Math 135: Randomized Midterm Problem Solutions

Felix Guo
Luke Ning

1. *Proof.* Assume that $S \cap T = S$, let $s \in S$, then $s \in S \cap T \Rightarrow s \in T$. So we have shown for that for any $s \in S$, $s \in T$, and thus $S \subseteq T$.

   The converse is also true. Assume that $S \subseteq T$. To prove that $S = S \cap T$, we have to show that both $S \subseteq S \cap T$ and $S \supseteq S \cap T$.

   Let $s \in S$. Since $S \subseteq T$, $s \in T$. This implies that $s \in S \cap T$. This proves that $S \subseteq S \cap T$.

   Remark. For any two sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

   Therefore, $S \cap T \subseteq S$ and we have shown that $S = S \cap T$. $\square$

2. (a) *Proof.* We look at the contrapositive of this statement: If $2 \mid x$ or $2 \mid y$, then $2 \mid xy$.
   Assume $2 \mid x$, then for some $k \in \mathbb{Z}$, $2k = x$.
   Remark. If $a \mid b$ then $a \mid bc$ where $a, b, c \in \mathbb{Z}$
   Since $2 \mid 2$, then $2 \mid (2k)y \Rightarrow 2 \mid xy$.
   For the case where $2 \mid y$, the proof is similar to above. $\square$

   (b) *Proof.* We look at the contrapositive of the statement: If $2 \mid xy$, then $2 \mid x$ or $2 \mid y$.
   Assume $2 \mid xy$, that means that either x is even, or y is even (or both).
   If x is even, then $2 \mid x$. If x is not even, then y must be even, and $2 \mid y$. $\square$

   (c) *Proof.* We look at the contrapositive of the statement: If $10 \mid x$ or $10 \mid y$, then $10 \mid xy$.
   Assume $10 \mid x$, then for some $k \in \mathbb{Z}$, $10k = x$.
   Since $10 \mid 10$, then $10 \mid (10k)y \Rightarrow 10 \mid xy$.
   For the case where $10 \mid y$, the proof is similar to above. $\square$

   (d) This statement is false. We can prove by counter-example.

   Proof. Notice that $10 \nmid 5$ and $10 \nmid 2$, but $10 \mid (5 * 2)$. $\square$

3. *Proof.* Let k, $k+1$, $k+2$, $k+3$ be four consecutive numbers (where $k \in \mathbb{Z}$).

   The product can be written as: $k(k+3)(k+1)(k+2)$
   $= (k^2 + 3k)(k^2 + 3k + 2)$
   $= ((k^2 + 3k + 1) - 1)((k^2 + 3k + 1) + 1)$

   Let $n = k^2 + 3k + 1$, $n \in \mathbb{Z}$.

   Then the above statement becomes:
   $= (n-1)(n+1)$
   $= n^2 - 1$, which is one less than a perfect square. $\square$

4. (a) *Proof.* Since $n \in \mathbb{N}$, $n \geq 1 \iff n + 1 \geq 1 + 1 \iff n + 1 \geq 2$ $\square$

   (b) *Proof.* To prove the existence of such an n, we find one.
   Let $n = 6$, notice that $\frac{5x-6}{3} \in \mathbb{Z}$ and $\frac{5(6)-6}{3} = 8 \in \mathbb{Z}$.
   Remark. Any n where $3 \mid n$ would be a valid solution. Proof left as an exercise to the reader.

   $\square$

5. If Jane did not go to medical school, then Jane is not a doctor.

6. *Proof.* Assume $xy + 2x - 3y - 6 < 0$
$\iff x(y+2) - 3(y+2) < 0$
$\iff (x-3)(y+2) < 0.$

   If $x < 3$, we are done.

   If $x \not< 3$, then $(x-3) > 0 \Rightarrow (y+2) < 0 \Rightarrow y < -2$.

   Notice that x cannot equal 3 since this would violate $(x-3)(y+2) < 0$ (which was our hypothesis). $\square$

7. *Proof.*

$$\sum_{i=3}^{8} 2^i = 2^3 \sum_{i=0}^{5} 2^i = 2^3(1+2+4+8+16+32) = 8 * 65 = 520$$

$$\prod_{j=1}^{5} \frac{j}{3} = \frac{\prod_{j=1}^{5} j}{3^5} = \frac{1*2*3*4*5}{3^5} = \frac{40}{81}$$

$\square$

8. *Proof.* $\lfloor x \rfloor = 7 \Rightarrow 7 \leq x < 8$.

   $\lceil x \rceil = 7 \Rightarrow 6 < x \geq 7$.

   The only x that satisfies these two equations is $x = 7$. $\square$

9. (a) The statement is false.

   Proof. Let $n = 1$, $\frac{-1}{3}$ is not an integer. $\square$

   (b) First, we look at the case where a is even.

   Proof. Let $a = 2k$ for some $k \in \mathbb{Z}$.
   $a^3 + a + 2$
   $= (2k)^3 + 2k + 2$
   $= 8k^3 + 2k + 2$
   $= 2(k^3 + k + 1)$
   Since we've factored out a 2, $2 \mid a^3 + a + 2$. $\square$

   Now, for the case where a is odd.

   Proof. Let $a = 2k + 1$ for some $k \in \mathbb{Z}$.
   $a^3 + a + 2$
   $= (2k+1)^3 + (2k+1) + 2$
   $= (8k^3 + 12k^2 + 8k + 4)$
   $= 2(4k^3 + 6k^2 + 4k + 2)$
   Since we've factored out a 2, $2 \mid a^3 + a + 2$. $\square$

   (c) *Proof.* Every prime number is either odd, or equals to 2. Let p be a prime number.
   If p is 2, then $p + 7 = 2 + 7 = 9 = 3 * 3$, a composite number.
   If p is odd, then $p = 2k + 1$ for some $k \in \mathbb{Z}$.
   Then, $p + 7 = 2k + 1 + 7 = 2k + 8 = 2(k + 4)$, a compositive number. $\square$

   (d) This statement is false.

   Proof. Choose $x = 3$.
   Clearly $|x-3| + |x-7| = |0| + |-4| = 4$ which is less than 10. $\square$

(e) *Proof.* We want to factor out a perfect square from 123456. 4 looks like an easy choice. $123456 = 4(30864)$. Notice now that if $m = 30864$, then $123456m = 4(30864)^2 = (2 * 30864)^2$, a perfect square.

This can be generalized to other cases as well. As long as a perfect square can be factored out of the original number, we can always find an m that satisfies the above condition. $\square$

(f) This statement is false.

Proof. We will prove the negation: $\forall k \in \mathbb{Z}, 8 \mid (4k^2 + 12k + 8)$.
If k is even, $k = 2n$ for some $n \in \mathbb{Z}$.
Then, $4k^2 + 12k + 8$
$= 4(2n)^2 + 12(2n) + 8$
$= 16n^2 + 24n + 8$
$= 8(2n^2 + 3n + 1)$
Therefore, $8 \mid (4k^2 + 12k + 8)$.

If k is odd, $k = 2n + 1$ for some $n \in \mathbb{Z}$.
Then, $4k^2 + 12k + 8$
$= 4(2n + 1)^2 + 12(2n + 1) + 8$
$= 4(4n^2 + 4n + 1) + (24n + 12) + 8$
$= 16n^2 + 16n + 4 + 24n + 12 + 8$
$= 16n^2 + 40n + 24 = 8(2n^2 + 5n + 3)$
Therefore, $8 \mid (4k^2 + 12k + 8)$. $\square$

10. *Proof.* Assume $xy = 0$.

By contradiction, we also assume that $x \neq 0$ and $y \neq 0$.

Then $xy \neq 0$, which contradicts our assumption. Therefore, $x = 0$ or $y = 0$. $\square$

11.

A	B	C	$(A \wedge B) \Rightarrow \neg C$
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

12. *Proof.* By induction:

Base Case: $m = 0$, determine $P(0) : \binom{0}{0} = \frac{0!}{0!0!} = 2^0$

Inductive Hypothesis:

Let $m \geq 0$ be an integer such that $P(m)$ is true:

$$\binom{m}{0} + \binom{m}{1} + ... + \binom{m}{m} = 2^m$$

Inductive Step:

Remark. $\binom{m}{0} = \binom{m+1}{0}$ and $\binom{m}{m} = \binom{m+1}{m+1}$.

Now by using Pascal's identity,

$$\binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + ... + \binom{m+1}{m} + \binom{m+1}{m+1}$$

$$= \binom{m+1}{0} + \binom{m}{0} + \binom{m}{1} + \binom{m}{1} + \binom{m}{2} + ... + \binom{m}{m-1} + \binom{m}{m} + \binom{m+1}{m+1}$$

$$= \binom{m}{0} + \binom{m}{0} + \binom{m}{1} + \binom{m}{1} + \binom{m}{2} + ... + \binom{m}{m-1} + \binom{m}{m} + \binom{m}{m}$$

By **Inductive Hypothesis**:

$$= 2 * \sum_{i=0}^{m} \binom{m}{i} = 2 * 2^m = 2^{m+1}$$

We have shown that $P(m) \Rightarrow P(m+1)$, hence $P(n)$ for all $n \geq 0$ is true by Principle of Mathematical Induction (POMI). ☐

Adapted from: `https://math.stackexchange.com/questions/519832/proving-by-induction-that-sum-k-0nn-choose-k-2n`

13. (a) You have proven that there exists values a and b such that the statement is true, but not that the statement is true for all a and b.

    (b) You have proven the converse of the statement but not the original statement.

    (c) You have not shown that $\frac{ka^2}{b^2}$ is an integer, so the final implication is not necessarily true.

14. *Proof.* By Induction:

Base Case: $k = 0$, determine $P(0)$:

$$\sum_{j=0}^{0} \binom{n+j}{j} = \binom{n}{0} = 1$$

$$\text{and } \binom{n+0+1}{0} = \binom{n+1}{0} = 1$$

so P(0) is true.

Inductive Hypothesis: Assume that for some $k \geq 0$, P(k) is true.

$$\text{i.e. assume that } \sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$$

Inductive Step: Prove that $P(k+1)$ is true:

$$\text{i.e. prove that } \sum_{j=0}^{k+1} \binom{n+j}{j} = \binom{n+k+2}{k+1}$$

$$\text{LHS: } \sum_{j=0}^{k+1} \binom{n+j}{j} = \sum_{j=0}^{k} \binom{n+j}{j} + \binom{n+k+1}{k+1}$$

By **Inductive Hypothesis:**

$$\sum_{j=0}^{k} \binom{n+j}{j} + \binom{n+k+1}{k+1} = \binom{n+k+1}{k} + \binom{n+k+1}{k+1}$$

Now by using Pascal's identity,

$$\binom{n+k+1}{k} + \binom{n+k+1}{k+1} = \binom{n+k+2}{k+1}$$

And so we have shown that,

$$\sum_{j=0}^{k+1} \binom{n+j}{j} = \binom{n+k+2}{k+1}$$

We have shown that $P(k) \Rightarrow P(k+1)$ for all $k \geq 0$, hence $P(k)$ is true by Principle of Mathematical Induction (POMI). $\square$

15. *Proof.* This is trivial:

    $(a \mid b) \wedge (b \mid c) \Rightarrow (a \mid c)$ by Transitivity of Divisibility.

    $(a \mid c) \wedge (c \mid d) \Rightarrow (a \mid d)$ by Transitivity of Divisibility. $\square$

16. We must prove both directions of the if and only if:

    Proof. $\Rightarrow$

    Assume that $n \in (A - B)$.

    Then $2 \mid n$ and $4 \nmid n$. By way of contradiction, assume that $n = 2k$ where k is an *even* number, then $n = 2(2j)$ for some $j \in \mathbb{Z}$. It follows that $n = 4j$ and $4 \mid n$. This is a contradiction so it must follow that $n = 2k$ for some *odd* integer k. $\square$

    Proof. $\Leftarrow$

    Assume that $n = 2k$ for some *odd* integer k.

    This implies that $2 \mid n$ and that $n \in A$. By way of contradiction, assume that $4 \mid n$. Then $n = 4j$ for some integer j. This would imply that $k = 2j$ and that k is *even*. This is a contradiction and thus, $4 \nmid n$.

    Since $4 \nmid n$, $n \notin B$ and that $n \in (A - B)$. $\square$

17. To prove equality, we must show that both sides are subsets of each other.

    Proof. Let $s \in (S \cup T) - (S \cap T)$.

    This implies that $s \in S$ or $s \in T$, but **not in both** as $s \notin (S \cap T)$.

    If $s \in S$, then $s \notin T$ and $S \in (S - T)$.

    If $s \notin S$, then $s \in T$ and $s \in (T - S)$.

    Thus, $s \in (S - T) \cup (T - S)$ and

    $$(S \cup T) - (S \cap T) \subseteq (S - T) \cup (T - S)$$

    To prove the other direction:

    Let $s \in (S - T) \cup (T - S)$.

    This implies that $s \in S \wedge s \notin T$ or $s \in T \wedge s \notin S$ but s **cannot be in both**.

    The universe that contains all S and T is $S \cup T$. We can eliminate items that are in both sets by removing $S \cap T$. Thus, $s \in (S \cup T) - (S \cap T)$ and

    $$(S - T) \cup (T - S) \subseteq (S \cup T) - (S \cap T)$$

    Since both sides are subsets of each other:

    $$(S \cup T) - (S \cap T) = (S - T) \cup (T - S)$$

    $\square$