Lecture 12: Analytical examples of Bayesian inference

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Bayesian parameter estimation

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Example: Coin toss

- We run a coin toss experiment N times and we wish to figure out the probability of heads.

- The data we have observe are: $x_1, x_2, \dots, x_N$

$$x_i \in \{\underset{H}{\underset{\parallel}{0}}, \underset{T}{\underset{\parallel}{1}}\}$$

- For notational convenience we will be writing:

$$x_{1:N} = (x_1, \dots, x_N)$$

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Example: Coin toss

- The probability of success of the coin toss: θ

- How can we describe our uncertainty about it?

$\theta \in [0,1] \xrightarrow[\text{ent}]{\text{max}} \theta \sim U([0,1])$ prior

- Each coin toss experiment corresponds is:

$X_i | \theta \sim \text{Bernoulli}(\theta)$ likelihood

The likelihood of the data

$$x_i \mid \theta \sim \text{Bernoulli}(\theta)$$

$$p(x_i \mid \theta) = \theta^{x_i}(1-\theta)^{1-x_i} = \begin{cases} \theta, & x_i = 1 \\ 1-\theta, & x_i = 0 \end{cases}$$

$$p(x_{1:N} \mid \theta) \overset{x_i \text{ ind.}}{=} p(x_1 \mid \theta) \cdot p(x_2 \mid \theta) \cdots p(x_N \mid \theta)$$

$$= \prod_{i=1}^{N} p(x_i \mid \theta)$$

$$= \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{N} x_i} \cdot (1-\theta)^{N - \sum_{i=1}^{N} x_i}$$

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Bayes' rule applied

$$p(A|B) = \frac{p(AB)}{p(B)}.$$

$A =$ **the model parameters** $= \theta$

$B =$ **the data** $= x_{1:N}$

The joint probability density

Posterior state of knowledge - posterior

$$\boxed{p(\textbf{the model parameters}|\textbf{the data})} = \frac{\boxed{p(\textbf{the parameters and the data})}}{p(\textbf{the data})}$$

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Simplifying the joint

$$p(AB) = p(B|A)p(A)$$

$$A = \textbf{the model parameters} = \theta$$

$$B = \textbf{the data} = x_{1:N}$$

$$\underbrace{\boxed{p(\textbf{the parameters and the data})}}_{\text{Joint}} = \underbrace{\boxed{p(\textbf{the data}|\textbf{the parameters})}}_{\text{Likelihood}}\underbrace{\boxed{p(\textbf{the parameters})}}_{\text{Prior}}$$

$$\textbf{posterior} \propto \textbf{likelihood} \times \textbf{prior}$$

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The posterior for the coin toss example

posterior ∝ likelihood × prior

$$p(\theta \mid x_{1:N}) \propto p(x_{1:N} \mid \theta) p(\theta)$$

$$= \theta^{\sum_{i=1}^{N} x_i} (1-\theta)^{N - \sum_{i=1}^{N} x_i} \mathbb{1}_{[0,1]}(\theta)$$

$$\theta \mid x_{1:N} \sim \mathrm{Beta}\left(\alpha = \sum_{i=1}^{N} x_i + 1, \beta = N - \sum_{i=1}^{N} x_i + 1\right)$$

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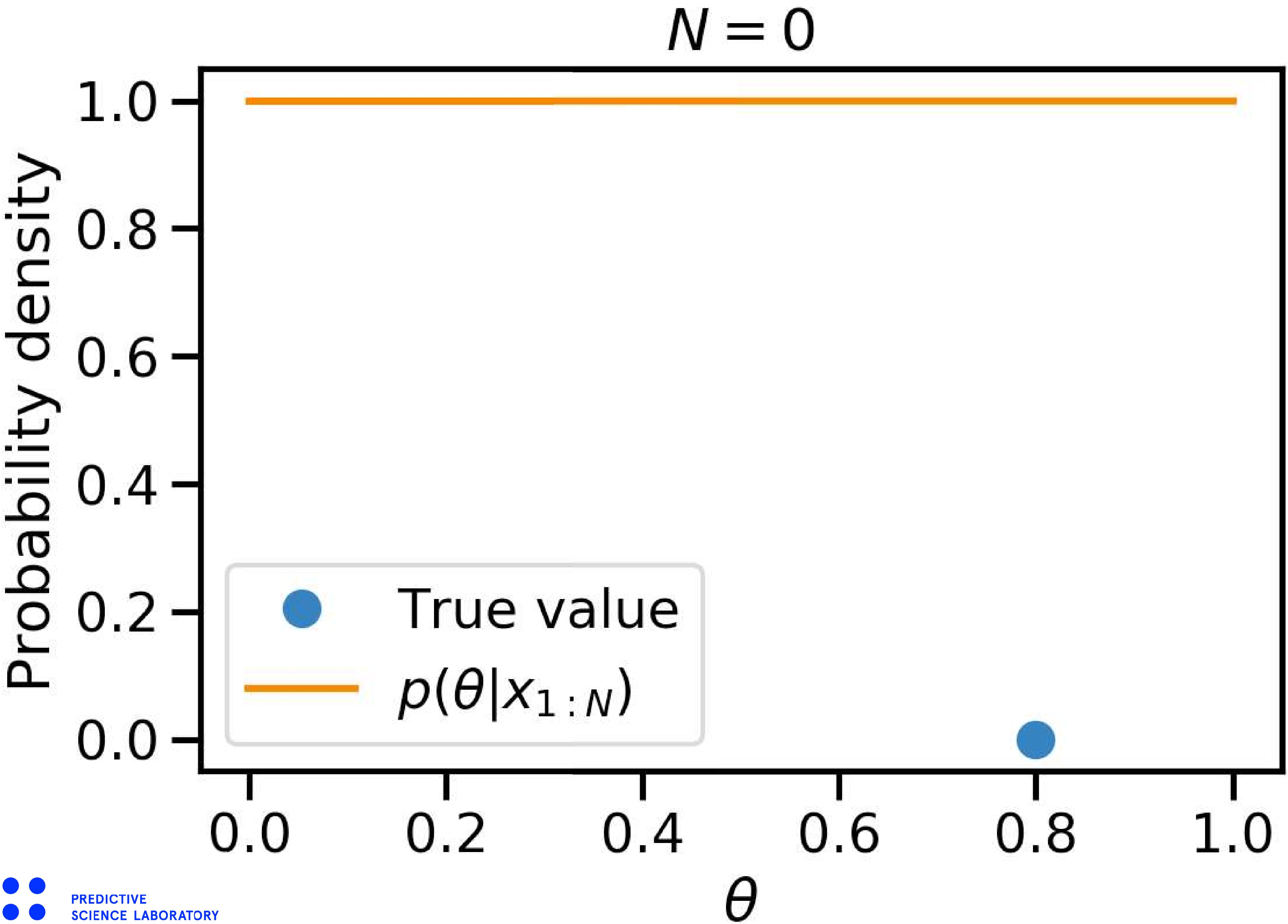
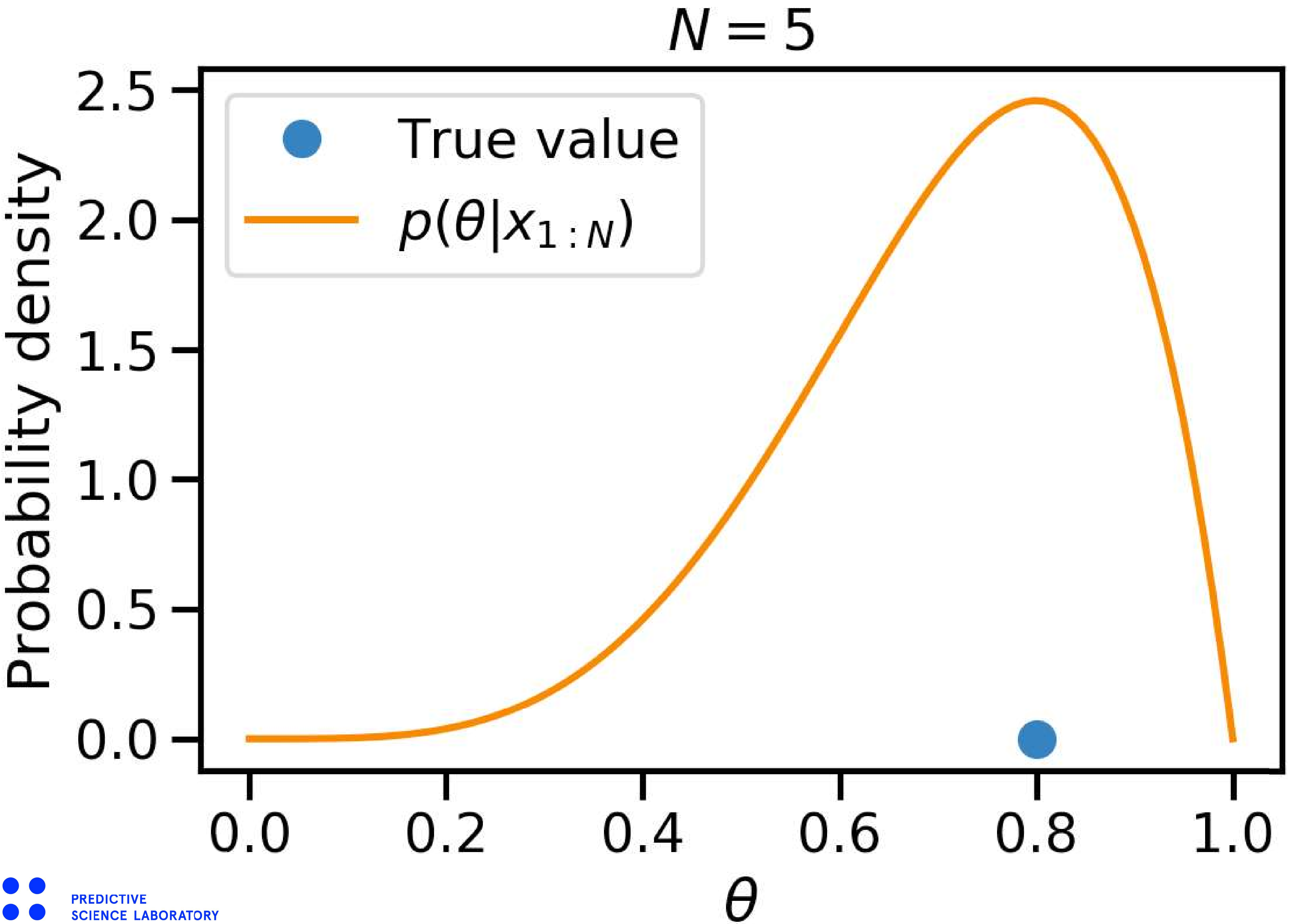


$N = 0$
1.0
0.8
0.6
0.4
0.2
0.0
Probability density
True value
$p(\theta|x_{1:N})$
0.0
0.2
0.4
0.6
0.8
1.0
θ



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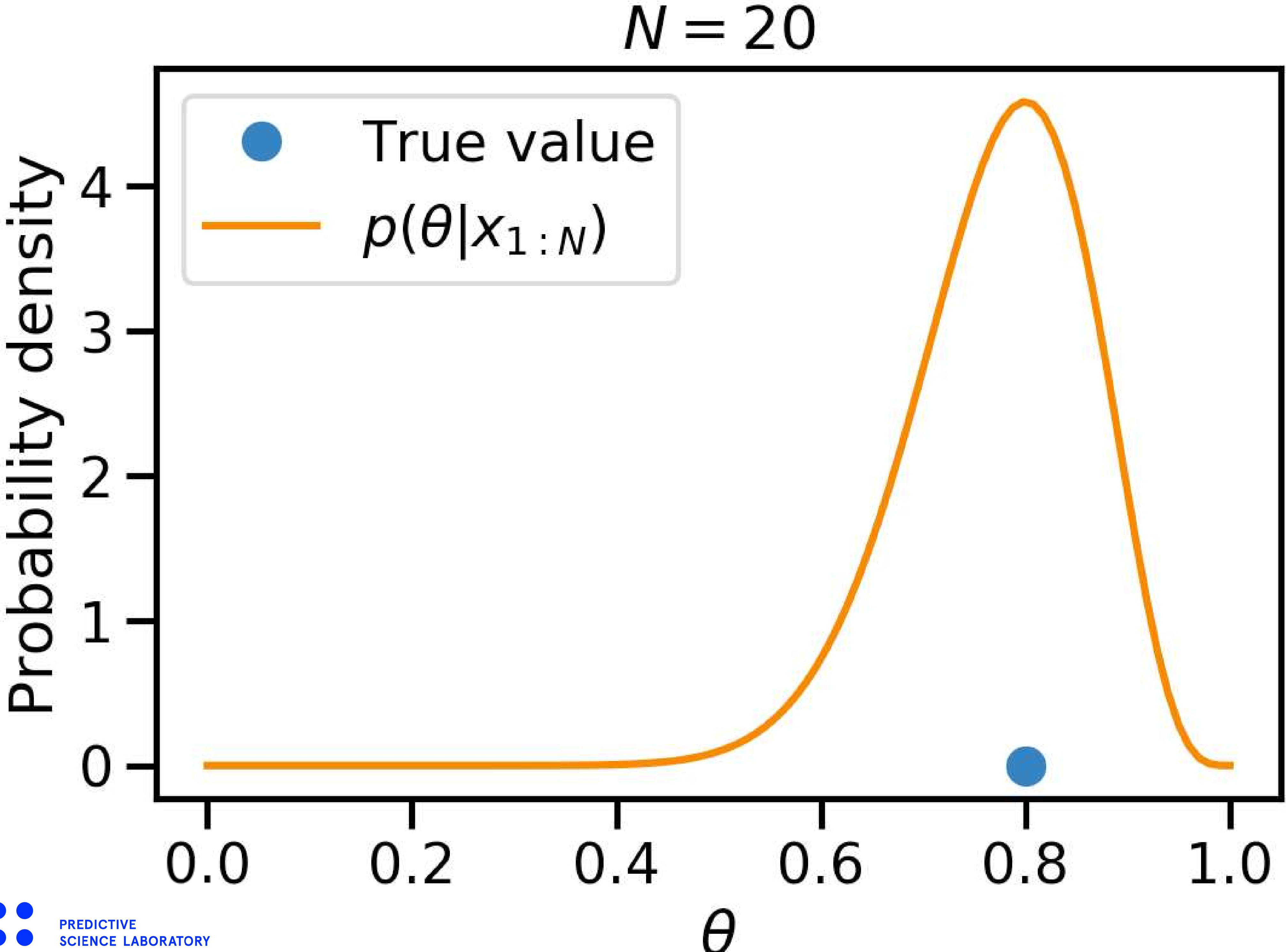

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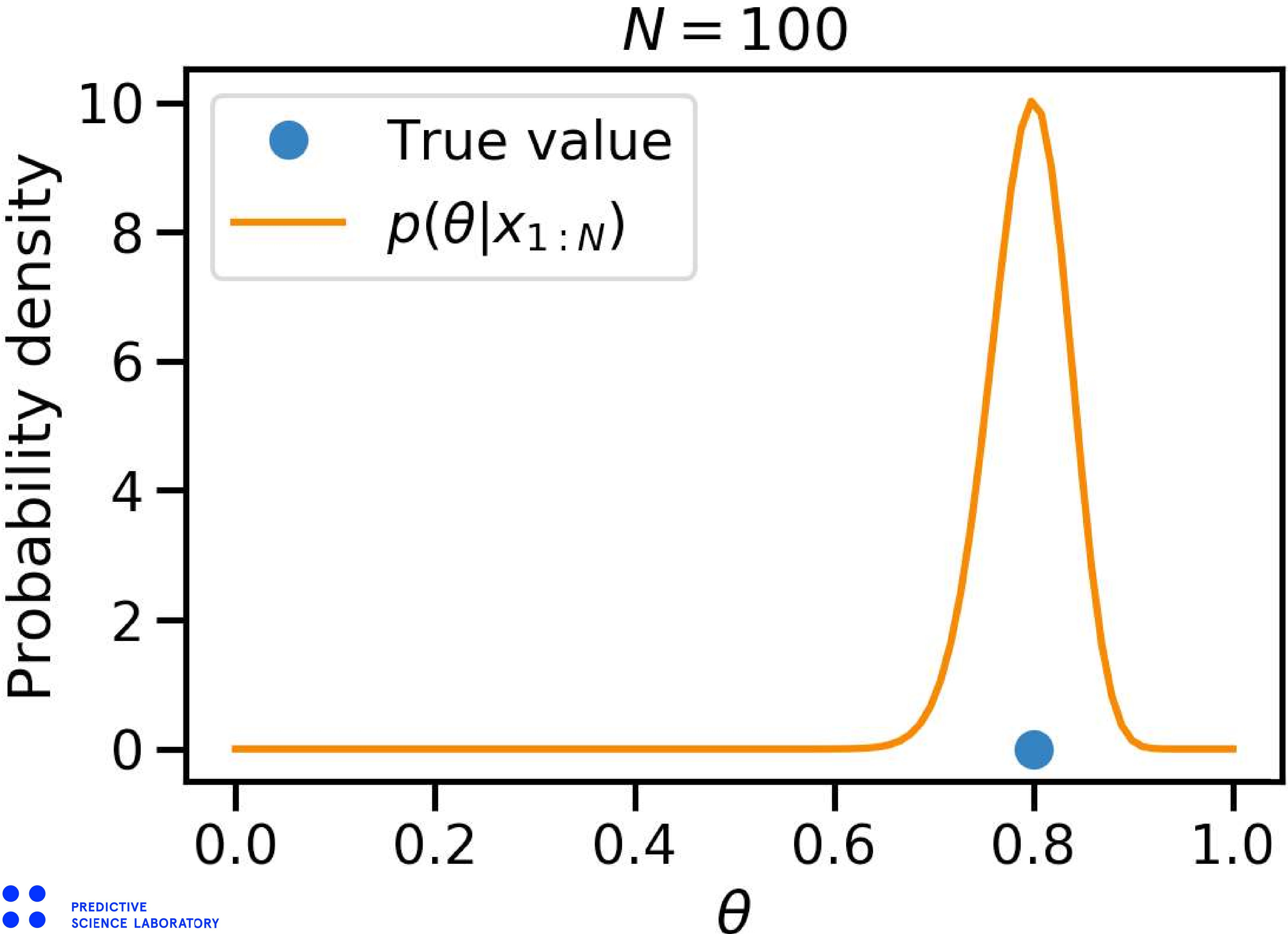

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