Review of "Asymptotic Analysis of Subglacial Plumes in Stratified Environment", Bradley et al, # JFM-21-S-0408

The study considers a one-dimensional model of a steady turbulent plume under an ice shelf with appreciable variation of the depth-dependent freezing point. Cases considered here have variable but shallow slope, and an ocean stratification that varies as a smoothed step function between two different temperature and salinity conditions. The dimensional equations are simplified using a 2-equation approximation to the thermodynamics, non-dimensionalised, then asymptotic simplifications are obtained in regions below, above and in the stratification exploiting a similar small magnitude of several model parameters. An analytical asymptotic approximation is derived for the lower layer, and jump conditions across the sharply stratified region. The asymptotically simplified model has to be treated numerically in the upper layer, but approximate solutions for the melt rate are justified by blending asymptotic limits for the start and end of the region.

The manuscript is clearly written, and the approach provides an elegant solution to this aspect of the parameterisation problem associated with subglacial plumes (subject to some physical caveats regarding application of the original model). The topic fits squarely within the remit of JFM, with a derivation of asymptotic solutions to a fluid mechanical model. I have separated my comments below into main scientific comments and suggestions, some technical comments, and minor issues of presentation. I suspect some of the technical comments relate to typos, and others could possibly be my error whilst briefly verifying the calculation, but I flag them to cross check. There are also a couple of calculation steps that need clarification.

Main comments/suggestions:

1. The physics of the model used here is likely to break down in regions where ΔT is negative. Supercooling of the plume is likely to lead frazil ice formation if there are many nucleation sites, which significantly changes plume dynamics and ice precipitation rates (c.f. Jenkins and Bombosch, 1995, https://doi.org/10.1029/94JC03227). There is likely still useful insight from the solutions of the case with pure basal freeze on, but you need clearer caveats when discussing the results.

2. Page 7, use of 2-equation formulation. This is a reasonable thing to do given that current data are insufficient to constrain the 3 equation formulation with adequate precision, but I would add some discussion of the realism. I.e. the impact of slow salt transfer is explicitly removed, but accounted for approximately by adjusting the rate of heat transfer using a modified value of St. From past experience with these models, running a model with a 3 equation formulation would likely lead to some quantitative differences for temperatures close to $0°C$ or below.

3. In figures 3 and 4, the transition across pycnocline triggers supercooling by the end of the pycnocline region, which is a physically interesting scenario but presumably not general. Can you discuss criteria for this to occur? And also illustrate a case where it does not occur, to see how well the scalings hold?

4. Following figure 7. It might be interesting to also investigate varying l_t. How far can you push your approximation of a localised jump in stratification?

Technical comments

5. The velocity scale in (2.26) seems to be missing a factor of $g\alpha$ inside the square root. And the density in (2.27) is missing a factor of ρ_0.

6. Equation (2.28). For the penultimate equality, should this be $(4E_0\alpha/3 + C_d)/\alpha$ multiplying $\cos\alpha$?

7. Table 2. Definition of P_T. Is there a sign error before Γ? (Based on this arising from $T_a + \Gamma S_a$ after calculating the difference between the ambient and freezing point temperatures).

8. Equation (3.11) - is there a missing prime for first Z_b? Also, I couldn't recreate (3.13) - possibly due to a typo.

9. Equation (3.59). It wasn't clear how you removed the constant of integration to yield this equation? The current asymptotic matching conditions as $\chi \to -\infty$ don't seem to immediately preclude this constant being non-zero. E.g. there could be a non-zero later order correction to p that has not yet been calculated.

10. Equations (3.62). Are there some typos here? I didn't manage to entirely cancel some parts of the constant prefactors when substituting into (3.60)-(3.63).

11. Page 22. Going from (4.9) to (4.10). Shouldn't you also expand Z' as a series in epsilon, because it depends on X? This would seemingly modify (4.10).

12. In (4.16), does $\kappa_3 = \lambda$? If so, should the prefactor for $\dot{m}$ (inside the square root) be τ^4 rather than τ^3? U has one-half power of τ from buoyancy, one-half from length, then ΔT is linear in τ, which matches $\sqrt{\tau^4}$.

13. In appendix B, there seemed to be a few algebraic disagreements with my attempted verification, which may be typos. Equation (B11) - where did the $\kappa^{1/3}$ come from? (B12) - should first term on right hand side be proportional to $[Z_b'(X_0)]^3$ rather than squared, and were there some factors of $[Z_b'(X_0)]^2$ missing for later terms? (B13) - should this have $-4K_1^2K_2$ rather than 3? (B14) - seems to have some spurious X values, and several missing factors of $1/3$ from the integration.

Minor clarifications/comments on presentation/typos

15. Page 3, discussion of Coriolis effects. Presumably flows on alongslope lengthscales larger than the deformation radius will feel some effects of the Coriolis force (unless geometrically confined). Add a comment on this.

16. Figure 1 (caption and its discussion). Would be good to emphasise that the shallow slope approximation means you can use vertical depth and depth normal to the ice face interchangeably (as the cartoon and caption suggest you are doing). Also, the $Z = -Z_{gl}$ label at the top was initially confusing until I had got further in the text, and might be explained in the caption (i.e. you have shifted co-ordinates, so that the grounding line is at $Z = 0$ rather than $Z = Z_{gl}$).

17. Page 4. 4th paragraph of section 2, description of boundary and ice values. I don't think your description of the boundary layer structure makes physical sense given the equations. A more usual description has T_p in the plume, T_b at the ice-ocean interface/boundary (which differs from T_p due to temperature gradients across the molecular sublayer in the ocean), and T_i the interior ice temperature (beyond a thermal boundary layer in the ice dominated by conduction and ice advection relative to the ice-ocean boundary). Salinity behaves similarly, except that the salinity is discontinuous on the ice and ocean sides of the interface (due to neglect of solutal diffusion through the ice).

18. If Q_T is actually intended as a heat flux, then equation (2.4) and (2.6) are missing a factors of the heat capacity c_p. There also appear to be typos in (2.5) and (2.6) where T_i and S_i should be T_b and S_b.

19. In equations (2.5)-(2.8). Note that you are implicitly assuming $U > 0$ and $Z_b' > 0$ here (I.e. No modulus signs are needed to control the sign of entrainment/turbulent flux if the flow were in another direction).

20. Variable T_0 is used twice for the freezing point relation in (2.12) and several subsequent equations/discussion in text, and the ambient stratification in (2.25). See also table 1.

21. Table 1. Unit errors for Γ, S_0 and S_1. Also, this is non-consequential, but your reference seawater density seemed quite low (e.g. vs Jenkins & Bombosch, 1995).

22. Page 7, discussion between equations (2.14) and (2.15). Could make the physical implication of the second step clearer: it seems you are effectively approximating $T_b \sim T_f$ using the plume freezing temp in (2.14), then calculating $\dot{m}$ using the single transfer coefficient.

23. Page 7, sentence following (2.16). Does $T_b = T_f$ in the inline equation?

24. Figure 2 caption. $Z_{gl} = 750\mathrm{m}$. Is this a typo?

25. page 8. Section 2.2, first paragraph. "no information about the ambient stratification" should presumably refer to no gradient in the ambient density.

26. page 9. Paragraph including (2.28), discussion of numerical instability. Are you referring to cases where the buoyancy term proportional to gradients in plume thickness is included (rather than neglected as above)? Would clarify here.

27. Line following (2.30). Typo on B_T and B_S; should be β.
28. Sentence concluding following (2.31). Would reword this - a first read suggested you were referring to decay of the temperature at the grounding line.
29. Equation (2.37) - should this be X_0 rather than Z_0 in the first stratification term? Also, last ΔT in (2.38) was missing a hat.
30. Page 10, end of 5th paragraph. Γ is dimensional, so can't have $\Gamma \ll 1$. Need to form a dimensionless ratio.
31. Page 12, first paragraph. Discussion of deceleration in figure 3. It looks like the reversal of sign of surface buoyancy flux (due to basal freezing) is the cause of deceleration here (there is no ambient stratification here, so entrainment does not change the sign of the buoyancy force on its own).
32. Equation (3.28). Should this be $I(X_0)$ rather than $I(X)$.
33. Figure 4. Why is the blue shading asymmetric about zero here? You seem to use a different approximation for the pycnocline region for the melt rate in section 4.
34. Page 18, second paragraph. "Continuing this argument ..." This logic wasn't clear to me. But can't you deduce the same result directly from (3.40) given $U > 0$ and $Z' > 0$ have been assumed to avoid using modulus signs for these terms in your entrainment formula.
35. Page 18. Sentence following (3.47). Modify to "… is also useful …"
36. (3.56)-(3.57) - are you missing some minus signs multiplying χ?
37. Figure 5 caption. Does the cyan line show numerical solutions of the stopping region equations?
38. Page 20, 4th line. Should "pressure" be "buoyancy"?
39. Page 21, 2nd paragraph of §4.3. "for ~~this~~ the region"
40. page 22, equation (4.11). Can you justify the prefactor a bit more carefully. Presumably scaling $U\Delta T$ by $Z'(X)^{3/2}$ based on the similarity scaling?
41. Discussion of figure 6. Is there an obvious explanation for overestimating peak melt rates slightly? Is this tied to the relative smallness of small parameters? Also, stating some indicative percentage errors might be useful here for modellers trying to understand the uncertainty.
42. Page 25, section 5.2 second line. " approximations, for *the* case …".
43. Page 26, last paragraph. "with with defined derivatives".
44. Figure 8. Looks like the (b) and (c) labels don't match the caption or text. Can you also indicate the pycnocline depths for this figure?
45. Page 28, end of section 5. It looks like there is a significant error for the plume intrusion depth (where $U \to 0$) which seems worthy of comment.
46. Equation (A1) - power and brackets broken on first term of RHS.
47. Appendix B, equations (B3)-(B10) should presumably depend on Y rather than X. Also, don't need equation numbers for lines (B4), (B6), (B7), (B8).
48. Equation (C1) was missing an equality.