## Linear Mappings

→ mappings on vector spaces that preserve their structure; allow us to define concept of a coordinate

→ consider vector spaces $V, W$. A mapping ($\Phi$) preserves structure if:

$$\Phi(x,y) = \Phi(x) + \Phi(y)$$
$$\Phi(\lambda x) = \lambda \Phi(x)$$

for all $x, y \in V$ and $\lambda \in \mathbb{R}$

→ a mapping $\Phi : V \to W$ is a linear mapping if

$$\forall x, y \in V \, \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda x + \psi y) = \lambda \Phi(x) + \psi \Phi(y)$$

## Special Mappings

→ consider a mapping $\Phi : V \to W$

- injective if $\forall x, y \in V : \Phi(x) = \Phi(y) \to x = y$
- surjective if $\Phi(V) = W$
- bijective if both

→ if $\Phi$ is surjective, then all elements in $W$ can be reached from $V$ using the mapping. A bijective mapping can be undone; there exists a mapping $\Psi : W \to V$ such that $\Psi \circ \Phi(x) = x$.

- Isomorphism $\Phi : V \to W$ linear and bijective
- Endomorphism $\Phi : V \to W$ linear
- Automorphism $\Phi : V \to W$ linear and bijective
- $id_V : V \to V$, $x \mapsto x$ as identity mapping in $V$.

Example:

→ the mapping $\Phi : \mathbb{R}^2 \to \mathbb{C}$, $\Phi(x) = x_1 + ix_2$ is a homomorphism:

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = (x_1 + y_1) + i(x_2 + y_2) = x_1 + ix_2 + y_1 + iy_2$$
$$= \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \Phi\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$\Phi\left(\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \lambda x_1 + \lambda i x_2 = \lambda(x_1 + ix_2) = \lambda \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$