Lecture 4: Continuous Random Variables

Professor Ilias Bilionis
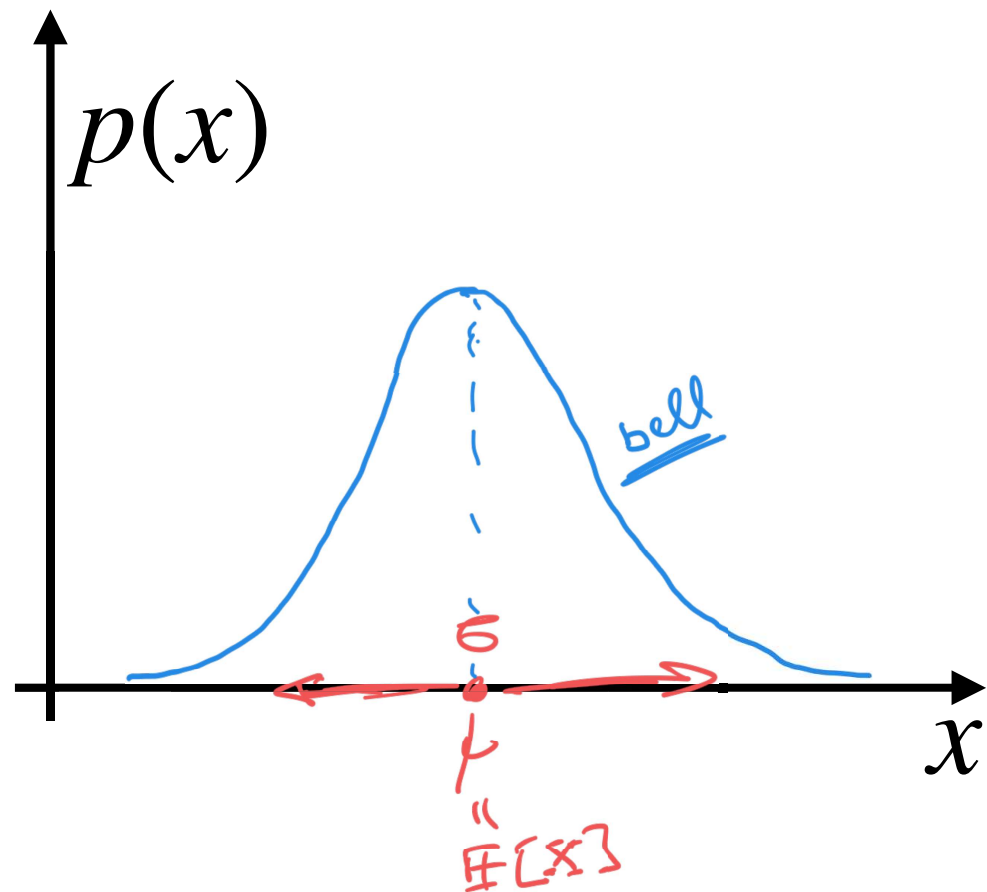
The Gaussian distribution

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The Gaussian (or normal) distribution

- Models a random variable that takes values any real value.

- The values are concentrated around a value μ.

- The variance σ^2 determines how spread out the function values are around the mean.

- We write:

$$X \sim N(\mu, \sigma^2)$$

Expectation $\equiv$ mean

$p(x)$

bell

σ

μ

$\mathbb{E}[X]$

x

The PDF and CDF of the normal distribution
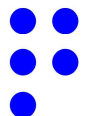
- Consider:

$$X \sim N(\mu, \sigma^2)$$

- The PDF of the normal rises naturally from the central limit theory and the maximum entropy principle (later).
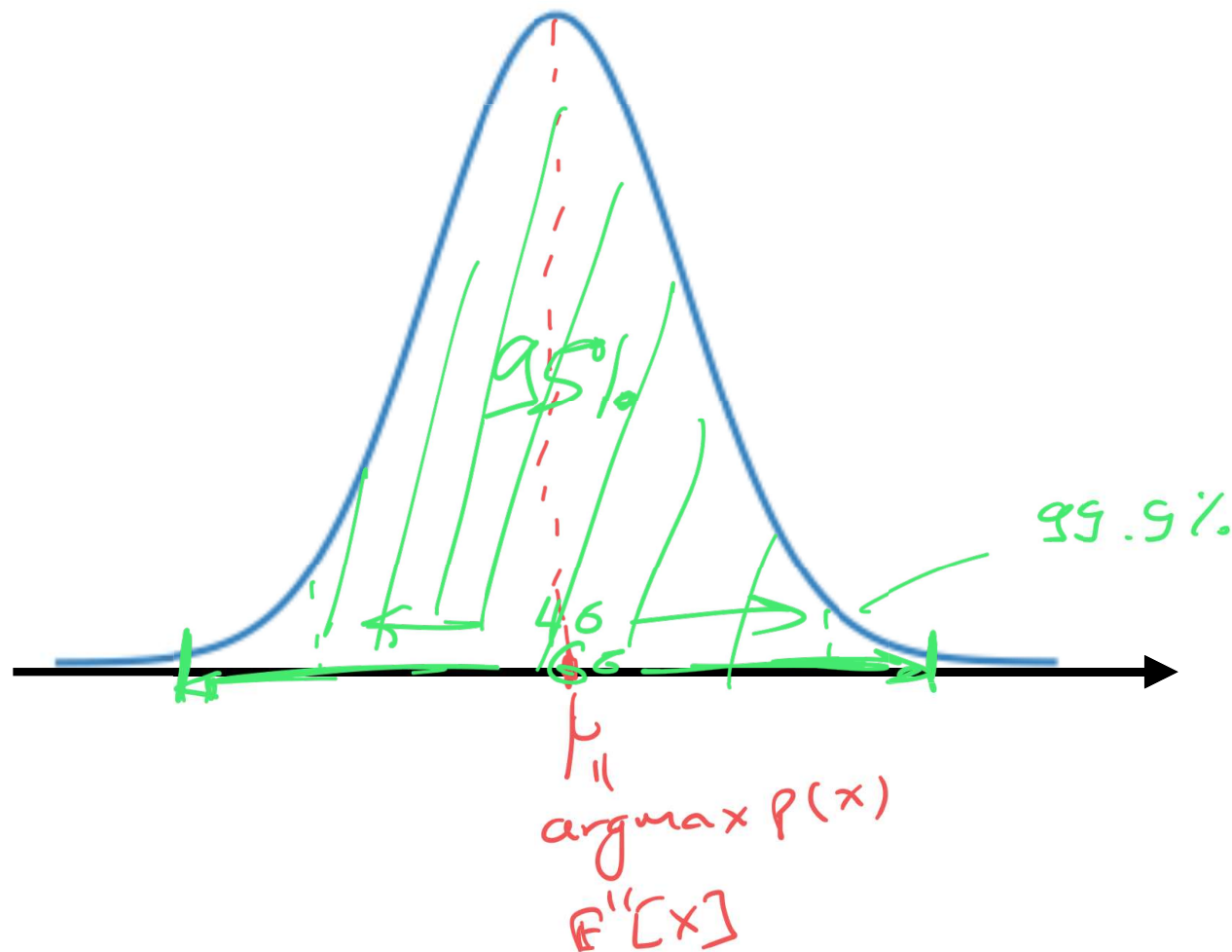
- The PDF is:

$$p(x) = N(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The CDF is not analytically available.

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} N(\tilde{x} \mid \mu, \sigma^2) d\tilde{x}$$

Some useful things to know about the normal

The standard normal distribution
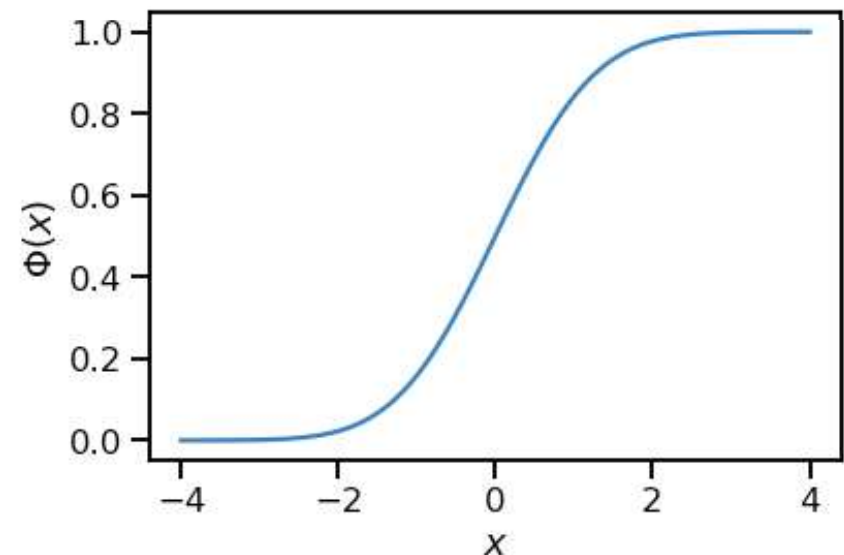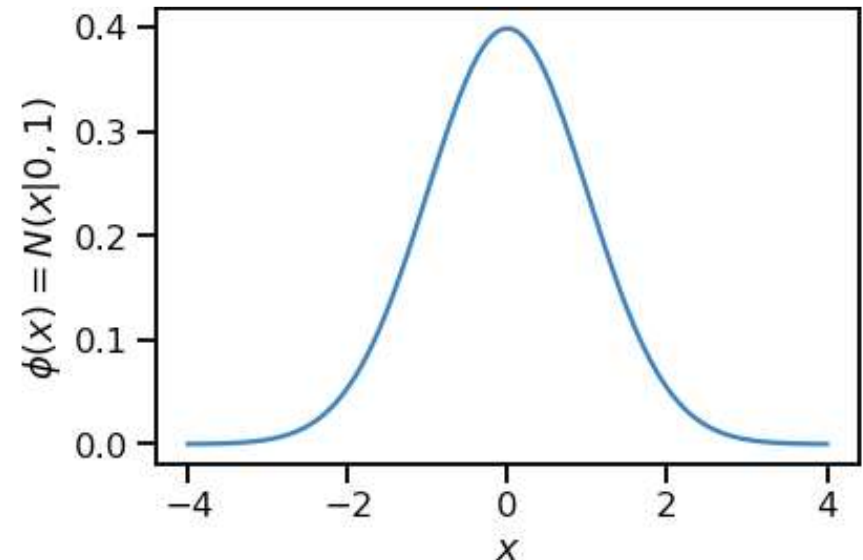
- Let

$$Z \sim N(0,1).$$

- The PDF of the standard normal is:

$$\phi(z) := N(z \,|\, 0,1) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2} \right\}$$

- The CDF of the standard normal is:

$$\Phi(z) := \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \phi(z') dz',$$

is also not analytically available.

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Connections between the normal and the standard normal

- Take a standard normal $Z \sim N(0,1)$ and two numbers μ and σ^2.

- Make the random variable $X = \mu + \sigma Z$.

- Then $X \sim N(\mu, \sigma^2)$.
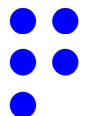
- Proof requires the change variables formula, but note:

$$\mathbb{E}[X] = \mathbb{E}[\mu + \sigma Z] = \mu + \cancel{\mathbb{E}[\sigma Z]} = \mu + \sigma \underbrace{\mathbb{E}[Z]}_{0} = \mu$$

$$\mathbb{V}[X] = \mathbb{V}[\mu + \sigma Z] = \mathbb{V}[\sigma Z] = \sigma^2 \underbrace{\mathbb{V}[Z]}_{1} = \sigma^2$$

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Connections between the normal and the standard normal

- Take a normal $X \sim N(\mu, \sigma^2)$.

- Make the random variable $Z = \dfrac{X - \mu}{\sigma}$.

- Then $Z \sim N(0,1)$.

- Proof requires the change variables formula, but note:

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma} \mathbb{E}[X - \mu] = \frac{1}{\sigma}\left\{\mathbb{E}[X] - \mu\right\} = 0.$$

$$\mathbb{V}[Z] = \mathbb{V}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma^2} \mathbb{V}[X - \mu] = \frac{1}{\sigma^2} \mathbb{V}[X] = 1.$$

Connections between the normal and the standard normal

- Take a normal $X \sim N(\mu, \sigma^2)$.

- We can now write:

$$p(X \leq x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of $N(\mu, \sigma^2)$

CDF of $N(0,1)$

$\sim N(0,1)$

- The proof is good practice of probability laws:

$$p(X \leq x) = p(X - \mu \leq x - \mu) = p\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$
$$= p\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

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