Non-linear systems

Modeling Spread of Infectious disease using the SIRS Model

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MAT3150 Differential Equations and Numerical Methods: Written Submission 2

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October 20, 2020

1.0) Determining critical points and classification

1.1) Problem Definition

There are a vast amount models which can be used to model the spread of infectious diseases. One of which is the SIRS model, which is divided into the following components

$$
\begin{aligned}
S(t) &= \text{number of population members that are susceptible to the disease at time t.}\\
I(t) &= \text{number of population members that are infected with the disease at time t.}\\
R(t) &= \text{number of population that are recovered and now immune at time t.}
\end{aligned}
$$

There are a select few assumptions that are made. One is that the population size *N* is constant over the time period. Also, the rate at which susceptible members become infected is given by the product *βIS*, where *β* is a measure of the contact rate between infected and susceptible individuals. The rate at which infected members recover from the infection is given by *νI.* The final assumption is that the recovered members are not permanently immune. This leads them to rejoin the susceptible group at a rate of *γR.* Furthermore, *S(t) + I(t) + R(t) = N*. This means the components can be written as proportions of the population, such as $s = \frac{S}{N}$, $i = \frac{I}{N}$ and $r = \frac{R}{N}$. Based on the following assumptions, the below system of non-linear differential equations can be formulated

$$
\begin{aligned}
s' &= -\beta is + \gamma r\\
i' &= \beta is - \nu i\\
r' &= \nu i - \gamma r
\end{aligned}
$$

Since *s(t), i(t)* and *r(t)* are proportions of the population, it holds that *s(t) + i(t) + r(t) = 1*. This can be rearranged for *r(t) = 1 − s(t) − i(t)*. Thus, the above system can be converted as follows

$$
\begin{aligned}
s' &= -\beta is + \gamma - \gamma s - \gamma i\\
i' &= \beta is - \nu i
\end{aligned}
$$

Suppose *t* is time in days the parameters have values of *β = 0.05, ν = 0.02* and *γ = 0.01.* The system can be solved for its critical points which can then be classified.

1.2) Solving the nonlinear system for critical points

Substituting the parameter values of *β, ν* and *γ* while also setting the derivatives to 0 yields the following set of nonlinear equations

$$\begin{array}{lllllllll} 0 & = & -0.05is & + & 0.01 & - & 0.01s & - & 0.01i \qquad (1) \\ 0 & = & 0.05is & - & 0.02i & & & & \qquad (2) \end{array}$$

Equation (2) can be factored to determine the values of *i* and *s* required for the solution

$$\begin{aligned} 0 &= 0.05is - 0.02i \\ &= i(0.05s - 0.02) \\ \Rightarrow \quad i = 0 \quad &\vee \quad s = \frac{0.02}{0.05} = 0.4 \end{aligned}$$

Substituting i = 0 into equation (1) is done to determine the first critical point, as below

$$\begin{aligned} 0 &= -0.05(0)s + 0.01 - 0.01s - 0.01(0) \\ &= 0.01 - 0.01s \\ \Rightarrow \quad s &= \frac{0.01}{0.01} = 1 \end{aligned}$$

Thus, the first critical point of the system is *(s,i) = (1,0)*. This means that 100 percent of the population is susceptible to getting the disease at time *t*, while 0 percent of the population is infected/infectious with the disease and 0 percent have recovered and are immune at time *t*.

Substituting s = 0.4 into equation (1) is done to determine the second critical point, as below

$$\begin{aligned} 0 &= -0.05(0.4)i + 0.01 - 0.01(0.4) - 0.01i \\ &= -0.02i + 0.01 - 0.004 - 0.01i \\ &= -0.03i + 0.006 \\ \Rightarrow \quad i &= \frac{0.006}{0.03} = 0.2 \end{aligned}$$

Thus, the second critical point of the system is *(s,i) = (0.4,0.2)*. This means that 40 percent of the population is susceptible to getting the disease at time *t,* while 20 percent of the population is infected/infectious with the disease and 40 percent of the population have recovered from the disease and are immune from reinfection at time *t*.

The classification of critical points of the system can be determined by evaluating the Jacobian at the specific critical points. The Jacobian of the system is defined by the following matrix

$$\boldsymbol{J(s,i)} = \begin{bmatrix} \dfrac{\partial F_1}{\partial s} & \dfrac{\partial F_1}{\partial i} \\ \\ \dfrac{\partial F_2}{\partial s} & \dfrac{\partial F_2}{\partial i} \end{bmatrix}$$

Where $F_1 = -0.05is + 0.01 - 0.01s - 0.01i$ & $F_2 = 0.05is - 0.02i$

1.3) Classifying the critical points

To determine the Jacobian of the system, the partial derivatives for each function with respect to *s* and *i* need to be determined, as below

$$\frac{\partial F_1}{\partial s} = -0.05i - 0.01$$
$$\frac{\partial F_1}{\partial i} = -0.05s - 0.01$$
$$\frac{\partial F_2}{\partial s} = 0.05i$$
$$\frac{\partial F_2}{\partial i} = 0.05s - 0.02i$$

Substituting the partial derivatives into the Jacobian formula yields the following result

$$\boldsymbol{J(s,i)} = \begin{bmatrix} -0.05i - 0.01 & -0.05s - 0.01 \\ 0.05i & 0.05s - 0.02i \end{bmatrix}$$

Now, the critical points can be classified by substituting them into the Jacobian, as follows

$$\boldsymbol{J(1,0)} = \begin{bmatrix} \frac{-1}{100} & \frac{-3}{50} \\ 0 & \frac{3}{100} \end{bmatrix}$$

Since this is a triangular matrix, $\lambda_1 = \frac{-1}{100}$ and $\lambda_2 = \frac{3}{100}$. Since the critical point has two real eigenvalues, one positive and one negative, the critical point is an unstable saddle point. The same method can be applied to determine the type and stability of the second point, as below

$$\boldsymbol{J(0.4,0.2)} = \begin{bmatrix} \frac{-1}{50} & \frac{-3}{100} \\ \frac{1}{100} & 0 \end{bmatrix}$$

The eigenvalues for the above matrix are $\lambda_{1/2} = -0.01 \pm 0.014i$. Since the eigenvalues are complex, the above critical point is a spiral. In addition, it is a stable spiral as the real part of the eigenvalue is negative.

Since *(1,0)* is an unstable critical point, all nearby solutions will move away from the point if we consider plausible initial conditions for *s* and *i*, for example non-negative proportions and *i(t) + s(t) < 1*. The solution *(0.4,0.2)* is stable, thus we can expect long term behavior of the system to approach this critical point for plausible initial conditions.

2.0) Phase portrait for the system

2.1) Matlab Plot of phase portrait

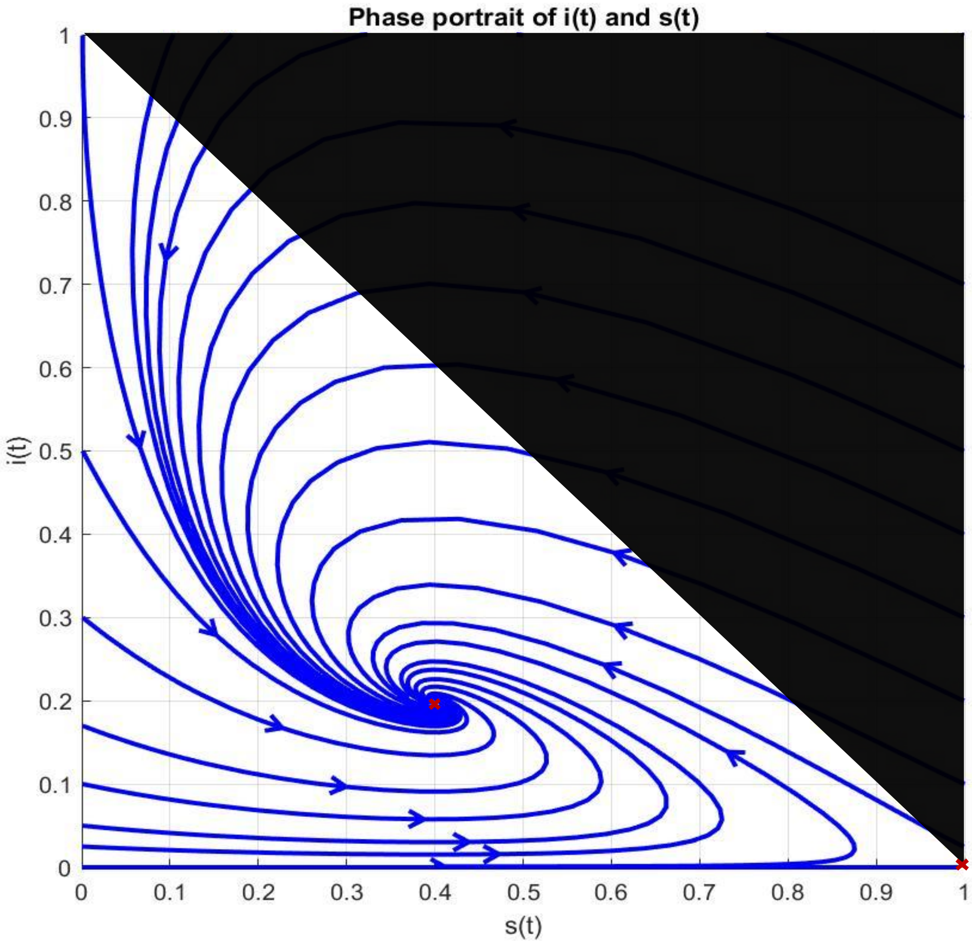

Figure 1.0: System phase portrait

2.2) Notes on the phase portrait

The part of the graph that has been redacted represents non-feasible initial conditions. In other words, *i(t) + s(t) > 1* which does not make sense as *i(t)* and *s(t)* are proportions and it does not make sense to have proportions of a population add to > 100%. The interval of *s(t)* and *i(t)* are both between 0 and 1 as it would not make sense to have proportions of a population above or below these values.

3.0) Approximation for system curves

3.1) Matlab plot for Runge-Kutta approximation

The Runge-Kutta method can be used to approximate s(t), i(t) and r(t). A step size of 1 was used to produce the graph below over the interval $0 \le t \le 500$ and initial conditions (*s(0)*, *i(0)*) = (0.8, 0.1)

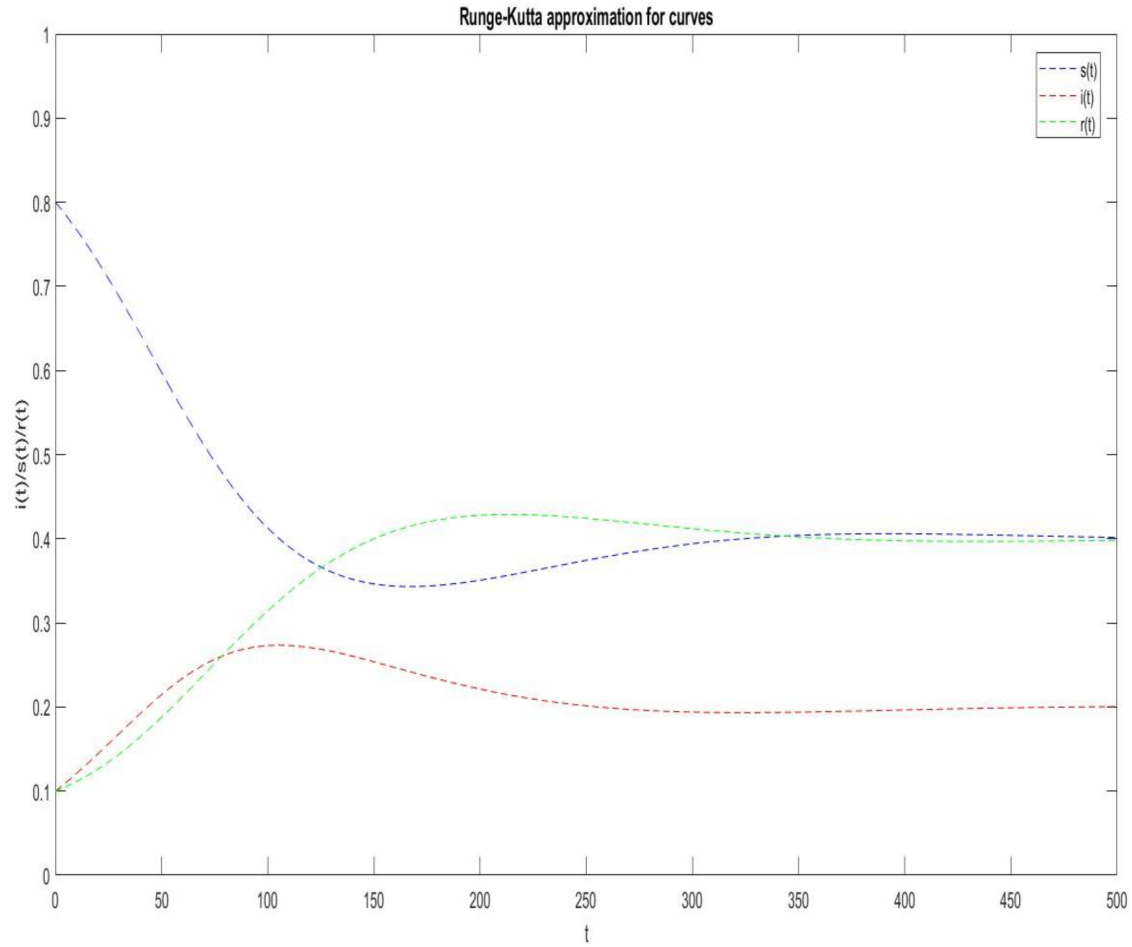

Figure 2.0: Runge-Kutta approximation for curves

3.2) Explanation

For the first 100 days, the proportion of the population susceptible to getting the disease significantly lowers, while the proportion of the infected population and the proportion of the population that have recovered significantly rises. However, after this, the recovered population proportion continues to rise for another 100 days while the infected proportion of the population slightly lowers. The proportion of the population that is susceptible to the disease slightly increases. All curves flatten after this, where the infected population stays at roughly 20%, while the susceptible portion and recovered proportion stays at roughly 40%. This is consistent with the phase portrait.

4.0) One critical point in the system

4.1) Solving for the threshold of β

Recall that the system can be condensed into the below nonlinear system of differential equations

$$\begin{array}{lllllllll} s' & = & -\beta is & + & \gamma & - & \gamma s & - & \gamma i \\ i' & = & \beta is & - & \nu i & & & & \end{array}$$

Setting the derivatives to 0, and setting ν and γ to retain their previous values, allows to determine the threshold of β that will yield one critical point in the system, as follows

$$\begin{array}{lllllllll} 0 & = & -\beta is & + & 0.01 & - & 0.01s & - & 0.01i \\ 0 & = & \beta is & - & 0.02i & & & & \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Notice that, in equation (2), i is a common factor in terms of the right hand side of the equation. In other words, $i = 0$ will always make the equation true. This means that for every integer value of β, the system will always have at least the solution of (*s,i*) = (1,0). Thus, a trivial value to yield one critical point is $\beta = 0$. However, a value of *s* = 1 will also yield one critical point because as shown before, substituting *s* = 1 into equation (1) yields *i* = 0, for (*s,i*) = (1,0). To make *s* = 1 in equation (2), the value of $\beta = 0.02$ is used as 0.02*i* can be factored out. This is shown below, as follows

$$\begin{array}{lllll} 0 & = & 0.02is & - & 0.02i \\ & = & 0.02i(s-1) & & \\ \Rightarrow \quad i = 0 & \vee & s = 1 & & \end{array}$$

Substituting *s* = 1 into equation (1) yields the one critical point of the system, as below

$$\begin{array}{lllllllll} 0 & = & -0.02is & + & 0.01 & - & 0.01s & - & 0.01i \\ & = & -0.02(1)i & + & 0.01 & - & 0.01(1) & - & 0.01i \\ & = & -0.02i & + & 0.01 & - & 0.01 & - & 0.01i \\ \Rightarrow \quad i = 0 & & & & & & & & \end{array}$$

Thus, the one critical point for the system when $\beta = 0.02$ is (*s,i*) = (1,0). The Jacobian can be used to classify this critical point as below

$$\boldsymbol{J}(\mathbf{1},\mathbf{0}) = \begin{bmatrix} \frac{-1}{100} & \frac{-3}{50} \\ 0 & \frac{3}{100} \end{bmatrix}$$

Since this is a triangular matrix, $\lambda_1 = \frac{-1}{100}$ and $\lambda_2 = \frac{3}{100}$. Since the critical point has two real eigenvalues, one positive and one negative, the critical point is an unstable saddle point. This means 100 percent of the population is susceptible to getting the disease while 0 percent have recovered and 0 percent are infected/infectious.