Lecture 15: Advanced topics in Bayesian linear regression

Professor Ilias Bilionis

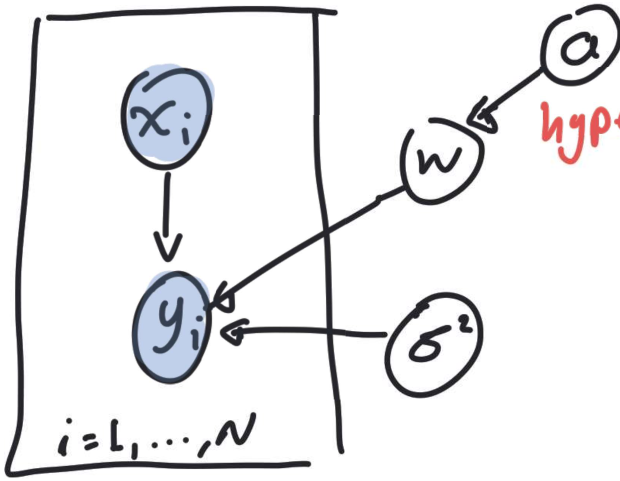
The evidence approximation

PREDICTIVE
SCIENCE LABORATORY

Open questions

- How do I quantify the measurement noise?
- How do we avoid overfitting?
- How do I quantify epistemic uncertainty induced by limited data?
- How do I choose any remaining parameters?
- How do I choose which basis functions to keep?

Hyper-priors



hyper-parameter

Prior: $\alpha \sim p(\alpha)$

$$\underline{w}|\alpha \sim p(\underline{w}|\alpha) = N(\underline{w}|0, \alpha^{-1}\underline{\underline{I}})$$

$$\sigma \sim p(\sigma)$$

Likelihood: $y_i | x_i, \underline{w}, \sigma^2 \sim N(\underline{\varphi}^T(x_i)\underline{w}, \sigma^2)$

$$p(y_{1:N}|x_{1:N}, \underline{w}, \sigma^2) = \prod_{i=1}^{N} \ldots$$

PREDICTIVE SCIENCE LABORATORY

Posterior over hyper-parameters and the evidence approximation

$$p(\underline{w}, \alpha, \sigma \mid x_{1:N}, y_{1:N}) \propto p(y_{1:N} \mid x_{1:N}, \underline{w}, \sigma)\, p(\underline{w} \mid \alpha)\, p(\alpha)\, p(\sigma)$$

$$p(\alpha, \sigma \mid x_{1:N}, y_{1:N}) = \int p(\underline{w}, \alpha, \sigma \mid x_{1:N}, y_{1:N})\, dw$$

$$\propto \int p(y_{1:N} \mid x_{1:N}, \underline{w}, \sigma)\, p(\underline{w} \mid \alpha)\, \boxed{p(\alpha)\, p(\sigma)}\, d\underline{w}$$

$$= \int p(y_{1:N} \mid x_{1:N}, \underline{w}, \sigma)\, p(\underline{w} \mid \alpha)\, d\underline{w} \;\; p(\alpha)\, p(\sigma)$$

$$p(\underline{w} \mid x_{1:N}, y_{1:N}, \alpha, \sigma) \propto p(y_{1:N} \mid x_{1:N}, \underline{w}, \sigma) \cdot p(\underline{w} \mid \alpha)$$

$$\mathcal{N}(\underline{w} \mid \underline{m}(\alpha, \sigma), \underline{\underline{S}}(\alpha, \sigma))$$

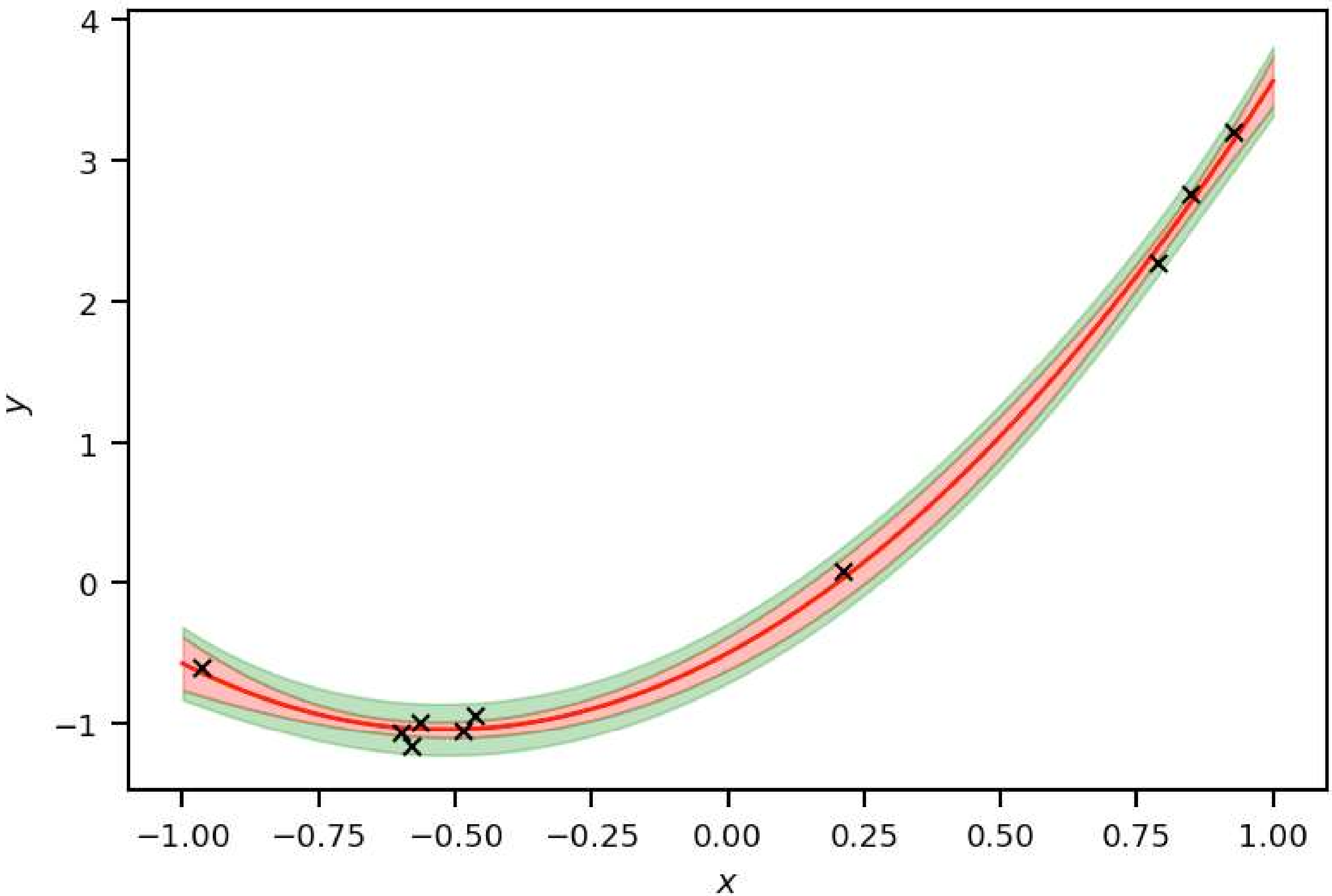
$$= \quad Z(\alpha, \sigma)\, p(\alpha)\, p(\sigma)$$

PREDICTIVE
SCIENCE LABORATORY

Evidence approximation : Find α, σ by maximizing their posterior (with $\underline{w}$ integrated out).

$$\alpha^*, \sigma^* = \arg\max_{\alpha, \sigma} \; p(\alpha, \sigma \mid x_{1:N}, y_{1:N})$$

Example



PREDICTIVE
SCIENCE LABORATORY