymbol{\alpha}}_J^0$.
A2. Compute normalized outer weight vectors $\boldsymbol{\alpha}_1^0, \boldsymbol{\alpha}_2^0, \ldots, \boldsymbol{\alpha}_J^0$ as

$$\boldsymbol{\alpha}_j^0 = \left[\left(\tilde{\boldsymbol{\alpha}}_j^0\right)^t \boldsymbol{\Sigma}_{jj}^{-1} \tilde{\boldsymbol{\alpha}}_j^0\right]^{-1/2} \boldsymbol{\Sigma}_{jj}^{-1} \tilde{\boldsymbol{\alpha}}_j^0.$$

For $s = 0, 1, \ldots$ **(until convergence)**
For $j = 1, 2, \ldots, J$

B. Computing the inner component v_j^s
Compute the inner component according to the selected scheme:

$$v_j^s = \sum_{k<j} c_{jk} w\left[\operatorname{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \boldsymbol{x}_j, \left(\boldsymbol{\alpha}_k^{s+1}\right)^t \boldsymbol{x}_k\right)\right]\left(\boldsymbol{\alpha}_k^{s+1}\right)^t \boldsymbol{x}_k + \sum_{k>j} c_{jk} w\left[\operatorname{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \boldsymbol{x}_j, \left(\boldsymbol{\alpha}_k^s\right)^t \boldsymbol{x}_k\right)\right]\left(\boldsymbol{\alpha}_k^s\right)^t \boldsymbol{x}_k$$

where $w(x) = 1$ for the Horst scheme, x for the factorial scheme and $\operatorname{sign}(x)$ for the centroid scheme.

C. Computing the outer weight vector $\boldsymbol{\alpha}_j^{s+1}$
Compute the outer weight vector

$$\boldsymbol{\alpha}_j^{s+1} = \left[\operatorname{Cov}\left(\boldsymbol{x}_j, v_j^s\right)^t \boldsymbol{\Sigma}_{jj}^{-1} \operatorname{Cov}\left(\boldsymbol{x}_j, v_j^s\right)\right]^{-1/2} \boldsymbol{\Sigma}_{jj}^{-1} \operatorname{Cov}\left(\boldsymbol{x}_j, v_j^s\right).$$

End
End

FIGURE 1.
A PLS algorithm for population GCCA.

Proposition 2. *Let* $\boldsymbol{\alpha}_j^s$, $j = 1, \ldots, J$, $s = 0, 1, 2, \ldots$, *be a sequence of outer weight vectors generated by the PLS algorithm for population GCCA. The following function is defined for outer weight vectors* $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_J$:

$$f(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_J) = \sum_{j,k=1, j\neq k}^{J} c_{jk} g\left[\rho\left(\boldsymbol{\alpha}_j^t \boldsymbol{x}_j, \boldsymbol{\alpha}_k^t \boldsymbol{x}_k\right)\right]. \tag{11}$$

The following inequalities hold:

$$\forall s \quad f\left(\boldsymbol{\alpha}_1^s, \boldsymbol{\alpha}_2^s, \ldots, \boldsymbol{\alpha}_J^s\right) \leq f\left(\boldsymbol{\alpha}_1^{s+1}, \boldsymbol{\alpha}_2^{s+1}, \ldots, \boldsymbol{\alpha}_J^{s+1}\right). \tag{12}$$

Proof: See Appendix A. □

The essential feature of this algorithm is that each replacement is optimal, and sequential, that is to say that $\boldsymbol{\alpha}_j^s$ must be replaced by $\boldsymbol{\alpha}_j^{s+1}$ before replacing $\boldsymbol{\alpha}_{j+1}^s$. This is the essence of the Gauss–Seidel algorithm for solving a system of linear equations and of several other iterative algorithms such as the Wold and Hanafi's PLS algorithms, the MAXDIFF algorithm (Ten Berge, 1988) and the multivariate eigenvalue problem (Chu & Watterson, 1993). This sequential approach leads to the monotonic convergence of these algorithms.

Because the main objective is to estimate the outer weights $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_J$ from a finite sample, it is now necessary to write the population stationary equations at the sample level.

3. Stationary Equations at the Sample Level Using Shrinkage Estimates of the Covariance Matrices $\boldsymbol{\Sigma}_{jj}$

Let us consider J blocks $\mathbf{X}_1, \ldots, \mathbf{X}_J$ of centered variables measured on a set of n individuals. A row of $\mathbf{X}_j$ represents a realization of the row-random vector $\boldsymbol{x}_j^t$, a column $\boldsymbol{x}_{jh}$ of $\mathbf{X}_j$ is considered as a variable observed on n individuals, x_{jhi} is the value of variable $\boldsymbol{x}_{jh}$ for individual i. Now, $\mathbf{C} = \{c_{jk}\}$ is a design matrix describing a network of relationships between blocks: $c_{jk} = 1$ for two connected blocks, and 0 otherwise.

The sample covariance matrices are $\mathbf{S}_{jj} = \frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j$ and $\mathbf{S}_{jk} = \frac{1}{n}\mathbf{X}_j^t\mathbf{X}_k$. In case of high multicollinearity or when the number of individuals is smaller than the number of variables, the sample covariance matrix $\mathbf{S}_{jj}$ is a poor estimate of the true covariance matrix $\boldsymbol{\Sigma}_{jj}$ for a very simple reason pointed out by Schäfer and Strimmer (2005): the number of eigenvalues close or equal to zero is much higher in the sample covariance matrix than in the true covariance matrix. One suggestion of Ledoit and Wolf (2004) for finding a better estimate of the true covariance matrix $\boldsymbol{\Sigma}_{jj}$ is to consider the class of linear combinations $\{\hat{\mathbf{S}}_{jj} = \tau_j\mathbf{I} + (1-\tau_j)\mathbf{S}_{jj}, 0 \le \tau_j \le 1\}$ of the identity matrix $\mathbf{I}$ and the sample covariance matrix $\mathbf{S}_{jj}$. The matrix $\hat{\mathbf{S}}_{jj}$ is called a shrinkage estimate of $\boldsymbol{\Sigma}_{jj}$ and τ_j is the shrinkage constant. Furthermore, Schäfer and Strimmer, using the unbiased estimate $\mathbf{S}'_{jj} = \frac{n}{n-1}\mathbf{S}_{jj}$ of $\boldsymbol{\Sigma}_{jj}$ instead of $\mathbf{S}_{jj}$, give a formula for finding an optimal shrinkage constant, i.e., minimizing the mean square error $\mathrm{MSE} = E[\|\hat{\mathbf{S}}_{jj} - \boldsymbol{\Sigma}_{jj}\|^2]$, where $\|\cdot\|$ is the Frobenius norm (formula 6, p. 8 of Schäfer & Strimmer, 2005). In the same reference, they also give a formula for estimating this optimal shrinkage constant from the data (formula for Target A, p. 11 and Appendix A):

$$\hat{\tau}_j^* = \frac{\sum_{k\neq l=1}^{p_j} \widehat{\mathrm{Var}}(s'_{j,kl}) + \sum_{k=1}^{p_j} \widehat{\mathrm{Var}}(s'_{j,kk})}{\sum_{k\neq l=1}^{p_j} (s'_{j,kl})^2 + \sum_{k=1}^{p_j} (s'_{j,kk} - 1)^2},$$

where $s'_{j,kl}$ is an entry of $\mathbf{S}'_{jj}$ and, setting $w_{jkli} = (x_{jki} - \bar{x}_{jk})(x_{jli} - \bar{x}_{jl})$, $\widehat{\mathrm{Var}}(s'_{j,kl}) = \frac{n}{(n-1)^3}\sum_{i=1}^{n}(w_{jkli} - \bar{w}_{jkl})^2$ is the empirical unbiased variance of $s'_{j,kl}$.

We also introduce for each block an outer weight vector $\mathbf{a}_j$, an outer component $\mathbf{y}_j = \mathbf{X}_j\mathbf{a}_j$ and an inner component $\mathbf{z}_j$ defined as

$$\mathbf{z}_j = \sum_{k=1, k\neq j}^{J} c_{jk}\, w\big[\mathrm{Cov}(\mathbf{y}_j, \mathbf{y}_k)\big]\mathbf{y}_k. \tag{13}$$

These vectors $\mathbf{a}_j, \mathbf{y}_j, \mathbf{z}_j$ are the sample versions of the previously defined $\boldsymbol{\alpha}_j, \eta_j, \nu_j$. For the Horst scheme, the inner component $\mathbf{z}_j = \sum_{k=1,k\neq j}^{J} c_{jk}\mathbf{y}_k$; for the factorial scheme, $\mathbf{z}_j = \sum_{k=1,k\neq j}^{J} c_{jk}[\mathrm{Cov}(\mathbf{y}_j, \mathbf{y}_k)]\mathbf{y}_k$; and for the centroid scheme, $\mathbf{z}_j = \sum_{k=1,k\neq j}^{J} c_{jk} \times \mathrm{sign}[\mathrm{Cov}(\mathbf{y}_j, \mathbf{y}_k)]\mathbf{y}_k$. For easier readability, we prefer to keep the same notation for the population covariance and the sample covariance.

In this section, we consider a sample version of the stationary equations (9) where $\boldsymbol{\Sigma}_{jj}$ is replaced by $\hat{\mathbf{S}}_{jj} = \tau_j\mathbf{I} + (1-\tau_j)\mathbf{S}_{jj}$. This leads to the following sample stationary equations:

$$\mathbf{a}_j = \left[\mathbf{z}_j^t\mathbf{X}_j\left[\tau_j\mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j\right]^{-1}\mathbf{X}_j^t\mathbf{z}_j\right]^{-1/2}\left[\tau_j\mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j\right]^{-1}\mathbf{X}_j^t\mathbf{z}_j,$$
$$j = 1, \ldots, J \tag{14}$$

and to normalization constraints on the outer weight vectors $\mathbf{a}_j$ which depend upon the values of the shrinkage constants τ_j:

$$\mathbf{a}_j^t\left[\tau_j\mathbf{I}+(1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j\right]\mathbf{a}_j=1,\quad j=1,\ldots,J. \tag{15}$$

Let us introduce some useful terminology inspired by the PLS approach: the situation corresponding to $\tau_j=0$ is called "mode B", the one corresponding to $\tau_j=1$ is called "new mode A" and the case $0<\tau_j<1$ is called "mode Ridge". Let us have a closer look at these three modes.

Mode B $(\tau_j=0)$

For $\tau_j=0$, the normalization constraint is $\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)=1$ and the stationary equation (14) becomes

$$\mathbf{a}_j=n^{1/2}\left[\mathbf{z}_j^t\mathbf{X}_j\left(\mathbf{X}_j^t\mathbf{X}_j\right)^{-1}\mathbf{X}_j^t\mathbf{z}_j\right]^{-1/2}\left(\mathbf{X}_j^t\mathbf{X}_j\right)^{-1}\mathbf{X}_j^t\mathbf{z}_j. \tag{16}$$

This vector of outer weights $\mathbf{a}_j$ is proportional to the vector of the regression coefficients in the multiple regression of $\mathbf{z}_j$ on $\mathbf{X}_j$. It is worth pointing out that due to the inversion of the intra-block covariance matrices, this way of computing the outer weight vector cannot be applied to an ill-conditioned block. This way of computing the outer weights is the usual mode B of the PLS approach.

New Mode A $(\tau_j=1)$

For $\tau_j=1$, the normalization constraint becomes $\|\mathbf{a}_j\|=1$ and the stationary equation (14) is written as

$$\mathbf{a}_j=\mathbf{X}_j^t\mathbf{z}_j/\left\|\mathbf{X}_j^t\mathbf{z}_j\right\|. \tag{17}$$

We may note that the outer component $\mathbf{y}_j=\mathbf{X}_j\mathbf{a}_j$ is the first PLS component in the PLS regression (Wold, Martens, & Wold, 1983) of the inner component $\mathbf{z}_j$ on block $\mathbf{X}_j$. In this paper, this way of computing the outer weights is called "new mode A". In the original mode A of the PLS approach, the outer weights are computed in the same way as formula (17), but normalized so that the outer component $\mathbf{y}_j=\mathbf{X}_j\mathbf{a}_j$ is standardized. This "new mode A" shrinks the intra-block covariance matrix to the identity. This shrinkage is probably too strong, but is useful for very high-dimensional data because it avoids the inversion of the intra-block covariance matrix.

Mode Ridge $(0<\tau_j<1)$

For $0<\tau_j<1$, (14) may also be written as

$$\mathbf{a}_j=\frac{n}{1-\tau_j}\left[\mathbf{z}_j^t\mathbf{X}_j\left[\tau_j\mathbf{I}+(1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j\right]^{-1}\mathbf{X}_j^t\mathbf{z}_j\right]^{-1/2}\left[\left(\mathbf{X}_j^t\mathbf{X}_j+\frac{n\tau_j}{1-\tau_j}\mathbf{I}\right)\right]^{-1}\mathbf{X}_j^t\mathbf{z}_j. \tag{18}$$

The vector of outer weights $\mathbf{a}_j$ is proportional to the vector of the regression coefficients in the ridge regression of $\mathbf{z}_j$ on $\mathbf{X}_j$ with a ridge constant equal to $n\tau_j/(1-\tau_j)$. We call this new way of computing the outer weights "mode Ridge". This "mode Ridge" allows a gradual shrinkage of the intra-block covariance matrix towards the identity.

Three interesting properties of the outer component $\mathbf{y}_j=\mathbf{X}_j\mathbf{a}_j$ are established in Qannari and Hanafi (2005):

(a) $\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j/\|\mathbf{a}_j\|)$ is an increasing function of τ_j.
(b) $\mathrm{Cov}(\mathbf{X}_j\mathbf{a}_j/\|\mathbf{a}_j\|,\mathbf{z}_j)$ is an increasing function of τ_j.
(c) $\mathrm{Cor}(\mathbf{X}_j\mathbf{a}_j,\mathbf{z}_j)$ is a decreasing function of τ_j.

These properties serve as a useful guide to choosing the shrinkage constant τ_j. They show that, for the outer weights $\mathbf{a}_j$ defined in (14), an increase of $\mathrm{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{z}_j)$ is balanced by a decrease of $\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j/\|\mathbf{a}_j\|)$. Furthermore, property (b) shows that the variance term dominates over the correlation term. Therefore, if the user is favoring stability (high variance) compared to correlation, $\tau_j = 1$ is the natural choice. If the user wants to give priority to the correlation between $\mathbf{y}_j = \mathbf{X}_j\mathbf{a}_j$ and its neighboring components, property (c) shows that $\tau_j = 0$ is the best choice. For a compromise between variance and correlation, the shrinkage constant τ_j can be determined by using the Schäfer and Strimmer (2005) formula. This automatic estimation of the shrinkage constant allows one to come closer to the correlation criterion, even in the case of high multicollinearity or when the number of individuals is smaller than the number of variables. In PLS path modeling, Fornell and Bookstein (1982) propose to favor the variance when a block is reflective (block variables reflect some underlying unobservable variable) and to favor the correlation when a block is formative (block variables are used to produce a score).

4. Regularized Generalized Canonical Correlation Analysis (RGCCA)

By considering the J data blocks $\mathbf{X}_1, \ldots, \mathbf{X}_J$, the design matrix $\mathbf{C} = \{c_{jk}\}$, the function g and the shrinkage constants $\tau_1, \ldots, \tau_J$ described in Section 3, we define regularized generalized canonical correlation analysis (RGCCA) as the following optimization problem:

$$\begin{aligned} &\underset{\mathbf{a}_1,\ldots,\mathbf{a}_J}{\text{Maximize}} \quad \sum_{j,k=1, j\neq k}^{J} c_{jk}\, g\big(\mathrm{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)\big) \\ &\text{subject to the constraints} \quad \tau_j\|\mathbf{a}_j\|^2 + (1-\tau_j)\,\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j) = 1, \quad j = 1, \ldots, J. \end{aligned} \tag{19}$$

The stationary equations obtained by canceling the derivatives of the Lagrangian function related to optimization problem (19) are exactly the stationary equations (14).

4.1. The PLS Algorithm for RGCCA

There is no known analytical solution to optimization problem (19). However, writing the PLS algorithm for population GCCA at the sample level straightforwardly yields a monotonically convergent algorithm. This algorithm is described in Figure 2.

The procedure begins by an arbitrary choice of initial values $\mathbf{a}_1^0, \mathbf{a}_2^0, \ldots, \mathbf{a}_J^0$ (Step A in Figure 2). Suppose outer weight vectors $\mathbf{a}_1^{s+1}, \mathbf{a}_2^{s+1}, \ldots, \mathbf{a}_{j-1}^{s+1}$ are constructed for blocks $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{j-1}$. The outer weight vector $\mathbf{a}_j^{s+1}$ is computed by considering the inner component $\mathbf{z}_j^s$ for block $\mathbf{X}_j$ given in Step B in Figure 2, and the formula given in Step C in Figure 2. The procedure is iterated until convergence of the bounded criterion, which is due to Proposition 3 given below.

Proposition 3. *Let* $\mathbf{a}_j^s$, $j = 1, \ldots, J$, $s = 0, 1, 2, \ldots$, *be a sequence of outer weight vectors generated by the PLS algorithm for RGCCA. The following function is defined*:

$$h(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_J) = \sum_{j,k=1, j\neq k}^{J} c_{jk}\, g\big[\mathrm{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)\big] \tag{20}$$

A. Initialisation

A1. Choose J arbitrary vectors $\tilde{\mathbf{a}}_1^0, \tilde{\mathbf{a}}_2^0, \ldots, \tilde{\mathbf{a}}_J^0$.

A2. Compute normalized outer weight vectors $\mathbf{a}_1^0, \mathbf{a}_2^0, \ldots, \mathbf{a}_J^0$ as

$$\mathbf{a}_j^0 = \left[\left(\tilde{\mathbf{a}}_j^0\right)^t \left[\tau_j \mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j \right]^{-1} \tilde{\mathbf{a}}_j^0 \right]^{-1/2} \left[\tau_j \mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j \right]^{-1} \tilde{\mathbf{a}}_j^0.$$

For $s = 0, 1, \ldots$ **(until convergence)**
For $j = 1, 2, \ldots, J$

B. Computing the inner component $\mathbf{X}_j$

Compute the inner component according to the selected scheme:

$$\mathbf{z}_j^s = \sum_{k<j} c_{jk}\, w\left[\operatorname{Cov}\left(\mathbf{X}_j\mathbf{a}_j^s, \mathbf{X}_k\mathbf{a}_k^{s+1}\right)\right]\mathbf{X}_k\mathbf{a}_k^{s+1} + \sum_{k>j} c_{jk}\, w\left[\operatorname{Cov}\left(\mathbf{X}_j\mathbf{a}_j^s, \mathbf{X}_k\mathbf{a}_k^{s}\right)\right]\mathbf{X}_k\mathbf{a}_k^{s}$$

where $w(x) = 1$ for the Horst scheme, x for the factorial scheme and $\operatorname{sign}(x)$ for the centroid scheme.

C. Computing the outer weight vector for block $\mathbf{X}_j$

Compute the outer weight vector

$$\mathbf{a}_j^{s+1} = \left[\left(\mathbf{z}_j^s\right)^t \mathbf{X}_j \left[\tau_j \mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j \right]^{-1} \mathbf{X}_j^t\mathbf{z}_j^s \right]^{-1/2} \times \left[\tau_j \mathbf{I} + (1-\tau_j)\frac{1}{n}\mathbf{X}_j^t\mathbf{X}_j \right]^{-1} \mathbf{X}_j^t\mathbf{z}_j^s.$$

End
End

FIGURE 2.
A PLS algorithm for RGCCA.

for vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_J$ *verifying normalization constraints* (15). *The following inequalities hold*:

$$\forall s \quad h\left(\mathbf{a}_1^s, \mathbf{a}_2^s, \ldots, \mathbf{a}_J^s\right) \leq h\left(\mathbf{a}_1^{s+1}, \mathbf{a}_2^{s+1}, \ldots, \mathbf{a}_J^{s+1}\right). \tag{21}$$

Proof: The proof is similar to that of Proposition 2. □

The proposed algorithm has two limitations:

(1) There is no proof that the algorithm converges towards a fixed point of the stationary equations, although it has always been the case in our simulations.
(2) There is no guarantee that the algorithm converges towards a global optimum of the criterion. Krämer (2007) has given an example of convergence to a local optimum.

TABLE 1.
Special cases of RGCCA for multi-block data analysis.

Method	Criterion to be maximized	Scheme	Normalization	Shrinkage constants
SUMCOR	$\sum_{j,k,j\neq k} \mathrm{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)$	Horst	$\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)=1,\ j=1,\ldots,J$	$\tau_j=0,$ $j=1,\ldots,J$
SSQCOR	$\sum_{j,k,j\neq k} \mathrm{Cor}^2(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)$	Factorial	$\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)=1,\ j=1,\ldots,J$	$\tau_j=0,$ $j=1,\ldots,J$
SABSCOR	$\sum_{j,k,j\neq k} \lvert\mathrm{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)\rvert$	Centroid	$\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)=1,\ j=1,\ldots,J$	$\tau_j=0,$ $j=1,\ldots,J$
SUMCOV	$\sum_{j,k,j\neq k} \mathrm{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)$	Horst	$\lVert\mathbf{a}_j\rVert=1,\ j=1,\ldots,J$	$\tau_j=1,$ $j=1,\ldots,J$
SSQCOV	$\sum_{j,k,j\neq k} \mathrm{Cov}^2(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)$	Factorial	$\lVert\mathbf{a}_j\rVert=1,\ j=1,\ldots,J$	$\tau_j=1,$ $j=1,\ldots,J$
SABSCOV	$\sum_{j,k,j\neq k} \lvert\mathrm{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)\rvert$	Centroid	$\lVert\mathbf{a}_j\rVert=1,\ j=1,\ldots,J$	$\tau_j=1,$ $j=1,\ldots,J$

5. Special Cases of RGCCA

The great power and flexibility of RGCCA allow a large spectrum of methods to be recovered as special cases. In the following, we list the main ones.

5.1. Multi-block Data Analysis

In multi-block data analysis, all blocks $\mathbf{X}_j$, $j=1,\ldots,J$ are connected and many criteria exist with the objective of finding components $\mathbf{y}_j=\mathbf{X}_j\mathbf{a}_j$ with useful properties. Many of them are listed in Hanafi and Kiers (2006) and in Tenenhaus and Hanafi (2010). In this section, we only consider methods related to optimization problem (19) with all $c_{jk}=1$ for $j\neq k$. In Table 1 various methods which are all special cases of RGCCA are listed.

The SUMCOR criterion has been proposed by Horst (1961) and the SSQCOR criterion by Kettenring (1971). The SUMCOV criterion is in fact a special case of the MAXDIFF criterion proposed by Van de Geer (1984), and the SSQCOV criterion is a special case of the MAXDIFF B criterion proposed by Hanafi and Kiers (2006). The SABSCOR criterion has been studied by Hanafi (2007) and the SABSCOV criterion by Krämer (2007). These two last criteria are not common in the multi-block data analysis literature and are simply a special case of RGCCA. From the point of view of optimization problem (19), the first three criteria correspond to shrinkage constants τ_j equal to 0 for all blocks and the last three criteria correspond to shrinkage constants equal to 1 for all blocks.

We may also consider the situation where the shrinkage constants are 0 for some blocks and 1 for others. Therefore, optimization problem (19), with all $c_{jk}=1$ for $j\neq k$, becomes

$$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_J}{\text{Maximize}} \quad \sum_{j,k=1,j\neq k}^{J} g\left[\mathrm{Cor}(\mathbf{X}_j\mathbf{a}_j,\mathbf{X}_k\mathbf{a}_k)\left(\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)\right)^{\tau_j/2}\left(\mathrm{Var}(\mathbf{X}_k\mathbf{a}_k)\right)^{\tau_k/2}\right]$$

$$\text{subject to the constraints} \quad \tau_j\lVert\mathbf{a}_j\rVert^2+(1-\tau_j)\,\mathrm{Var}(\mathbf{X}_j\mathbf{a}_j)=1 \quad \text{with } \tau_j=0 \text{ or } 1,\ j=1,\ldots,J. \tag{22}$$

TABLE 2.
Special cases of RGCCA for the two block situation.

Method	Criterion	Constraints	Values of τ_1 and τ_2
Inter-Battery Factor Analysis ($\approx$ PLS regression)	Maximize $\text{Cov}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_2\mathbf{a}_2)$	$\|\mathbf{a}_1\| = 1$ $\|\mathbf{a}_2\| = 1$	$\tau_1 = 1$ $\tau_2 = 1$
Canonical Correlation Analysis	Maximize $\text{Cor}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_2\mathbf{a}_2)$	$\text{Var}(\mathbf{X}_1\mathbf{a}_1) = 1$ $\text{Var}(\mathbf{X}_2\mathbf{a}_2) = 1$	$\tau_1 = 0$ $\tau_2 = 0$
Redundancy analysis of $\mathbf{X}_1$ with respect to $\mathbf{X}_2$	Maximize $\text{Cor}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_2\mathbf{a}_2)\,\text{Var}(\mathbf{X}_1\mathbf{a}_1)^{1/2}$	$\|\mathbf{a}_1\| = 1$ $\text{Var}(\mathbf{X}_2\mathbf{a}_2) = 1$	$\tau_1 = 1$ $\tau_2 = 0$
Redundancy analysis of $\mathbf{X}_2$ with respect to $\mathbf{X}_1$	Maximize $\text{Cor}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_2\mathbf{a}_2)\,\text{Var}(\mathbf{X}_2\mathbf{a}_2)^{1/2}$	$\text{Var}(\mathbf{X}_1\mathbf{a}_1) = 1$ $\|\mathbf{a}_2\| = 1$	$\tau_1 = 0$ $\tau_2 = 1$

If $\tau_j = 0$ (mode B), the main objective in the construction of the outer component $\mathbf{y}_j = \mathbf{X}_j\mathbf{a}_j$ is to maximize its correlation with its neighboring components. Conversely, if $\tau_j = 1$ (new mode A), the objective is to construct a component $\mathbf{y}_j = \mathbf{X}_j\mathbf{a}_j$ which well explains its own block $\mathbf{X}_j$ (first priority) and at the same time is well correlated to its neighboring components (second priority).

5.2. *Regularized Canonical Correlation Analysis*

For the two block situation, optimization problem (19) becomes:

$$\begin{aligned} &\underset{\mathbf{a}_1,\mathbf{a}_2}{\text{Maximize}} \quad \text{Cov}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_2\mathbf{a}_2) \\ &\text{subject to the constraints} \quad \tau_j\|\mathbf{a}_j\|^2 + (1-\tau_j)\,\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1, \quad j = 1, 2. \end{aligned} \tag{23}$$

This problem has been introduced under the name of "Regularized Canonical Correlation Analysis" for handling high-dimensional data or highly correlated data in order to stabilize the solution (see Vinod, 1976; Leurgans et al., 1993; Shawe-Taylor & Cristianini, 2004). When one block is reduced to only one variable, optimization problem (23) is equivalent to the simple continuum regression approach proposed by Qannari and Hanafi (2005). Using the stationary equations (14) we get:

- $\mathbf{a}_1$ is the eigenvector of the matrix $[\tau_1\mathbf{I} + (1-\tau_1)\mathbf{S}_{11}]^{-1}\mathbf{S}_{12}[\tau_2\mathbf{I} + (1-\tau_2)\mathbf{S}_{22}]^{-1}\mathbf{S}_{21}$ related to the largest eigenvalue λ_1^2,
- $\mathbf{a}_2$ is the eigenvector of the matrix $[\tau_2\mathbf{I} + (1-\tau_2)\mathbf{S}_{22}]^{-1}\mathbf{S}_{21}[\tau_1\mathbf{I} + (1-\tau_1)\mathbf{S}_{11}]^{-1}\mathbf{S}_{12}$ related to the largest eigenvalue λ_1^2.

This problem covers a situation which goes from Tucker's (1958) inter-battery factor analysis (or equivalently PLS regression with one component) to canonical correlation analysis while passing through redundancy analysis. The special case $0 \leq \tau_1 \leq 1$ and $\tau_2 = 1$ which corresponds to a regularized version of redundancy analysis has been studied by Takane and Hwang (2007) and by Bougeard, Hanafi, and Qannari (2008) under the name "Continuum redundancy-PLS regression". For various extreme cases $\tau_1 = 0$ or 1 and $\tau_2 = 0$ or 1, we get the methods detailed in Table 2, which corresponds exactly to the framework proposed by Burnham, Viveros, and MacGregor (1996).

5.3. *Hierarchical Models*

The models described in Figure 3 have been called hierarchical models by Wold (1982) and are very common in the chemometric literature. In this paper, the left blocks in each panel

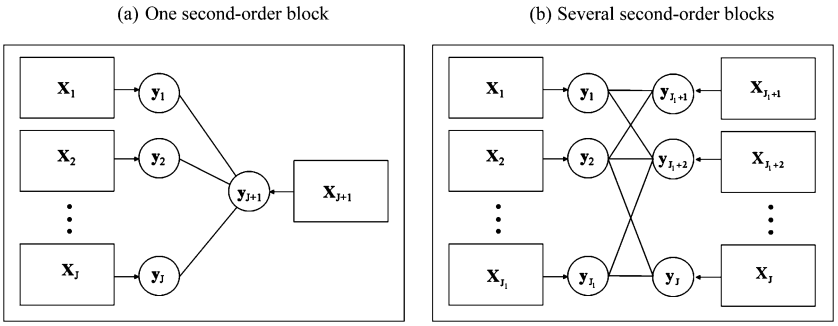

FIGURE 3.
Hierarchical models.

TABLE 3.
Special cases of RGCCA for the hierarchical model with one second-order block.

Method	Criterion	Constraints
Hierarchical inter-battery factor analysis ($\approx$ Hierarchical PLS regression)	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} g(\text{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1}))$	$\|\mathbf{a}_j\| = 1,$ $j = 1,\ldots,J+1$
Hierarchical Canonical Correlation Analysis	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} g(\text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1}))$	$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1,$ $j = 1,\ldots,J+1$
Hierarchical Redundancy analysis of the $\mathbf{X}_j$'s with respect to $\mathbf{X}_{J+1}$	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} g(\text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1}) \times \text{Var}(\mathbf{X}_j\mathbf{a}_j)^{1/2})$	$\|\mathbf{a}_j\| = 1,$ $j = 1,\ldots,J$ $\text{Var}(\mathbf{X}_{J+1}\mathbf{a}_{J+1}) = 1$
Hierarchical Redundancy analysis of $\mathbf{X}_{J+1}$ with respect to the $\mathbf{X}_j$'s	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} g(\text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1}) \times \text{Var}(\mathbf{X}_{J+1}\mathbf{a}_{J+1})^{1/2})$	$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1,$ $j = 1,\ldots,J$ $\|\mathbf{a}_{J+1}\| = 1$

are called first-order blocks and are often considered as predictor blocks. The right blocks are called second-order blocks and are often considered as response blocks. We consider J_1 first-order blocks $\mathbf{X}_1,\ldots,\mathbf{X}_{J_1}$ and $J - J_1$ second-order block $\mathbf{X}_{J_1+1},\ldots,\mathbf{X}_J$. We will discuss two cases: the one second-order block case (Figure 3a) and the several second-order blocks case (Figure 3b).

5.3.1. Hierarchical Model with One Second-order Block The following optimization problem:

$$\begin{aligned} &\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \quad \sum_{j=1}^{J} g\big(\text{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1})\big) \\ &\text{subject to the constraints:} \quad \tau_j\|\mathbf{a}_j\|^2 + (1-\tau_j)\,\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1, \quad j = 1,\ldots,J+1 \end{aligned} \tag{24}$$

includes many useful methods that are described in Table 3.

To the best of our knowledge, the methods corresponding to g being equal to the identity (Horst scheme) or to the absolute value (Centroid scheme) are new. When g is equal to the square

TABLE 4.
Special cases of RGCCA for hierarchical multi-block data analysis.

Method	Criterion	Constraints
SUMCOR (Horst, 1961)	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} \text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1})$ or $\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J} \lvert\text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1})\rvert$	$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1,$ $j = 1,\ldots,J+1$
Generalized CCA (Carroll, 1968a, 1968b)	$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \sum_{j=1}^{J_1} \text{Cor}^2(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1}) + \sum_{j=J_1+1}^{J} \text{Cov}^2(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1})$	$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1,$ $j = 1,\ldots,J_1, J+1$ $\lVert\mathbf{a}_j\rVert = 1,$ $j = J_1+1,\ldots,J$

function (factorial scheme), optimization problem (24) is similar to "concordance analysis" proposed by Hanafi and Lafosse (2001). An application of concordance analysis using the shrinkage constants $\tau_j = 0$ for $j = 1,\ldots,J$ and $\tau_{J+1} = 1$ (line 4 of Table 3 with g being equal to the square function) has been proposed by Bougeard et al. (2007). These authors note that, due to the choice of the shrinkage constants, this method is oriented towards the construction of predictor block components correlated as much as possible with a stable (large variance) response block component. If the objective is to build good prediction of the response block with stable predictor block components, then the methods corresponding to line 3 of Table 3 would be more adapted. An illustration of the latter method is shown in the application section.

When the second-order block $\mathbf{X}_{J+1}$ is equal to the concatenation of the first-order blocks $\mathbf{X}_1,\ldots,\mathbf{X}_J$ ($\mathbf{X}_{J+1} = [\mathbf{X}_1,\ldots,\mathbf{X}_J]$ and is called the super-block), this hierarchical model leads to the Horst's and Carroll's GCCA (see Table 4).

The proofs that the optimization problems given in the second column of Table 4 yield the same optimal solutions $\mathbf{a}_1,\ldots,\mathbf{a}_J$ (and, for Carroll's GCCA, the same global component $\mathbf{X}_{J+1}\mathbf{a}_{J+1}$) as the methods given in the first column of Table 4 are given in Appendix B. The interest of these methods, compared to the previous multi-block data analysis methods, lies in the computation of a global component related to the super-block. Furthermore, one might notice that "Multiple Co-inertia Analysis" proposed by Chessel and Hanafi (1996) is in fact a special case of Carroll's GCCA with only the covariance criterion being used ($J_1 = 0$ in Table 4). Finally, it is worth mentioning that Dahl and Naes (2006) and Takane, Hwang, and Abdi (2008) incorporated other ridge type regularization into Carroll's GCCA.

5.3.2. Hierarchical Model with Several Second-order Blocks All second-order blocks are not necessarily connected to all first-order blocks. Therefore, in this section, we need to introduce the design coefficient c_{jk} equal to 1 if the second-order block $\mathbf{X}_k$ is related to the first-order block $\mathbf{X}_j$ and 0 otherwise. The optimization problem

$$\begin{aligned} &\underset{\mathbf{a}_1,\ldots,\mathbf{a}_J}{\text{Maximize}} \quad \sum_{j=1}^{J_1}\sum_{k=J_1+1}^{J} c_{jk}\, g\big(\text{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k)\big) \\ &\text{subject to the constraints:} \quad \tau_j\lVert\mathbf{a}_j\rVert^2 + (1-\tau_j)\,\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1, \quad j = 1,\ldots,J \end{aligned} \tag{25}$$

corresponds to this kind of design. It contains "generalized orthogonal multiple co-inertia analysis" proposed by Vivien and Sabatier (2003) as a particular case. The first step of this method

consists in the following optimization problem:

$$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_J}{\text{Maximize}} \quad \sum_{j=1}^{J_1}\sum_{k=J_1+1}^{J} \text{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_k\mathbf{a}_k) \tag{26}$$

$$\text{subject to the constraints:} \quad \|\mathbf{a}_j\| = 1, \quad j = 1,\ldots,J.$$

Therefore, this method is a special case of RGCCA applied to hierarchical data, using the Horst scheme and new mode A for all blocks. The other steps of this method consist in applying (26) to deflated blocks obtained through various strategies described in Vivien and Sabatier (2003).

5.4. PLS Path Modeling

Wold (1982) proposed the PLS approach to structural equation modeling (SEM) as a component-based alternative to the covariance-based SEM proposed by Jöreskog (1970). The final version of the PLS approach (also called PLS path modeling) is described in detail in Wold (1985). This algorithm has been modified and extended by Lohmöller (1989) and by Krämer (2007). The most recent review of the PLS approach can be found in Tenenhaus, Esposito Vinzi, Chatelin, and Lauro (2005). In PLS path modeling, we consider that each block $\mathbf{X}_j$ is the expression of an unobservable latent variable (LV) and that structural relations (i.e., multiple regression equations) exist between the latent variables. We consider that two blocks $\mathbf{X}_j$ and $\mathbf{X}_k$ are connected if the associated latent variables are related: LV($\mathbf{X}_j$) explains LV($\mathbf{X}_k$) or vice versa. Let us define the design matrix $\mathbf{C}$: $c_{jk} = 1$ if blocks $\mathbf{X}_j$ and $\mathbf{X}_k$ are connected in the structural equation model and $= 0$ otherwise. The PLS algorithm relies on two modes for the latent variable outer estimations (A or B) and four schemes for the latent variable inner estimations (Horst, centroid, factorial and structural). When mode B and the Horst, centroid, or factorial schemes are selected, the inner and outer latent variable estimations of PLS path modeling are identical to the inner and outer components of RGCCA. Mode A of PLS path modeling and new mode A of RGCCA lead, however, to different outer components. Both approaches will be compared in further studies.

6. The Russett Example

RGCCA is somewhere between multi-block data analysis and component-based structural equation modeling (Tenenhaus, 2008; Tenenhaus & Hanafi, 2010), and therefore uses ingredients of both approaches. This double filiation is illustrated on the Russet data (Russett, 1964) previously studied in Gifi (1990).

6.1. Data Description

Three blocks of variables have been defined for 47 countries. The first block $\mathbf{X}_1 =$ {GINI, FARM, RENT} is related to "Agricultural Inequality": GINI = Inequality of land distribution, FARM = % farmers that own half of the land (the percent larger than 50 is selected), RENT = % farmers that rent all their land. The second block $\mathbf{X}_2 =$ {GNPR, LABO} describes "Industrial Development": GNPR = Gross national product per capita in 1955, LABO = % of labor force employed in agriculture. The third one $\mathbf{X}_3 =$ {INST, ECKS, DEAT} measures "Political Instability": INST = Instability of Executive (1945–1961), ECKS = Number of violent internal war incidents (1946–1961), DEAT = Number of people killed as a result of civic group violence (1950–1962). An additional variable DEMO describes the political regime: stable democracy, unstable democracy, dictatorship. These data have been used in Gifi (1990) to illustrate nonlinear canonical correlation analysis. This method has been applied on the two blocks {GINI, FARM,

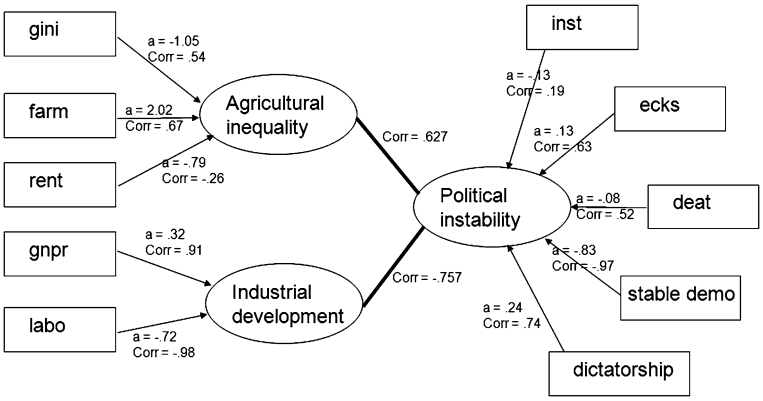

FIGURE 4.
Russett data: Factorial scheme + Mode B.

RENT, GNPR, LABO} and {INST, ECKS, DEAT, DEMO}. Optimal transformations of these variables are given in Gifi (p. 230). We have approximated these optimal transformations. GINI and FARM have not been transformed. The following parametric transformations have been used for the other variables: Ln(RENT+1), Ln(GNPR), Ln(LABO), Exp(INST−16.3), Ln(ECKS+1) and Ln(DEAT+1). The missing data of variable Ln(RENT+1) have been estimated (using multiple regression) for Australia, Nicaragua and Peru by the values 3.27, 2.39 and 2.61. Gifi had some regrets about their analysis. Here is what they wrote on page 226: "*We have chosen to use canonical correlation analysis for the two sets of variables GINI, FARM, RENT, GNPR, LABO versus INST, ECKS, DEAT, DEMO. This is not necessarily the most rational choice. Both Russett's discussion and the definition of the variables suggest that it may be preferable to use three-set canonical correlation analysis.*" Therefore, we have the opportunity in this paper to both repair Gifi's regrets and illustrate RGCCA.

6.2. *Drawing Conventions*

In RGCCA, we use drawing conventions similar to the PLS path modeling ones. Each block $\mathbf{X}_j$ is represented by an ellipse, each variable $\boldsymbol{x}_{jh}$ by a rectangle. The ellipses contain the block names, and the rectangles contain the variable names. Each variable is connected to its block by an arrow. For new mode A ($\tau = 1$), the arrow goes from the ellipse to the rectangle in order to symbolize that each outer weight is computed by simple regression of a block variable on the block inner component. For mode B ($\tau = 0$), the arrow goes from the rectangle to the ellipse in order to symbolize that the vector of outer weights for a block is computed by multiple regression of the block inner component on the block variables. Finally, for mode ridge, we use double-headed arrows in order to symbolize the continuum between new mode A and mode B. Two connected blocks are linked with a line. The resulting figure is called a model. Two submodels are also considered: the outer model is concerned with the relations between the block variables and their block component and the inner model is concerned with the relations between the block components. The model studied for the Russett data is shown in Figure 4 where the transformed variables are labeled using their original names. In this example, we have decided to connect Agricultural Inequality to Political Instability ($c_{13} = 1$), Industrial Development to Political Instability ($c_{23} = 1$) and to not connect Agricultural Inequality to Industrial Development ($c_{12} = 0$). The dummy variable "unstable democracy" has been left out because of redundancy. All the manifest variables have been standardized.

In Section 6.3, both centroid and factorial schemes are compared using mode B for all blocks (full correlation criterion) or new mode A for all blocks (full covariance criterion). We conclude

TABLE 5.
Comparison of factorial and centroid schemes for mode B.

Criterion to be maximized	Mode B + Factorial scheme	Mode B + Centroid scheme
$\text{Cor}^2(\mathbf{y}_1, \mathbf{y}_3) + \text{Cor}^2(\mathbf{y}_2, \mathbf{y}_3)$	**0.967**	0.966
$\lvert\text{Cor}(\mathbf{y}_1, \mathbf{y}_3)\rvert + \lvert\text{Cor}(\mathbf{y}_2, \mathbf{y}_3)\rvert$	1.384	**1.386**

that there is very little difference between these two schemes. In Section 6.4, the various modes new A, B and Ridge are compared. Some useful recommendations in selecting the most adequate mode will emerge from these comparisons. Finally in Section 6.5, we present a hierarchical version of Barker and Rayens PLS discriminant analysis in order to show the ability of RGCCA to straightforwardly produce a new method.

6.3. Comparison Between Centroid and Factorial Schemes

In this section, we want to compare the centroid and factorial schemes when all blocks are fixed to mode B or to new mode A. RGCCA algorithm described in Figure 2 is implemented in the publicly available R package RGCCA (R Development Core Team, 2009).

Using Mode B This package has been used for analyzing the model shown in Figure 4 using the factorial scheme. The mode B option is shown by building arrows going from the rectangles to the ellipses. For this model and this scheme, the outer components $\mathbf{y}_1 = \mathbf{X}_1\mathbf{a}_1$, $\mathbf{y}_2 = \mathbf{X}_2\mathbf{a}_2$ and $\mathbf{y}_3 = \mathbf{X}_3\mathbf{a}_3$ maximize the criterion

$$\text{Cor}^2(\mathbf{y}_1, \mathbf{y}_3) + \text{Cor}^2(\mathbf{y}_2, \mathbf{y}_3). \tag{27}$$

To ensure that the global optimum was reached, we tried 50,000 different random initial weights. They all led to the same maximum. If the centroid scheme is used instead of the factorial scheme, then the following criterion is maximized:

$$\big|\text{Cor}(\mathbf{y}_1, \mathbf{y}_3)\big| + \big|\text{Cor}(\mathbf{y}_2, \mathbf{y}_3)\big|. \tag{28}$$

Here too, to check for global optimality, we tried 50,000 different random initial weights and they all led to the same maximum. The Russett model analyzed with the centroid scheme and mode B is not shown. In Table 5, the factorial and centroid schemes with mode B are compared.

Practice supports theory: the square value criterion is larger for "Mode B + Factorial scheme" than for "Mode B + Centroid scheme," as it should, and we see the opposite case for the absolute value criterion. The very small difference between the criterion values in both situations confirms a well known result (Noonan & Wold, 1982): the values of the outer components are not very sensitive to the choice of the factorial or centroid schemes.

Using New Mode A In Figure 5, we report new mode A results for the factorial scheme. The new mode A option is shown by building arrows going from the ellipses to the rectangles.

Factorial and centroid schemes are compared in Table 6. We also checked that the maximum was reached for both criteria by trying 50,000 different random initial weights. Here too, practice supports theory and the values of the outer components are not very sensitive to the choice of the factorial or centroid schemes.

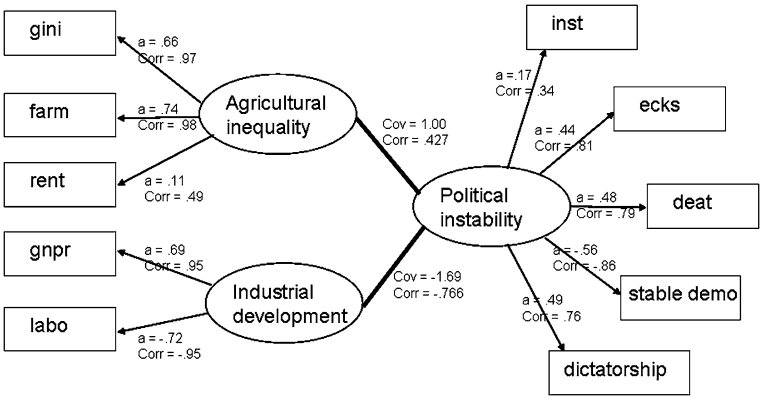

FIGURE 5.
Russett data: Factorial scheme + New mode A.

TABLE 6.
Comparison of factorial and centroid schemes for new mode A.

Criterion to be maximized	New mode A + Factorial scheme	New mode A + Centroid scheme
$\mathrm{Cov}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_3\mathbf{a}_3)^2 + \mathrm{Cov}(\mathbf{X}_2\mathbf{a}_2, \mathbf{X}_3\mathbf{a}_3)^2$[a]	**3.8711**	3.8676
$\lvert\mathrm{Cov}(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_3\mathbf{a}_3)\rvert + \lvert\mathrm{Cov}(\mathbf{X}_2\mathbf{a}_2, \mathbf{X}_3\mathbf{a}_3)\rvert$[a]	2.6952	**2.6964**

[a]with $\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| = 1$

6.4. *Comparison Between Modes*

In this section, we choose to compare the various modes when the factorial scheme is used. Using the idea of average variance explained (AVE), the following indicators of model quality are defined:

$$\text{For one block } \mathbf{X}_j: \quad \mathrm{AVE}(\mathbf{X}_j) = (1/p_j)\sum_{h=1}^{p_j}\mathrm{Cor}^2(\boldsymbol{x}_{jh}, \mathbf{y}_j).$$

$$\text{For all blocks:} \quad \mathrm{AVE}(\text{outer model}) = \left(1\Big/\sum_j p_j\right)\sum_j p_j\mathrm{AVE}(\mathbf{X}_j).$$

$$\text{For the inner model:} \quad \mathrm{AVE}(\text{inner model}) = \left(1\Big/\sum_{j<k} c_{jk}\right)\sum_{j<k} c_{jk}\mathrm{Cor}^2(\mathbf{y}_j, \mathbf{y}_k).$$

Using these AVEs, new mode A, mode B and mode Ridge associated with the factorial scheme are compared in Table 7.

As in the Russett example, all the manifest variables are standardized, the block AVEs and the AVE(outer model) are maximum when the block outer components $\mathbf{y}_j$ are the first principal components of each block $\mathbf{X}_j$. These maximal AVEs are given in Table 7 in the column labeled "mode PCA".

TABLE 7.
Russett data: AVE results summary for the factorial scheme.

	Mode PCA	New Mode A	Mode Ridge with optimal shrinkage constants $\tau_1 = 0.0924$, $\tau_2 = 0.0282$, $\tau_3 = 0.0879$	Mode B
AVE (Agricultural Inequality)	0.7432	0.7225	0.4566	0.2696
AVE (Industrial Development)	0.9075	0.9074	0.8985	0.8956
AVE (Political Instability)	0.5423	0.5412	0.4954	0.4387
AVE (outer model)	0.6756	0.6688	0.5644	0.4793
AVE (inner model)	0.3603	0.3851	0.4594	0.4834

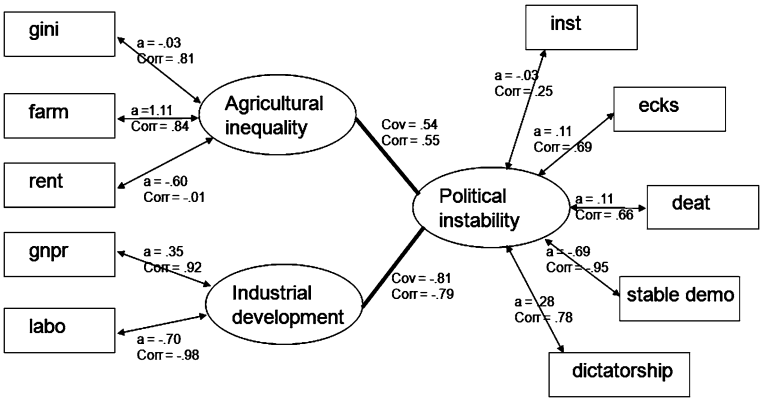

FIGURE 6.
Russett data: Factorial scheme + Mode Ridge (with optimal shrinkage constants: $\tau_1 = 0.0924$, $\tau_2 = 0.0282$, $\tau_3 = 0.0879$).

New Mode A For new mode A + factorial scheme, the maximized criterion is here written as

$$\text{Cor}^2(\mathbf{X}_1\mathbf{a}_1, \mathbf{X}_3\mathbf{a}_3)\,\text{Var}(\mathbf{X}_1\mathbf{a}_1)\,\text{Var}(\mathbf{X}_3\mathbf{a}_3) + \text{Cor}^2(\mathbf{X}_2\mathbf{a}_2, \mathbf{X}_3\mathbf{a}_3)\,\text{Var}(\mathbf{X}_2\mathbf{a}_2)\,\text{Var}(\mathbf{X}_3\mathbf{a}_3)$$

with the constraints $\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| = 1$. Therefore, this optimization problem leads to results between mode PCA and mode B. However, as we see in Table 7, column new mode A is very close to column mode PCA. In fact the block outer components computed using the new mode A are very close to the block first principal components. This is generally the result when the blocks are fairly unidimensional. The reason for this result has been mentioned in Section 3: when new mode A is used, the variance terms dominate over the correlation terms. This result is also true for the usual mode A of PLS path modeling (see Tenenhaus, 2008).

Mode B The AVE(inner model) is maximum for mode B as this is exactly the criterion maximized by mode B + factorial scheme. However, when mode B is used, the value of AVE(Agricultural Inequality) is small: the block component fails to explain its block.

Mode Ridge For mode Ridge, using the Schäfer & Strimmer formula, we have computed the optimal shrinkage constants $\tau_1 = 0.0924$, $\tau_2 = 0.0282$ and $\tau_3 = 0.0879$. Results for "Mode Ridge + Factorial scheme" are shown in Figure 6 and Table 7. To figure out the mode Ridge, double-headed arrows are used for connecting rectangles and ellipses.

In mode Ridge, the AVEs are located between those of new mode A and mode B. New mode A is favoring the outer model and mode B is favoring the inner model. Mode Ridge appears to be a useful compromise between new mode A and mode B. As compared to mode B, mode Ridge yields components with higher block AVEs, specifically for Agricultural Inequality, and comparable AVE(inner model).

Recommendations New mode A is recommended when the user wants a stable component (large variance) while taking into account the correlations with connected blocks. The user must, however, be aware that variance dominates over correlation. Mode B is recommended when the user wants to maximize correlations between connected components. This option can yield unstable solutions in case of multicollinearity and cannot be used when a data block is not of full rank. Mode Ridge is a good compromise between new mode A and mode B: the block component is simultaneously stable and as well correlated as possible with its connected components. Mode Ridge can be used when the data block is not of full rank. The shrinkage constants can be determined by using the Schäfer and Strimmer formula.

6.5. Hierarchical Barker and Rayens PLS-DA

It is possible to mix the modes of the various blocks. A special case of this problem is the PLS method for discrimination proposed by Barker and Rayens (2003). They consider a block $\mathbf{X}$ of explanatory variables and a block $\mathbf{Y}$ of dummy variables describing a categorical variable. They are looking for a component $\mathbf{Xa}$ (with $\mathbf{a}$ normalized) and a standardized component $\mathbf{Yb}$ maximizing the criterion

$$\operatorname{Cov}(\mathbf{Xa}, \mathbf{Yb}) = \operatorname{Cor}(\mathbf{Xa}, \mathbf{Yb}) \times \operatorname{Var}(\mathbf{Xa})^{1/2}. \tag{29}$$

The solution of the Barker and Rayens PLS-DA is therefore obtained by running a redundancy analysis of block $\mathbf{X}$ with respect to block $\mathbf{Y}$. Thus, Barker and Rayens PLS-DA is an intermediate solution between discriminant analysis (maximize $\operatorname{Cor}(\mathbf{Xa}, \mathbf{Yb})$ subject to $\mathbf{Xa}$ and $\mathbf{Yb}$ standardized) and PLS-DA (maximize $\operatorname{Cov}(\mathbf{Xa}, \mathbf{Yb}) = \operatorname{Cor}(\mathbf{Xa}, \mathbf{Yb}) \times \operatorname{Var}(\mathbf{Xa})^{1/2} \times \operatorname{Var}(\mathbf{Yb})^{1/2}$ subject to $\mathbf{a}$ and $\mathbf{b}$ normalized). Suppose now that several blocks $\mathbf{X}_1, \ldots, \mathbf{X}_J$ of explanatory variables are used to predict the block $\mathbf{Y}$ of dummy variables describing the categorical variable. It is then rather natural to extend the Barker and Rayens PLS-DA to a new method called hierarchical Barker and Rayens PLS-DA. Therefore, we look for components $\mathbf{X}_1\mathbf{a}_1, \ldots, \mathbf{X}_J\mathbf{a}_J$ (with $\|\mathbf{a}_j\| = 1$, $j = 1, \ldots, J$) and for a standardized component $\mathbf{Yb}$ maximizing the criterion

$$\sum_{j=1}^{J} g\big[\operatorname{Cov}(\mathbf{X}_j\mathbf{a}_j, \mathbf{Yb})\big] = \sum_{j=1}^{J} g\big[\operatorname{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{Yb}) \operatorname{Var}(\mathbf{X}_j\mathbf{a}_j)^{1/2}\big], \tag{30}$$

where g is the identity (Horst scheme), the square (factorial scheme), or the absolute value (centroid scheme). In this problem, new mode A is used for the explanatory blocks $\mathbf{X}_1, \ldots, \mathbf{X}_J$; and mode B is used for the response block $\mathbf{Y}$.

We may illustrate the hierarchical Barker and Rayens PLS discriminant analysis by relating the block formed by the Agricultural Inequality ($\mathbf{X}_1$) and the block formed by Industrial Development ($\mathbf{X}_2$) to the $\mathbf{Y}$ block formed by the political regimes "stable democracy" and "dictatorship". Here too, the dummy variable "unstable democracy" is left out because of redundancy. RGCCA is applied to the three blocks with $\mathbf{X}_1, \mathbf{X}_2$ as first-order blocks and $\mathbf{Y}$ as the second-order block. When the new mode A is used for all blocks, hierarchical PLS-DA is obtained. When mode B is used for all blocks, hierarchical factorial discriminant analysis is obtained. At last, when the new mode A is used for the $\mathbf{X}$ blocks and mode B for $\mathbf{Y}$, we obtain hierarchical redundancy analysis

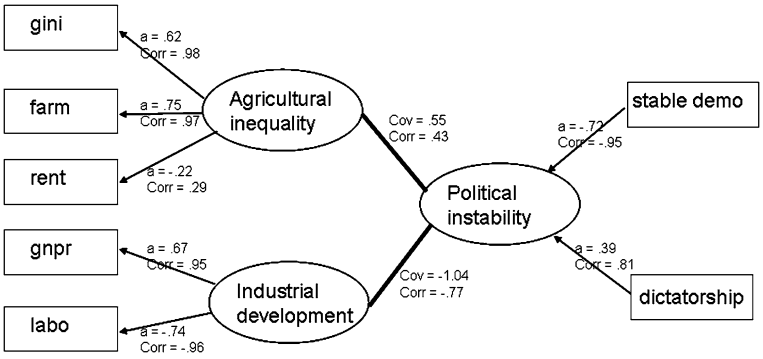

FIGURE 7.
Hierarchical Barker and Rayens PLS discriminant analysis.

of $\mathbf{X}_1, \mathbf{X}_2$ with respect to $\mathbf{Y}$ (i.e., hierarchical Barker & Rayens PLS-DA). Using the factorial scheme, this last method leads to maximizing the criterion

$$\mathrm{Cov}^2(\mathbf{X}_1\mathbf{a}_1, \mathbf{Yb}) + \mathrm{Cov}^2(\mathbf{X}_2\mathbf{a}_2, \mathbf{Yb}) \tag{31}$$

subject to the constraints:

$$\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \mathrm{Var}(\mathbf{Yb}) = 1. \tag{32}$$

This is equivalent to maximizing the criterion

$$\mathrm{Cor}^2(\mathbf{X}_1\mathbf{a}_1, \mathbf{Yb})\,\mathrm{Var}(\mathbf{X}_1\mathbf{a}_1) + \mathrm{Cor}^2(\mathbf{X}_2\mathbf{a}_2, \mathbf{Yb})\,\mathrm{Var}(\mathbf{X}_2\mathbf{a}_2) \tag{33}$$

subject to the constraints:

$$\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = 1. \tag{34}$$

We get the components from Figure 7 (* signifies standardized)

$$\mathbf{X}_1\mathbf{a}_1 = 0.62\mathrm{Gini}^* + 0.75\mathrm{Farm}^* - 0.22\mathrm{Rent}^*,$$
$$\mathbf{X}_2\mathbf{a}_2 = 0.67\mathrm{GNPR}^* - 0.74\mathrm{Labo}^*,$$
$$\mathbf{Yb} = -0.72(\text{Stable Democracy})^* + 0.39(\text{Dictatorship})^*.$$

The graphical display of the countries obtained by crossing $\mathbf{X}_1\mathbf{a}_1$ = Agricultural Inequality and $\mathbf{X}_2\mathbf{a}_2$ = Industrial Development and marked with their political regime in 1960 is shown in Figure 8. It may be noted that the upper left quadrant concentrates on dictatorships. It is difficult for a country to escape dictatorship when its industrial development is below-average and its agricultural inequality is above average. It is worth pointing out that some unstable democracies located in this quadrant (or close to it) became dictatorships for a period of time after 1960: Greece (1967–1974), Brazil (1964–1985), Chili (1973–1990), and Argentina (1966–1973).

7. Conclusion

Regularized Generalized Canonical Correlation Analysis is a very attractive and general framework for multi-block data analysis with good generalization in the case where each block

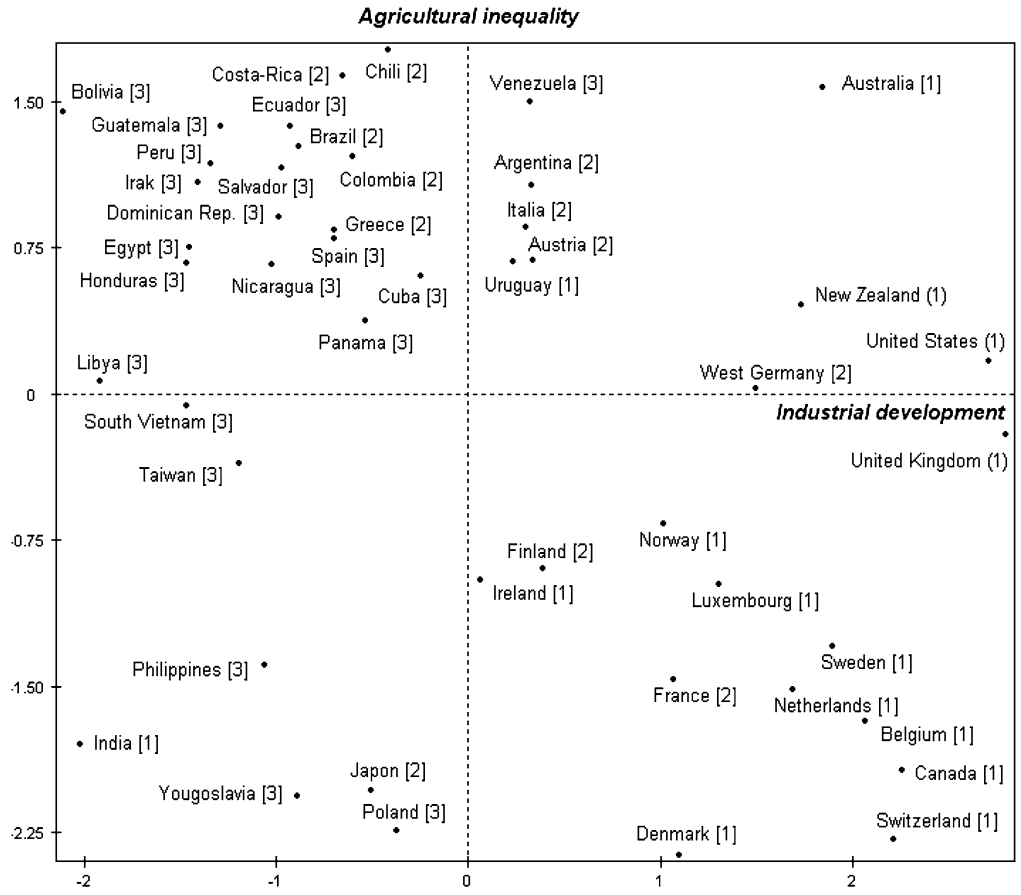

FIGURE 8.
Hierarchical Barker and Rayens PLS discriminant analysis. Graphical display of countries marked with their political regime in 1960 (1 = stable democracy, 2 = unstable democracy, 3 = dictatorship).

is connected to only a subset of blocks. By defining the design matrix, the shrinkage constants and the function g, RGGCA includes a remarkably large spectrum of methods as particular cases. In this article we have also tried to define a guideline for the choice of the shrinkage constants by providing interpretations on the properties of the resulting block components. If one wants to give priority to the correlation between a component and its neighboring components, mode B ($\tau_j = 0$) is the natural choice. Conversely, if one wants a component which explains its own block well, and yet remains correlated to its neighboring components, new mode A ($\tau_j = 1$) is the natural choice. For a better compromise between variance and correlation, mode Ridge is proposed in this paper. For each block, the determination of the shrinkage constant can be made fully automatic by using the analytical formula proposed by Schäfer and Strimmer (2005). In contrast to other common approaches that are based on cross-validation or bootstrap strategies, the computational effort of this technique is very attractive. Moreover, the regularization aspect of RGCCA allows one to come closer to mode B even in the case of high multicollinearity within blocks or when the number of individuals is smaller than the number of variables.

Several points have been eluded in this paper but will be investigated in future research. (i) Computing several components for each block: In this paper, we supposed that each block is summarized by one block component only. Using a deflation strategy, it is possible to compute

additional components for each block which are orthogonal to the previous ones. (ii) Model assessment by bootstrap strategy: RGCCA can be viewed as an estimation procedure for population GCCA. Therefore, using a bootstrap strategy, it is possible to compute confidence intervals for all the elements computed in RGCCA. (iii) Comparison with PLS path modeling: Wold (1982) mentions a drawback of the outer model estimation step of PLS path modeling: "*Most nonlinear iterative techniques of estimation are lacking an analytic proof of convergence. The proof of the pudding is in the eating*". This outer model estimation step can be replaced by RGCCA, a monotonically convergent algorithm. Simulations will be carried out in further studies to compare PLS path modeling (mode A) and RGCCA (new mode A) from a practical point of view. Nevertheless, this paper already suggests that the PLS approach of Herman Wold is much more powerful and founded on much more solid theoretical bases than it seemed to be at first. (iv) Nonlinear relationships between blocks: Finally, RGCCA captures only linear relations between blocks. To assess nonlinear relations, a kernel extension of RGCCA is currently under investigation (Tenenhaus, 2010).

Acknowledgements

The authors are very grateful to Mohamed Hanafi, Yoshio Takane, the editor, the associate editor and three anonymous reviewers for their constructive comments that contributed to improving the quality of the paper. The authors also thank Monika Jelizarow and Vincent Guillemot for their contributions in the programming.

Appendix A. Monotonic Convergence of the PLS Algorithm for Population GCCA

Hanafi (2007) has proven the monotonic convergence of the PLS algorithm of Wold (1985) when the options mode B for all blocks and the centroid or factorial schemes are used. In this appendix, we adapt the proof of Hanafi to the PLS algorithm for population GCCA described in Figure 1. The proof of Proposition 2 relies on the following lemma, where the outer weight vectors $\boldsymbol{\alpha}_j^s$ and the inner components v_j^s are defined in Figure 1:

Lemma A.1. *For $j = 1, \ldots, J$, $s = 0, 1, 2, \ldots$, let f_j^s be the function defined by*

$$f_j^s(\boldsymbol{\alpha}_j) = \sum_{k<j} c_{jk} g\left[\mathrm{Cov}\left((\boldsymbol{\alpha}_j)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s+1}\right)^t \mathbf{x}_k\right)\right] + \sum_{k>j} c_{jk} g\left[\mathrm{Cov}\left((\boldsymbol{\alpha}_j)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s}\right)^t \mathbf{x}_k\right)\right].$$

Then the following property holds:

$$\forall s \quad f_j^s\left(\boldsymbol{\alpha}_j^s\right) \leq f_j^s\left(\boldsymbol{\alpha}_j^{s+1}\right). \tag{35}$$

Proof of Lemma A.1: The function $f_j^s(\boldsymbol{\alpha}_j^s)$ may be written as

$$\begin{aligned} f_j^s\left(\boldsymbol{\alpha}_j^s\right) &= \sum_{k<j} c_{jk} g\left[\mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s+1}\right)^t \mathbf{x}_k\right)\right] + \sum_{k>j} c_{jk} g\left[\mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s}\right)^t \mathbf{x}_k\right)\right] \\ &= \sum_{k<j} c_{jk} w\left[\mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s+1}\right)^t \mathbf{x}_k\right)\right] \mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s+1}\right)^t \mathbf{x}_k\right) \\ &\quad + \sum_{k>j} c_{jk} w\left[\mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s}\right)^t \mathbf{x}_k\right)\right] \mathrm{Cov}\left(\left(\boldsymbol{\alpha}_j^s\right)^t \mathbf{x}_j, \left(\boldsymbol{\alpha}_k^{s}\right)^t \mathbf{x}_k\right) \end{aligned}$$

$$= \operatorname{Cov}\Bigg\{ \big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \sum_{k<j} c_{jk} w\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big)\big]\big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k + \sum_{k>j} c_{jk} w\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big)\big]\big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k \Bigg\}$$

$$= \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, v_j^s\big). \tag{36}$$

Using the definitions of v_j^s and $\boldsymbol{\alpha}_j^{s+1}$, the following inequality holds:

$$f_j^s\big(\boldsymbol{\alpha}_j^s\big) = \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, v_j^s\big) \leq \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, v_j^s\big) \tag{37}$$

and we get the equality

$$\begin{aligned}\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, v_j^s\big) &= \sum_{k<j} c_{jk} w\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big)\big] \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big) \\ &\quad + \sum_{k>j} c_{jk} w\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big)\big] \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big). \end{aligned} \tag{38}$$

Then, we have to consider the three schemes separately.

Horst scheme (g = identity and $w(x) = 1$)

For the Horst scheme, equality (38) becomes

$$\begin{aligned}\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, v_j^s\big) &= \sum_{k<j} c_{jk} \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big) \\ &\quad + \sum_{k>j} c_{jk} \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big) = f_j^s\big(\boldsymbol{\alpha}_j^{s+1}\big) \end{aligned} \tag{39}$$

and therefore $f_j^s(\boldsymbol{\alpha}_j^s) \leq f_j^s(\boldsymbol{\alpha}_j^{s+1})$.

Centroid scheme (g = absolute value and $w(x) = \operatorname{sign}(x)$)

For the centroid scheme

$$f_j^s\big(\boldsymbol{\alpha}_j^s\big) = \sum_{k<j} c_{jk} \big|\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big)\big| + \sum_{k>j} c_{jk} \big|\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big)\big| \tag{40}$$

and

$$\begin{aligned}&\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, v_j^s\big) \\ &= \sum_{k<j} c_{jk} \operatorname{sign}\big(\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big)\big]\big) \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^{s+1}\big)^t \boldsymbol{x}_k\big) \\ &\quad + \sum_{k>j} c_{jk} \operatorname{sign}\big(\big[\operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^s\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big)\big]\big) \operatorname{Cov}\big(\big(\boldsymbol{\alpha}_j^{s+1}\big)^t \boldsymbol{x}_j, \big(\boldsymbol{\alpha}_k^s\big)^t \boldsymbol{x}_k\big). \end{aligned} \tag{41}$$

Therefore, we get

$$f_j^s(\boldsymbol{\alpha}_j^s) = \left|f_j^s(\boldsymbol{\alpha}_j^s)\right| \le \operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, v_j^s\right) = \left|\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, v_j^s\right)\right|$$
$$\le \sum_{k<j} c_{jk}\left|\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right)\right| + \sum_{k>j} c_{jk}\left|\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right)\right|$$
$$= f_j^s(\boldsymbol{\alpha}_j^{s+1}). \tag{42}$$

Factorial scheme (g = square and $w(x) = x$)
For the factorial scheme

$$f_j^s(\boldsymbol{\alpha}_j^s) = \sum_{k<j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^s)^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right) + \sum_{k>j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^s)^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right) \tag{43}$$

and

$$\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, v_j^s\right) = \sum_{k<j} c_{jk}\left[\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right)\right] \operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right)$$
$$+ \sum_{k>j} c_{jk}\left[\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right)\right] \operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right). \tag{44}$$

The right term of (44) is the scalar product between two vectors. Using the Cauchy–Schwartz inequality and the fact that $c_{jk}^2 = c_{jk}$, we get

$$\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, v_j^s\right)$$
$$\le \left[\sum_{k<j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^s)^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right) + \sum_{k>j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^s)^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right)\right]^{1/2}$$
$$\times \left[\sum_{k<j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right) + \sum_{k>j} c_{jk} \operatorname{Cov}^2\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right)\right]^{1/2}. \tag{45}$$

From that we deduce

$$f_j^s(\boldsymbol{\alpha}_j^s) \le \left[f_j^s(\boldsymbol{\alpha}_j^s)\right]^{1/2}\left[f_j^s(\boldsymbol{\alpha}_j^{s+1})\right]^{1/2} \tag{46}$$

and therefore

$$f_j^s(\boldsymbol{\alpha}_j^s) \le f_j^s(\boldsymbol{\alpha}_j^{s+1}). \tag{47}$$

□

Proof of Proposition 2: Proposition 2 is deduced from Lemma A.1 and the following equality:

$$\sum_{j=1}^{J}\left[f_j^s(\boldsymbol{\alpha}_j^{s+1}) - f_j^s(\boldsymbol{\alpha}_j^s)\right] = \sum_{k<j} c_{jk} g\left[\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right)\right]$$
$$+ \sum_{k>j} c_{jk} g\left[\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\right)\right]$$
$$- \sum_{k<j} c_{jk} g\left[\operatorname{Cov}\left((\boldsymbol{\alpha}_j^{s})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\right)\right]$$

$$
\begin{aligned}
&- \sum_{k>j} c_{jk} g\big[\mathrm{Cov}\big((\boldsymbol{\alpha}_j^s)^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^s)^t \mathbf{x}_k\big)\big] \\
&= \frac{1}{2} \sum_{j,k=1, j\neq k}^{J} c_{jk} g\big[\mathrm{Cov}\big((\boldsymbol{\alpha}_j^{s+1})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s+1})^t \mathbf{x}_k\big)\big] \\
&\quad - \frac{1}{2} \sum_{j,k=1, j\neq k}^{J} c_{jk} g\big[\mathrm{Cov}\big((\boldsymbol{\alpha}_j^{s})^t \mathbf{x}_j, (\boldsymbol{\alpha}_k^{s})^t \mathbf{x}_k\big)\big] \\
&= \frac{1}{2}\big[f\big(\boldsymbol{\alpha}_1^{s+1}, \ldots, \boldsymbol{\alpha}_J^{s+1}\big) - f\big(\boldsymbol{\alpha}_1^{s}, \ldots, \boldsymbol{\alpha}_J^{s}\big)\big]. \qquad \square
\end{aligned}
$$

Appendix B. Proof of the Equivalences Mentioned in Table 4

B.1. SUMCOR

The stationary equations for the SUMCOR method (see first line of Table 1) are obtained by using Equation (13) with all $c_{jk} = 1$ for $j \neq k$, with the Horst scheme and Equation (14) with all $\tau_j = 0$. Pre-multiplying these equations by $\mathbf{X}_j$, they become

$$
\mathbf{y}_j = \mathbf{X}_j \mathbf{a}_j \propto \mathbf{P}_j \left(\sum_{k\neq j} \mathbf{y}_k \right), \tag{48}
$$

where $\propto$ means that the left side is the standardized version of the right side and where $\mathbf{P}_j = \mathbf{X}_j (\mathbf{X}_j^t \mathbf{X}_j)^{-1} \mathbf{X}_j^t$ is the orthogonal projection operator onto the $\mathbf{X}_j$-space. We now consider the stationary equations of the following optimization problem (mode B for all blocks and Horst scheme):

$$
\begin{aligned}
&\underset{\mathbf{a}_1, \ldots, \mathbf{a}_{J+1}}{\text{Maximize}} \quad \sum_{j=1}^{J} \mathrm{Cor}(\mathbf{X}_j \mathbf{a}_j, \mathbf{X}_{J+1} \mathbf{a}_{J+1}) \\
&\text{subject to the constraints:} \quad \mathrm{Var}(\mathbf{X}_j \mathbf{a}_j) = 1, \quad j = 1, \ldots, J+1,
\end{aligned} \tag{49}
$$

where $\mathbf{X}_{J+1} = [\mathbf{X}_1, \ldots, \mathbf{X}_J]$ is the super-block. In this situation $c_{j,J+1} = 1$ for $j = 1, \ldots, J$ and $c_{jk} = 0$ otherwise. The stationary equations for this optimization problem are

$$
\mathbf{y}_j = \mathbf{X}_j \mathbf{a}_j \propto \mathbf{P}_j (\mathbf{X}_{J+1} \mathbf{a}_{J+1}), \tag{50}
$$

$$
\mathbf{y}_{J+1} = \mathbf{X}_{J+1} \mathbf{a}_{J+1} \propto \mathbf{P}_{J+1} \left(\sum_{j=1}^{J} \mathbf{X}_j \mathbf{a}_j \right). \tag{51}
$$

However, as $\sum_{j=1}^{J} \mathbf{X}_j \mathbf{a}_j$ belongs to the space generated by the super-block $\mathbf{X}_{J+1}$, we get

$$
\mathbf{y}_{J+1} \propto \sum_{j=1}^{J} \mathbf{y}_j. \tag{52}
$$

Therefore, (50) becomes

$$
\mathbf{y}_j \propto \mathbf{P}_j \left(\sum_{k=1}^{J} \mathbf{y}_k \right) = \mathbf{y}_j + \mathbf{P}_j \left(\sum_{k\neq j} \mathbf{y}_k \right) \tag{53}
$$

and Equation (48) is found again. The equivalence mentioned in Table 4 is also valid if the centroid scheme is used because the sign of $\text{Cor}(\mathbf{X}_j\mathbf{a}_j, \mathbf{X}_{J+1}\mathbf{a}_{J+1})$ is equal to the sign of $\mathbf{y}_{J+1}^t\mathbf{P}_j\mathbf{y}_{J+1}$ and therefore always positive.

B.2. Carroll's Generalized CCA

In his 1968b paper, Carroll considers the "Mixed" correlation and covariance criterion which consists in the following optimization problem:

$$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_J,\mathbf{z}}{\text{Maximize}} \quad \sum_{j=1}^{J_1}\text{Cor}^2(\mathbf{X}_j\mathbf{a}_j,\mathbf{z}) + \sum_{j=J_1+1}^{J}\text{Cov}^2(\mathbf{X}_j\mathbf{a}_j,\mathbf{z}) \tag{54}$$

subject to the constraints:

$$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = \text{Var}(\mathbf{z}) = 1, \quad j = 1,\ldots,J_1 \quad \text{and} \quad \|a_j\| = 1, \quad j = J_1+1,\ldots,J.$$

The solution of this optimization problem is obtained for the eigenvector $\mathbf{z}$ associated with the largest eigenvalue of the matrix $\mathbf{Q} = \sum_{j=1}^{J_1}\mathbf{X}_j(\mathbf{X}_j^t\mathbf{X}_j)^{-1}\mathbf{X}_j^t + \sum_{j=J_1+1}^{J}\mathbf{X}_j\mathbf{X}_j^t$. As the eigenvector $\mathbf{z}$ belongs to the space generated by the columns of $\mathbf{X}_{J+1}$, optimization problem (54) is equivalent to the following optimization problem:

$$\underset{\mathbf{a}_1,\ldots,\mathbf{a}_{J+1}}{\text{Maximize}} \quad \sum_{j=1}^{J_1}\text{Cor}^2(\mathbf{X}_j\mathbf{a}_j,\mathbf{X}_{J+1}\mathbf{a}_{J+1}) + \sum_{j=J_1+1}^{J}\text{Cov}^2(\mathbf{X}_j\mathbf{a}_j,\mathbf{X}_{J+1}\mathbf{a}_{J+1}) \tag{55}$$

subject to the constraints:

$$\text{Var}(\mathbf{X}_j\mathbf{a}_j) = 1, \quad j = 1,\ldots,J_1,J+1 \quad \text{and} \quad \|a_j\| = 1, \quad j = J_1+1,\ldots,J.$$

And this shows the equivalence announced in Table 4.

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Manuscript Received: 13 DEC 2009
Final Version Received: 29 SEP 2010
Published Online Date: 17 MAR 2011