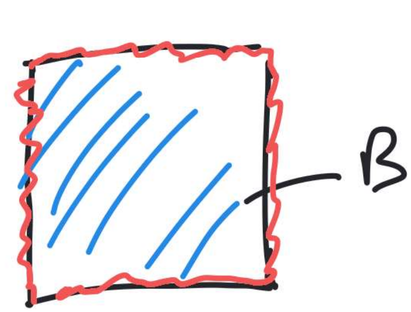
Lecture 26: Physics-informed deep neural networks

Professor Ilias Bilionis

Physics-informed regularization: Solving PDEs

From PDE to a loss function

$$\begin{cases} (1) & -\nabla\cdot\left[a(x)\nabla u(x)\right] = f(x),\quad x\in B \\ (2) & u\big|_{\partial B} = g \end{cases}$$



IDEA:

$$u(x) = \underbrace{N}_{\text{DNN}}(x;\underbrace{\theta}_{\text{param.}})$$

$$L(\theta) = \underbrace{\int_B \left\{\nabla\cdot\left[a(x)\nabla u(x)\right] + f(x)\right\}^2 dx}_{\text{Physics (PDE)}} + \lambda \underbrace{\int_{\partial B}\left[u(x) - g(x)\right]^2 dx}_{\text{Data}}$$

Solving the problem with stochastic gradient descent

$$L(\theta) = \int_B \left\{ \nabla \cdot \left[a(x) \nabla \overset{\theta}{u}(x) \right] + f(x) \right\}^2 dx + \lambda \int_{\partial B} \left[\overset{\theta}{u}(x) - g(x) \right]^2 dx$$

M_p, M_b integers

$X_{pj} \sim U(B)$ ind., $X_{Bj} \sim U(\partial B)$ ind.

$j = 1, \dots, M_p$, $\quad j = 1, \dots, M_B$

$$L(\theta) = \mathbb{E}\left[\frac{\overset{\text{area of } B}{|B|}}{M_p} \sum_{j=1}^{M_p} \left\{ \nabla \cdot \left[a(X_{pj}) \nabla u(X_{pj}) \right] + f(X_{pj}) \right\}^2 + \lambda \frac{|\partial B|}{M_B} \sum_{j=1}^{M_B} \left[u(X_{Bj}) - g(X_{Bj}) \right]^2 \right]$$

PREDICTIVE
SCIENCE LABORATORY

- θ_0
- Sample x_{pjk}, x_{Bjk} from X_{pj} and X_{Bj}
- $\theta_{k+1} = \theta_k - \alpha_k \left\{ \frac{|B|}{M_p} \sum_{j=1}^{M_p} \nabla_\theta \left\{ \quad \right\}\Big|_{\theta=\theta_k} + \lambda \frac{|\partial B|}{M_B} \sum_{j=1}^{M_B} \nabla_\theta \left\{ \quad \right\}\Big|_{\theta=\theta_k} \right\}$