6.058 Fourier Examples

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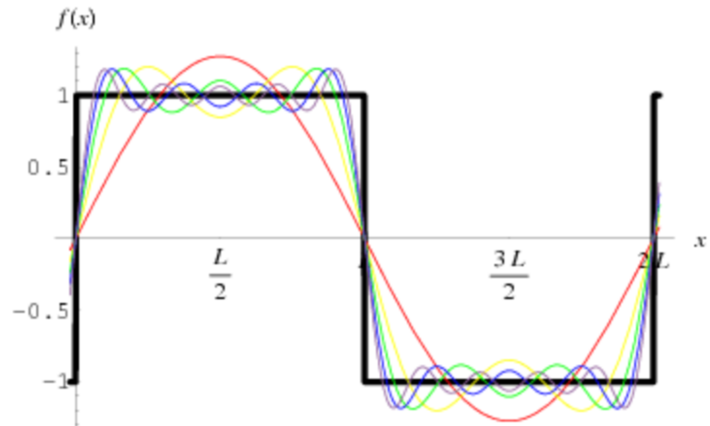
1 Fourier Series

We are going to take the Fourier Series of a square wave over the range 0 to 2L. A square wave, as is noted by the sine waves



on it, is a periodic function which is odd.

Since the function is odd, we only have to consider the odd part of the Fourier Series. We will calculate the odd Fourier series coefficients, b_n using

$$b_n = \frac{1}{L}\int_0^{2L} f(t)\sin(\frac{n\pi t}{L})dt$$

This equation reduces to

$$b_n = \frac{2}{L}\int_0^{L} f(t)\sin(\frac{n\pi t}{L})dt$$

$$b_n = \frac{4}{n\pi}\sin^2(\frac{1}{2}n\pi)$$

$$b_n = \frac{2}{n\pi}(1-(-1)^n)$$

The Fourier series is therefore

$$f(t) = \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}\sin(\frac{n\pi t}{L})$$

2 Continuous Fourier Transform

We shall consider the Fourier Transform of several very important signals.

For the signal $\delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt = 1$$

Since a δ in time domain maps to a constant in frequency domain, a constant in time maps to a δ in frequency. In particular, the Fourier transform of 1 is $2\pi\delta(t)$.

For the signal $\delta(t-t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty}\delta(t-t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\delta(0)e^{-j\omega t_0}dt = e^{-j\omega t_0}$$

The signal $e^{-at}u(t)$ for $Re(a) > 0$

$$X(jw) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{(-a-j\omega)t}dt = \left.\frac{e^{(-a-j\omega)t}}{(-a-j\omega)}\right|_0^\infty = \frac{1}{a+j\omega}$$

The signal $\cos(\omega_c t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} e^{-j\omega t} = \int_{-\infty}^{\infty} \frac{e^{j(\omega-\omega_c)t} + e^{j(\omega+\omega_c)t}}{2} = 2\pi\delta(\omega-\omega_c) + 2\pi\delta(\omega+\omega_c)$$

3 Discrete Fourier Transform

Now let's flex our muscles by working through a few simple examples in the discreet case. Let's start with the unit impulse:

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = 1$$

We see that the delta function gets smeared out over the whole space, just as in the continuous case. What about a delta function that's been translated?

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} \delta[n-k]e^{-j\Omega n} = e^{-j\Omega k}$$

Notice that both $1 = e^{j\Omega 0}$ and $e^{-j\Omega k}$ are both eigenfunctions. Thus, it seems that shifting the delta function changes the eigenfunction that is spread across the frequency domain. Now for something slightly harrier: what is the Fourier transform of the function equal to 1 for all n (x[n] = 1)?

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} e^{-j\Omega n}$$

This is a strange summation. For any non-zero Ω, the sum converges to zero, since this amounts to summing together uniformly distributed vectors on the unit circle (if you don't quite see this spend some time thinking about it; it's a neat exercise). But when $\Omega = 0$, we get an infinite summation. So $X(j\Omega) = 0$ for all $\Omega \neg 0$ but $X(j\Omega) = \infty$ when $\Omega = 0$; this smells like a Dirac delta function. To confirm this suspicion, lets see what happens when we integrate over it:

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{-j\Omega n} d\Omega = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-j\Omega n} d\Omega = 2\pi$$

Note that in the last step we employed the fact that

$$\int_{-\pi}^{\pi} e^{-j\Omega n} d\Omega = 0$$

in all cases except for $n = 0$, in which case

$$\int_{-\pi}^{\pi} e^{-j\Omega 0} d\Omega = \int_{-\pi}^{\pi} 1 d\Omega = 2\pi$$

Thus, we have a Dirac delta centered at the origin of the frequency domain yielding 2π when integrated over:

$$X(j\Omega) = 2\pi\delta(\Omega)$$

Above we noted that, when we are transforming delta functions, shifted delta functions return different eigenfunctions to be spread across the Fourier domain. Naturally, we might ask: what is the Fourier transform of an arbitrary eigenfunction spread across the time domain (say, $x[n] = e^{j\Omega_0 k}$)?

$$X(j\Omega) = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{j\Omega_0 k} e^{-j\Omega n} d\Omega = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{j\Omega_0 k} e^{-j\Omega n} d\Omega = 2\pi\delta(\Omega - \Omega_0)$$

4 System Functional

Suppose we have the RC circuit system from PSET 1 with the known impulse response of $h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$. We wish to find the output of this system to arbitrary input. In order to do this we must calculate $H(j\omega)$. From the above continuous identity that $e^{-at}u(t)$ with $Re(a) > 0$ has fourier transform $\frac{1}{a+j\omega}$ we find that

$$H(j\omega) = \frac{1}{RC}\frac{1}{\frac{1}{RC} + j\omega}$$

If we were to put $x(t) = \cos(6t)$ input the system. In order to do this we must first take the Fourier transform. We note that the Fourier series is $X(j\omega) = 2\pi\delta(\omega - 6) + 2\ pi\delta(\omega + 6)$. Thus, $Y(j\omega) = \frac{2\pi\delta(\omega-6)}{RC}\frac{1}{\frac{1}{RC}+j\omega} + \frac{2\ pi\delta(\omega+6)}{RC}\frac{1}{\frac{1}{RC}+j\omega}$

5 Real World Systems: RLC Circuit

The differential equation modeling an RLC circuit is

$$\frac{d^2}{dt^2}i(t) + \frac{R}{L}\frac{d}{dt}i(t) + \frac{i(t)}{LC} = \frac{1}{L}\frac{d}{dt}V(t)$$

where $i(t)$ is the current, R is the resistance, C is the capacitance, and L is the inductance, and $V(t)$ is the voltage supplied to the circuit.

Suppose we want to find the transfer function, $H(j\omega)$. We know that $Y(j\omega) = X(j\omega)H(j\omega)$ so finding $H(j\omega)$ reduces to finding $\frac{Y(j\omega)}{X(j\omega)}$. We note that a property of the Fourier Transform is that the Fourier transform of a derivative corresponds to pulling $j\omega$ out in front of a Fourier Transform. Thus, this equation, by the Fourier transform, reduces to

$$(j\omega)^2Y(j\omega) + \frac{R}{L}(j\omega)Y(j\omega) + \frac{1}{LC}Y(j\omega) = \frac{1}{L}(j\omega)X(j\omega)$$

where $Y(j\omega)$ is the Fourier transform of the current and $X(j\omega)$ is the Fourier Transform of the voltage.

Thus, we find that

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{j\omega}{L}}{(j\omega)^2 + \frac{R}{L}(j\omega) + \frac{1}{LC}} = H(j\omega)$$

6 Real World System: Low Pass Filter

Suppose we have a system in the real world that obeys the equation

$$y(t) = k\frac{dx(t)}{dt} + x(t)$$

Let us find the transfer function.

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega k + 1} = H(j\omega)$$

We shall plot the magnitude of this Fourier transform as we increase ω. We get an image similar to this: