Solution to the minimum-error

$$\min_{\underline{\underline{V}}, \underline{\underline{W}}, \underline{x}_0} \sum_{i=1}^{n} \left\| \underline{\underline{V}} \left(\underline{\underline{W}}^T \underline{x}_i \right) - \underline{x}_0 - \underline{x}_i \right\|^2$$

subject to $\underline{\underline{V}}$ and $\underline{\underline{W}}$ orthogonal matrices

$$\underline{v}_i^T \underline{v}_j = 0, \; i \neq j, \quad \underline{w}_i^T \underline{w}_j = 0, \; i \neq j$$

Solution : Necessary con. : $\underset{\text{param}}{\text{grad}}$ loss $= 0$

$$\Rightarrow \underline{x}_0 = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i \text{ (empirical mean)}$$

$$\underline{\underline{V}} = \underline{\underline{W}}$$

Empirical Covariance Matrix : $\underline{\underline{C}} = \frac{1}{n} \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_0)(\underline{x}_i - \underline{x}_0)^T$

$\underline{\underline{C}}$ $D \times D$ pos. def. $\Rightarrow$ positive eigenvalues $\lambda_i > 0$ and orthogonal eigenvectors. $\underline{u}_i \in \mathbb{R}^D$

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$$\lambda_1 > \lambda_2 > \dots > \lambda_D > 0$$

$$\underline{\underline{C}} \, \underline{u}_i = \lambda_i \underline{u}_i$$

$$\begin{aligned}
\underline{\underline{W}} &= [\underline{w}_1, \dots, \underline{w}_d] \\
&= [\sqrt{\lambda_1} \underline{u}_1, \sqrt{\lambda_2} \underline{u}_2, \dots, \sqrt{\lambda_d} \underline{u}_d] \\
&= \underset{D \times d}{\underline{\underline{U}}} \cdot \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})
\end{aligned}$$

Projection map :

$$\boxed{\begin{aligned}
\underline{z}_i = \underline{\underline{W}}^T \underline{x}_i &= \left(\underline{\underline{U}} \cdot \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \right)^T \underline{x}_i \\
&= \operatorname{diag}_d(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \cdot \underline{\underline{U}}^T \cdot \underline{x}_i \\
&= \sum_{j=1}^{d} \sqrt{\lambda_j} \cdot \underline{u}_j^T \underline{x}_i
\end{aligned}}$$

Reconstruction map :

$$\begin{aligned}
\tilde{\underline{x}}_i &= \underline{\underline{V}} \underline{z}_i + \underline{x}_0 = \underline{\underline{W}} \underline{z}_i + \underline{x}_0 \\
&= \underline{\underline{U}} \cdot \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \cdot \underline{z}_i + \underline{x}_0
\end{aligned}$$