Example: Determining a Basis

→ a vector subspace $U \subseteq \mathbb{R}^5$ spanned by the vectors:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad x_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5$$

→ we want to find which vectors are a basis for U.

→ Thus, we have to find if $x_1, \dots, x_4$ are linearly independent.

$$[x_1, x_2, x_3, x_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & 8 & 1 \end{bmatrix} \xrightarrow[\text{transformations}]{} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

→ x_1, x_2, x_4 are linearly independent; thus, $\{x_1, x_2, x_4\}$ is a basis for U.

Rank:

→ the number of linearly independent columns of a matrix equals the # of independent rows and is called the rank, denoted by $rk(A)$.

→ important properties:

- $rk(A) = rk(A^T)$
- the columns of $A \in \mathbb{R}^{m \times n}$ span subspace $W \subseteq \mathbb{R}^n$ with $\dim(U) = rk(A)$
- the rows of $A \in \mathbb{R}^{m \times n}$ span subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = rk(A)$
- for all $A \in \mathbb{R}^{n \times n}$ it holds that A is regular/invertible if $rk(A) = n$
- for all $A \in \mathbb{R}^{m \times n}$ and all $b \in \mathbb{R}^{m \times n}$ the linear eq. system $Ax = b$ can be solved if $rk(A) = rk(A|b)$, $A|b$ being the augmented system
- kernel — for $A \in \mathbb{R}^{m \times n}$ the subspace of solutions for $Ax = 0$ have dimension $n - rk(A)$
- matrix $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of said dimensions

Example

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} =$$

two linearly independent rows & columns

rank$(A) = 2$