Lecture 5: Collections of Random Variables

Professor Ilias Bilionis

Independent random variables

PREDICTIVE
SCIENCE LABORATORY

Independent random variables

- We say that the two random variables are independent conditional on I, and we write:

$$X \perp Y \mid I$$

if and only if conditioning on one does not tell you anything about the other, i.e.,

$$p(x|y, I) = p(x|I)$$

- When there is no ambiguity, we can drop I.

$$p(x|y) = p(x)$$

PREDICTIVE
SCIENCE LABORATORY

Independent random variables

- It is easy to show using Bayes' rule that the definition is consistent. $X \perp Y \mid I \Rightarrow Y \perp X \mid I$

- That is, if Y does not give you any information about X, then X does not give you any information about Y, i.e.,

$p(x|y) = p(x)$

$$\Rightarrow \quad p(y|x) = p(y).$$

Proof: $p(y|x) = \frac{p(x,y)}{p(x)} \underset{\text{Bayes rule}}{=} \ldots = p(y)$

Properties of independent random variables

- Assume X and Y are independent.

- Then, the joint pdf factorizes:

$$\boxed{p(x,y) = p(x)p(y).}$$

Proof: $X \perp Y \Rightarrow p(x|y) = p(x) \Leftrightarrow p(y|x) = p(y)$

$X \perp Y$

Bayes' rule: $p(x,y) = \underline{p(x|y)}\, p(y) = p(x)\, p(y)$ $\blacksquare$

Properties of independent random variables

- Assume X and Y are independent.

- The expectation of the product is the product of the expectation:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Proof:

$$\mathbb{E}[XY] = \iint xy\, p(x,y)\, dx\, dy = \iint xy\, p(x)\, p(y)\, dx\, dy$$

$$= \left(\int x\, p(x)\, dx\right) \cdot \left(\int y\, p(y)\, dy\right) \quad \text{(Fubini's Theorem)}$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

PREDICTIVE
SCIENCE LABORATORY

Properties of independent random variables

- Assume X and Y are independent.

- The covariance is zero:

$$\mathbb{C}[X,Y]=0.$$

<u>Proof</u>: $\mathbb{C}[X,Y] = \mathbb{E}[(X-\mathbb{E}[X])\cdot(Y-\mathbb{E}[Y])]$

$$= \mathbb{E}[XY - X\cdot\mathbb{E}[Y] - \mathbb{E}[X]\cdot Y + \mathbb{E}[X]\cdot\mathbb{E}[Y]]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X\,\underbrace{\mathbb{E}[Y]}_{\text{const.}}] - \mathbb{E}[\underbrace{\mathbb{E}[X]}_{\text{const.}}\cdot Y] + \mathbb{E}[\underbrace{\mathbb{E}[X]\cdot\mathbb{E}[Y]}_{\text{const.}}]$$

$$= \underbrace{\mathbb{E}[XY]}_{\substack{\overset{?}{=}\\ \mathbb{E}[X]\cdot\mathbb{E}[Y] \\ (X\perp Y)}} - \mathbb{E}[Y]\cdot\mathbb{E}[X] - \mathbb{E}[X]\cdot\mathbb{E}[Y] + \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$= 0$$

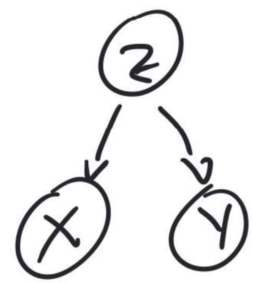
$X \perp Y \Rightarrow$ uncorrelated.

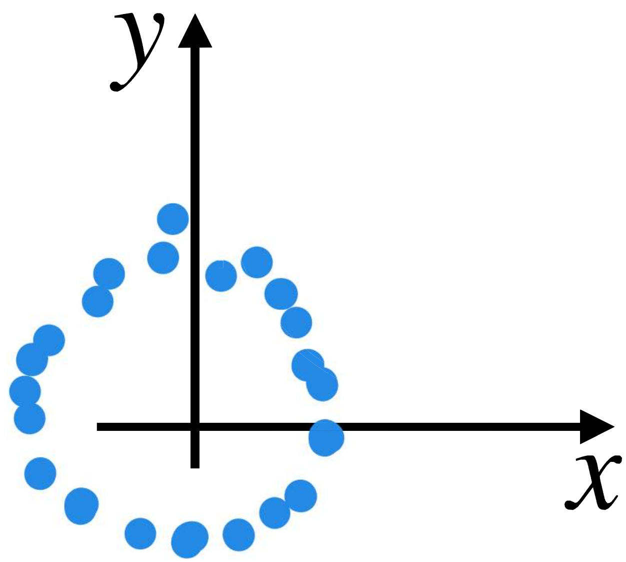
The reverse is not true! Uncorrelated variables do not have to be independent

$$Z \sim U([0, 2\pi])$$

$$X = \cos Z$$

$$Y = \sin Z$$





$$\mathbb{C}[X, Y] = 0$$

Properties of independent random variables

- Assume X and Y are independent.

- The variance of the sum of two independent random variables is the sum of the variance of the random variables:

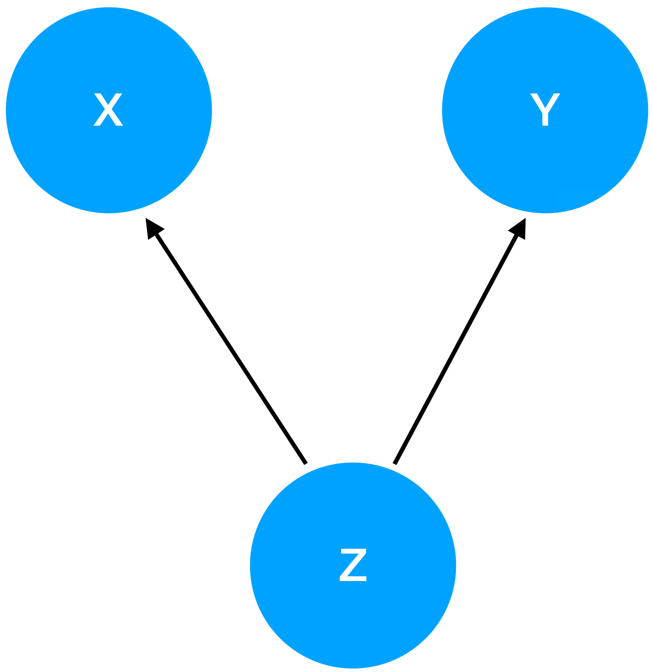
Must remember

$$\boxed{\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y].}$$

Proof: $\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2\,\cancelto{0}{\mathbb{C}[X,Y]}$

$= \mathbb{V}[X] + \mathbb{V}[Y]. \blacksquare$

Reading independence from causal graphs



- X and Y are not independent in general.

  $p(x,y) \neq p(x)p(y)$

- But X is independent of Y conditioned on Z.

  $p(x,y|z) = p(x|z)p(y|z)$

D-separation (advanced)

PREDICTIVE
SCIENCE LABORATORY