Fast Inverse Sqrt

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32 bits

$\underset{\text{not 0, must be 1}}{1}.011010 \times 2^{10110}$ $\underset{\text{not 0}}{3}.7645 \times 10^{17}$

1 sign bit, 8 exponent bits, 23 mantissa bits

[-127, 128] (no 1 because first digit is always 1).

Let E be the value of the exponent.

Let M be the value of the mantissa.

Let B be the value of the binary digits of the float.

Let V be the value encoded by B

$$B = \underbrace{2^{23} \cdot E}_{\text{shifts bits into place}} + M$$

$$V = \underbrace{\left(1 + \frac{M}{2^{23}}\right)}_{\substack{M \text{ is a 23 bit number but should be } 0-1. \\ +1 \text{ to get mantissa for standard form}}} \times 2^{\overbrace{E-127}^{-127 \text{ because it includes negatives}}}$$

$$\log_2(V) = \log_2\left(1 + \frac{M}{2^{23}}\right) + E - 127$$

Note: $\log(1+x) \approx x + \mu$ where μ is a constant

You can find the best value of μ by just trying different ones to find the best fit.

$$\Rightarrow \log_2(V) = \frac{M}{2^{23}} + \mu + E - 127$$

$$= \frac{M + 2^{23}E}{2^{23}} + \mu - 127$$

$$= \frac{1}{2^{23}}\left(M + 2^{23}E\right) + \mu - 127$$

$$\log_2(v) = \frac{1}{2^{23}} B + \mu - 127$$

The value of the bit representation of v is related to $\log_2(v)$.

Note: $\frac{1}{\sqrt{v}} = v^{-\frac{1}{2}}$ which is what we want to calculate.

$$\log\left(\frac{1}{\sqrt{v}}\right) = \log\left(v^{-\frac{1}{2}}\right) = \frac{-1}{2}\log(v)$$

Define our answer as Γ such that

$$\Gamma = \frac{1}{\sqrt{v}}$$

$$\Rightarrow \log_2(\Gamma) = \log_2\left(\frac{1}{\sqrt{v}}\right) = \frac{-1}{2}\log_2(v)$$

$$\Rightarrow \log_2(\Gamma) = \frac{-1}{2}\log_2(v) \qquad (1)$$

We actually want the bit representation of Γ so it can be returned to floating point format.

Recall: $\log_2(v) = \frac{1}{2^{23}} B + \mu - 127$

The same will be true for Γ. We can sub. this into eqn. 1 to solve for the bits of the answer.

$$\frac{1}{2^{23}} B_I + \mu - 127 = \frac{-1}{2}\left(\frac{1}{2^{23}} B_v + \mu - 127\right)$$

$$\frac{1}{2^{23}} B_I = 127 - \mu - \frac{1}{2}\left(\frac{1}{2^{23}} B_v + \mu - 127\right)$$

$$= \underbrace{(127 - \mu)} - \frac{1}{2}\left(\frac{1}{2^{23}} B_v\right) - \frac{1}{2}\underbrace{(\mu - 127)}$$

$$= (127 - \mu) + \frac{1}{2}(127 - \mu) - \frac{1}{2}\left(\frac{1}{2^{23}} B_v\right)$$

$$\frac{1}{2^{23}} B_I = \frac{3}{2}(127 - \mu) - \frac{1}{2}\left(\frac{1}{2^{23}} B_v\right)$$

$$B_I = \frac{3}{2} 2^{23}(127 - \mu) - \frac{1}{2} B_v$$

Note: $\frac{3}{2} \cdot 2^{23}$

$= 3 \cdot \frac{2^{23}}{2^1} = 3 \cdot 2^{22}$

$$B_I = \underbrace{3 \cdot 2^{22}(127 - \mu)}_{\text{Integer constant}} - \underbrace{\frac{1}{2} B_v}_{\frac{-1}{2}\text{ the Bit representation of the input}}$$

You can use Newton's method to improve the approximation.

Also, it turns out $\mu = 0.043$, on average gives the best approximation of $\log_2(1+x)$

$3 \cdot 2^{22}(127 - \mu)$ isn't actually an integer but has so many digits it can be rounded to the nearest integer without a big loss in accuracy.

<u>Newton's method:</u>

Some $f(x_1)$, take the tangent at x_1

$y = f'(x_1) + c$

$y - y_1 = f'(x_1)(x - x_1)$

$$y = \underbrace{f'(x_1)}_{m}x \underbrace{- f'(x_1)x_1 + y_1}_{c}$$

Where does this cross y axis? $y=0$

$$\underbrace{f'(x_1)}x - \underbrace{f'(x_1)x_1 + y_1} = 0$$

$$x = \frac{f'(x_1)x_1 - y_1}{f'(x_1)} = x_1 - \frac{y_1}{f'(x_1)}$$

note: $y_1 = f(x_1)$

$x = x_2$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

we want an equation for which finding the root is the same as finding $\frac{1}{\sqrt{x}}$

① Suppose $y = \frac{1}{\sqrt{x}}$ $y=0$ there are no x

② Suppose $x = \frac{1}{\sqrt{y}}$ $y=0$ $x \to \infty$

③ Introduce some n where we want to find $\frac{1}{\sqrt{n}}$

$y = x - \frac{1}{\sqrt{n}}$

when $y=0$

when $y = 0$

$x = \frac{1}{\sqrt{n}}$

function definition includes $\frac{1}{\sqrt{n}}$ → no good

(4) $x = \frac{1}{\sqrt{n}}$ we need n by itself because we don't want to do any operation on n.

$\sqrt{n} = \frac{1}{x}$

$n = \frac{1}{x^2}$ Now we want it to be the 0 of a function.

$0 = \frac{1}{x^2} - n$

$y = \frac{1}{x^2} - n$ root of this is $\frac{1}{\sqrt{n}}$

$f(x) = \frac{1}{x^2} - n$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

Remember $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_2 = x_1 - \frac{x_1^3}{-2}\left(\frac{1}{x_1^2} - n\right)$$

$$= x_1 + \frac{1}{2}\left(x_1 - x_1^3 n\right)$$

$$= x_1 + \frac{1}{2}x_1 - x_1^3 n = \frac{3}{2}x_1 - \frac{1}{2}x_1^3 n$$

$$= x_1\left(\frac{3}{2} - \frac{1}{2}x_1^2 n\right)$$