Lecture 11: Selecting prior information

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The principle of maximum entropy for continuous random variables

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The naïve extension of information entropy to continuous distributions and why it doesn't always work

$$\mathbb{H}[p(X)] = -\int p(x) \log p(x)\, dx$$

$$Y = T(X)$$

$$\mathbb{H}[p(Y)] \neq \mathbb{H}[p(X)]$$

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The correct information entropy for continuous distributions

$$\mathbb{H}[p(X)] := -\int p(x) \log \frac{p(x)}{q(x)} dx$$

density of maximal uncertainty

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Mathematical statement of the principle of maximum entropy for continuous distributions

$$\max \mathbb{H}[p(X)] = \max -\int p(x) \log \frac{p(x)}{q(x)} dx$$

subject to

$$\int p(x) dx = 1$$

and

$$\mathbb{E}[f_k(X)] = F_k, \quad k = 1, \dots, K$$

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The general solution to the maximum entropy problem for continuous distributions

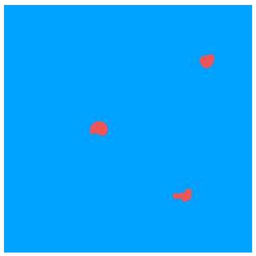
$$p(x) = \frac{q(x)}{Z} \exp\left\{ \sum_{k=1}^{K} \lambda_k f_k(x) \right\}$$

$$Z = \int q(x) \exp\left\{ \sum_{k=1}^{K} \lambda_k f_k(x) \right\} dx$$

$$\boxed{F_k = \frac{\partial Z}{\partial \lambda_k}, \quad k = 1, \ldots, K}$$

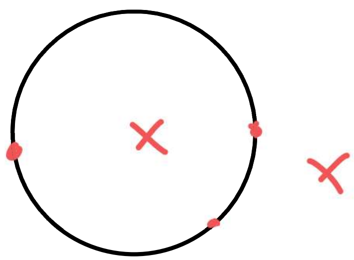
A few comments on the maximal uncertainty density $q(x)$

Example: Particle in a box:



$$q(x) = \begin{cases} 1, & \text{in box} \\ 0, & \text{other.} \end{cases} \propto 1_{Box}(x)$$

Example: Particle restricted on circular guide:



$$q(x) \propto 1_{guide}(x)$$

Mathematical theory for finding the maximal uncertainty density $q(x)$

- Principle of transformation groups.
- Theory of Haar measures.

Example 1

- X takes values in $[a, b]$

- $q(x) = 1$

$X \sim U([a,b])$

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Example 2

- X takes values in $\mathbb{R}$
- $q(x) = 1$
- $\mathbb{E}[X] = \mu$
- $\mathbb{V}[X] = \sigma^2$

$$X \sim N(\mu, \sigma^2)$$

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Example 3

- X takes values in $[0,\infty)$
- $q(x) = 1$
- $\mathbb{E}[X] = \mu$

$$X \sim \mathrm{Exp}\left(\frac{1}{\mu}\right)$$

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A final note on the use of maximum entropy for finding priors

- The principle of maximum entropy is a great tool for assigning “objective” priors.
- However:
  - The cost of “theorizing” and “computing” for finding the ideal distributions should be taken into account. This was called “type-2 reasoning” by I. J. Good.
  - Sometimes, you have subjective information. You should not be afraid to use it.