Lecture 11: Selecting prior information

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The principle of maximum entropy for discrete random variables

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Prequel to the principle of maximum entropy

- You have a discrete random variable X.
- You know what values it takes, say $x_1, \dots, x_N$.
- You also have some **testable information** about it.
- The principle of maximum entropy states that we should assign to X the probability distribution that maximizes the entropy subject to the constraints imposed by the testable information.

Mathematical definition of testable information

$$\mathbb{E}[f_k(X)] = f_k$$

known function

known value

$k = 1, \dots, K$

Is this definition broad enough?

I = “the expected value of X is μ”

$$\mathbb{E}[X] = \mu$$

$$K = 1, \quad f_1(x) = x, \quad F_1 = \mu.$$

Is this definition broad enough?

I = "the expected value of X is μ and the variance of X is σ^2"

$$\boxed{\mathbb{E}[X]=\mu}\,; \quad \mathbb{V}[X]=\sigma^2$$

$$\sigma^2=\mathbb{V}[X]=\mathbb{E}[X^2]-(\mathbb{E}[X])^2=\mathbb{E}[X^2]-\mu^2$$

$$\Rightarrow \boxed{\mathbb{E}[X^2]=\sigma^2+\mu^2}$$

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Mathematical statement of the principle of maximum entropy

You should assign to X the pmf $p(x)$ that

$$\max \mathbb{H}[p(X)] = \max -\sum_{i=1}^{N} p(x_i) \log p(x_i)$$

subject to

$$\mathbb{E}[f_k(X)] = F_k, \quad \text{for } k=1,\dots,K$$

$$\parallel$$

$$\sum_{i=1}^{N} f_k(x_i) p(x_i)$$

and

$$\sum_{i=1}^{N} p(x_i) = 1$$

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The general solution to the maximum entropy problem

$$p(X=x_i) = \frac{1}{Z} \exp\left\{ \sum_{k=1}^{K} \lambda_k f_k(x_i) \right\}$$

need to be det.

$$Z = \sum_{i=1}^{N} \exp\left\{ \sum_{k=1}^{K} \lambda_k f_k(x_i) \right\}$$

$$\boxed{F_k = \frac{\partial Z}{\partial \lambda_k}}$$

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Example 1

- X takes N different values (no other constraints)

$$p(X = x_i) = \frac{1}{N}$$

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Example 2

- X takes two values 0 and 1.

- $\mathbb{E}[X] = \theta$.

$X \sim \text{Bernoulli}(\theta)$

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Example 3

- X takes values $0, 1, 2, \ldots, N$.
- $\mathbb{E}[X] = \mu$.
- X is the number of successful trials in N sequential experiments (potentially correlated)/

$$X \sim B\left(N, \frac{\mu}{N}\right)$$

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Example 4

- X takes values $0,1,2,\ldots$
- $\mathbb{E}[X] = \mu$.
- X is the number of successful trials in an infinite number of sequential experiments (potentially correlated).

$$X \sim \text{Poisson}(\mu)$$