Variance Reduction for Monte Carlo Option Pricing

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Abstract

Option pricing is the task of assigning a fair value to an option contract. An option contract is a type of financial asset derived from an underlying asset, typically a stock. The contract offers the owner the right to either buy or sell the underlying asset at a specified price, on or before an expiry date. In exchange for this right, the buyer pays a premium on the underlying asset betting the asset reaches the specified price by the expiry date. The seller, in exchange for receiving a premium on the asset, takes on the obligation to fulfill the contract should the underlying asset meet the price by the expiry date. Thus determining the fair value of an option contract depends heavily on accurate estimates of the future price of the underlying asset. The evolution of this asset price over time can be modeled as geometric Brownian motion (GBM), which is simply a variation of a random walk simulated with Monte Carlo methods, or with a closed form partial differential equation (PDE) like the Black-Scholes equation, or with a combination of the two methods known as Quasi-Monte Carlo (QMC) simulation where stochastic motion is added to the PDE system. In this paper we review the basics of option pricing theory and Monte Carlo methods relating to it. We then evaluate a suite of variance reduction techniques specifically for the arithmetic Asian style of option on a Geometric Brownian motion model (whose distribution does not have a closed form solution), and propose a combination of techniques that best reduce the variance of the Monte Carlo estimator. Since a low variance increases confidence in our estimated price we've chosen it as our primary metric for comparison.

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1. Introduction

An option contract between a buyer (*holder*) and seller (*writer*) is a type of financial asset know as a derivative. This is because the value of the option is derived from an underlying assets' value over a fixed time horizon. All option contracts consist of an underlying asset, a right to buy (*call*) or a right to sell (*put*), an expiry date, and a strike price. The strike price is the price which the underlying asset must reach before the call or put option can be exercised. Option contracts also come in four popular variants which have different rules on the right to exercise it. **American** options allow for the contract to be exercised at any time before the expiration date, **European** options can only be exercised on the expiration date, **Asian** options when exercised are paid out by the difference between the strike and the average asset price over the life of the contract, and **barrier** options where the payoff also depends on a predefined barrier that the asset price may cross.

Formally, let V denote the price of an option whose value at maturity depends on a stochastic process S_t, typically modeled as Geometric Brownian Motion (GBM) under the Black-Scholes framework. The option payoff is generally expressed as,

$$V = \max\{f(S) - K, 0\},$$

where K is the strike price and $f(S)$ is the a function specific to the option style. For American options, $f(S) = S_t$ for any t such that $S_t - K > 0$, European options $f(S) = S_T$, Asian options $f(S) = (1/T)\int_{t_0}^{T} S_t dt$, and barrier options the payout depends on whether the path S_t crosses a predefined barrier level during the contract life (Hull, 2015; Glasserman, 2004).

For many such contracts, particularly those with path-dependent payoffs like Asian or barrier options, a closed-form analytical solution is not available. In such cases, Monte Carlo (MC) methods provide a flexible numerical approach to approximate the fair price of the option defined $v = \mathbb{E}^{\mathbb{Q}}[V]$, the expected value of its future payoff V under the risk-neutral probability measure $\mathbb{Q}$, by simulating a large number of sample paths of the underlying asset and averaging the discounted payoffs (Glasserman, 2004). The stochastic nature of these methods introduces statistical error and the convergence rate of the standard MC is

$O(N^{-1/2})$. This means that reducing the standard error by a factor of 10 requires increasing the sample size by a factor of 100. At scale this becomes computationally expensive for long-life span or complex option contracts.

As such, a key concern in Monte Carlo option pricing is the variance of the estimator. High variance estimators require more simulations to reach a given accuracy, which may be infeasible in time-sensitive applications. Variance reduction techniques aim to reduce this variance without introducing bias, thereby improving computational efficiency while maintaining pricing fidelity (Caflisch, 1998; Glasserman et al., 1999).

Despite its broad applicability, standard Monte Carlo methods suffer from slow convergence and high variance, paritularly Asian options with long monitoring periods or deep in-/out-of-the-money strike prices (ITM and OTM respectively). To address this inefficiency, significant research has focused on **variance reduction techniques**. Among the most effective is the **control variate method**, which leverages the exact solution for the geometric-average Asian option – available under Black-Scholes assumptions – to substantially variance when pricing the arithmetic-average counter part (Kemna & Vorst, 1990; Fu et al., 1999). Other approaches include **antithetic variates** and **importance sampling**, each offering unique benefits depending on the options' "moneyness" and monitoring frequency (Glasserman et al., 1999).

In parallel, **Quasi-Monte Carlo** (QMC) methods have gained prominence as an alternative to standard Monte Carlo. By using low-discrepant sequences, such as Sobol or Halton, QMC techniques achieve faster convergence guarantees in practice, often approaching $O(N^{-1})$ instead of the traditional $O(N^{-1/2})$ (Caflisch, 1998). When combined with path-construction strategies like Brownian bridge or principal component analysis (PCA), QMC methods can significantly outperform classical simluations for high-dimensional path-dependent options (Imai & Tan, 2006; Sabino, 2007).

This paper investigates and compares several variance reduction methods for pricing Asian options using Monte Carlo simulation. Through theoretical discussions and empirical experiments, we asses the impact of each method on estimator accuracy and computational efficiency. The objective is to identify combinations of techniques that enable fast and reliable pricing for path-dependent options where standard methods fall short.

2. Methodology

The Black-Scholes PDE model provides a theoretical framework for how asset prices are assumed to evolve over time in a risk-neutral world. One of its key assumptions is that the price of the underlying asset follows a geometrix Brownian motion (GBM). The mathematical representation of this GBM is often expressed by the following stochastic differential equation,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

However, for option pricing under the risk-neutral frameowrk (which is consistent with the Black-Scholes approach), the drift term μ is replaced by the risk-free interest rate r:

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t$$

where dS_t is the change in the asset price at time t, S_t is the asset price at time t, r is the risk-free interest rate, σ is the volatility of the asset price, dW_t is the increment of the standard Wiener process (Brownian motion), representing the random shocks.

These core variables in the Black-Scholes formula are used to model the random walk in a Monte Carlo simulation designed to value options under similar assumptions. The current stock price S_0 is the starting point for all the simulated price paths. Each simulation begins with the current price. The risk free interest rate, r is used as the drift rate in the risk-neutral random walk and it represents the expected growth rate of the asset price. The volatility term σ determines the magnitude of the random fluctuations in the asset price paths. Higher volatility leads to wider price swings in the simulations. The time to expiry T defines the time horizon over which the random walk is simulated. The simulation generates price paths up to the options' expiry date. The strike price, K, is not directly used in simulating the random walk of the underlying asset, but is crucial for calculating the payoff of the option at expiration for each simulated path.

The general Monte Carlo process then is as follows. We first simulate price paths. Numerous possible future price paths of the underlying asset are generated from the current price S_0 up to the expiration date T, driven by the drift r and the volatility σ. Next, at the expiration time T for each simulated price path, the payoff of the option is calculated based on the strike price, K. For a call option, the payoff is $\max(S_t - K, 0)$ and for a put options its $\max(K - S_T, 0)$, where S_T is the simulated asset price at expiration. Lastly, the calculated payoffs for each path are then discounted back to the present value using the risk-free interest rate r. The average of these discounted payoffs across all the simulated paths proves an estimate of the options fair value. The pseudo code for a basic MC method for option pricing is given in Algorithm 1.

In our experiments we run up to $10,000$ random walks, over a time horizon of one year. We break up the year

Algorithm 1 Monte Carlo Pricing for Arithmetic Asian Option

```
Set parameters: S_0 (initial asset price), K (strike price), r (risk-free rate), σ (volatility), T (time to maturity), N (number of time steps), M (number of simulations)
Δt ← T/N
payoffSum ← 0
for i = 1 to M do
    Initialize asset price path: S ← S_0
    pathSum ← S_0
    for j = 1 to N do
        Generate Z ~ N(0,1)
        S ← S exp((r − ½σ²)Δt + σ√(Δt) Z)
        pathSum ← pathSum + S
    end for
    Compute average price: S_avg ← pathSum/(N+1)
    Compute payoff: payoff ← max(S_avg − K, 0)
    Accumulate payoff: payoffSum ← payoffSum + payoff
end for
Compute option price: Price ← e^{−rT} payoffSum/M
return Price
```

into 256 steps in our random walk approximating the 252 trading days in the year. The number 256 is chosen as an approximation due to its property of being a power of 2, which helps in some of the modeling. We test each of the MC variants below on a time horizon of one year, with this step size of $\Delta t = 1/256$ up to $10,000$ paths. We test on three types of option contracts with strike price at ($K = 100$), above ($K = 120$), and below ($K = 80$) the original asset price. These are known as at the money, in the money and out the money options.

For each method we produce graphs of the evolution of the option price and margine of error as a function of our number of simulations. The margin of error is directly correlated to the variance of the method, so we aim for accurate methods with minimal margins of error. We also compare the variance reduction ratio of the various methods against the vanilla MC method and display these results in a table.

2.1. Vannila Monte Carlo

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2.2. Antithetic Variates

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2.3. Control Variates

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2.4. Importance Sampling

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2.5. Quasi Monte Carlo

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3. Results

- Classic w/ $K = 100$
- AV w/ K = 100

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4. Discussion

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5. Author Contribution

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Software and Data

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Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none that require special highlight.

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A. You *can* have an appendix here.

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