General Form for System of Equations

Say: products $N_1 \ldots N_n$, resources $R_1, \ldots R_m$ to produce unit N_j, a_{ij} of resources R_i are needed where $i = 1, \ldots m$ and $j = 1, \ldots n$.
Find optimal production plan to create x_j units of product N_j if b_i units of resource and ideally none is left over.

$x_1, \ldots, x_n$ units of product 'unknowns'

$a_{i1}x_1 + \ldots + a_{in}x_n$ units of resource

$(x_1, \ldots x_n)$ an element of set $\mathbb{R}^n$

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

where $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$

→ every n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfies the above set of equations is a solution of the system.

Example of infinite solutions

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 - x_2 + 2x_3 = 2 \\ 2x_1 \quad + 3x_3 = 5 \end{cases} \qquad \begin{aligned} 2x_1 &= 5 - 3x_3 \\ 2x_2 &= 1 + x_3 \end{aligned}$$

→ we define $x_3 = a \in \mathbb{R}$ as a free variable, such that

$$\left(\frac{5}{2} - \frac{3}{2}a, \frac{1}{2} + \frac{1}{2}a, a\right), a \in \mathbb{R}$$

→ for any system of linear equations, the solution will either be none, one, or infinitely many

→ when we intersect these planes (satisfy all linear equations at once), we obtain a solution that is a plane, line, point, or empty (they have no common intersection).

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \Leftarrow$$

$$\Rightarrow \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$