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# Stochastic Lotka–Volterra systems with Lévy noise

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ABSTRACT

This paper is concerned with stochastic Lotka–Volterra models perturbed by Lévy noise. Firstly, stochastic logistic models with Lévy noise are investigated. Sufficient and necessary conditions for stochastic permanence and extinction are obtained. Then three stochastic Lotka–Volterra models of two interacting species perturbed by Lévy noise (i.e., predator–prey system, competition system and cooperation system) are studied. For each system, sufficient and necessary conditions for persistence in the mean and extinction of each population are established. The results reveal that firstly, both persistence and extinction have close relationships with Lévy noise; Secondly, the interaction rates play very important roles in determining the persistence and extinction of the species.



## 1. Introduction

In recent years, stochastic Lotka–Volterra system driven by Brownian motion has been received great attention and has been studied extensively (see, e.g. [2,6–8,10,12,14–17,19–21]). A classical stochastic two-species Lotka–Volterra system can be expressed as follows

$$\begin{cases} dx_1(t) = x_1(t)\big[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\big]dt + \sigma_1 x_1(t)\,dB(t), \\ dx_2(t) = x_2(t)\big[r_2 - a_{21}x_1(t) - a_{22}x_2(t)\big]dt + \sigma_2 x_2(t)\,dB(t), \end{cases} \tag{1}$$

with initial conditions

$$x_1(0) = x_{10} > 0, \qquad x_2(0) = x_{20} > 0, \tag{2}$$

where $r_i$, $a_{ij}$ and $\sigma_i$ $(i, j = 1, 2)$ are constants, $B(t)$ is a standard Brownian motion. Rudnicki and Pichór [19] investigated system (1) in the predator–prey case (that is $r_1 > 0$, $r_2 < 0$, $a_{12} > 0$ and $a_{21} < 0$). The authors considered the convergence of densities of the distributions of the solutions. Li and Mao [10] and Jiang et al. [8] studied system (1) in the competition case (that is $r_1 > 0$, $r_2 > 0$, $a_{12} > 0$ and $a_{21} > 0$). They considered the persistence, extinction, global attractivity and stationary distribution of the system. System (1) with time delays was developed and discussed in [2,7,20]. Model (1) under Markov switching was analyzed in [14,21].

On the other hand, population systems may suffer sudden environmental perturbations, such as epidemics, earthquakes, hurricanes, etc. These phenomena cannot be described by stochastic system (1). Introducing Lévy jumps into the underlying population models may be a reasonable way to describe these phenomena [3,4,13]. Bao et al. [3] did pioneering work in this area. They proposed the following stochastic competitive Lotka–Volterra population dynamics with jumps

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$$dx_i(t) = x_i(t^-)\left[\left(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_i(t^-)\right)dt + \sigma_i(t)\,dB(t) + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\right], \quad i = 1,\dots,n, \tag{3}$$

where $x(t^-)$ is the left limit of $x(t)$, $N$ is a Poisson counting measure with characteristic measure $\lambda$ on a measurable subset $\mathbb{Y}$ of $(0,+\infty)$ with $\lambda(\mathbb{Y}) < +\infty$, and $\tilde{N}(dt,du) = N(dt,du) - \lambda(du)\,dt$. The authors considered the existence and uniqueness, boundedness, tightness, Lyapunov exponents and extinction of the positive solutions. Particularly, for the following logistic model

$$dx(t) = x(t^-)\left[\left(r(t) - \sum_{j=1}^{n} a(t)x(t^-)\right)dt + \sigma(t)\,dB(t) + \int_{\mathbb{Y}} \gamma(t,u)\tilde{N}(dt,du)\right], \tag{4}$$

under the assumption that

$$1 + \gamma(t,u) > 0, \quad u \in \mathbb{Y}, \quad t \geqslant 0, \tag{5}$$

the authors [3] investigated its explicit solution, Lyapunov exponent, extinction, stochastic permanence and invariant measure. Then Bao and Yuan [4] developed the following general Lotka–Volterra population dynamics with jumps

$$dX(t) = X(t^-)\left[\left(R - AX(t^-)\right)dt + \Gamma X(t^-)\,dB(t) + \int_{\mathbb{Y}} H\left(X(t^-),u\right)\tilde{N}(dt,du)\right], \tag{6}$$

where

$$X = (x_1,\dots,x_n)^T, \qquad R = (r_1,\dots,r_n)^T, \qquad A = (a_{ij})_{n\times n}, \qquad \Gamma = (\sigma_{ij})_{n\times n}.$$

Under some simple conditions, the authors showed that both jump processes and Brownian motion can suppress the explosion of the solution. These important results reveal that jump processes may have significant impacts on the properties of systems. Moreover, the authors discussed the asymptotic pathwise estimation of (6).

So far, to the best of our knowledge, there are only three papers [3,4,13] which deal with population systems driven by Lévy jumps. Many important problems have not been investigated yet. Firstly, Bao et al. [3] has established sufficient conditions for extinction and stochastic permanence of system (4). However, their conditions are some restrictive. Thus in Section 2, we establish a new sufficient condition for stochastic permanence which is much weaker than [3]. Particularly, if system (4) is time independent, we show that our conditions for extinction and stochastic permanence are sufficient and necessary.

Secondly, it is well known that Lotka–Volterra system is one of the most important models in both ecology and mathematical ecology. Thus it is important to investigate Lotka–Volterra model with Lévy noise to see the effects of Lévy noise on the persistence and extinction of each species. However, these effects are not clear yet. Motivated by these, in Section 3, we study the following stochastic Lotka–Volterra model of two interacting species with jumps

$$(\mathrm{M}):\quad \begin{cases} dx_1(t) = x_1(t^-)\left\{\left[r_1 - a_{11}x_1(t^-) - a_{12}x_2(t^-)\right]dt + \sigma_1\,dB(t) + \int_{\mathbb{Y}} \gamma_1(u)\tilde{N}(dt,du)\right\}, \\ dx_2(t) = x_2(t^-)\left\{\left[r_2 - a_{21}x_1(t^-) - a_{22}x_2(t^-)\right]dt + \sigma_2\,dB(t) + \int_{\mathbb{Y}} \gamma_2(u)\tilde{N}(dt,du)\right\}, \end{cases} \tag{7}$$

The following three types of model (M) will be dealt with:

(I) Cooperation system (M), that is $r_1 > 0$, $r_2 > 0$, $a_{12} < 0$, $a_{21} < 0$;
(II) Predator–prey system (M), that is $r_1 > 0$, $r_2 < 0$, $a_{12} > 0$, $a_{21} < 0$;
(III) Competition system (M), that is $r_1 > 0$, $r_2 > 0$, $a_{12} > 0$, $a_{21} > 0$.

For each type of (M), under some simple assumptions, sufficient and necessary conditions for persistence in the mean and extinction of each population are established. Particularly, a competitive exclusion principle for the competition system (M) is derived. Form these results, we can see clearly that: Firstly, Lévy noise can change the persistence and extinction of each type of (M) significantly; Secondly, both persistence and extinction of each type of (M) have close relationships with $a_{ij}$.

The following additional restrictions on (7) are natural for biological meanings: $1 + \gamma_i(u) > 0$, $u \in \mathbb{Y}$, $i = 1, 2$. As a standing hypothesis we assume in this paper that $N$ and $B$ are independent. Throughout this paper, $K$ stands for a generic positive constant whose values may be different at different places. And the up limit includes $-\infty$.

## 2. Logistic model

To begin with, let us consider system (4) in time independent case:

$$dx(t) = x(t^-)\left[\left(r - ax(t^-)\right)dt + \sigma\, dB(t) + \int_{\mathbb{Y}} \gamma(u)\tilde{N}(dt, du)\right], \tag{8}$$

where $\gamma(u) > -1$ for $u \in \mathbb{Y}$. We need the following assumption and lemma.

**Assumption 1.** There is a positive constant $c$ such that $\int_{\mathbb{Y}}[\ln(1+\gamma(u))]^2\lambda(du) < c$.

**Lemma 1.** *(See Bao et al. [3].) Consider system (8).*

(i) *For any $p \in [0, 1]$, there is a constant $K$ such that $\limsup_{t\to+\infty} E(x(t))^p \leqslant K$.*
(ii) *If Assumption 1 holds and $b < 0$, where*

$$b = r - 0.5\sigma^2 - \int_{\mathbb{Y}} \left(\gamma(u) - \ln\left(1+\gamma(u)\right)\right)\lambda(du),$$

*then $x(t)$ is extinctive almost surely (a.s.), i.e., $\lim_{t\to+\infty} x(t) = 0$, a.s.*

**Definition 1.** (See e.g. Bao et al. [3].) A population, $x(t)$, is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\left\{x(t) \geqslant \alpha_1\right\} \geqslant 1 - \varepsilon, \qquad \liminf_{t\to+\infty} \mathcal{P}\left\{x(t) \leqslant \alpha_2\right\} \geqslant 1 - \varepsilon. \tag{9}$$

**Theorem 1.** *For system (8), if $b > 0$, then $x(t)$ is stochastically permanent.*

**Proof.** Clearly,

$$\lim_{\theta\to 0^+} \left\{0.5\theta\sigma^2 + \int_{\mathbb{Y}} \left[\frac{1}{\theta(1+\gamma(u))^\theta} - \frac{1}{\theta}\right]\lambda(du)\right\} = \int_{\mathbb{Y}} \ln\left(\frac{1}{1+\gamma(u)}\right)\lambda(du) = -\int_{\mathbb{Y}} \ln\left(1+\gamma(u)\right)\lambda(du).$$

Thus if $b > 0$, we can find a sufficiently small $\theta > 0$ such that

$$r - 0.5\sigma^2 - \int_{\mathbb{Y}} \gamma(u)\lambda(du) - \left\{0.5\theta\sigma^2 + \int_{\mathbb{Y}} \left[\frac{1}{\theta(1+\gamma(u))^\theta} - \frac{1}{\theta}\right]\lambda(du)\right\} > 0. \tag{10}$$

Define $y = 1/x$ for $x > 0$. Then by Itô's formula (see e.g. [9, Theorem 2.5]),

$$\begin{aligned} dy(t) = {} & y(t)\left[-r + \sigma^2 + \int_{\mathbb{Y}} \left(\frac{1}{1+\gamma(u)} - 1 + \gamma(u)\right)\lambda(du)\right]dt + a\,dt \\ & + \sigma y(t)\,dB(t) + y(t)\int_{\mathbb{Y}} \left(\frac{1}{1+\gamma(u)} - 1\right)\tilde{N}(dt, du). \end{aligned}$$

Define $z = y^\theta$, where $\theta$ satisfies (10). In view of Itô's formula,

$$\begin{aligned} dz(t) = {} & \theta y^{\theta-2}(t)\left\{-y^2(t)\left[r - 0.5\sigma^2 - \int_{\mathbb{Y}} \gamma(u)\lambda(du) - 0.5\theta\sigma^2 - \int_{\mathbb{Y}} \left(\frac{1}{\theta(1+\gamma(u))^\theta} - \frac{1}{\theta}\right)\lambda(du)\right] + ay(t)\right\}dt \\ & - \theta\sigma y^\theta(t)\,dB(t) + y^\theta(t)\int_{\mathbb{Y}} \left[\frac{1}{(1+\gamma(u))^\theta} - 1\right]\tilde{N}(dt, du). \end{aligned}$$

Let $\eta$ be sufficiently small such that

$$r - 0.5\sigma^2 - \int_{\mathbb{Y}} \gamma(u)\lambda(du) - \left\{0.5\theta\sigma^2 + \int_{\mathbb{Y}} \left[\frac{1}{\theta(1+\gamma(u))} - \frac{1}{\theta}\right]\lambda(du)\right\} > \eta/\theta. \tag{11}$$

Define $V = e^{\eta t} z = e^{\eta t} y^{\theta}$. Making use of Itô's formula leads to

$$\begin{aligned} dV(t) &= \theta e^{\eta t} y^{\theta-2}(t)\Bigg\{-y^2(t)\Bigg[r - 0.5\sigma^2 - \int_{\mathbb{Y}} \gamma(u)\lambda(du) - 0.5\theta\sigma^2 \\ &\quad - \int_{\mathbb{Y}} \left(\frac{1}{\theta(1+\gamma(u))^{\theta}} - \frac{1}{\theta}\right)\lambda(du) - \eta/\theta\Bigg] + ay(t)\Bigg\} dt \\ &\quad - \theta\sigma e^{\eta t} y^{\theta}(t)\, dB(t) + e^{\eta t} y^{\theta}(t) \int_{\mathbb{Y}} \left[\frac{1}{(1+\gamma(u))^{\theta}} - 1\right]\tilde{N}(dt, du) \\ &=: e^{\eta t} K\big(y(t)\big)\, dt - \theta\sigma e^{\eta t} y^{\theta}(t)\, dB(t) + e^{\eta t} y^{\theta}(t) \int_{\mathbb{Y}} \left[\frac{1}{(1+\gamma(u))^{\theta}} - 1\right]\tilde{N}(dt, du). \end{aligned}$$

By (11), $K(y)$ is upper bounded in $(0, +\infty)$, namely $K := \sup_{y>0} K(y) < +\infty$. Thus,

$$dV(t) \leqslant K e^{\eta t}\, dt - \theta\sigma e^{\eta t} y^{\theta}(t)\, dB(t) + e^{\eta t} y^{\theta}(t) \int_{\mathbb{Y}} \left[\frac{1}{(1+\gamma(u))^{\theta}} - 1\right]\tilde{N}(dt, du).$$

Integrating then taking expectations, we can observe that $\limsup_{t\to+\infty} E[x^{-\theta}(t)] \leqslant K/\eta = K_1$. For any given $\varepsilon > 0$, set $\alpha_1 = \varepsilon^{1/\theta}/K_1^{1/\theta}$. An application of Chebyshev's inequality gives

$$\mathcal{P}\big\{x(t) < \alpha_1\big\} = \mathcal{P}\big\{x^{-\theta}(t) > \alpha_1^{-\theta}\big\} \leqslant E\big[x^{-\theta}(t)\big]/\alpha_1^{-\theta} = \alpha_1^{\theta} E\big[x^{-\theta}(t)\big].$$

Consequently $\limsup_{t\to+\infty} \mathcal{P}\{x(t) < \alpha_1\} \leqslant \varepsilon$, in other words, $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \geqslant \alpha_1\} \geqslant 1 - \varepsilon$.

The proof of the second part in (9) follows from Chebyshev's inequality and (i) in Lemma 1. This completes the proof. □

**Remark 1.** Clearly, if Assumption 1 holds, then Lemma 1 and Theorem 1 establish the sufficient and necessary conditions for stochastic permanence and extinction of system (8).

**Remark 2.** Note that $b \leqslant r - 0.5\sigma^2$, thus Theorem 1 reveals an important fact that: Lévy noise makes the population extinct. For example, consider the following equation

$$dx = x(r - ax)\, dt + \sigma x\, dB(t), \quad x(0) = x_0 > 0. \tag{12}$$

It is well known that (see e.g. [12]) if $r - 0.5\sigma^2 > 0$, then the solution of (12) is stochastically permanent. However, Theorem 1 tells us that if $\int_{\mathbb{Y}} (\gamma(u) - \ln(1 + \gamma(u)))\lambda(du) > r - 0.5\sigma^2$, then the solution of (8) is extinctive with probability one.

**Remark 3.** It is useful to point out that Theorem 1 can be generalized to the non-autonomous case. For Eq. (4), under (5), similar to the proof of Theorem 1 we have:

(a) If $\int_{\mathbb{Y}} [\ln(1 + \gamma(u, t))]^2 \lambda(du) < c$ for all $t \geqslant 0$ and $\limsup_{t\to+\infty} t^{-1} \int_0^t b(s)\, ds < 0$, then $x(t)$ is extinctive a.s., where $b(t) = r(t) - 0.5\sigma^2(t) - \int_{\mathbb{Y}} (\gamma(t, u) - \ln(1 + \gamma(t, u)))\lambda(du)$.
(b) If there is a constant $\varsigma > 0$ such that $b(t) \geqslant \varsigma$, then $x(t)$ is stochastically permanent.

Bao et al. [3] showed that under (5), if there is a constant $\varsigma > 0$ such that

$$\rho(t) := r(t) - \sigma^2(t) - \int_{\mathbb{Y}} \frac{\gamma^2(t, u)}{1 + \gamma(t, u)} \lambda(du) \geqslant \varsigma$$

for $t \geqslant 0$, then $x(t)$ represented by Eq. (4) is stochastically permanent. Note that

$$b(t) = \rho(t) + 0.5\sigma^2(t) + \int_{\mathbb{Y}} \left[\frac{1}{1 + \gamma(t, u)} - 1 - \ln \frac{1}{1 + \gamma(t, u)}\right]\lambda(du) \geqslant \rho(t),$$

that is to say, our conditions are much weaker than that of [3].

## 3. Two-species Lotka–Volterra models

In this section, let us consider system (M). For simplicity, define

$$R_+^2 = \{a = (a_1, a_2) \in R^2 \mid a_i > 0,\ i = 1, 2\}, \qquad b_i = r_i - 0.5\sigma_i^2 - \int_{\mathbb{Y}} \big(\gamma_i(u) - \ln\big(1 + \gamma_i(u)\big)\big)\lambda(du);$$

$$\Delta = a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta_1 = b_1 a_{22} - b_2 a_{12}, \qquad \Delta_2 = b_2 a_{11} - b_1 a_{21};$$

$$\overline{f(t)} = t^{-1}\int_0^t f(s)\,ds, \qquad \overline{f}^* = \limsup_{t\to+\infty} \overline{f(t)}, \qquad \overline{f}_* = \liminf_{t\to+\infty} \overline{f(t)}.$$

**Assumption 2.** There is a constant $c > 0$ such that $\int_{\mathbb{Y}} [\ln(1 + \gamma_i(u))]^2 \lambda(du) < c$, $i = 1, 2$.

Now let us prepare a useful lemma which will be proved in Appendix A.

**Lemma 2.** *Suppose that $Z(t) \in C(\Omega \times [0, +\infty), R_+)$ and let Assumption 2 hold.*

(I) *If there exist two positive constants $T$ and $\delta_0$ such that*

$$\ln Z(t) \leqslant \delta t - \delta_0 \int_0^t Z(s)\,ds + \alpha B(t) + \sum_{i=1}^2 \delta_i \int_0^t \int_{\mathbb{Y}} \ln\big(1 + \gamma_i(u)\big)\tilde{N}(ds, du) \text{ a.s.} \tag{13}$$

*for all $t \geqslant T$, where $\alpha$, $\delta_1$ and $\delta_2$ are constants, then*

$$\begin{cases} \overline{Z}^* \leqslant \delta/\delta_0 \text{ a.s.}, & \text{if } \delta \geqslant 0; \\ \lim_{t\to+\infty} Z(t) = 0 \text{ a.s.}, & \text{if } \delta < 0. \end{cases}$$

(II) *If there exist three positive constants $T$, $\delta$ and $\delta_0$ such that*

$$\ln Z(t) \geqslant \delta t - \delta_0 \int_0^t Z(s)\,ds + \alpha B(t) + \sum_{i=1}^2 \delta_i \int_0^t \int_{\mathbb{Y}} \ln\big(1 + \gamma_i(u)\big)\tilde{N}(ds, du) \text{ a.s.}$$

*for all $t \geqslant T$, then $\overline{Z}_* \geqslant \delta/\delta_0$ a.s.*

### 3.1. Cooperation system (M)

Now we are in the position to study the cooperation system (M), that is $r_1 > 0$, $r_2 > 0$, $a_{12} < 0$, $a_{21} < 0$. First of all, let us recall a useful lemma.

**Assumption 3.** $\Delta = a_{11}a_{22} - a_{12}a_{21} > 0$.

**Lemma 3.** *Let Assumption 3 hold, then for any given initial value $(x_{10}, x_{20}) \in R_+^2$, the cooperation system* (M) *has a unique solution $(x_1(t), x_2(t))$ on $t \geqslant 0$ and the solution will remain in $R_+^2$ a.s.*

**Proof.** Clearly, if $\Delta = a_{11}a_{22} - a_{12}a_{21} > 0$, then there exist two positive constants $c_1$ and $c_2$ such that

$$-2\tau := \lambda_{\max}\big(CA + A^T C\big) < 0, \tag{14}$$

where $A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$, $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ and $\lambda_{\max}(CA + A^T C)$ is the largest eigenvalue of $CA + A^T C$. The following proof is similar to that of Bao and Yuan [4, Theorem 2.1] by defining $V(x) = c_1 x_1 + c_2 x_2$, $x \in R_+^2$ and hence is omitted. □

**Assumption 4.** There is a positive constant $\kappa$ such that $\int_{\mathbb{Y}} \gamma_i^2(u)\lambda(du) \leqslant \kappa$ for $i = 1, 2$.

**Lemma 4.** *Under Assumptions 3 and 4, the solution $x(t)$ of the cooperation system* (M) *obeys*

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{\ln t} \leqslant 1, \text{ a.s.}, \quad i = 1, 2. \tag{15}$$

**Proof.** Applying Itô's formula to $e^t V(x) = e^t(c_1x_1 + c_2x_2)$ gives

$$\begin{aligned}
d\big[e^t V\big(x(t)\big)\big] &= e^t V\big(x(t)\big)\,dt + e^t dV\big(x(t)\big) \\
&\leqslant e^t\big[c_1x_1(t) + c_2x_2(t) + r_1c_1x_1(t) + r_2c_2x_2(t) - \tau\big(x_1^2(t) + x_2^2(t)\big)\big]\,dt \\
&\quad + e^t\bigg[c_1\sigma_1x_1(t)\,dB(t) + c_2\sigma_2x_2(t)\,dB(t) + c_1x_1(t)\int_{\mathbb{Y}}\gamma_1(u)\tilde{N}(dt,du) + c_2x_2(t)\int_{\mathbb{Y}}\gamma_2(u)\tilde{N}(dt,du)\bigg] \\
&\leqslant Ke^t\,dt + e^t\big[c_1\sigma_1x_1(t)\,dB(t) + c_2\sigma_2x_2(t)\,dB(t)\big] \\
&\quad + e^t\bigg[c_1x_1(t)\int_{\mathbb{Y}}\gamma_1(u)\tilde{N}(dt,du) + c_2x_2(t)\int_{\mathbb{Y}}\gamma_2(u)\tilde{N}(dt,du)\bigg].
\end{aligned}$$

Consequently

$$\limsup_{t\to+\infty} EV\big(x(t)\big) \leqslant K. \tag{16}$$

This, together with $|x| \leqslant x_1 + x_2 \leqslant V(x)/\min\{c_1, c_2\}$, means

$$\limsup_{t\to+\infty} E\big|x(t)\big| \leqslant K/\min\{c_1, c_2\} =: K_2. \tag{17}$$

On the other hand, by Itô's formula

$$\begin{aligned}
EV\big(x(t+1)\big) &\leqslant EV\big(x(t)\big) + E\int_t^{t+1}\big[r_1c_1x_1(s) + r_2c_2x_2(s) - \tau\big(x_1^2(s) + x_2^2(s)\big)\big]\,ds \\
&\leqslant EV\big(x(t)\big) + qE\int_t^{t+1}\big|x(s)\big|\,ds - \tau E\int_t^{t+1}\big|x(s)\big|^2\,ds,
\end{aligned}$$

where $q = \sqrt{2}\max\{r_1c_1, r_2c_2\}$. Note that $EV(x(t+1)) \geqslant 0$, then

$$\limsup_{t\to+\infty} E\int_t^{t+1}\big|x(s)\big|^2\,ds \leqslant (K + qK_2)/\tau =: K_3. \tag{18}$$

Moreover, by virtue of Itô's formula

$$\begin{aligned}
E\Big(\sup_{t\leqslant u\leqslant t+1} V\big(x(u)\big)\Big) &\leqslant EV\big(x(t)\big) + qE\int_t^{t+1}\big|x(s)\big|\,ds \\
&\quad + c_1\sigma_1\bigg(\sup_{t\leqslant u\leqslant t+1}\bigg|\int_t^u x_1(s)\,dB(s)\bigg|\bigg) + c_2\sigma_2\bigg(\sup_{t\leqslant u\leqslant t+1}\bigg|\int_t^u x_2(s)\,dB(s)\bigg|\bigg) \\
&\quad + c_1\bigg(\sup_{t\leqslant u\leqslant t+1}\bigg|\int_t^u x_1(s)\int_{\mathbb{Y}}\gamma_1(u)\tilde{N}(ds,du)\bigg|\bigg) \\
&\quad + c_2\bigg(\sup_{t\leqslant u\leqslant t+1}\bigg|\int_t^u x_2(s)\int_{\mathbb{Y}}\gamma_2(u)\tilde{N}(ds,du)\bigg|\bigg).
\end{aligned}\tag{19}$$

Denote

$$M_1(t) = \int_t^u x_1(s)\,dB(s), \qquad M_2(t) = \int_t^u x_2(s)\,dB(s),$$

$$M_3(t) = \int_t^u x_1(s)\int_{\mathbb{Y}}\gamma_1(u)\tilde{N}(ds,du), \qquad M_4(t) = \int_t^u x_2(s)\int_{\mathbb{Y}}\gamma_2(u)\tilde{N}(ds,du).$$

In view of the Burkholder–Davis–Gundy inequality (see e.g. [1, pp. 264–265]) and the Hölder inequality, we derive that

$$E\Bigl(\sup_{t\leqslant u\leqslant t+1}\bigl|M_1(u)\bigr|\Bigr)\leqslant JE\Biggl(\int_t^{t+1}x_1^2(s)\,ds\Biggr)^{0.5}\leqslant J\Biggl(E\int_t^{t+1}x_1^2(s)\,ds\Biggr)^{0.5}\leqslant J\Biggl(E\int_t^{t+1}\bigl|x(s)\bigr|^2\,ds\Biggr)^{0.5};$$

$$E\Bigl(\sup_{t\leqslant u\leqslant t+1}\bigl|M_2(u)\bigr|\Bigr)\leqslant J\Biggl(E\int_t^{t+1}x_2^2(s)\,ds\Biggr)^{0.5}\leqslant J\Biggl(E\int_t^{t+1}\bigl|x(s)\bigr|^2\,ds\Biggr)^{0.5};$$

$$\begin{aligned}E\Bigl(\sup_{t\leqslant u\leqslant t+1}\bigl|M_3(u)\bigr|\Bigr)&\leqslant JE\Biggl(\int_t^{t+1}\int_{\mathbb{Y}}x_1^2(s)\gamma_1^2(u)N(ds,du)\Biggr)^{0.5}\\&\leqslant J\Biggl(E\int_t^{t+1}\int_{\mathbb{Y}}x_1^2(s)\gamma_1^2(u)N(ds,du)\Biggr)^{0.5}\\&=J\Biggl(\int_{\mathbb{Y}}\gamma_1^2(u)\lambda(du)\Biggr)^{0.5}\Biggl(E\int_t^{t+1}x_1^2(s)\,ds\Biggr)^{0.5}\\&\leqslant J\Biggl(\int_{\mathbb{Y}}\gamma_1^2(u)\lambda(du)\Biggr)^{0.5}\Biggl(E\int_t^{t+1}\bigl|x(s)\bigr|^2\,ds\Biggr)^{0.5}.\end{aligned}$$

Similarly,

$$E\Bigl(\sup_{t\leqslant u\leqslant t+1}\bigl|M_4(u)\bigr|\Bigr)\leqslant J\Biggl(\int_{\mathbb{Y}}\gamma_2^2(u)\lambda(du)\Biggr)^{0.5}\Biggl(E\int_t^{t+1}\bigl|x(s)\bigr|^2\,ds\Biggr)^{0.5}.$$

Substituting the above inequalities into (19) and then making use of (16), (17), (18) and Assumption 4, we can see that

$$\limsup_{t\to+\infty}E\Bigl(\sup_{t\leqslant u\leqslant t+1}V\bigl(x(u)\bigr)\Bigr)\leqslant K+qK_2+\bigl[c_1\sigma_1J+c_2\sigma_2J+c_1J\kappa^{0.5}+c_2J\kappa^{0.5}\bigr]K_3^{0.5}.$$

Therefore there is a positive constant $K_4$ such that

$$E\Bigl(\sup_{k\leqslant u\leqslant k+1}\bigl|x(u)\bigr|\Bigr)\leqslant K_4,\quad k=1,2,\ldots.$$

Let $\varepsilon>0$ be arbitrary, making use of the famous Chebyshev inequality leads to

$$P\Bigl\{\sup_{k\leqslant t\leqslant k+1}\bigl|x(t)\bigr|>k^{1+\varepsilon}\Bigr\}\leqslant\frac{K_4}{k^{1+\varepsilon}},\quad k=1,2,\ldots.$$

In view of the well-known Borel–Cantelli lemma, we get that there is a $k_0$ such that for almost all $\omega\in\Omega$, if $k\geqslant k_0$ and $k\leqslant t\leqslant k+1$, $\sup_{k\leqslant t\leqslant k+1}|x(t)|\leqslant k^{1+\varepsilon}$. That is to say

$$\frac{\ln|x(t)|}{\ln t}\leqslant\frac{(1+\varepsilon)\ln k}{\ln k}=1+\varepsilon.$$

Then the desired assertion follows from $\varepsilon\to 0$. □

Now we are in the position to give our main result of this part. Define

$$\beta_i=0.5\sigma_i^2+\int_{\mathbb{Y}}\bigl(\gamma_i(u)-\ln\bigl(1+\gamma_i(u)\bigr)\bigr)\lambda(du),\quad i=1,2;$$

$$B=\beta_1r_2-\beta_2r_1;\qquad C_1=a_{22}r_1-a_{12}r_2;\qquad C_2=a_{11}r_2-a_{21}r_1;$$

$$D_1=a_{22}\beta_1-a_{12}\beta_2;\qquad D_2=a_{11}\beta_2-a_{21}\beta_1;$$

$$\kappa_1=\begin{cases}r_1/\beta_1, & B\leqslant 0;\\ C_1/D_1, & B\geqslant 0;\end{cases}\qquad \kappa_2=\begin{cases}C_2/D_2, & B\leqslant 0;\\ r_2/\beta_2, & B\geqslant 0.\end{cases}$$

Then $b_i=r_i-\beta_i$, $\Delta_i=C_i-D_i$, $i=1,2$, $B\leqslant 0\Rightarrow\kappa_1\geqslant\kappa_2$, $B\geqslant 0\Rightarrow\kappa_2\geqslant\kappa_1$.

**Theorem 2.** *For cooperation system* (M), *let Assumptions 2, 3 and 4 hold.*

(I) *Suppose* $B \leqslant 0$.
 (i) *If* $1 < \kappa_2$, *then both* $x_1$ *and* $x_2$ *are stable in the mean a.s.*:

$$\lim_{t\to+\infty} \overline{x_1(t)} = \frac{\Delta_1}{\Delta}, \qquad \lim_{t\to+\infty} \overline{x_2(t)} = \frac{\Delta_2}{\Delta}, \text{ a.s.}$$

 (ii) *If* $\kappa_2 < 1 < \kappa_1$, *then* $x_1$ *is stable in the mean a.s.*: $\lim_{t\to+\infty} \overline{x_1(t)} = b_1/a_{11}$, *a.s. At the same time,* $x_2$ *is extinctive a.s.*
 (iii) *If* $\kappa_1 < 1$, *then both* $x_1$ *and* $x_2$ *are extinctive a.s.*

(II) *Suppose* $B > 0$.
 (iv) *If* $1 < \kappa_1$, *then both* $x_1$ *and* $x_2$ *are stable in the mean a.s.*:

$$\lim_{t\to+\infty} \overline{x_1(t)} = \frac{\Delta_1}{\Delta}, \qquad \lim_{t\to+\infty} \overline{x_2(t)} = \frac{\Delta_2}{\Delta}, \text{ a.s.}$$

 (v) *If* $\kappa_1 < 1 < \kappa_2$, *then* $x_1$ *is extinctive a.s. and* $x_2$ *is stable in the mean a.s.*: $\lim_{t\to+\infty} \overline{x_2(t)} = b_2/a_{22}$, *a.s.*
 (vi) *If* $\kappa_2 < 1$, *then both* $x_1$ *and* $x_2$ *are extinctive a.s.*

**Proof.** Applying Itô's formula to (M) gives

$$\ln x_1(t) - \ln x_1(0) = b_1 t - a_{11}\int_0^t x_1(s)\,ds - a_{12}\int_0^t x_2(s)\,ds + \sigma_1 B(t) + Q_1(t), \tag{20}$$

$$\ln x_2(t) - \ln x_2(0) = b_2 t - a_{21}\int_0^t x_1(s)\,ds - a_{22}\int_0^t x_2(s)\,ds + \sigma_2 B(t) + Q_2(t), \tag{21}$$

where

$$Q_1(t) = \int_0^t \int_{\mathbb{Y}} \ln\bigl(1+\gamma_1(u)\bigr)\tilde{N}(ds,du),$$

$$Q_2(t) = \int_0^t \int_{\mathbb{Y}} \ln\bigl(1+\gamma_2(u)\bigr)\tilde{N}(ds,du).$$

We only give the proof of (I), the proof of (II) is analogous. Clearly, $\kappa_1 = r_1/\beta_1 \geqslant \kappa_2 = C_2/D_2$. From (20)$\times a_{22}$–(21)$\times a_{12}$, we obtain

$$\begin{aligned} &a_{22}t^{-1}\ln\bigl(x_1(t)/x_1(0)\bigr) - a_{12}t^{-1}\ln\bigl(x_2(t)/x_2(0)\bigr) \\ &\quad = C_1 - D_1 - \Delta\overline{x_1(t)} + t^{-1}\bigl[a_{22}\sigma_1 B(t) - a_{12}\sigma_2 B(t) + a_{22}Q_1(t) - a_{12}Q_2(t)\bigr]. \end{aligned} \tag{22}$$

Similarly, compute (21)$\times a_{11}$–(20)$\times a_{21}$,

$$\begin{aligned} &a_{11}t^{-1}\ln\bigl(x_2(t)/x_2(0)\bigr) - t^{-1}a_{21}\ln\bigl(x_1(t)/x_1(0)\bigr) \\ &\quad = C_2 - D_2 - \Delta\overline{x_2(t)} + t^{-1}\bigl[a_{11}\sigma_2 B(t) - a_{21}\sigma_1 B(t) + a_{11}Q_2(t) - a_{21}Q_1(t)\bigr]. \end{aligned} \tag{23}$$

In view of (20) and (21), for sufficiently large $t$,

$$\frac{\ln(x_1(t)/x_1(0))}{t} \leqslant b_1 + \varepsilon - a_{11}\overline{x_1(t)} - a_{12}\overline{x_2}^* + \sigma_1 B(t)/t + Q_1(t)/t;$$

$$\frac{\ln(x_2(t)/x_2(0))}{t} \leqslant b_2 + \varepsilon - a_{21}\overline{x_1}^* - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t.$$

Let

$$\rho_1 = b_1 + \varepsilon - a_{12}\overline{x_2}^*; \qquad \rho_2 = b_2 + \varepsilon - a_{21}\overline{x_1}^*.$$

Thus

$$\frac{\ln(x_1(t)/x_1(0))}{t} \leqslant \rho_1 - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t; \tag{24}$$

$$\frac{\ln(x_2(t)/x_2(0))}{t} \leqslant \rho_2 - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t. \tag{25}$$

(i) By (15), for arbitrarily $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geqslant T$

$$-a_{12}t^{-1}\ln\big(x_2(t)/x_2(0)\big) \leqslant -a_{12}\big[t^{-1}\ln x_2\big]^* + \varepsilon \leqslant \varepsilon.$$

Substituting this inequality into (22) yields

$$a_{22}t^{-1}\ln\big(x_1(t)/x_1(0)\big) \geqslant C_1 - D_1 - \varepsilon - \Delta\overline{x_1(t)} + t^{-1}\big[a_{22}\sigma_1 B(t) - a_{12}\sigma_2 B(t) + a_{22}Q_1(t) - a_{12}Q_2(t)\big]. \tag{26}$$

Since $C_1/D_1 \geqslant C_2/D_2 > 1$, then we can let $\varepsilon$ be sufficiently small such that $C_1 - D_1 - \varepsilon > 0$. Then using (II) in Lemma 2 and the arbitrariness of $\varepsilon$, we can see that

$$\overline{x_1}_* \geqslant \frac{C_1 - D_1}{\Delta} = \frac{\Delta_1}{\Delta}. \tag{27}$$

Consequently $\rho_1 > 0$ (otherwise, (24) and Lemma 2 would lead to $\overline{x_1}^* = 0$). In the same way, by (23),

$$\overline{x_2}_* \geqslant \frac{\Delta_2}{\Delta}. \tag{28}$$

Therefore $\rho_2 > 0$. Applying (I) in Lemma 2 to (24) and (25), we obtain

$$\overline{x_1}^* \leqslant \rho_1/a_{11}, \qquad \overline{x_2}^* \leqslant \rho_2/a_{22}.$$

That is to say

$$a_{11}\overline{x_1}^* + a_{12}\overline{x_2}^* \leqslant b_1 + \varepsilon; \qquad a_{21}\overline{x_1}^* + a_{22}\overline{x_2}^* \leqslant b_2 + \varepsilon. \tag{29}$$

Solving the above two inequalities and then applying the arbitrariness of $\varepsilon$, we have $\overline{x_1}^* \leqslant \Delta_1/\Delta$, $\overline{x_2}^* \leqslant \Delta_2/\Delta$. Then the required assertion follows from the above inequalities, (27) and (28).

(ii) Since $C_1/D_1 > 1$, then (27) holds. That is to say, $\overline{x_1}_* > \Delta_1/\Delta$. Therefore $\rho_1 > 0$. Thus inequality (29) holds. If $\omega \in \{\overline{x_2(\omega)}^* > 0\}$, then applying Lemma 2 to (25) yields

$$\overline{x_2(\omega)}^* \leqslant \frac{\rho_2}{a_{22}} = \frac{b_2 + \varepsilon - a_{21}\overline{x_1(\omega)}^*}{a_{22}}.$$

Substituting (29) into the above inequality, we can see that

$$0 < \Delta\overline{x_2(\omega)}^* \leqslant a_{11}b_2 - a_{21}b_1 + a_{11}\varepsilon - a_{21}\varepsilon = C_2 - D_2 + a_{11}\varepsilon - a_{21}\varepsilon.$$

Since $\varepsilon$ is arbitrarily small, then $1 \leqslant C_2/D_2$, which is a contradiction with $1 > C_2/D_2$. Consequently, $\mathcal{P}\{\omega\colon \overline{x_2}^* > 0\} = 0$, which means that $\overline{x_2}^* = 0$ a.s.

Now, substituting (29) into (25), we get

$$\begin{aligned}\frac{\ln(x_2(t)/x_2(0))}{t} &\leqslant b_2 + \varepsilon - \frac{a_{21}}{a_{11}}\big(b_1 + \varepsilon - a_{12}\overline{x_2}^*\big) - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t\\ &= \big[C_2 - D_2 + \varepsilon(t) + a_{11}\varepsilon - a_{21}\varepsilon\big]/a_{11} + \sigma_2 B(t)/t + Q_2(t)/t,\end{aligned}$$

where $\varepsilon(t) = a_{12}a_{21}\overline{x_2}^* - a_{11}a_{22}\overline{x_2(t)}$. Since $1 > C_2/D_2$, then $\overline{x_2}^* = 0$. Consequently $\varepsilon(t) \to 0$. An application of Lemma 2 again gives $\lim_{t\to+\infty} x_2(t) = 0$, a.s. Then by (20), for sufficiently large $t$, we have

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \leqslant b_1 + \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t, \tag{30}$$

$$t^{-1}\ln\frac{y_1(t)}{x_1(0)} \geqslant b_1 - \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t, \tag{31}$$

where $0 < \varepsilon < b_1$. Applying (I) and (II) in Lemma 2 to (30) and (31) respectively, we can observe that

$$\frac{b_1 - \varepsilon}{a_{11}} \leqslant \overline{x_1}_* \leqslant \overline{x_1}^* \leqslant \frac{b_1 + \varepsilon}{a_{11}}, \text{ a.s.}$$

Making use of the arbitrariness of $\varepsilon$ results in $\lim_{t\to+\infty}\overline{x_1(t)} = b_1/a_{11}$ a.s.

(iii) First of all, let us show $\lim_{t\to+\infty} x_2(t) = 0$ a.s.

If $\overline{x_1}^* > 0$, then $\rho_1 > 0$. Similar to the proof of (ii), we have $\lim_{t\to+\infty} x_2(t) = 0$, a.s.

If $\overline{x_1}^* = 0$. Then it follows from (25) that

$$\frac{\ln(x_2(t)/x_2(0))}{t} \leqslant b_2 + \varepsilon - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t$$

for sufficiently large $t$. An application of $1 > C_2/D_2 > r_2/\beta_2$ (then $b_2 < 0$) and Lemma 2, we can observe that $\lim_{t\to+\infty} x_2(t) = 0$, a.s.

Now let us prove $\lim_{t\to+\infty} x_1(t) = 0$ a.s. Note that $\lim_{t\to+\infty} x_2(t) = 0$, a.s., then it follows from (24) that for sufficiently large $t$,

$$\frac{\ln(x_1(t)/x_1(0))}{t} \leqslant b_1 + \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t.$$

Then the required assertion follows from $1 > r_1/\beta_1$ (i.e., $b_1 < 0$) and Lemma 2. □

**Remark 4.** Chen [5] introduced the following classical persistence definition: $x(t)$ is said to be persistent in the mean if $\bar{x}_* > 0$. Then it is easy to see that if $x(t)$ is stable in the mean, then it is persistent in the mean.

**Remark 5.** Theorem 2 has an interesting biological interpretation. Suppose that $B = \beta_1 r_2 - \beta_2 r_1 < 0$, then $x_1$ has a larger growth rate and smaller noises than $x_2$. Consequently, the persistent ability of $x_1$ is stronger than that of $x_2$.

(i) Suppose that $\kappa_1 < 1$. Then $r_1 < \beta_1$ and $r_2 < \beta_2$. That is to say, the environmental noises are so large that both $x_1$ and $x_2$ cannot resist. Consequently both $x_1$ and $x_2$ are extinctive.
(ii) Suppose that $\kappa_2 < 1 < \kappa_1$. Then $r_1 > \beta_1$ and $C_2 < D_2$. In other words, the environmental noises of $x_1$ are small and $x_1$ is persistent. However, the environmental noises of $x_2$ are sufficiently large such that the cooperation from $x_1$ is not enough, then $x_2$ is extinctive.
(iii) Suppose that $\kappa_2 > 1$. Then $r_1 > \beta_1$ and $C_2 > D_2$. In other words, the environmental noises are small such that $x_1$ is persistent. Under the cooperation of $x_1$, $x_2$ also is persistent.

When $B > 0$, the biological interpretation can be obtained similarly by symmetry.

### 3.2. Predator–prey system (M)

In this part, we suppose that $r_1 > 0$, $r_2 < 0$, $a_{12} > 0$, $a_{21} < 0$. To begin with, let us establish two lemmas.

**Lemma 5.** *For any given initial value $(x_{10}, x_{20}) \in R_+^2$, the predator–prey system* (M) *has a unique solution $(x_1(t), x_2(t))$ on $t \geqslant 0$ and the solution will remain in $R_+^2$ a.s.*

**Proof.** The proof is similar to that of Bao and Yuan [4] by defining $V(x) = -a_{21}x_1 + a_{12}x_2$, $x \in R_+^2$ and hence is omitted. □

**Theorem 3.** *Consider the predator–prey system* (M), *let Assumption 2 hold.*

(i) *If $b_1 < 0$, then both $x_1$ and $x_2$ are extinctive a.s.*;
(ii) *If $b_1 > 0$ and $\Delta_2 < 0$, then $x_2$ is extinctive a.s. and $x_1$ is stable in the mean a.s., i.e., $\lim_{t\to+\infty} \overline{x_1(t)} = b_1/a_{11}$, a.s.*;
(iii) *If $\Delta_2 > 0$, then both $x_1$ and $x_2$ are stable in the mean a.s.*:

$$\lim_{t\to+\infty} \overline{x_1(t)} = \frac{\Delta_1}{\Delta}, \qquad \lim_{t\to+\infty} \overline{x_2(t)} = \frac{\Delta_2}{\Delta}, \text{ a.s.} \tag{32}$$

**Proof.** (i) By (20),

$$t^{-1}\ln\frac{x_1(t)}{x_{10}} \leqslant b_1 - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t. \tag{33}$$

It then follows from (I) in Lemma 2 that $\lim_{t\to+\infty} x_1(t) = 0$, a.s. When this is used in (21) and notice that $a_{21} < 0$, we can see that

$$t^{-1}\ln\frac{x_2(t)}{x_{20}} \leqslant b_2 + \varepsilon - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t$$

for sufficiently large $t$, where $\varepsilon > 0$ and $b_2 + \varepsilon < 0$. Therefore by (I) in Lemma 2, $\lim_{t\to+\infty} x_2(t) = 0$, a.s.

(ii) Notice that $b_1 > 0$, an application of (I) in Lemma 2 to (33) gives $\overline{x_1}^* \leqslant b_1/a_{11}$. When this inequality is used in (21) and notice that $a_{21} < 0$, one can observe that for sufficiently large $t$,

$$\ln x_2(t) - \ln x_2(0) \leqslant b_2 t - a_{21}\overline{x_1}^* t + \varepsilon t - a_{22}\int_0^t x_2(s)\,ds + \sigma_2 B(t) + Q_2(t)$$
$$\leqslant (\Delta_2/a_{11} + \varepsilon)t - a_{22}\int_0^t x_2(s)\,ds + \sigma_2 B(t) + Q_2(t),$$

where $\Delta_2/a_{11} + \varepsilon < 0$. Then in view of Lemma 2, $\lim_{t\to+\infty} x_2(t) = 0$, a.s. Similar to the proof of Theorem 3, we get $\lim_{t\to+\infty} \overline{x_1(t)} = b_1/a_{11}$, a.s.

(iii) By the famous stochastic comparison theorem [18] and Lemma 2, we have

$$\limsup_{t\to+\infty} t^{-1}\ln x_1(t) \leqslant 0 \text{ a.s.} \tag{34}$$

By (21)$\times a_{11}$–(20)$\times a_{21}$, we can see that

$$t^{-1}a_{11}\ln\frac{x_2(t)}{x_{20}} - t^{-1}a_{21}\ln\frac{x_1(t)}{x_{10}}$$
$$= \Delta_2 - \Delta\overline{x_2(t)} - t^{-1}a_{21}\sigma_1 B(t) + t^{-1}a_{11}\sigma_2 B(t) - t^{-1}a_{21}Q_1(t) + t^{-1}a_{11}Q_2(t). \tag{35}$$

When (34) is used in (35), one can observe that

$$t^{-1}a_{11}\ln\frac{x_2(t)}{x_{20}} \geqslant \Delta_2 - \varepsilon - \Delta\overline{x_2(t)} - t^{-1}a_{21}\sigma_1 B(t) + t^{-1}a_{11}\sigma_2 B(t) - t^{-1}a_{21}Q_1(t) + t^{-1}a_{11}Q_2(t)$$

for sufficiently large $t$, where $\Delta_2 > \varepsilon > 0$. In view of (II) in Lemma 2 and the arbitrariness of $\varepsilon$, we have $\overline{x_2}_* \geqslant \Delta_2/\Delta$. That is to say for every $0 < \varepsilon < a_{12}\Delta_2/\Delta$, there exists a $T > 0$ such that

$$a_{12}\overline{x_2(t)} \geqslant a_{12}\overline{x_2}_* - \varepsilon \geqslant a_{12}\Delta_2/\Delta - \varepsilon, \quad t > T.$$

Substituting the above inequality into (20) gives

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \leqslant b_1 - a_{12}\Delta_2/\Delta + \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t$$
$$= a_{11}\Delta_1/\Delta + \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t$$

for sufficiently large $t$. Notice that $\Delta_1 > 0$, then applying Lemma 2 and the arbitrariness of $\varepsilon$, one can observe that $\overline{x_1}^* \leqslant \Delta_1/\Delta$. When this inequality is used in (21), we get

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)} \leqslant b_2 - a_{21}\Delta_1/\Delta + \varepsilon - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t$$
$$= a_{22}\Delta_2/\Delta + \varepsilon - a_{22}\overline{x_2(t)} + \sigma_2 B(t)/t + Q_2(t)/t$$

for sufficiently large $t$. Making use of Lemma 2 and the arbitrariness of $\varepsilon$ again, we obtain $\overline{x_2}^* \leqslant \Delta_2/\Delta$. Substituting this inequality into (20), one can see that

$$t^{-1}\ln\frac{x_1(t)}{x_{10}} \geqslant b_1 - a_{12}\Delta_2/\Delta - \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t$$
$$= a_{11}\Delta_1/\Delta - \varepsilon - a_{11}\overline{x_1(t)} + \sigma_1 B(t)/t + Q_1(t)/t$$

for sufficiently large $t$. Using (II) in Lemma 2 and the arbitrariness of $\varepsilon$ again, one can observe that $\overline{x_1}_* \geqslant \Delta_1/\Delta$. Then we obtain the required assertion (32). □

**Remark 6.** Consider the following classical predator–prey system:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)\big[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\big], \\ \dfrac{dx_2(t)}{dt} = x_2(t)\big[r_2 - a_{21}x_1(t) - a_{22}x_2(t)\big], \end{cases} \tag{36}$$

where $r_1 > 0$, $r_2 < 0$, $a_{12} > 0$, $a_{21} < 0$. It is well known that if $r_2 a_{11} - r_1 a_{21} > 0$, then model (36) has a positive equilibrium $(\frac{r_1 a_{22} - r_2 a_{12}}{\Delta}, \frac{r_2 a_{11} - r_1 a_{21}}{\Delta})$ which is globally asymptotically stable. When Eq. (36) is subject to environmental noises, it is of great importance to investigate whether system (M) still has some stability. Note that system (M) does not have a positive equilibrium. Thus the solution of (M) will not tend to a fixed positive point. However, Theorem 3 demonstrates that the solution of (M) could be stable in time average.

*3.3. Competition system (M)*

In this part, we shall consider the competition system (M), that is $r_1 > 0$, $r_2 > 0$, $a_{12} > 0$, $a_{21} > 0$. For competition system (M), Bao et al. [3, Theorem 4.6] have proved that if $b_i < 0$, then $\lim_{t\to+\infty} x_i(t) = 0$ a.s., $i = 1, 2$. Then it is interesting to investigate what happens if $b_1 > 0$ and $b_2 > 0$. In this part, we always assume that $b_1 > 0$ and $b_2 > 0$. To begin with, let us recall an important result.

**Lemma 6.** *(See Bao et al. [3].) For any given initial value* $(x_{10}, x_{20}) \in R_+^2$, *competition system* (M) *has a unique solution on* $t \geqslant 0$ *and the solution will remain in* $R_+^2$ *a.s.*

It is easy to prove the following result.

**Lemma 7.** *For competition system* (M), *we have* $\limsup_{t\to+\infty} t^{-1} \ln x_i(t) \leqslant 0$ *a.s.*, $i = 1, 2$.

For competition system (M), it is easy to see that if $\Delta > 0$, then $\Delta_1 < 0$ and $\Delta_2 < 0$ cannot hold simultaneously.

**Theorem 4.** *For competition system* (M), *let Assumptions 2 and 3 hold.*

(i) *If* $\Delta_1 > 0$ *and* $\Delta_2 < 0$, *then* $x_2$ *is extinctive and* $\lim_{t\to+\infty} \overline{x_1(t)} = b_1/a_{11}$, *a.s.*;
(ii) *If* $\Delta_1 < 0$ *and* $\Delta_2 > 0$, *then* $x_1$ *is extinctive and* $\lim_{t\to+\infty} \overline{x_2(t)} = b_2/a_{22}$, *a.s.*;
(iii) *If* $\Delta_1 > 0$ *and* $\Delta_2 > 0$, *then* $\lim_{t\to+\infty} \overline{x_1(t)} = \Delta_1/\Delta$, $\lim_{t\to+\infty} \overline{x_2(t)} = \Delta_2/\Delta$, *a.s.*

**Proof.** The proof is similar to Theorem 2 and hence is omitted. □

**Remark 7.** From Theorem 4 we can see that the interaction rate $a_{ij}$ plays an important role in determining extinction and stable in the mean (or persistence in the mean) of system (M), $i, j = 1, 2$. In fact, notice that $\Delta_1 = b_1 a_{22} - b_2 a_{12}$, $\Delta_2 = b_2 a_{11} - b_1 a_{21}$ and $b_1 > 0$, $b_2 > 0$. If one of $a_{ij}$, $i, j = 1, 2$, $i \neq j$ is large and the other is small such that $\Delta_1 \Delta_2 < 0$, for example, $a_{21}$ is large and $a_{12}$ is small such that $\Delta_1 > 0$, $\Delta_2 < 0$, then $x_1$ is stable in the mean a.s. while $x_2$ is extinctive a.s.; If both $a_{11}$ and $a_{22}$ are sufficiently large such that $\Delta_1 > 0$ and $\Delta_2 > 0$, then $x_1$ and $x_2$ can stably coexist and both of them are stable in the mean a.s. This is similar to the famous competitive exclusion principle.

## 4. Conclusions and remarks

This paper is concerned with several stochastic Lotka–Volterra systems with Lévy jumps. To begin with, a logistic equation is investigated. Sufficient conditions for stochastic permanence are established. It is shown that our condition for stochastic permanence is much weaker than that of [3]. Particularly, under simple assumptions, the conditions for stochastic permanence and extinction are sufficient and necessary. Then three stochastic Lotka–Volterra systems (i.e., cooperation, predator–prey and competition systems) with Lévy jumps are considered. For each system, sufficient and necessary conditions for stable in the mean and extinction of each population are established. Our results reveal some interesting facts which are neglected by all relevant known references.

(i) Theorem 1 reveals that the Lévy noise can make the population extinct;
(ii) For cooperation system (M), Theorem 2 reveals that the Lévy noise of $x_i$ is unfavorable for the persistence of both $x_1$ and $x_2$, $i = 1, 2$. From the viewpoint of biology, this is reasonable. Model (M) is a cooperation system. Since the Lévy noise of $x_i$ is unfavorable for the persistence of $x_i$, then $x_j$ will get less support. That is to say, the Lévy noise of $x_i$ is unfavorable for the persistence of $x_1$ and $x_2$, $i, j = 1, 2$, $j \neq i$.
(iii) For predator–prey system (M), Theorem 3 reveals that the Lévy noise of the prey population $x_1$ is unfavorable for persistence of both $x_1$ and $x_2$; The Lévy noise of the predator population $x_2$ is unfavorable for persistence of $x_2$ but is favorable for persistence of $x_1$. The biological interpretation is similar to Theorem 2.
(iv) For competition system (M), Theorem 4 reveals that the Lévy noise of population $x_i$ is unfavorable for persistence of $x_i$ and is favorable for the persistence of $x_j$, $i, j = 1, 2$, $j \neq i$. The biological interpretation is similar to Theorem 2.
(v) For system (M), the coefficients $a_{ij}$, $i, j = 1, 2$, play very important roles in determining the persistence and extinction of $x_1$ and $x_2$.

From Theorems 1, 2, 3 and 4, we can see that the Lévy noise may change the properties of population systems significantly. Remark 2 tells us that the original stochastic logistic equation which only depends on the Brownian motion is permanent. However, when the Lévy noise is taken into account, Theorem 1 reveals that the system is extinctive if the intensity of the Lévy noise exceeds a threshold, and the system is permanent if the intensity of the Lévy noise is small than the threshold. For cooperation system, consider the following system which only depends on the Brownian motion

$$\begin{cases} dx_1(t) = x_1(t)\{[r_1 - a_{11}x_1(t) - a_{12}x_2(t)]\,dt + \sigma_1\,dB(t)\}, \\ dx_2(t) = x_2(t)\{[r_2 - a_{21}x_1(t) - a_{22}x_2(t)]\,dt + \sigma_2\,dB(t)\}, \end{cases} \tag{37}$$

where $a_{12} < 0$ and $a_{21} < 0$. Notice that (37) is a special case of cooperation model (7), thus similar to the proof of Theorem 2 we have: if $\Delta > 0$, $r_1/\sigma_1^2 > r_2/\sigma_2^2$ and $a_{11}r_2 - a_{21}r_1 > 0.5(a_{11}\sigma_2^2 - a_{21}\sigma_1^2)$, then both $x_1$ and $x_2$ represented by (37) are stable in the mean. However, when the Lévy noises are taken into account, the properties of $x_1$ and $x_2$ represented by (7) become much more complicated, depending on the intensities of the Lévy noises. If the intensities are small, then both $x_1$ and $x_2$ are persistent; If the intensities are large, then both $x_1$ and $x_2$ are extinction; If the intensities are middle, then one population is extinctive and the other is persistent. For predator–prey system and competition system, the results can be obtained similarly.

Some interesting topics deserve further investigation. It is interesting to consider some more realistic but complex systems, for example, hybrid population models with jumps. The motivation is that some population models may subject to abrupt changes in their parameters, for example, several authors (see e.g. [14,21]) have claimed that the growth rates of some species in summer will be much different from those in winter, and one can model these abrupt changes by using a continuous-time Markov chain $\rho(t)$ with a finite state space $1, \ldots, m$. Hence one can investigate the following stochastic hybrid population model with jumps

$$\begin{cases} dx_1(t) = x_1(t^-)\Big\{[r_1(\rho(t)) - a_{11}(\rho(t))x_1(t^-) - a_{12}(\rho(t))x_2(t^-)]\,dt \\ \qquad + \sigma_1(\rho(t))\,dB(t) + \displaystyle\int_{\mathbb{Y}} \gamma_1(u, \rho(t))\tilde{N}(dt, du)\Big\}, \\ dx_2(t) = x_2(t^-)\Big\{[r_2(\rho(t)) - a_{21}(\rho(t))x_1(t^-) - a_{22}(\rho(t))x_2(t^-)]\,dt \\ \qquad + \sigma_2(\rho(t))\,dB(t) + \displaystyle\int_{\mathbb{Y}} \gamma_2(u, \rho(t))\tilde{N}(dt, du)\Big\}. \end{cases}$$

Another problem of interest is to consider the stochastic permanence of (M). We leave these investigations for future work.

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## Appendix A

**Proof of Lemma 2.** Under Assumption 2, it is easy to see that

$$\left\langle \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(u))\tilde{N}(ds, du) \right\rangle = t\int_{\mathbb{Y}} (\ln(1 + \gamma_i(u)))^2 \lambda(du) < ct.$$

By the famous strong law of large numbers for local martingales (see e.g. [11]), we have $\lim_{t\to+\infty} t^{-1}\sum_{i=1}^2 \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(u))\tilde{N}(ds, du) = 0$, a.s. Since $\lim_{t\to+\infty} t^{-1}B(t) = 0$ a.s., then for arbitrary $\varepsilon > 0$, there exists a $T_1 > 0$ such that for $t > T_1$

$$-\varepsilon t < \alpha B(t) + \sum_{i=1}^2 \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(u))\tilde{N}(ds, du) < \varepsilon t. \tag{38}$$

(I) Suppose that $\delta > 0$. Set $\phi(t) = \int_0^t Z(s)\,ds$ for all $t \geqslant T$, then $d\phi/dt = z(t) \geqslant 0$ for all $t \geqslant T$. Substituting $\phi$ and (38) into (13) yields

$$\ln\frac{d\phi}{dt} \leqslant \Lambda t - \delta_0\phi, \quad t \geqslant T_2 = \max\{T, T_1\},$$

where $\Lambda = \delta + \varepsilon$. In other words, we have shown that $e^{\delta_0\phi}\frac{d\phi}{dt} \leqslant e^{\Lambda t}$, $t \geqslant T_2$. Integrating the above inequality from $T_2$ to $t$ gives $\phi(t) \leqslant \delta_0^{-1}\ln\{e^{\delta_0\phi(T_2)} + \delta_0\Lambda^{-1}(e^{\Lambda t} - e^{\Lambda T_2})\}$. Consequently,

$$\overline{Z}^* = \limsup_{t\to+\infty} \overline{Z(t)} \leqslant \delta_0^{-1} \limsup_{t\to+\infty} \ln\{3\delta_0\Lambda^{-1}e^{\Lambda t}\}^{1/t} = \Lambda/\delta_0 = (\delta+\varepsilon)/\delta_0.$$

Using the arbitrariness of $\varepsilon$ we get $\overline{Z}^* \leqslant \delta/\delta_0$.

If $\delta < 0$, then $\limsup_{t\to+\infty} t^{-1}\ln Z(t) \leqslant \delta+\varepsilon < 0$, where $\varepsilon$ is sufficiently small such that $\delta+\varepsilon<0$. Therefore, $\lim_{t\to+\infty} Z(t) = 0$, a.s.

(II) The proof of (II) is similar to the first part of (I). This completes the proof. □

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