Lecture 5: Collections of Random Variables

Professor Ilias Bilionis

Joint probability density function

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Discrete rvs: Joint probability mass function

- Consider two discrete random variables X and Y.
- The **joint probability mass function** of the pair (X, Y) is the function $p(x, y)$ giving the probability that $X = x$ and $Y = y$:

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Properties of the joint pmf

- It is nonnegative:

$$p(x,y) \geq 0$$

- If you sum over all the possible values of all random variables, you should get one:

$$\sum_x \sum_y p(x,y) = 1$$

Properties of the joint pmf

- If you marginalize over the values of one of the random variables you get the pmf of the other:

$$\sum_y p(x,y) = p(x)$$

(Sum rule: $p(A) = \sum_i p(A, B_i)$, where $p(B_1 \text{ or } B_2 \text{ or} \dots) = 1$
$p(B_i, B_j) = 0,\ i \neq j$.)

$A = \{X = x\},\ B_i = \{Y = y_i\}$

- and

$$\sum_x p(x,y) = p(x)$$

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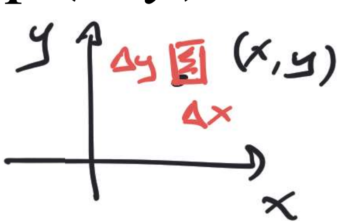
Continuous rvs: Joint probability density function

- Consider two continuous random variables X and Y.

- The **joint probability density function** of the pair (X, Y) is the function $p(x, y)$ giving the probability that $X = x$ and $Y = y$:

$$p(x,y) \approx \frac{p(x \le X \le x + \Delta x, y \le Y \le y + \Delta y)}{\Delta x \Delta y}$$



$, \; p(x,y) \geq 0 \; , \; \iint p(x,y)\,dx\,dy = 1$

Properties of the jpdf

- If you **marginalize** over the values of one of the random variables you get the pdf of the other:

$$\int p(x,y)\,dy = p(x)$$

and

$$\int p(x,y)\,dx = p(y)$$

Note on notation

- We will not distinguish between the notation of discrete and continuous random variables.

$p(x) \begin{cases} \text{pdf} \\ \text{pmf} \end{cases}, \quad p(x,y) \begin{cases} \text{jpdf} \\ \text{jpmf} \\ \dots \end{cases}$

- We will always use the integral sign to indicate marginalization understanding that it is a summation over all possible values if we have a discrete random variable.

$\int \cdot \, dx \begin{cases} \sum_x & \text{if } X \text{ disrete} \\ \int \cdot \, dx & \text{if } X \text{ continuous.} \end{cases}$

- We will only say joint pdf instead of joint pmf.

$\text{pdf} \begin{cases} \text{pmf} & \text{if } X \text{ discrete} \\ \text{pdf} & \text{if } X \text{ is cont.} \end{cases}$

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Conditioning random variables on one another

- Take two random variables X and Y with joint pdf $p(x, y)$.

- Suppose that you observe $Y = y$ and you want to update your state of knowledge about X.

$$p(A|B) = \frac{p(A,B)}{p(B)}$$

- The **conditional pdf** gives you this info:

$$A = \{X = x\}, \quad B = \{Y = y\}$$

$$p(X = x \mid Y = y) = \frac{p(X = x, Y = y)}{p(Y = y)}$$

$$\boxed{p(x|y) = \frac{p(x,y)}{p(y)}}$$

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The expectation of a sum of random variables

- Take two random variables X and Y with joint pdf $p(x, y)$.

- The expectation of their sum is:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Proof: $\mathbb{E}[X+Y] = \iint (x+y)\, p(x,y)\, dx\, dy$

$= \iint x\, p(x,y)\, dx\, dy + \iint y\, p(x,y)\, dx\, dy$

$= \int x \Big(\underbrace{\int p(x,y)\, dy}_{p(x)}\Big) dx + \int y \Big(\underbrace{\int p(x,y)\, dx}_{p(y)}\Big) dy$

$= \int x\, p(x)\, dx + \int y\, p(y)\, dy = \mathbb{E}[X] + \mathbb{E}[Y]$

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