Prophet Inequalities via the Expected Competitive Ratio

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Joint work with Tomer Ezra, Stefano Leonardi, Rebecca Reiffenhäuser (Sapienza University of Rome), and Matteo Russo (Georgia Tech)

UNIVERSIDAD
DE CHILE

A bit about myself

PostDoc, Universidad de Chile - ICMD team, 2022 - 2023

PhD, TU Munich - OR Group (Andreas S. Schulz) & AdONE (Math, School of Management, CS), 2018 - 2022

Undergrad, NTUA - Electrical and Computer Engineering, 2012 - 2018

<u>Focus</u>: Computer Science, Mathematics, Networks

PhD research related to **Economics and Computation**:

- (Algorithmic) mechanism design (e.g. auction theory, pricing).
- Online decision-making / optimal stopping.

Prophet Inequality

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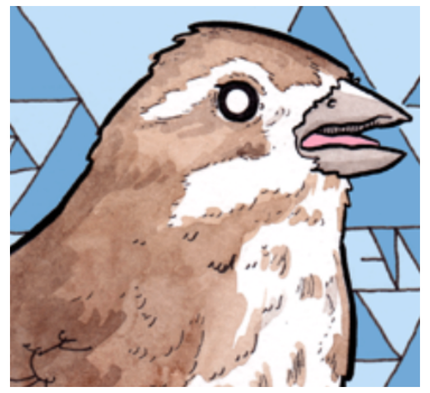
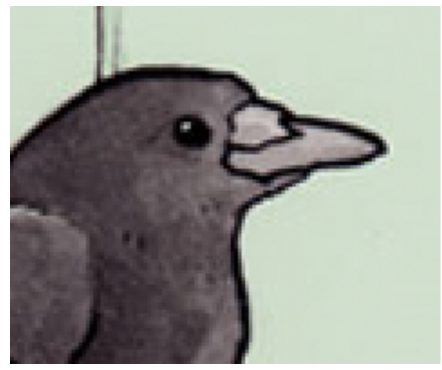
$U[4,7]$



$U[2,9]$

$U[6,8]$



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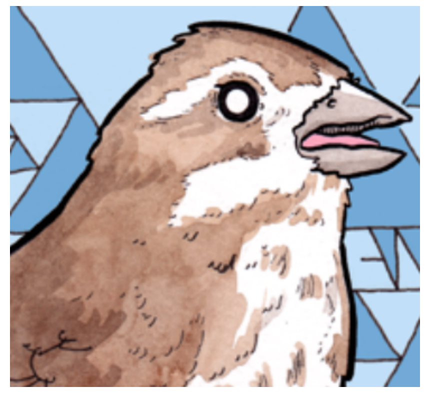

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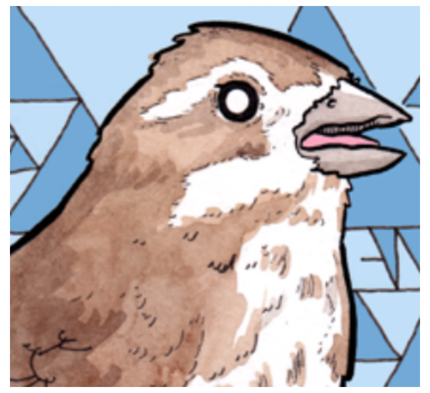

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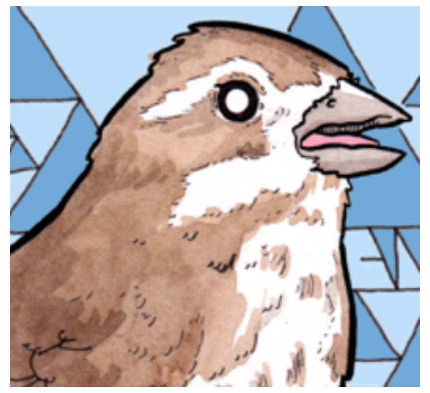

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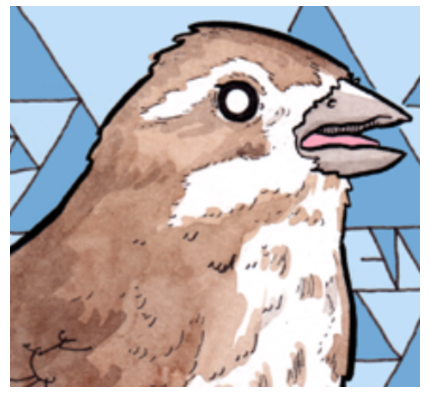
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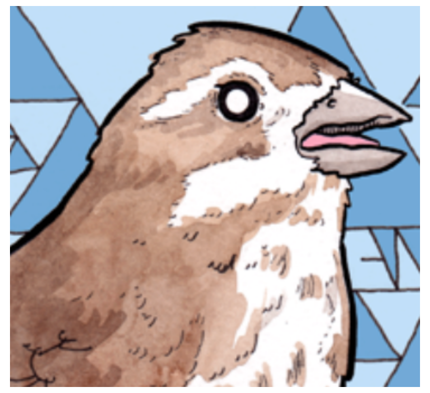

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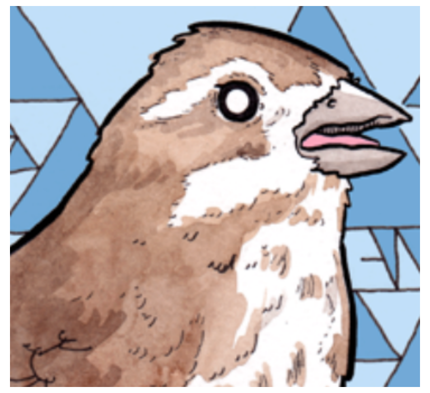

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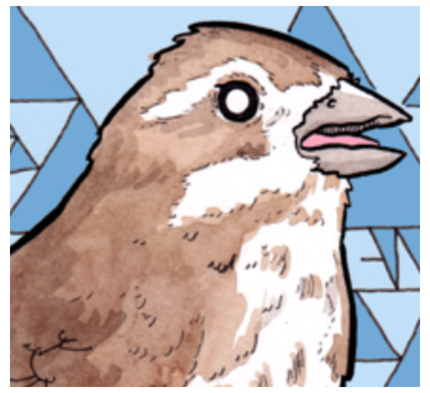

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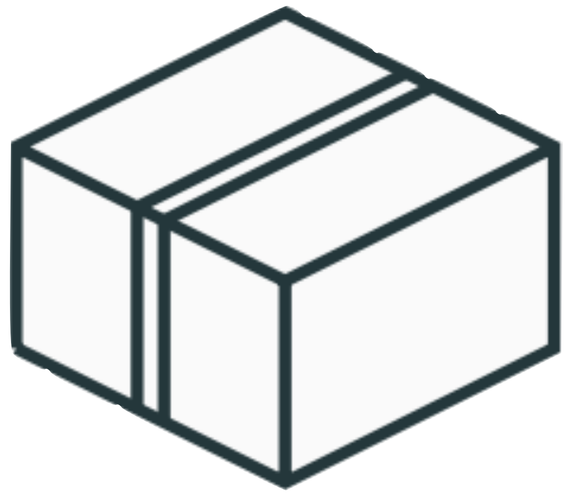

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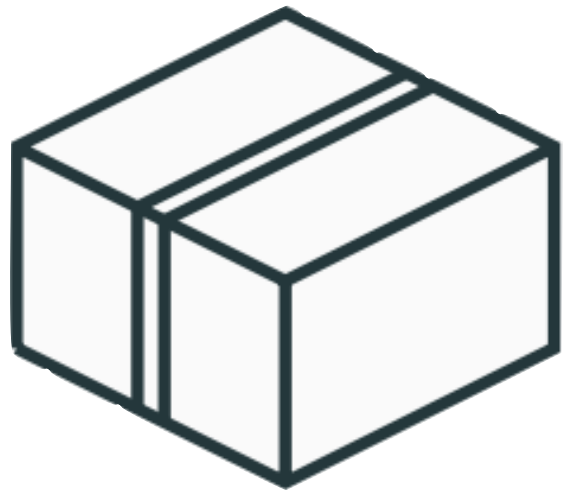

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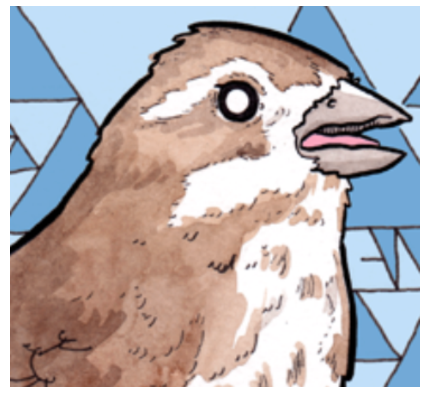

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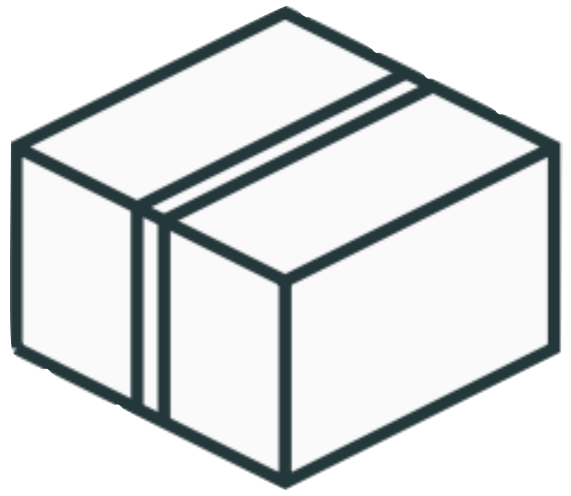

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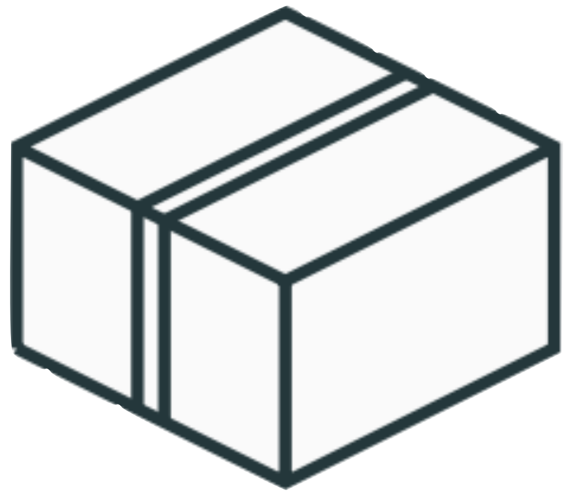

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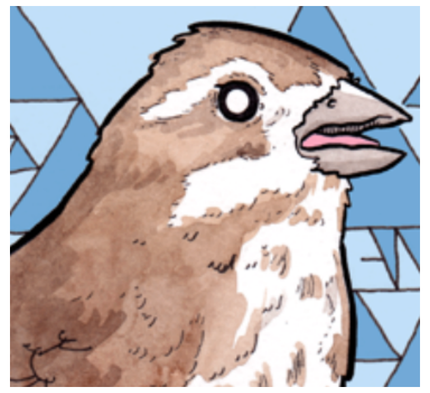

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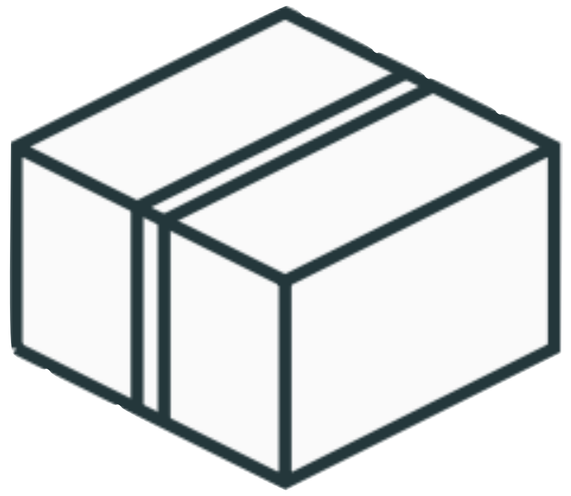

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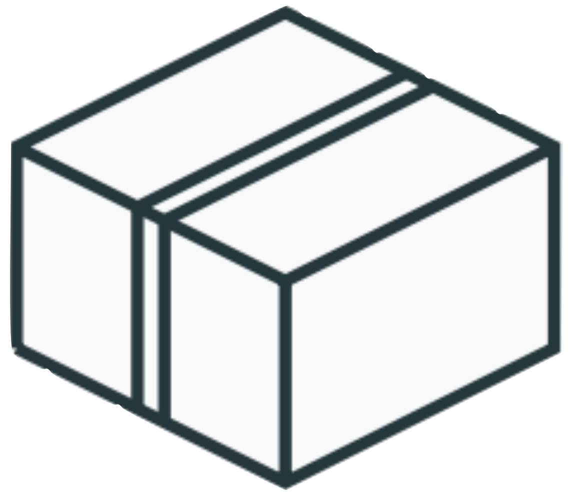

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Thm In fact, it is a fixed threshold strategy! [Samuel-Cahn '84; Kleinberg, Weinberg '12]

Bayesian online selection problems

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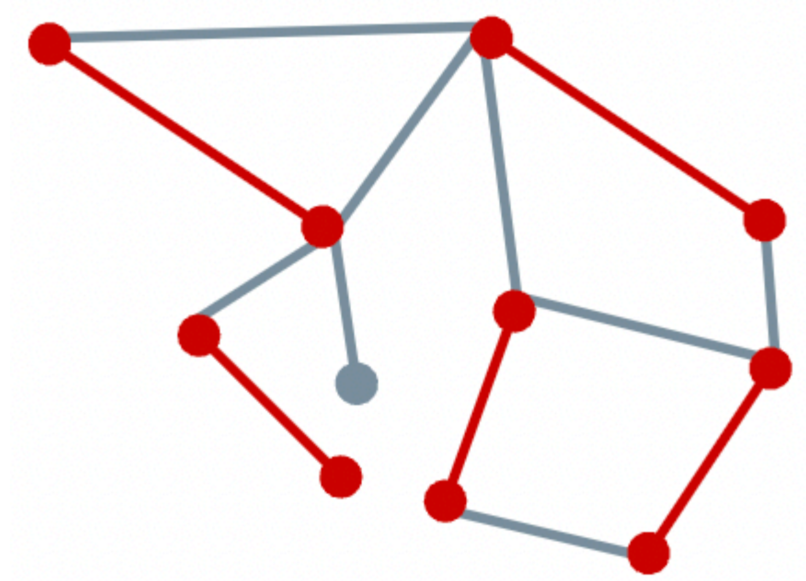
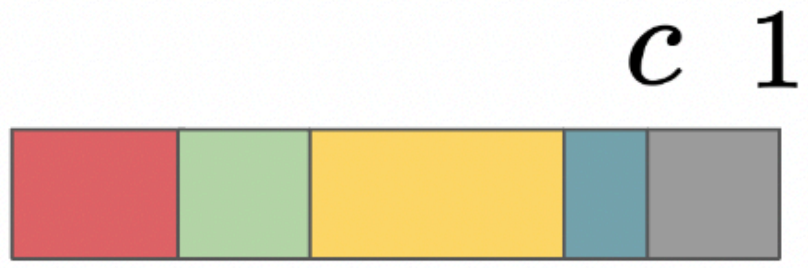
Matching



Knapsack



Matroid

$$\mathcal{M} = (E, \mathcal{I})$$

Prophet inequalities literature

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- Arrival order of the elements
  [Hill, Kertz '82], [Yan '11], [Ehsani, Hajiaghayi, Kesselheim, Singla '18], [Correa, Saona, Ziliotto '21], [Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]

- Combinatorial settings
  [Alaei '11], [Kleinberg, Weinberg '12], [Gravin, Feldman, Lucier '15], [Dütting, Feldman, Kesselheim, Lucier '17], [Rubinstein, Singla '17], [Ezra, Feldman, Gravin, Tang '20], [Feldman, Svensson, Zenklusen '21], [Jiang, Ma, Zhang '22]

- Samples from unknown distributions
  [Azar, Kleinberg, Weinberg '14], [Correa, Dütting, Fischer, Schewior '19], [Rubinstein, Wang, Weinberg '20], [Correa, Cristi, Epstein, Soto '20], [Kaplan, Naori, Raz '20], [Caramanis, Dütting, Faw, Fusco, Lazos, Leonardi, Papadigenopoulos, Pountourakis, Reiffenhäuser '22]

- Connections to posted price mechanisms
  [Hajiaghayi, Kleinberg, Sandholm '07], [Chawla, Hartline, Malec, Sivan '10], , [Dütting, Feldman, Kesselheim, Lucier '17], [Correa, Pizarro, Verdugo '19]

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This work: We initialize the study of the **expected ratio** $\mathsf{EoR} := \mathbb{E}\left[\dfrac{\mathsf{ALG}}{\mathsf{OPT}}\right]$.

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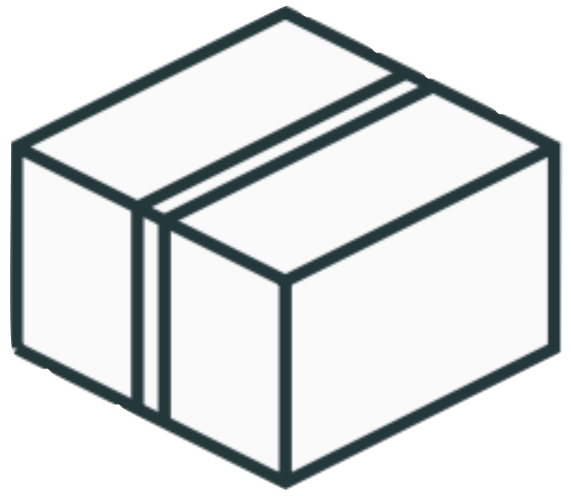

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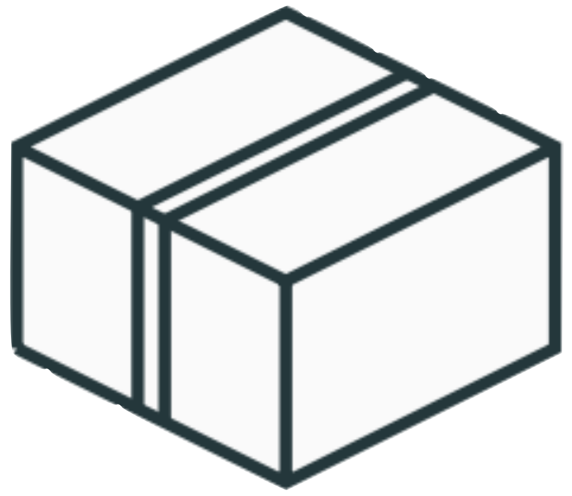

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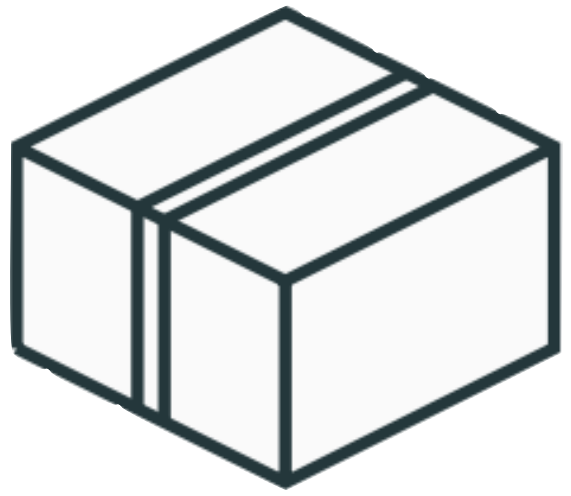

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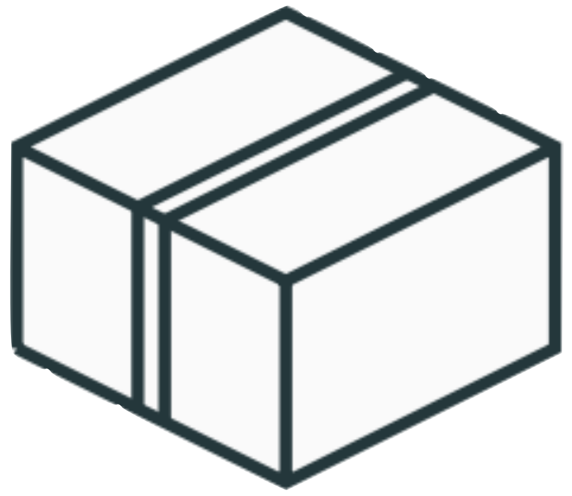

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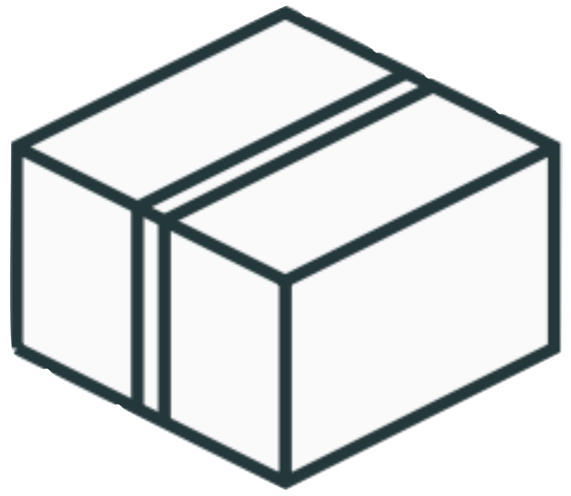

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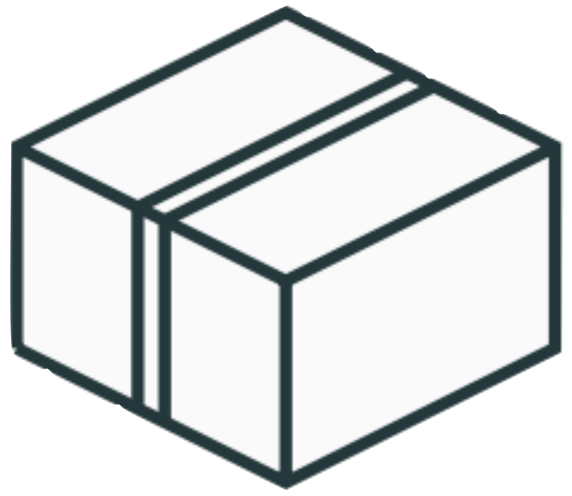

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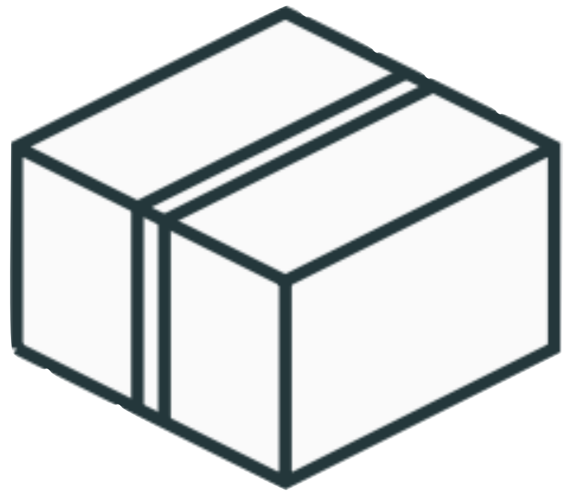

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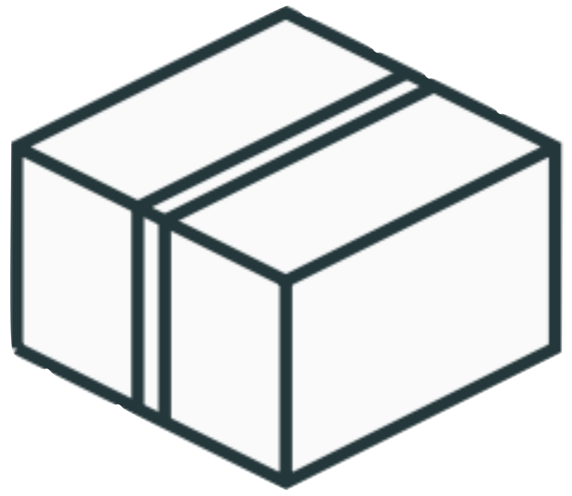

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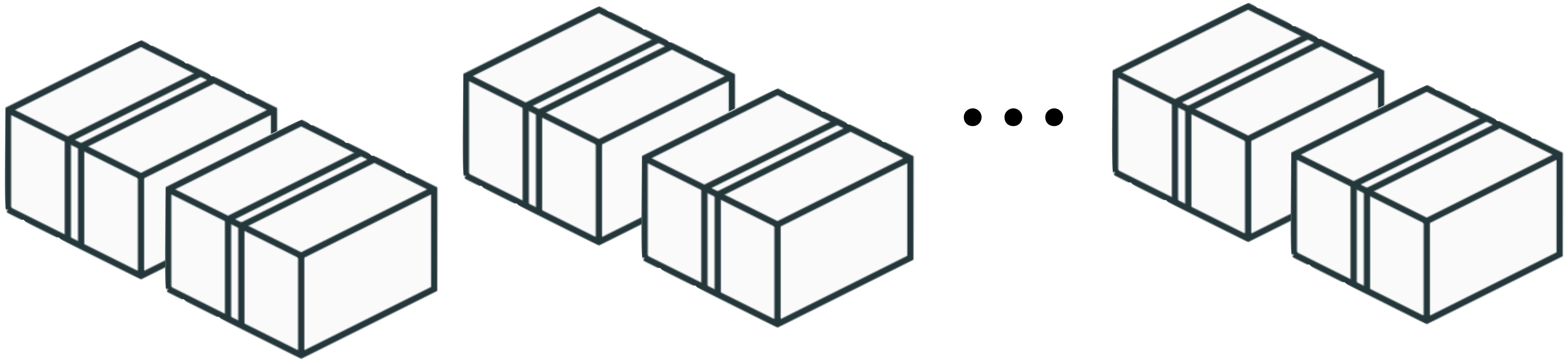

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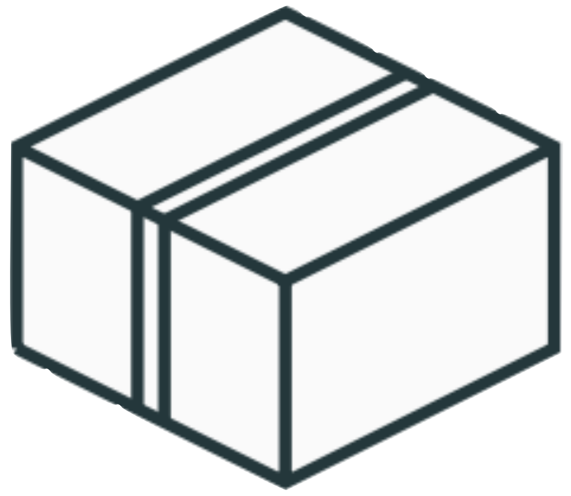

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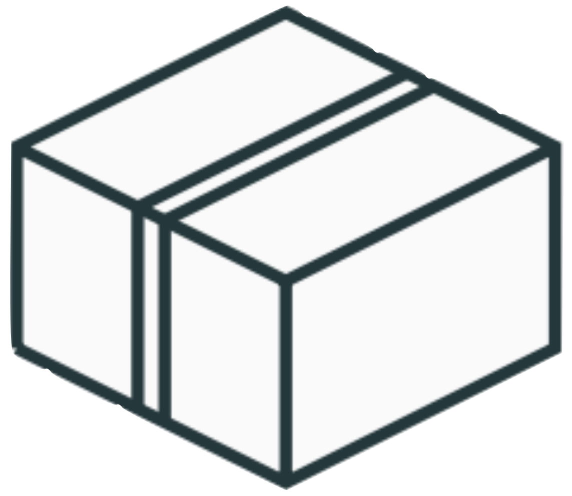

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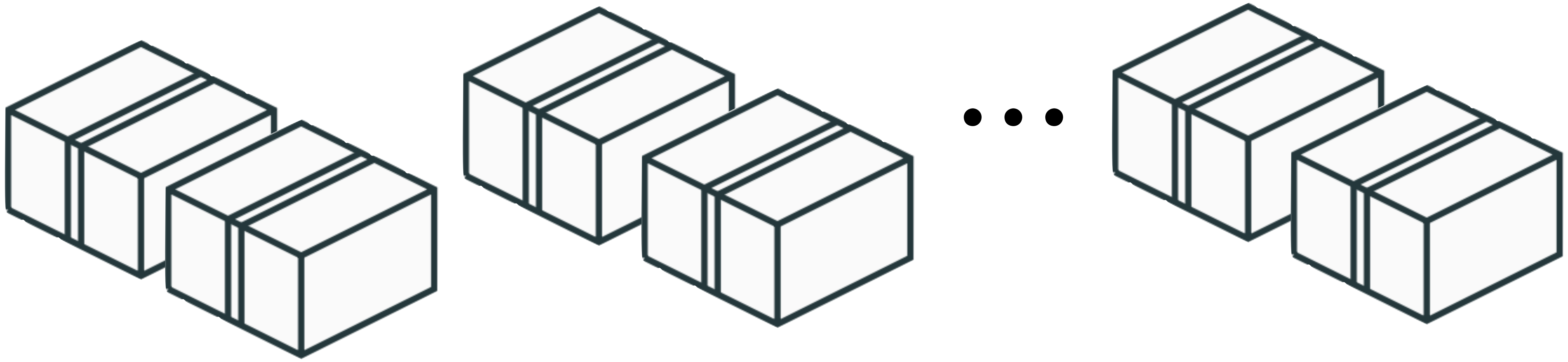

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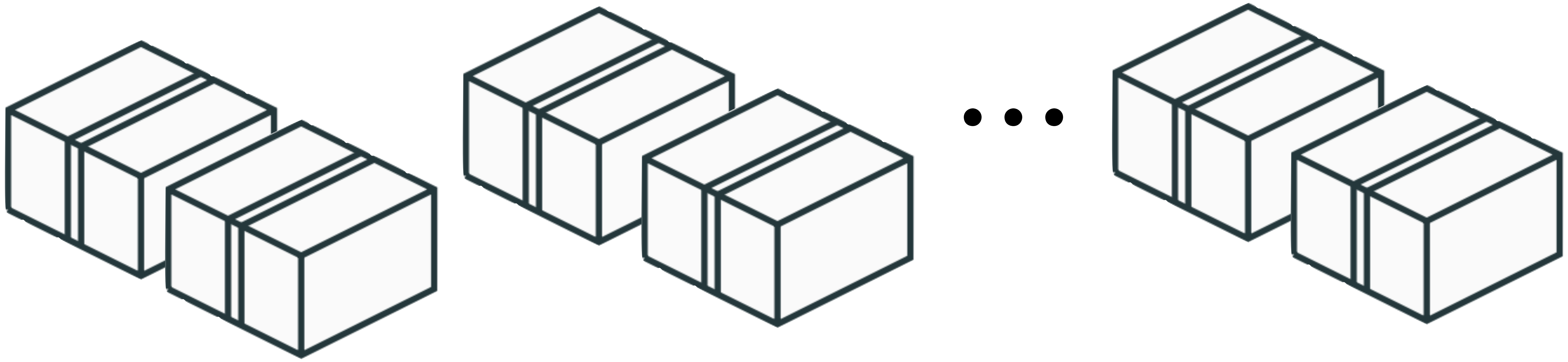

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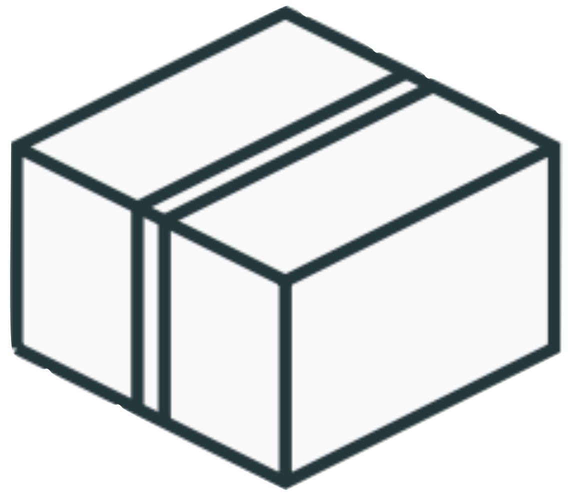

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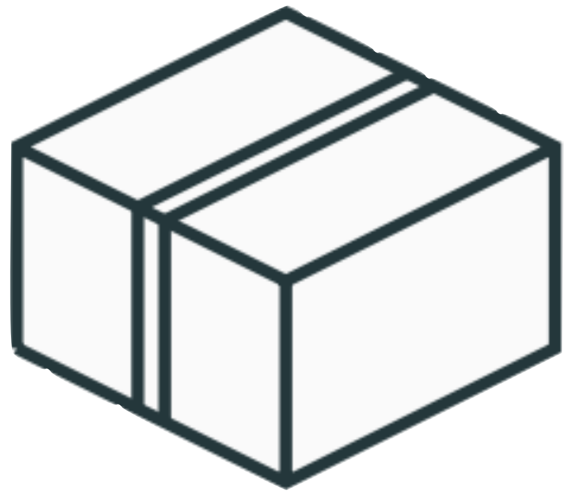

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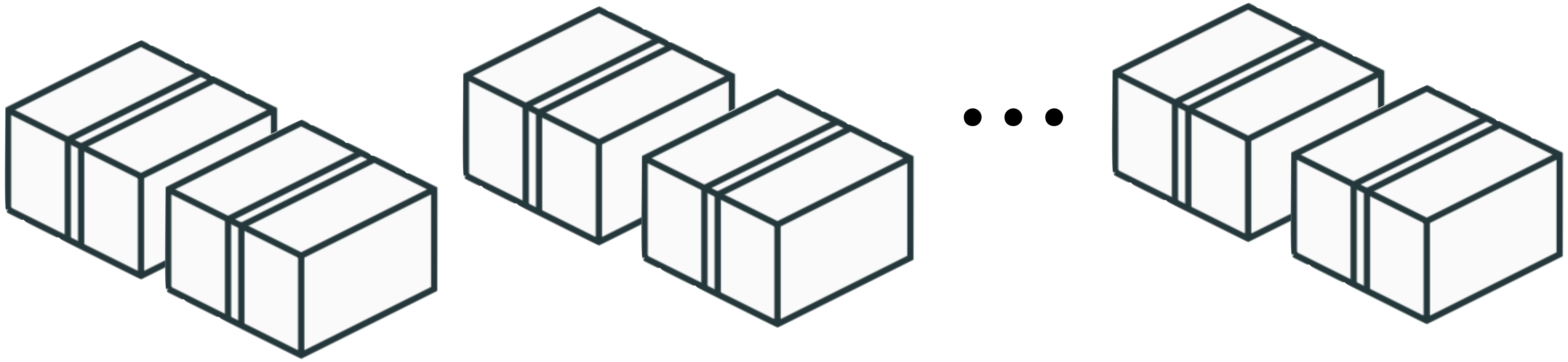

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Jensen's ineq

$$\textbf{PbM} \leq \frac{1}{2^n} \qquad \textbf{EoR} \geq \frac{2}{3}$$

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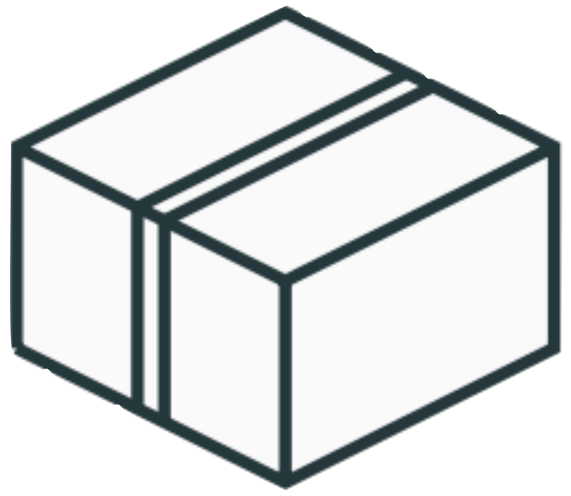

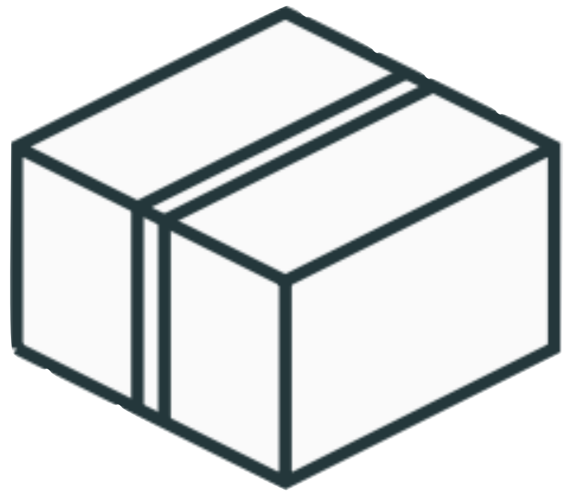

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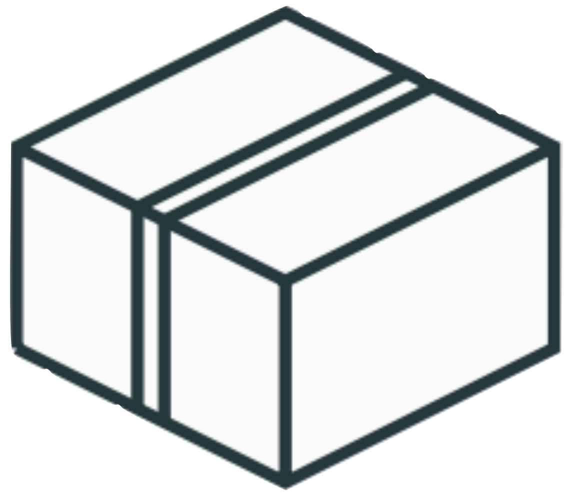

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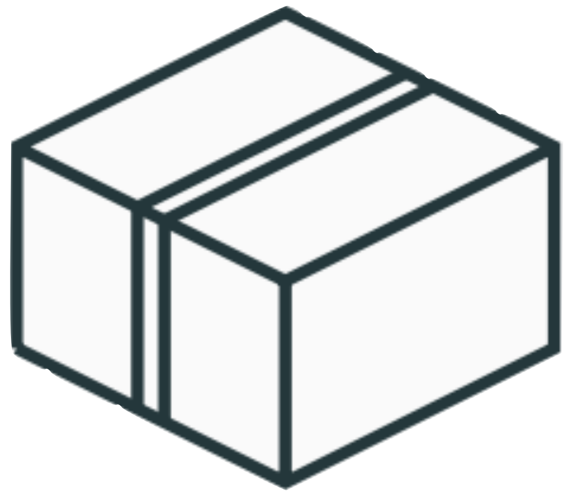

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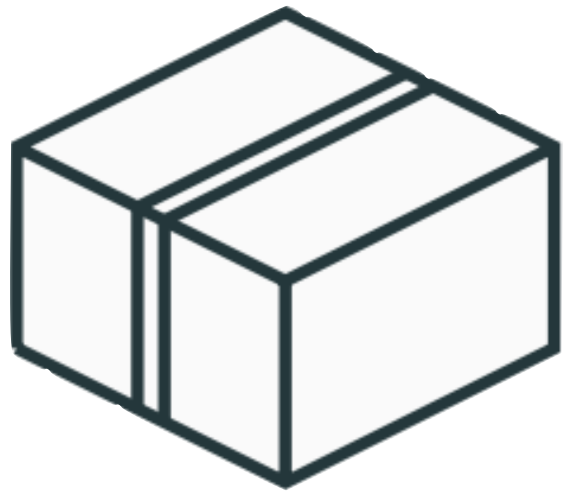

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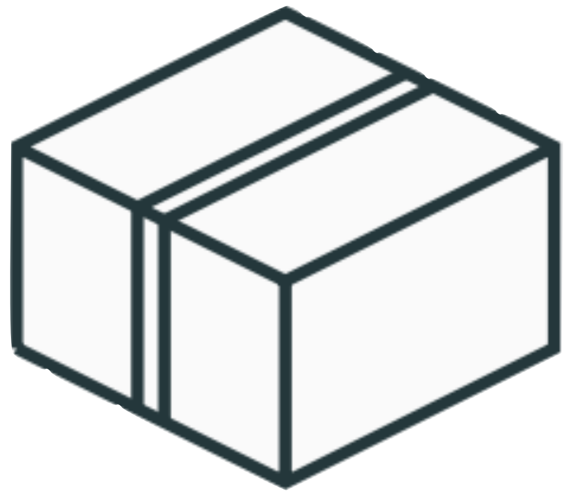

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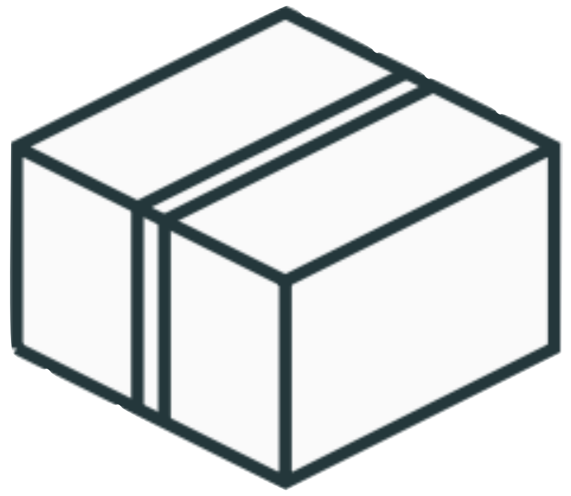

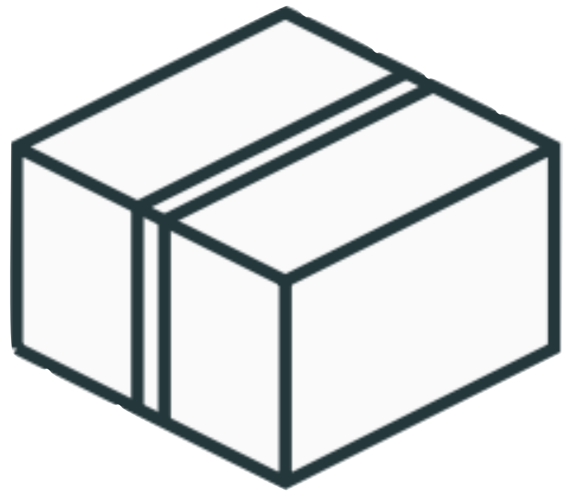

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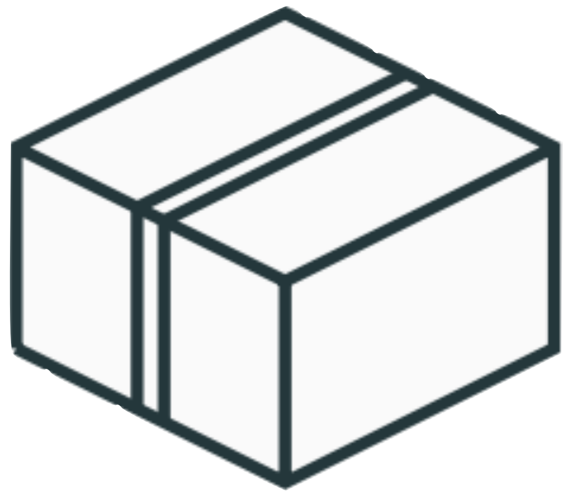

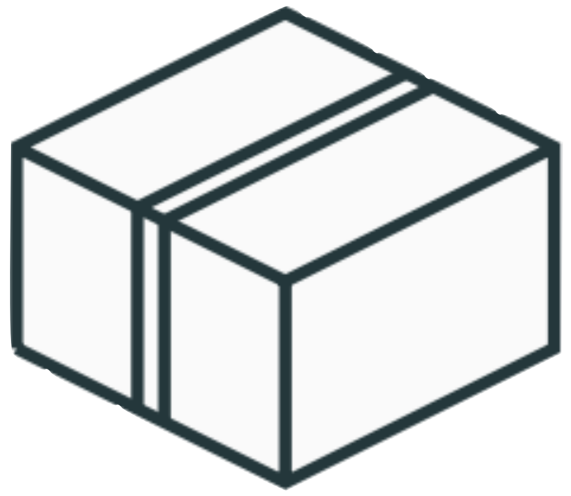

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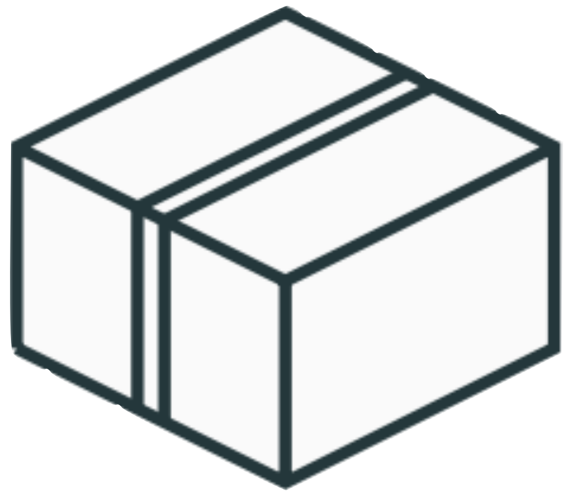

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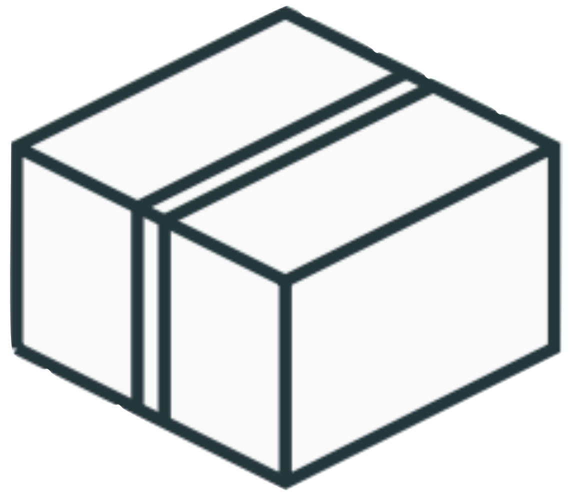

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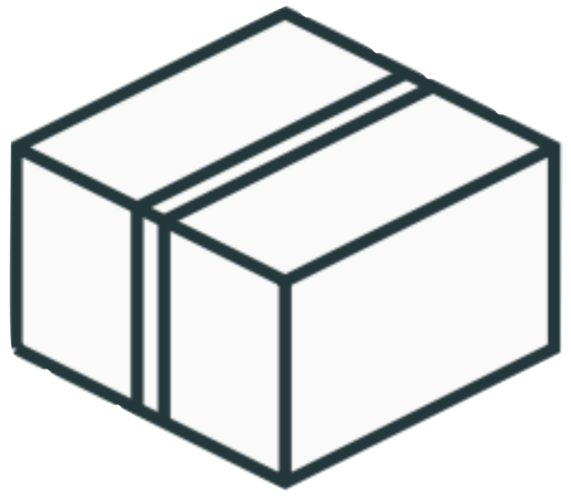

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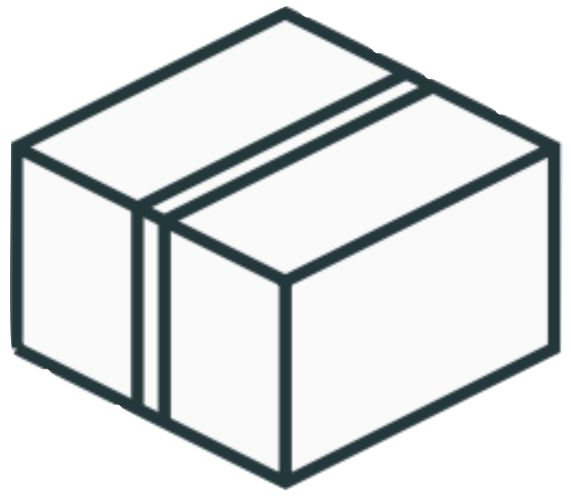

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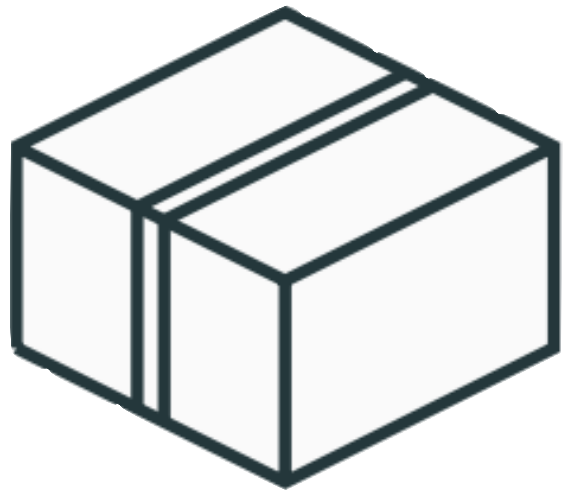

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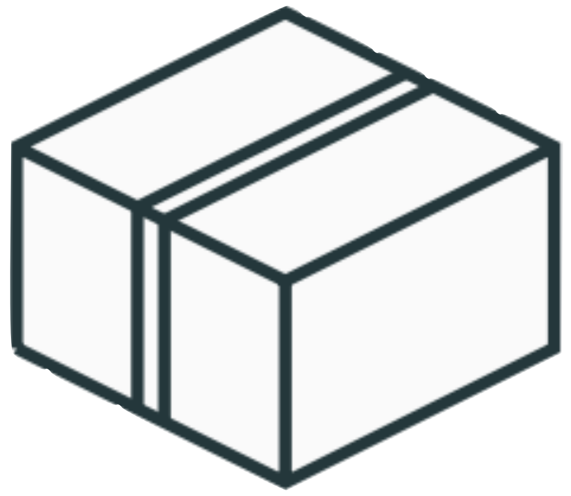

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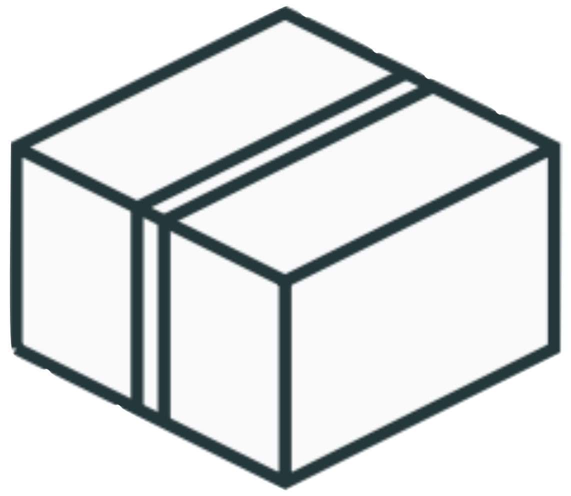

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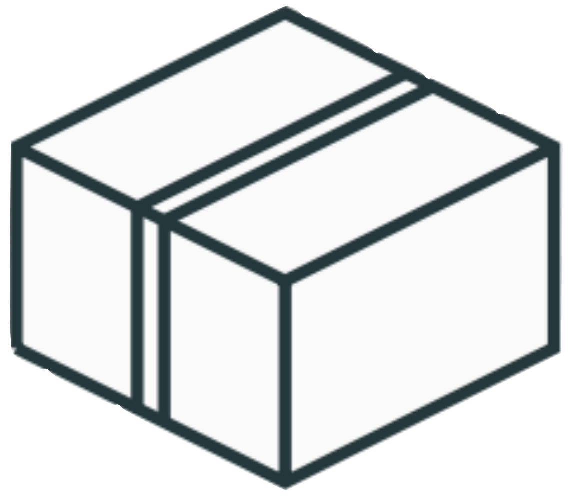

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This gives **EoR** $> 1-\varepsilon$ but **RoE** $< \varepsilon$!

What is the relation between RoE and EoR
in settings with general combinatorial constraints ?

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Analogously: $\mathrm{RoE}(\mathbf{F}) := \inf_{D} \sup_{ALG} \frac{\mathbb{E}\left[a(w)\right]}{\mathbb{E}\left[f(w)\right]}$ and $\mathrm{PbM}(\mathbf{F}) := \inf_{D} \sup_{ALG} \Pr\left[a(w) = f(w)\right]$.

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In fact, the statement says sth stronger:

Thm : For each product distr. D, we can construct a new product distr. D' for which EoR is abritrarily close to the PbM of the original distribution.

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Corollary: The gap between RoE and EoR is at least $2/e$, since

$$\mathrm{RoE}(\mathbf{F}) = \frac{1}{2} > \frac{1}{e} = \mathrm{EoR}(\mathbf{F}) \text{ for fixed order.}$$

Reduction RoE $\to$ EoR : some tools

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Useful **Lemma** : $f(w) \leq f(\overline{w}) + \sum_{e \in E} w_e \cdot 1\left[w_e > \tau\right]$.

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If $\mathbb{E}[f(\overline{w})] \leq c \cdot \tau$ then:

"Catch the superstar"

else:

"Run the Combinatorial Algorithm"

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$$\text{Then, for Case 1: } \mathbb{E}\left[\frac{a(w)}{f(w)}\right] \geq \Pr[\mathscr{E}_1] \cdot \Lambda(c) = O(1)\ .$$

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All in all, for Case 2: $\mathbb{E}\left[\dfrac{a(w)}{f(w)}\right] \geq \Pr[\mathcal{E}_0] \cdot \alpha \cdot \Gamma(k,\delta) = O(1)$.

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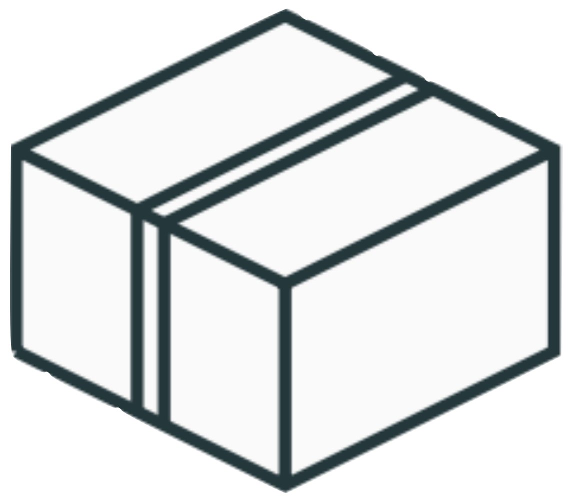

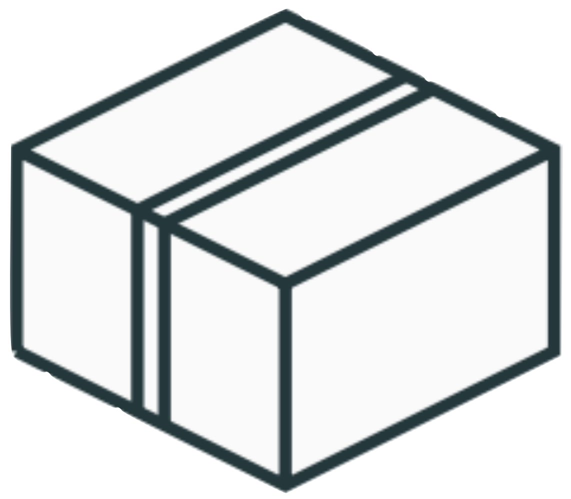

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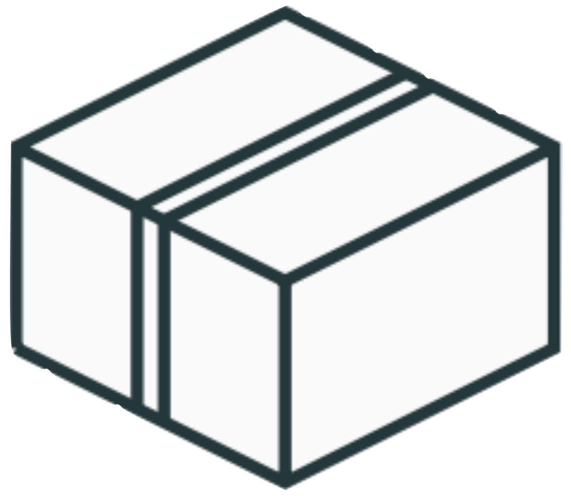

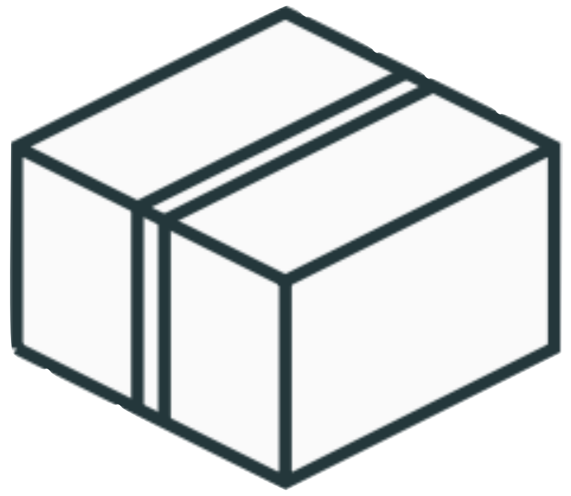

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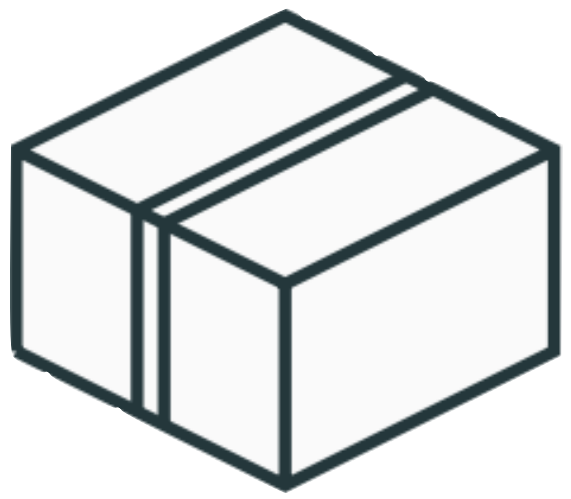
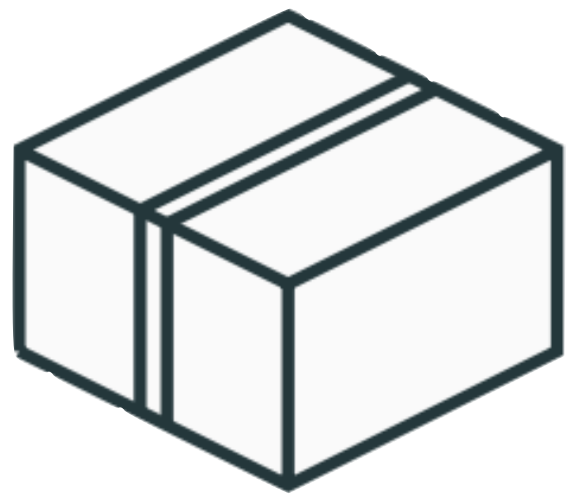


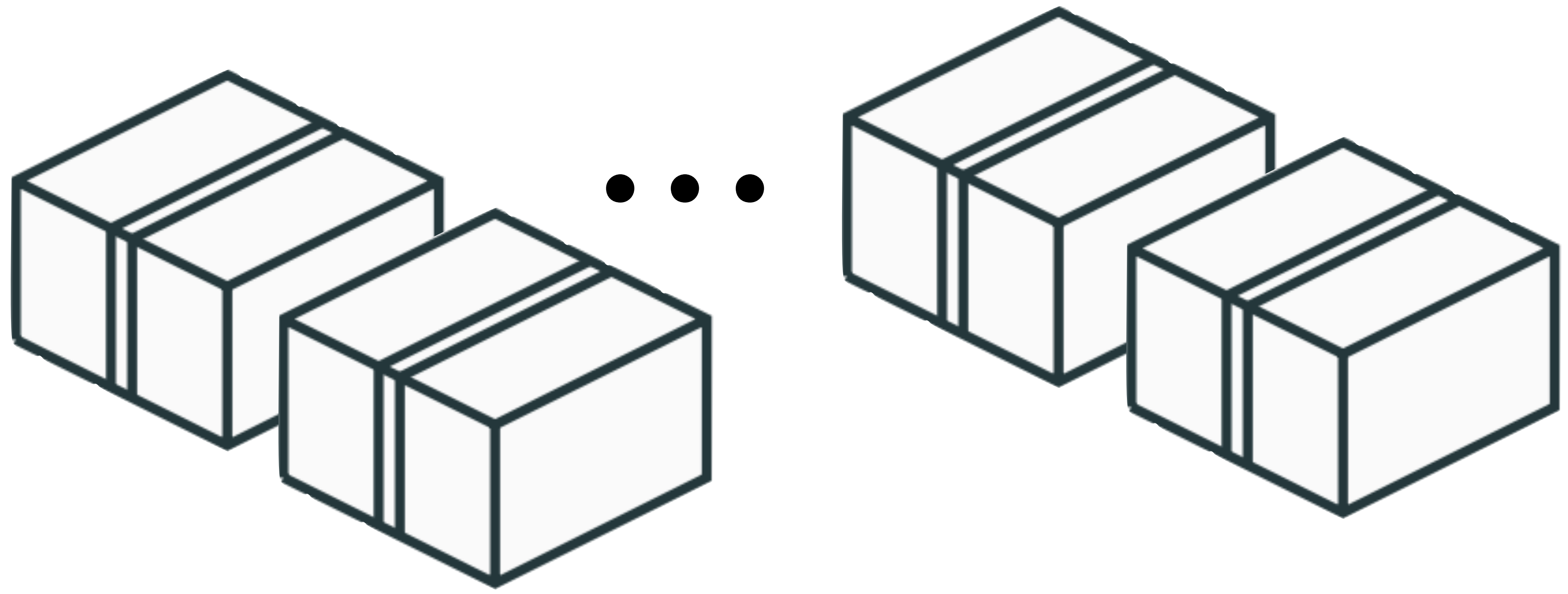


$w_1 = 1$

$$w_2 = \begin{cases} \varepsilon, \text{ w.p. } 1/2 \\ \dfrac{1}{\varepsilon}, \text{ w.p. } 1/2 \end{cases}$$

No algo can do $> \varepsilon$ – approx. with prob. $> 1/2$.

No $(1-\varepsilon)$ – approximation with $O(1)$ prob.



For each pair: $w_{1,i} = 1$ $\quad w_{2,i} = \begin{cases} 0, \text{ w.p. } 1/2 \\ 2, \text{ w.p. } 1/2 \end{cases}$

What's the best we can hope for the ex-post opt ?

Observation: Our reduction achieves an $O(\alpha)$ – approximation of the ex-post OPT with constant probability (independent of α).

Maximizing the EoR implies the above and is the best we can achieve (up to constant terms).

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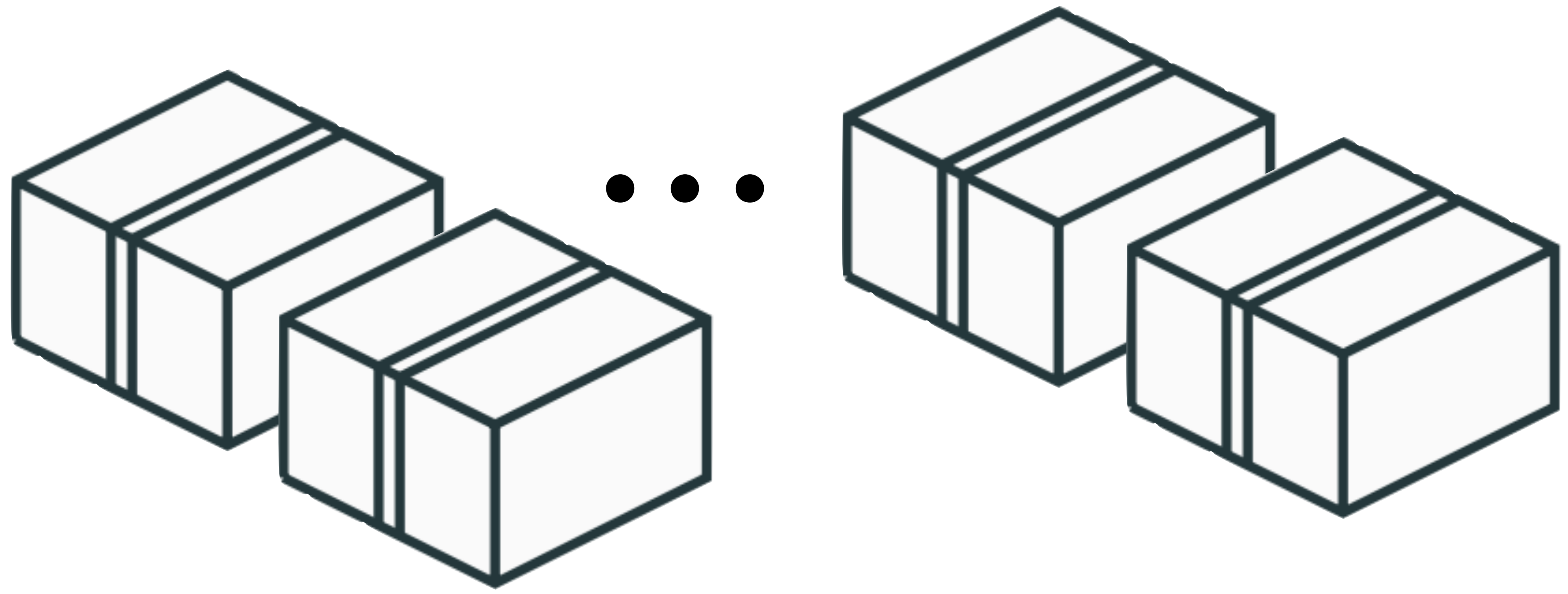

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For $(1-\varepsilon)$ – approx. need to guess max in $> 2/3$ pairs $\rightarrow$ arbitrarily small prob.

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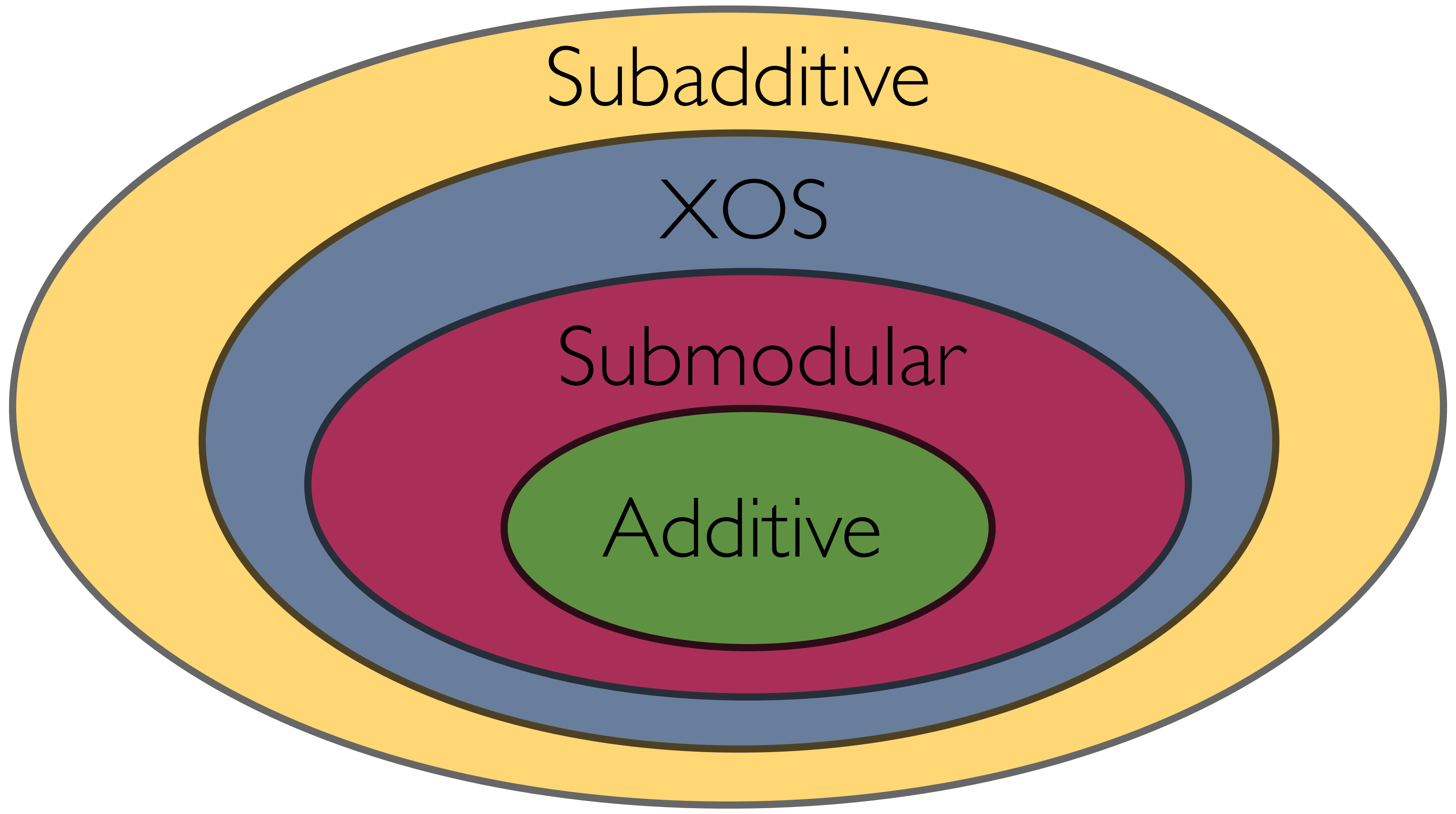
- We can extend the same techniques up to **XOS weight functions** (again, losing an extra constant factor).



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- What can we say when we have more samples from each distribution ?

Thank you for your attention!