Linear Independence

→ linear combination: consider V and finite number of vectors $x_1, \dots, x_n \in V$.
Then, every $v \in V$ of the form

$$v = \lambda_1 x_1 + \dots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i \in V$$

where $\lambda_1, \dots, \lambda_k$ is a linear combination of vectors $x_1, \dots, x_k$.

→ linear independence: consider V with $k \in \mathbb{N}$ and $x_1, \dots, x_k \in V$. If there is a non-trivial combination such that $0 = \sum_{i=1}^{k} \lambda_i x_i$ with at least one $\lambda_i \neq 0$, vectors $x_1, \dots, x_k$ are linearly dependent. If only trivial solution $(\lambda_1, \dots, \lambda_k = 0)$ the vectors are linearly independent.

REMARK: the following are useful to find if vectors are linearly independent:
→ k vectors are either linearly dependent or independent
→ if at least one of the vectors is 0 then they are linearly dependent, or if two vectors are identical
→ vectors $\{x_1, \dots, x_k : x_i \neq 0, i = 1, \dots, k\}$, $k \geq 2$ are linearly dependent if at least one of them is a linear combination of the others
→ use Gaussian elimination

Example: Consider $\mathbb{R}^4$ with

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

→ to check if they are linearly dependent, follow general approach:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0$$

→ write vectors as columns of matrix and apply elementary row operations to identify pivot:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column in the solution matrix is a pivot column. Thus, there is no non-trivial solution as 0 is required for each constant $\lambda_1, \lambda_2, \lambda_3$ to solve.
The vectors are thus linearly independent.