

Lecture 8: The Monte Carlo method for estimating expectations

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The law of large numbers

The strong law of large numbers

- Take an infinite series of independent random variables X_1, X_2, \dots with the same distribution (it doesn't matter what distribution).

- The *sample average*:

$$\frac{X_1 + \dots + X_N}{N} \longrightarrow \mu$$

$$\bar{X}_N = \frac{1}{N} \sum_{i=1}^N X_i \rightarrow \mu \text{ a.s.}$$

almost surely
(measure theory)

where $\mu = \mathbb{E}[X_i]$ (as $N \rightarrow \infty$).

The Monte Carlo method for estimating integrals

- Take a random variable $X \sim p(x)$ and some function $g(x)$.

- We want to estimate the expectation:

$$I = \mathbb{E}[g(X)] = \int g(x)p(x)dx$$

- Make independent identical copies of X : $X_1, X_2, \dots \sim p(x)$

- Consider the also the independent iden. dist.:

$$Y_1 = g(X_1), Y_2 = g(X_2), \dots$$

- By the strong law of large numbers:

$$I_N = \frac{Y_1 + \dots + Y_N}{N} \longrightarrow \mathbb{E}[Y_i] = \mathbb{E}[g(X_i)] = I \text{ a.s.}$$

Example: 1D expectation

(This is Example 3.4 of Robert & Casella (2004))

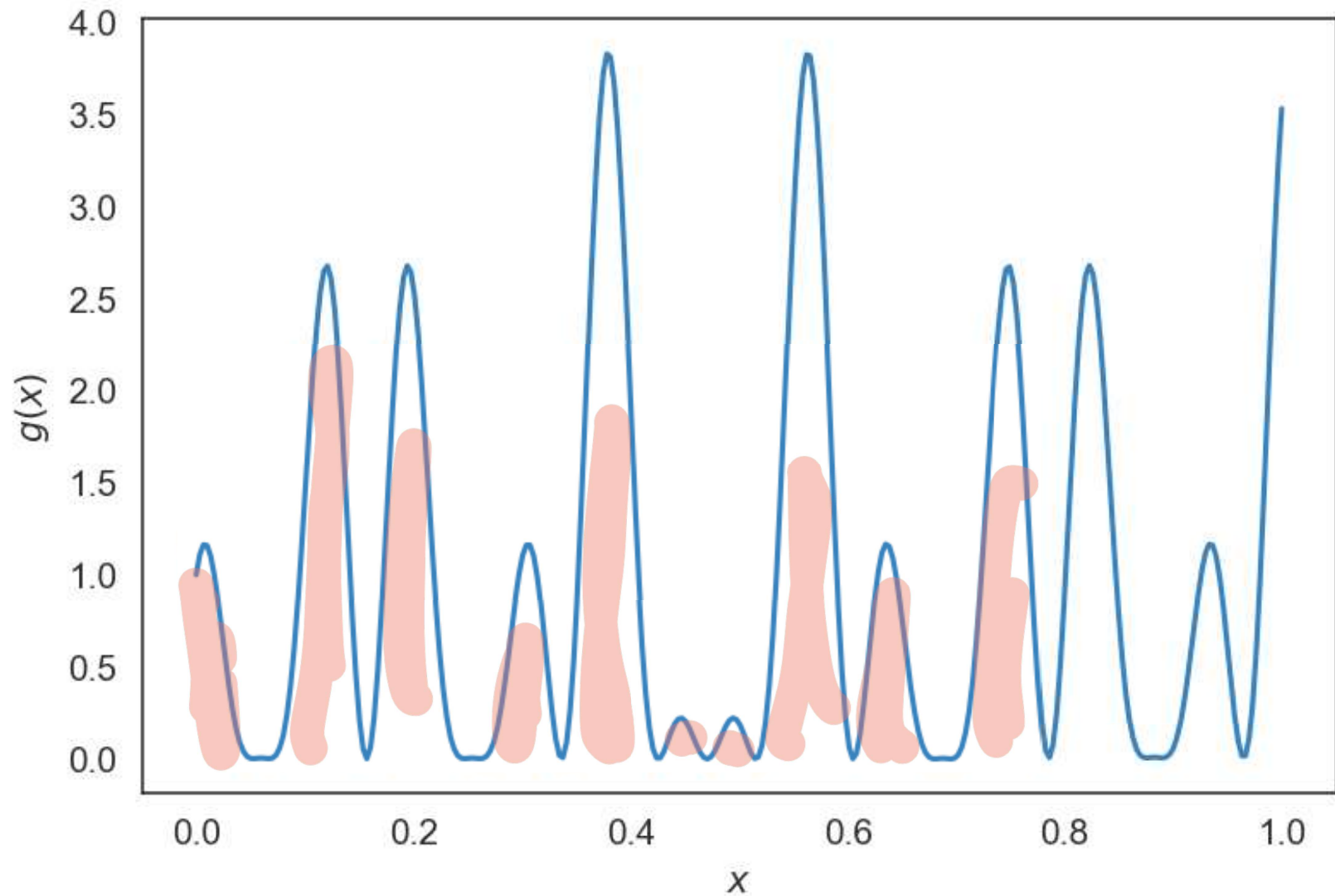
$$X \sim \mathcal{U}([0, 1])$$

$$g(x) = (\cos(50x) + \sin(20x))^2$$

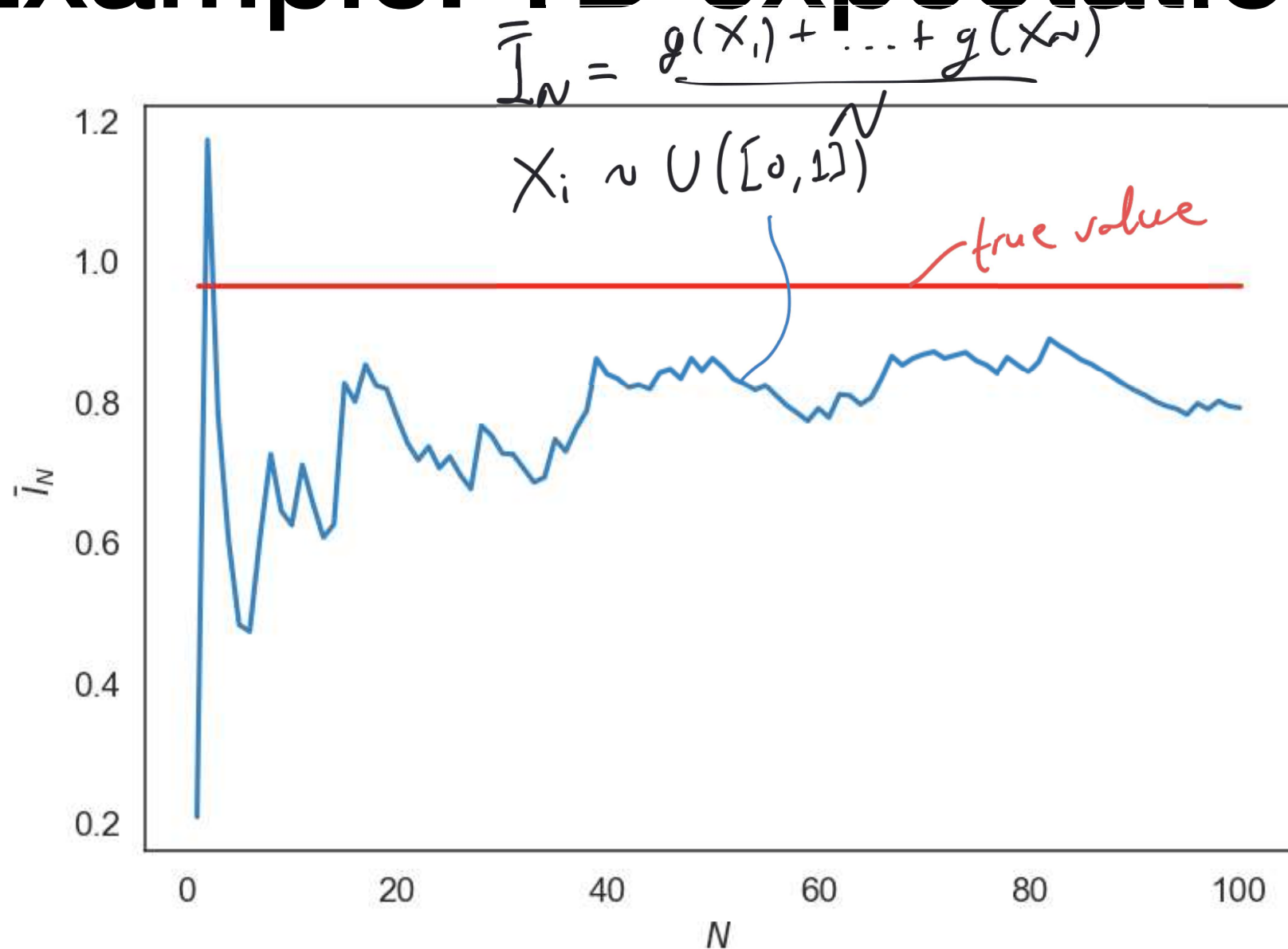
The correct value for the integral is:

$$\mathbb{E}[g(X)] = 0.965$$

Example: 1D expectation



Example: 1D expectation



Example: 1D expectation

