

# Probability & Statistics with Applications to Computing

Alex Tsun

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# Notes

## Information

This book was written in Summer of 2020 during an offering of “CSE 312: Foundations of Computing II”, which is essentially probability and statistics for computer scientists. The curriculum was based off of this course as well as Stanford University’s “CS 109: Probability for Computer Scientists”. I strongly believe coding applications (which are included in Chapter 9) are essential to teach to show why this class is a core CS requirement, but also it helps keeps the students engaged and excited. This textbook is currently being used at the University of Washington (Autumn 2020).

## Resources

- Course Videos ([YouTube Playlist](#)): Mostly under 5 minutes long, serves generally as a quick **review** of each section.
- Course Slides ([Google Drive](#)): Contains Google Slides presentations for each section, used in the videos.
- Course Website (UW CSE 312): Taught at the University of Washington during Summer 2020 and Autumn 2020 quarters by instructors Alex Tsun and Professor Anna Karlin.  
<https://courses.cs.washington.edu/courses/cse312/20su/>
- This Textbook: Available online at TODO.
- Key Theorems and Definitions: At the end of this book.
- Distributions (2 pages): At the end of this book.

## About the Author

Alex Tsun grew up in the bay area, with a family full of software engineers (parents and older brother). He completed Bachelor’s degrees in computer science, statistics, and theoretical mathematics at the University of Washington in 2018, before attending Stanford University for his Master’s degree in AI and Theoretical CS. During his six years as a student, he served as a TA for this course a total of 13 times. After graduating in June 2020, he returned to UW to be the instructor for the course CSE 312 during Summer 2020. This book was written in during that quarter during a “remodelling” of the course led by me, with lots of help from my TAs. This course introduced a lot more theoretical material to prepare students better for future ML courses they might take, as well as applications to CS such as Naive Bayes, bloom filters, distinct elements, Markov Chain Monte Carlo, bootstrapping, and multi-armed bandits.

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## Chapter 0: Prerequisites

### 0.1: Intro to Set Theory

#### 0.1.1 Sets and Cardinality

Before we start talking about probability, we must learn a little bit of set theory. These notations and concepts will be used across almost every chapter, and are key to understanding probability theory.

##### Definition 0.1.1: Set

A **set**  $S$  is an unordered collection of objects with no duplicates. They can be finite or infinite.

Some examples of sets are:

- $\{3.2, 8.555, 13.122, \pi\}$
- $\{\text{apple, orange, watermelon}\}$
- $[0, 1]$  (all real numbers between 0 and 1)
- $\{1, 2, 3, \dots\}$  (all positive integers)
- $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$  (a set of sets)

##### Definition 0.1.2: Cardinality

The **cardinality** of  $S$  is denoted  $|S|$ , which is the number of elements in the set.

##### Definition 0.1.3: Empty Set

There is only one set of cardinality 0 (containing no elements), the **empty set**, denoted by  $\emptyset = \{\}$

#### Example(s)

Calculate the cardinality of the sets:

1.  $\{\text{apple, orange, watermelon}\}$
2.  $\{1, 1, 1, 1, 1\}$
3.  $[0, 1]$
4.  $\{1, 2, 3, \dots\}$
5.  $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
6.  $\{\emptyset, \{1\}, \{1, 1\}, \{1, 1, 1\}, \dots\}$

*Solution* To calculate the cardinality of a set, we have to determine the number of elements in the set.

1. For the set {apple, orange, watermelon}, we have three distinct elements, so the cardinality is 3. That is  $|\{\text{apple, orange, watermelon}\}| = 3$
2. For {1, 1, 1, 1, 1}, there are five 1s, but recall that set's don't contain duplicates, so actually this set only contains 1, and is equal to the set {1}. This means that it's cardinality is 1, that is  $|\{1, 1, 1, 1, 1\}| = 1$
3. For the set [0, 1], all the values between 0 and 1 (inclusive) we have an infinite number of elements. This means that the cardinality of this set is infinity, that is  $|\{0, 1\}| = \infty$
4. For the set {1, 2, 3,  $\dots$ }, the set of all positive integers, we have an infinite number of elements. This means that the cardinality of this set is infinity, that is  $|\{1, 2, 3, \dots\}| = \infty$ .
5. For the set  $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$  (a set of sets), there are four distinct elements that are each a different set. This means that the cardinality is 4, that is  $|\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}| = 4$ .
6. Finally, for the set of  $\{\emptyset, \{1\}, \{1, 1\}, \{1, 1, 1\}, \dots\}$ , we do have an infinite number of sets, each of which is an element. But are these distinct? Upon further consideration, all the the sets containing various numbers of 1s are equivalent, as duplicates don't matter. So there is the set containing 1 and the empty set. So the cardinality is 2, that is  $|\{\emptyset, \{1\}, \{1, 1\}, \{1, 1, 1\}, \dots\}| = |\{\emptyset, \{1\}\}| = 2$ .

□

## 0.1.2 Subsets and Equality

### Definition 0.1.4: In and Not In

If  $x$  is in a set  $S$ , we write  $x \in S$ , If  $x$  is not in set  $S$ , we write  $x \notin S$ .

### Definition 0.1.5: Subset

We write  $A \subseteq B$  to mean  $A$  is a **subset** of  $B$ , that is for any  $x \in A$ , it must be the case that  $x \in B$ .

### Definition 0.1.6: Superset

We write  $A \supseteq B$  to mean that  $A$  is a **superset** of  $B$  (equivalent to  $B \subseteq A$ ).

### Definition 0.1.7: Set Equality

We say two sets  $A, B$  are equal, denoted  $A = B$ , if and only if both  $A \subseteq B$  and  $B \subseteq A$ .

### Example(s)

Let us define  $A = \{1, 3\}$ ,  $B = \{3, 1\}$ ,  $C = \{1, 2\}$  and  $D = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, 1, 2\}$ . Determine whether the following are true or false:

- $1 \in A$

- $1 \subseteq A$
- $\{1\} \subseteq A$
- $\{1\} \in A$
- $3 \notin C$
- $A \in B$
- $A \subseteq B$
- $C \in D$
- $C \subseteq D$
- $\emptyset \in D$
- $\emptyset \subseteq D$
- $A = B$
- $\emptyset \subseteq \emptyset$
- $\emptyset \in \emptyset$

*Solution*

- $1 \in A$ . True, because 1 is an element in  $A$ .
- $1 \subseteq A$ . False, because 1 is a value, not a set, so it cannot be a subset of a set.
- $\{1\} \subseteq A$ . True, because every element of the set  $\{1\}$  is an element of  $A$ .
- $\{1\} \in A$ . False, because  $\{1\}$  is a set, and  $A$  contains no sets as elements.
- $3 \notin C$ . True, because the value 3 is not one of the elements of  $C$ .
- $A \in B$ . False, because  $A$  is a set, and there are no elements of  $B$  which are sets,  $A \notin B$ .
- $A \subseteq B$ . True, because every element of  $A$  is an element of  $B$ .
- $C \in D$ . True, because  $C$  is an element of  $D$ .
- $C \subseteq D$ . True, because each of the elements of  $C$  are also elements of  $D$ .
- $\emptyset \in D$ . True, because the empty set is an element of  $D$ .
- $\emptyset \subseteq D$ . True, by definition, the empty set is a subset of any set. This is because if this were not the case, there would have to be an element of  $\emptyset$  which was not in  $D$ . But there are no elements in  $\emptyset$ , so the statement is true (vacuously).
- $A = B$ . True,  $A \subseteq B$ , as every element of  $A$  is an element of  $B$  and  $B \subseteq A$ , as every element of  $B$  is an element of  $A$ , so since this relationship is in both directions, we have  $A = B$ .
- $\emptyset \subseteq \emptyset$ . True, because the empty set is a subset of every set (vacuously).
- $\emptyset \in \emptyset$ . False, because the empty set contains no elements, so the empty set cannot be an element of it.

□

## Chapter 0: Prerequisites

### 0.2: Set Operations

#### 0.2.1 Set Operations

##### Definition 0.2.1: Universal Set

Let  $A, B$  be sets and  $\mathcal{U}$  be a **universal set**, so that  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{U}$ . The universal set contains all elements we would ever care about.

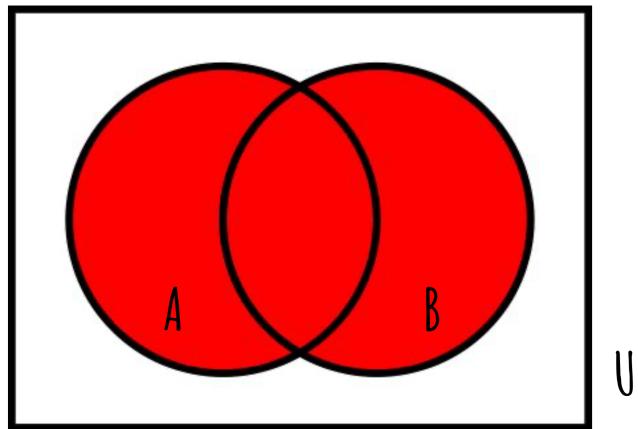
##### Example(s)

1. If we were talking about the set of fruits a supermarket might sell  $S$ , we might have  $S = \{\text{apple, watermelon, pear, strawberry}\}$  and  $\mathcal{U} = \{\text{all fruits}\}$ . We might want to know which fruits the supermarket doesn't sell, which would be denoted  $S^C$  (defined later). This requires a universal set of all fruits that we can check with to see which are missing from  $S$ .
2. If we were talking about the set of kinds of cars Bill Gates owns, that might be the set  $T$ . There must be a universal set  $\mathcal{U}$  of possible kinds of cars that exist, if we wanted to list out which ones he was missing  $T^C$ .

##### Definition 0.2.2: Set Operation: Union

The **union** of  $A$  and  $B$  is denoted  $A \cup B$ . It contains elements in  $A$  or  $B$ , or both (without duplicates). So  $x \in A \cup B$  if and only if  $x \in A$  or  $x \in B$ .

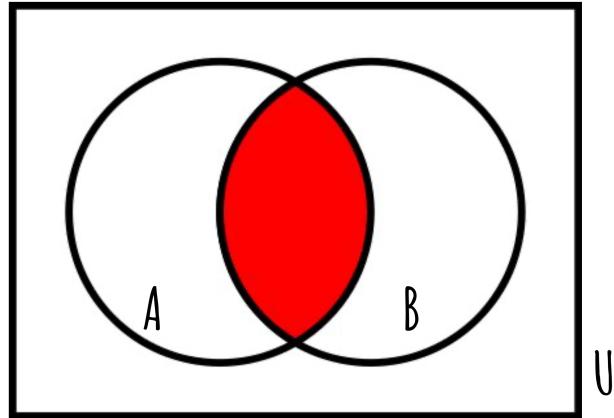
The image below shows in red the union of  $A$  and  $B$ :  $A \cup B$ . The outer rectangle is the universal set  $\mathcal{U}$ .



### Definition 0.2.3: Set Operation: Intersection

The **intersection** of  $A$  and  $B$  is denoted  $A \cap B$ . It contains elements in  $A$  and  $B$ . So  $x \in A \cap B$  if and only if  $x \in A$  and  $x \in B$ .

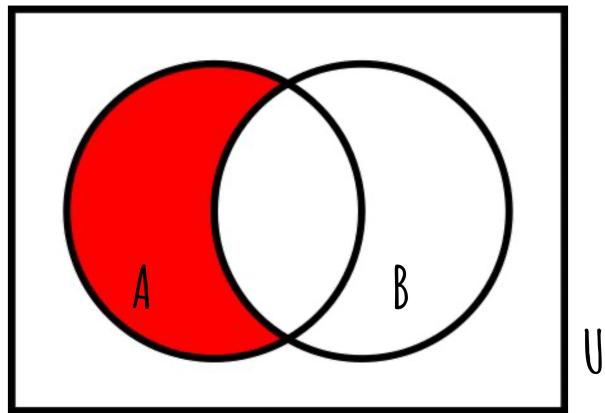
The image below shows in red the intersection of  $A$  and  $B$ :  $A \cap B$ . The outer rectangle is the universal set  $\mathcal{U}$ .



### Definition 0.2.4: Set Operation: Set Difference

The **set difference** of  $A$  with  $B$  is denoted,  $A \setminus B$ . It contains elements of  $A$  which are not in  $B$ . So  $x \in A \setminus B$  if and only if  $x \in A$  and  $x \notin B$ .

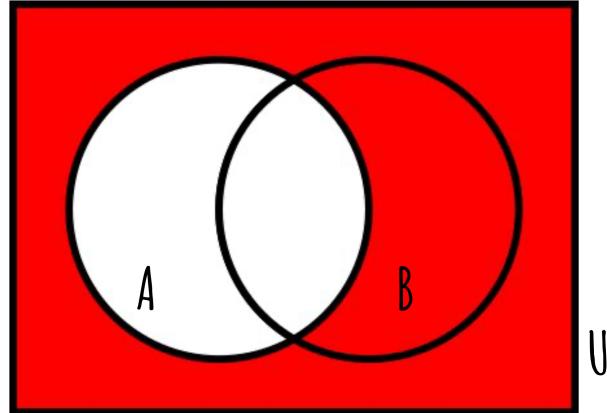
The image below shows in red the set different of  $A$  with  $B$ :  $A \setminus B$ . The outer rectangle is the universal set  $\mathcal{U}$ .



**Definition 0.2.5: Set Operation: Complement**

The **complement** of  $A$  (with respect to  $\mathcal{U}$ ) is denoted  $A^C = \mathcal{U} \setminus A$ . It contains elements of  $\mathcal{U}$ , the universal set, which are not in  $A$ .

The image below shows in red the complement of  $A$  with respect to  $\mathcal{U}$ :  $A^C = \mathcal{U} \setminus A$ .

**Example(s)**

Let  $A = \{1, 3\}$ ,  $B = \{2, 3, 4\}$ , and  $\mathcal{U} = \{1, 2, 3, 4, 5\}$ . Solve for:  $A \cap B$ ,  $A \cup B$ ,  $B \setminus A$ ,  $A \setminus B$ ,  $(A \cup B)^C$ ,  $A^C$ ,  $B^C$ , and  $A^C \cap B^C$ .

*Solution*

- $A \cap B = \{3\}$ , since 3 is the only element in both  $A$  and  $B$ .
- $A \cup B = \{1, 2, 3, 4\}$ , as these are all the elements in either  $A$  or  $B$ . Note we dropped the duplicate 3, since sets cannot contain duplicates.
- $B \setminus A = \{2, 4\}$ , as these are the elements of  $B$  which are not in  $A$ .
- $A \setminus B = \{1\}$ , as this is the only element of  $A$  which is not an element of  $B$ .
- $(A \cup B)^C = \{5\}$ , as by definition  $(A \cup B)^C = \mathcal{U} \setminus (A \cup B)$  and 5 is the only element of  $\mathcal{U}$  which is not an element of  $A \cup B$ .
- $A^C = \{2, 4, 5\}$ , as by definition  $A^C = \mathcal{U} \setminus A$ , and these are the elements of  $\mathcal{U}$  which are not elements of  $A$ .
- $B^C = \{1, 5\}$ , as by definition  $B^C = \mathcal{U} \setminus B$ , and these are the elements of  $\mathcal{U}$  which are not elements of  $B$ .
- $A^C \cap B^C = \{5\}$ , because the only element in both  $A^C$  and  $B^C$  is 5 (see the above).

□

## Chapter 0: Prerequisites

### 0.3: Sum and Product Notation

#### 0.3.1 Summation Notation

Suppose that we want to write the sum:  $1 + 2 + 3 + 5 + 6 + 7 + 8 + 9 + 10$ . We can write out each element, but it becomes tedious. We could use dots, to signify this as:  $1 + 2 + \dots + 9 + 10$ , but this can become vague if the pattern isn't as clear. Instead, we can use summation notation as shorthand for summations of values. Here we are referring to the sum of each element  $i$ , where  $i$  will take on every value in the range starting with 1 and ending with 10.

$$1 + 2 + 3 + \dots + 10 = \sum_{i=1}^{10} i$$

Note that  $i$  is just a dummy variable. We could have also used  $j$ ,  $k$ , or any other letter. What if we wanted to sum numbers that weren't consecutive integers?

As long as there is some pattern, we can write it compactly! For example, how could we write  $16 + 25 + 36 + \dots + 81$ ? In the first equation below (0.3.1),  $j$  takes on the values from 4 to 9, and the square of each of these values will be summed together. Note that this is equivalent to  $k$  taking on the values of 1 to 6, and adding 3 to each of the values before squaring and summing them up (0.3.2).

$$16 + 25 + 36 + \dots + 81 = \sum_{j=4}^9 j^2 \tag{0.3.1}$$

$$= \sum_{k=1}^6 (k + 3)^2 \tag{0.3.2}$$

If you know what a for-loop is (from computer science), this is exactly the following (in Java or C++). This first loop represents the first sum with dummy variable  $j$ .

```
int sum = 0
for (int j = 4; j <= 9; j++) {
    sum += (j * j)
}
```

This second loop represents the second sum with dummy variable  $k$ , and is equivalent to the first.

```
int sum = 0
for (int k = 1; k <= 6; k++) {
    sum += ((k + 3) * (k + 3))
}
```

This brings us to the following definition of summation notation:

**Definition 0.3.1: Summation Notation**

Let  $x_1, x_2, x_3, \dots$  be a sequence of numbers. Then, the following notation represents the “sub-sum”:

$$x_a + x_{a+1} + \dots + x_{b-1} + x_b = \sum_{i=a}^b x_i$$

Furthermore, if  $S$  is a set, and  $f : S \rightarrow \mathbb{R}$  is a function defined on  $S$ , then the following notation sums over all elements  $x \in S$  of  $f(x)$ :

$$\sum_{x \in S} f(x)$$

Note that the sum over no terms (the empty set) is defined as 0.

**Example(s)**

Write out the following sums:

- $\sum_{k=3}^7 k^{10}$
- $\sum_{y \in S} (2^y + 5)$ , for  $S = \{3, 6, 8, 11\}$
- $\sum_{t=6}^8 4$
- $\sum_{z=2}^1 \sin(z)$
- $\sum_{x \in T} 13x$ , for  $T = \{-1, -3, 5\}$ .

*Solution*

- For,  $\sum_{k=3}^7 k^{10}$ , we raise each value of  $k$  from 3 to 7 to the power of 10 and sum them together. That is:

$$\sum_{k=3}^7 k^{10} = 3^{10} + 4^{10} + 5^{10} + 6^{10} + 7^{10}$$

- Then, if we let  $S = \{3, 6, 8, 11\}$ , for  $\sum_{y \in S} (2^y + 5)$ , raise 2 to the power of each value  $y$  in  $S$  and add 5, and then sum the results together. That is

$$\sum_{y \in S} (2^y + 5) = (2^3 + 5) + (2^6 + 5) + (2^8 + 5) + (2^{11} + 5)$$

- For the sum of a constant,  $\sum_{t=6}^8 4$ , we add the constant, 4 for each value  $t = 6, 7, 8$ . This is equivalent to just adding 4 together three times.

$$\sum_{t=6}^8 4 = 4 + 4 + 4$$

- Then, for a range with no values, the sum is defined as 0, for  $\sum_{z=2}^1 \sin(z)$ , because there are no values

from 2 to 1, we have:

$$\sum_{z=2}^1 \sin(z) = 0$$

- Finally, if we let  $T = \{-1, -3, 5\}$ , for  $\sum_{x \in T} 13x$ , we multiply each value of  $x$  in  $T$  by 13 and then sum them up.

$$\begin{aligned} \sum_{x \in T} 13x &= 13(-1) + 13(-3) + 13(5) \\ &= 13(-1 + -3 + 5) \\ &= 13 \sum_{x \in T} x \end{aligned}$$

Notice that we can actually factor out the 13; that is, we could sum all values of  $x \in T$  first, and then multiply by 13. This is one of a few properties of summations we can see below!

□

Further, the associative and distributive properties hold for sums. If you squint hard enough, you can kind of see why they're true! We'll also see some examples below too, since the notation can be confusing at first.

#### Fact 0.3.1: The Associative and Distributive Properties of Sums

We have the associative property (0.3.3) and distributive property (0.3.4, 0.3.5) for sums.

$$\sum_{x \in A} f(x) + \sum_{x \in A} g(x) = \sum_{x \in A} (f(x) + g(x)) \quad (0.3.3)$$

$$\sum_{x \in A} \alpha f(x) = \alpha \sum_{x \in A} f(x) \quad (0.3.4)$$

$$\left( \sum_{x \in A} f(x) \right) \left( \sum_{y \in B} g(y) \right) = \sum_{x \in A} \sum_{y \in B} f(x)g(y) \quad (0.3.5)$$

The last property is like FOIL - if you multiply  $(x + x^2 + x^3)(1/y + 1/y^2)$  (left-hand side) for example, you would have to sum over every possible combination  $x/y + x/y^2 + x^2/y + x^2/y^2 + x^3/y + x^3/y^2$  (right-hand side).

The proof of these are left to the reader, but see the examples below for some intuition!

#### Example(s)

“Prove” the following by writing out the sums:

- $\sum_{i=5}^7 i + \sum_{i=5}^7 i^2 = \sum_{i=5}^7 (i + i^2)$
- $\sum_{j=3}^5 2j = 2 \sum_{j=3}^5 j$
- $(\sum_{i=1}^2 f(a_i))(\sum_{j=1}^3 g(b_j)) = \sum_{i=1}^2 \sum_{j=1}^3 f(a_i)g(b_j)$

*Solution*

- Looking at the associative property, we know the following:

$$\sum_{i=5}^7 i + \sum_{i=5}^7 i^2 = (5 + 6 + 7) + (5^2 + 6^2 + 7^2) = (5 + 5^2) + (6 + 6^2) + (7 + 7^2) = \sum_{i=5}^6 (i + i^2)$$

- Also, using the distributive property we know:

$$\sum_{j=3}^5 2j = 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 2(3 + 4 + 5) = 2 \sum_{j=3}^5 j$$

- This one is similar to FOIL. Finally, we have:

$$\begin{aligned} \left( \sum_{i=1}^2 f(a_i) \right) \left( \sum_{j=1}^3 g(b_j) \right) &= (f(a_1) + f(a_2))(g(b_1) + g(b_2) + g(b_3)) \\ &= f(a_1)g(b_1) + f(a_1)g(b_2) + f(a_1)g(b_3) + f(a_2)g(b_1) + f(a_2)g(b_2) + f(a_2)g(b_3) \\ &= \sum_{i=1}^2 \sum_{j=1}^3 f(a_i)g(b_j) \end{aligned}$$

□

### 0.3.2 Product Notation

Similarly, we can define product notation to handle multiplications.

#### Definition 0.3.2: Product Notation

Let  $x_1, x_2, x_3, \dots$  be a sequence of numbers. Then, the following notation represents the “sub-product”  $x_a \cdot x_{a+1} \cdot \dots \cdot x_{b-1} \cdot x_b$ :

$$\prod_{i=a}^b x_i$$

Further, if  $S$  is a set, and  $f : S \rightarrow \mathbb{R}$  is a function defined on  $S$ , then the following notation multiplies over all elements  $x \in S$  of  $f(x)$ :

$$\prod_{x \in S} f(x)$$

Note that the product over no terms is defined as 1 (not 0 like it was for sums).

#### Example(s)

Write out the following products:

- $\prod_{a=4}^7 a$
- $\prod_{x \in S} 8$  for  $S = \{3, 6, 8, 11\}$

- $\prod_{z=2}^1 \sin(z)$
- $\prod_{b=2}^5 9^{1/b}$

*Solution*

- For  $\prod_{a=4}^7 a$ , we multiply each value  $a$  in the range 4 to 7 and have:

$$\prod_{a=4}^7 a = 4 \cdot 5 \cdot 6 \cdot 7$$

- Then if, we let  $S = \{3, 6, 8, 11\}$ , for  $\prod_{x \in S} 8$ , we multiply 8 for each value in the set,  $S$  and have:

$$\prod_{x \in S} 8 = 8 \cdot 8 \cdot 8 \cdot 8$$

- Then for  $\prod_{z=2}^1 \sin(z)$ , we have the empty product, because there are no values in the range 2 to 1, so we have:

$$\prod_{z=2}^1 \sin(z) = 1$$

- Finally for  $\prod_{b=2}^5 9^{1/b}$ , we have each value of  $b$  from 2 to 5 of  $9^{1/b}$ , to get

$$\begin{aligned} \prod_{b=2}^5 9^{1/b} &= 9^{1/2} \cdot 9^{1/3} \cdot 9^{1/4} \cdot 9^{1/5} \\ &= 9^{1/2+1/3+1/4+1/5} \\ &= 9^{\sum_{b=2}^5 1/b} \end{aligned}$$

□

Also, if you were to do the same examples as we did for sums replacing  $\prod$  with  $\sum$ , you just multiply instead of add! They are almost identical, except the empty sum is 0 and the empty product is 1.

## Chapter 1: Combinatorial Theory

### 1.1: So You Think You Can Count?

Before we jump into probability, we must first learn a little bit of combinatorics, or more informally, counting. You might wonder how this is relevant to probability, and we'll see how very soon. You might also think that counting is for kindergarteners, but it is actually a lot harder than you think!

To motivate us, let's consider how easy or difficult it is for a robber to randomly guess your PIN code. Every debit card has a PIN code that their owners use to withdraw cash from ATMs or to complete transactions. How secure are these PINs, and how safe can we feel?

#### 1.1.1 Sum Rule

First though, we will count baby outfits. Let's say that a baby outfit consists of *either* a top *or* a bottom (but not both), and we have 3 tops and 4 bottoms. How many baby outfits are possible? We simply add  $3 + 4 = 7$ , and we have found the answer using the sum rule.

##### Definition 1.1.1: Sum Rule

If an experiment can either end up being one of  $N$  outcomes, or one of  $M$  outcomes (where there is no overlap), then the number of possible outcomes of the experiment is:

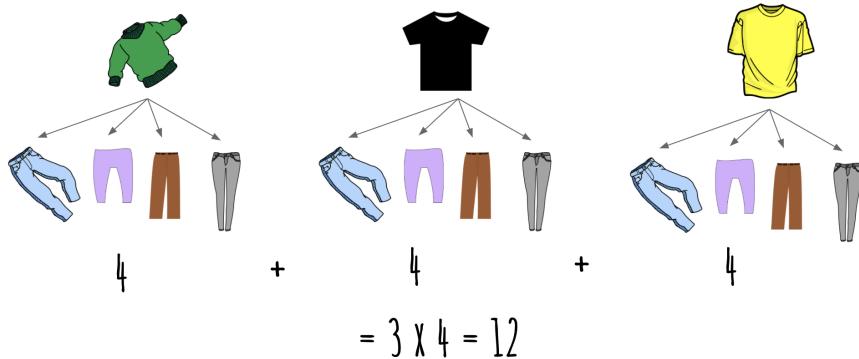
$$N + M$$

We'll see some examples of the Sum Rule combined with the Product Rule (next), so that they can be a bit more complex!

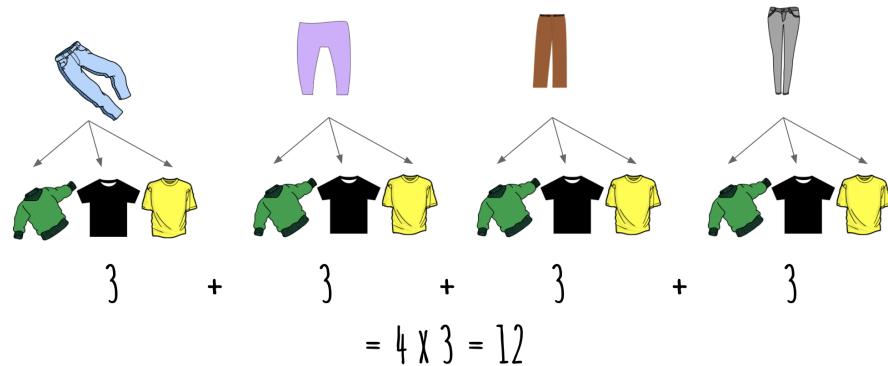
#### 1.1.2 Product Rule

Now we will count real outfits. Let's say that a real outfit consists of *both* a top *and* a bottom, and again, we still have 3 tops and 4 bottoms. then how many outfits are possible?

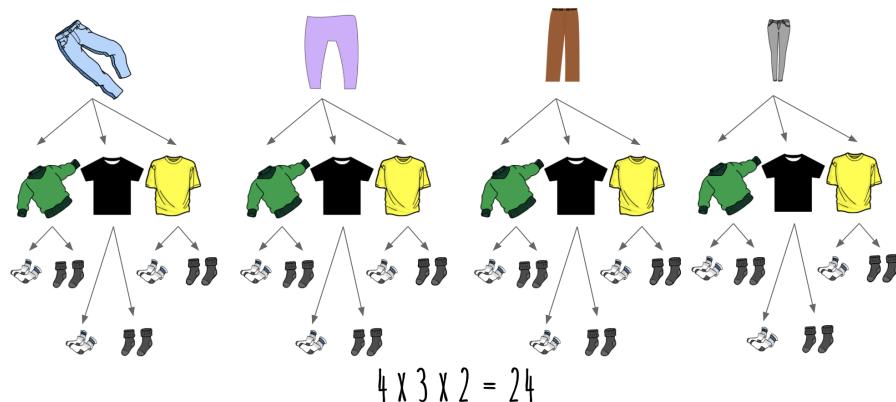
Well, we can consider this from first picking out a top. Once we have our top, we have 4 choices for our bottom. This means we have 4 choices of bottom for each top, which we have 3 of. So, we have a total of  $4 + 4 + 4 = 3 \times 4 = 12$  outfit choices.



We could also do this in reverse and first pick out a bottom. Once we have our bottom, we have 3 choices for our top. This means we have 3 choices of top for each bottom, which we have 4 of. So, we still have a total of  $3 + 3 + 3 + 3 = 4 \times 3 = 12$  outfit choices. (This makes sense - the number of outfits should be the same no matter how I count!)



What if we also wanted to add socks to the outfit, and we had 2 different pairs of socks? Then, for each of the 12 choices outlined above, we now have 2 choices of sock. This brings us to a total of 24 possible outfits.



This could be calculated more directly rather than drawing out each of these unique outfits, by multiplying our choices: 3 tops  $\times$  4 bottoms  $\times$  2 socks = 24 outfits.

**Definition 1.1.2: Product Rule**

If an experiment has  $N_1$  outcomes for the first stage,  $N_2$  outcomes for the second stage,  $\dots$ , and  $N_m$  outcomes for the  $m^{\text{th}}$  stage, then the total number of outcomes of the experiment is  $N_1 \times N_2 \times \dots \times N_m$ .

If this still sounds “simple” to you or you just want to practice, see the examples below! There are some pretty interesting scenarios we can count, and they are more difficult than you might expect.

**Example(s)**

Flamingos Fanny and Freddy have three offspring: Happy, Glee, and Joy. These five flamingos are to be distributed to seven different zoos so that no zoo gets both a parent and a child :(. It is not required that every zoo gets a flamingo. In how many different ways can this be done?

*Solution* There are two disjoint (mutually exclusive) cases we can consider that cover every possibility. We can use the sum rule to add them up since they don’t overlap!

1. **Case 1: The parents end up in the same zoo.** There are 7 choices of zoo they could end up at. Then, the three offspring can go to any of the 6 other zoos, for a total of  $7 \times 6 \times 6 \times 6 = 7 \times 6^3$  possibilities (by the product rule).
2. **Case 2: The parents end up in different zoos.** There are 7 choices for Fanny and 6 for Freddy. Then, the three offspring can go to any of the 5 other zoos, for a total of  $7 \times 6 \times 5^3$  possibilities.

The result, by the sum rule, is  $7 \times 6^3 + 7 \times 6 \times 5^3$ . (Note: This may not be the only way to solve this problem. Often, counting problems have two or more approaches, and it is instructive to try different methods to get the same answer. If they differ, at least one of them is wrong, so try to find out which one and why!)  $\square$

### 1.1.3 Permutations

Back to the example of the debit card. There are 10 possible digits for each of the 4 digits of a PIN. So how many possible 4-digit PINs are there? This can be solved as  $10 \times 10 \times 10 \times 10 = 10^4 = 10,000$ . So, there is a one in ten thousand chance that a robber can guess your pin code (randomly).

Let’s consider a stronger case where you must use each digit exactly once, so the PIN is exactly 10 digits long. How many such PINs exist?

Well, we have 10 choices for the first digit, 9 choices for the second digit, and so forth, until we only have 2 choices for the ninth digit, and 1 choice for the tenth digit. This means there are 362,880 possible PINs in this scenario as follows:

$$10 \times 9 \times \dots \times 2 \times 1 = \prod_{i=1}^{10} i = 362,880$$

This formula/pattern seems like it would appear often! Wouldn’t it be great if there were a shorthand for this?

**Definition 1.1.3: Permutation**

The number of orderings of  $N$  **distinct** objects, is called a permutation, and mathematically defined as:

$$N! = N \times (N - 1) \times (N - 2) \times \dots \times 3 \times 2 \times 1 = \prod_{j=1}^N j$$

$N!$  is read as “ $N$  factorial”. It is important to note that  $0! = 1$  since there is one way to arrange 0 objects.

**Example(s)**

A standard 52-card deck consists of one of each combination of: 13 different ranks (Ace, 2, 3,...,10, Jack, Queen, King) and 4 different suits (clubs, diamonds, hearts, spades), since  $13 \times 4 = 52$ . In how many ways a 52-card deck be dealt to thirteen players, four to each, so that every player has one card of each suit?

*Solution* This is a great example where we can try two equivalent approaches. Each person usually has different preferences, and sometimes one way is significantly easier to understand than another. Read them both, understand why they both make sense and are equal, and figure out which approach is more intuitive for you!

Let's assign each player one at a time. The first player has 13 choices for the club, 13 for the heart, 13 for the diamond, and 13 for the spade, for a total of  $13^4$  ways. The second player has  $12^4$  choices (since there are only 12 of each suit remaining). And so on, so the answer is  $13^4 \times 12^4 \times 11^4 \times \dots \times 2^4 \times 1^4$ .

Alternatively, we can assign each suit one at a time. For the clubs suit, there are  $13!$  ways to distribute them to the 13 different players. Then, the diamonds suit can be assigned in  $13!$  ways as well, and same for the other two suits. By the product rule, the total number of ways is  $(13!)^4$ . Check that this different order of assigning cards gave the same answer as earlier! (Expand the factorials.)  $\square$

**Example(s)**

A group of  $n$  families, each with  $m$  members, are to be lined up for a photograph. In how many ways can the  $nm$  people be arranged if members of a family must stay together?

*Solution* We first choose the ordering of the families, of which there are  $n!$ . Then, in the first family, we have  $m!$  ways to arrange them. The second family also has  $m!$  ways to be arranged. And so on. By the product rule, the number of orderings is  $n! \times (m!)^n$ .  $\square$

## 1.1.4 Complementary Counting

Now, let's consider an even trickier PIN requirement. Suppose we are still making a 10-digit PIN, but now at least one digit has to be repeated at least once. How many such PINs exist?

Some examples of this PIN would be 1111111111, 01234556788, or 9876598765, but the list goes on!

Let's try our "normal" approach. If we try this, we'll end up getting stuck. Consider placing the first digit - we have 10 choices. How many choices do we have for the second digit? Is this a repeated digit or not? We can try to find a product of counts of choices for each digit in different scenarios but this can become complicated as we move around which digits are repeated...

Another approach might be to count how many PINs **don't** satisfy this property, and subtract it from the total number of PINs. This strategy is called complementary counting, as we are counting the size of the complement of the set of interest. The number of possible 10-digit PINs, with no stipulations, is  $10^{10}$  (from the product rule, multiplying 10 choices with itself for each of 10 positions). Then, we found above that the 10-digit PINs with no repeats has  $10!$  possibilities (each digit used exactly once). Well, consider that the 10-digit PINs with at least one repeat will be all other possibilities (they could have one, two, or more repeats but certainly won't have none). This means that we can count this by taking the difference of all the possible 10-digit PINs and those with no repeats. That is:

$$10^{10} - 10!$$

HOW MANY SUCH PINs?  $10^{10} - 10!$  ALL 10-DIGIT PINs:  $10^{10}$

10-DIGIT PINs WITH AT LEAST ONE REPEAT: ???

10-DIGIT PINs WITH NO REPEATS:  $10!$

#### Definition 1.1.4: Complementary Counting

Let  $\mathcal{U}$  be a (finite) universal set, and  $S$  a subset of interest. Let  $S^C = \mathcal{U} \setminus S$  denote the set difference (complement of  $S$ ). Then,

$$|S| = |\mathcal{U}| - |S^C|$$

Informally, to find the number of ways to do something, we could count the number of ways to NOT to do that thing, and subtract it from the total. That is, the complement of the subset of interest is also of interest!

### 1.1.5 Exercises

1. Suppose we have 6 people who want to line up in a row for a picture, but two of them, A and B, refuse to sit next to each other. How many ways can they sit in a row?

**Solution:** There are two equivalent approaches. The first approach is to solve it directly. However, depending on where A sits, B has a different number of options (whether A sits at the end or the middle). So we have two disjoint (non-overlapping) cases:

- (a) Case 1: A sits at one of the two end seats. Then, A has 2 choices for where to sit, and B has 4. (See this diagram where A sits at the right end:  $\text{---} - A$ .) Then, there are  $4!$  ways for the remaining people to sit, for a total of  $2 \times 4 \times 4!$  ways.

(b) Case 2: A sits in one of the middle 4 seats. Then, A has 4 choices of seat, but B only has three choices for where to sit. (See this diagram where A sits in a middle seat:  $- A - \text{---}$ .) Again, there are  $4!$  ways to seat the rest, for a total of  $4 \times 3 \times 4!$  ways.

Hence our total by the sum rule is  $2 \times 4 \times 4! + 4 \times 3 \times 4! = 480$ .

The alternative approach is complementary counting. We can count the total orderings, of which there are  $6!$ , and subtract the cases where A and B *do* sit next to each other. There's a trick we can do to guarantee this: let's treat A and B as a *single entity*. Then, along with the remaining 4 people, there are only 5 entities. We order the entities in  $5!$  ways, but also multiply by  $2!$  since we could have the seating AB or BA. Hence, the number of ways they do sit together is  $2 \times 5! = 240$ , and the ways they do not sit together is  $6! - 240 = 720 = 240 = 480$ .

Decide which approach you liked better - oftentimes, one method will be easier than another!

## Chapter 1: Combinatorial Theory

### 1.2: More Counting

#### 1.2.1 $k$ -Permutations

Last time, we learned the foundational techniques for counting (the sum and product rule), and the factorial notation which arises frequently. Now, we'll learn even more "shortcuts"/"notations" for common counting situations, and tackle more complex problems.

We'll start with a simpler situation than most of the exercises from last time. How many 3-color mini rainbows can be made out of 7 available colors, with all 3 being different colors?



We choose an outer color, then a middle color and then an inner color. There are 7 possibilities for the outer layer, 6 for the middle and 5 for the inner (since we cannot have duplicates). Since order matters, we find that the total number of possibilities is 210, from the following calculation:

$$\begin{array}{c} 7 \quad \times \quad 6 \quad \times \quad 5 \quad = \quad 210 \\ \text{\# POSSIBLE} \quad \text{\# POSSIBLE} \quad \text{\# POSSIBLE} \quad \text{\# POSSIBLE} \\ \text{OUTER COLORS} \quad \text{MIDDLE COLORS} \quad \text{INNER COLORS} \quad \text{MINI-RAINBOWS} \end{array}$$

Let's manipulate our equation a little and see what happens.

$$\begin{aligned} 7 \cdot 6 \cdot 5 &= \frac{7 \cdot 6 \cdot 5}{1} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} && [\text{multiply numerator and denominator by } 4! = 4 \cdot 3 \cdot 2 \cdot 1] \\ &= \frac{7!}{4!} && [\text{def of factorial}] \\ &= \frac{7!}{(7-3)!} \end{aligned}$$

Notice that we are "picking" 3 out of 7 available colors - so order matters. This may not seem useful, but imagine if there were 835 colors and we wanted a rainbow with 135 different colors. You would have to multiply 135 numbers, rather than just three!

**Definition 1.2.1:  $k$ -Permutations**

If we want to arrange **only  $k$**  out of  $n$  distinct objects, the number of ways to do so is  $P(n, k)$  (read as “ $n$  pick  $k$ ”), where

$$P(n, k) = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

A **permutation** of  $n$  objects is an arrangement of each object (where order matters), so a  **$k$ -permutation** is an arrangement of  $k$  members of a set of  $n$  members (where order matters). The number of  $k$ -Permutations of  $n$  objects is just  $P(n, k)$ .

**Example(s)**

Suppose we have 13 chairs (in a row) with 9 TA's, and Professors Sunny, Rainy, Windy, and Cloudy to be seated. What is the number of seatings where every professor has a TA to his/her immediate left *and* right?

*Solution* This is quite a tricky problem if we don't choose the right setup. Imagine we first seat the 9 TA's - there are  $9!$  ways to do this. Then, there are 8 spots between them, so that if we place a professor there, they're guaranteed to have a TA to their immediate left and right. We can't place more than one professor in a spot. Out of the 8 spots, we **pick** 4 of them for the professors to sit (order matters, since the professors are different people). So the answer by the product rule is  $9! \cdot P(8, 4)$ .  $\square$

**1.2.2  $k$ -Combinations (Binomial Coefficients)**

What if order *doesn't* matter? For example, if I need to **choose** 3 out of 7 weapons on my online adventure? We'll tackle that now, continuing our rainbow example!

A kindergartener smears 3 different colors out of 7 to make a new color. How many smeared colors can she create?

Notice that there are  $3! = 6$  possible ways to order red, blue and orange, as you see below. However, all these rainbows produce the same “smeared” color!



Recall that there were  $P(7, 3) = 210$  possible mini-rainbows. But as we see from these rainbows, each “smeared” color is counted  $3! = 6$  times. So, to get our answer, we take the 210 mini-rainbows and divide by 6 to account for the overcounting since in this case, order doesn't matter.

The answer is,

$$\frac{210}{6} = \frac{P(7, 3)}{3!} = \frac{7!}{3!(7 - 3)!}$$

**Definition 1.2.2:  $k$ -Combinations (Binomial Coefficients)**

If we want to choose (order doesn't matter) **only**  $k$  out of  $n$  distinct objects, the number of ways to do so is  $C(n, k) = \binom{n}{k}$  (read as “ $n$  choose  $k$ ”), where

$$C(n, k) = \binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}$$

A  **$k$ -combination** is a selection of  $k$  objects from a collection of  $n$  objects, in which the order does not matter. The number of  $k$ -Combinations of  $n$  objects is just  $\binom{n}{k}$ .  $\binom{n}{k}$  is also called a **binomial coefficient** - we'll see why in the next section.

Notice, we can show from this that there is symmetry in the definition of binomial coefficients:

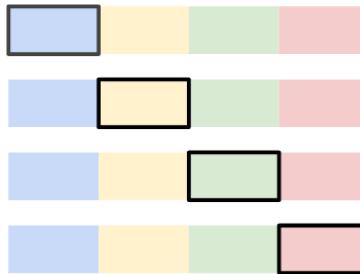
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

The algebra checks out - why is this true though, intuitively?

Let's suppose that  $n = 4$  and  $k = 1$ . We want to show  $\binom{4}{1} = \binom{4}{3}$ . We have 4 colors:



These are the possible ways to choose 1 color out of 4:



These are the possible ways to choose 3 colors out of 4:



Looking at these, we can see that the color choices in each row are complementary. Intuitively, choosing 1 color is the same as choosing  $4 - 1 = 3$  colors that we don't want - and vice versa. This explains the symmetry in binomial coefficients!

## Example(s)

There are 6 AI professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are to be chosen and divided into 4 pairs (one AI professor with one theory professor per pair), how many pairings are possible?

*Solution* We first choose 4 out of 6 AI professors, with order not mattering, and 4 out of 7 theory professors, again with order not mattering. There are  $\binom{6}{4} \cdot \binom{7}{4}$  ways to do this by the product rule. Then, for the first theory professor, we have 4 choices of AI professor to match with, for the second theory professor, we only have 3 choices, and so on. So we multiply by 4! to pair them off, and we get  $\binom{6}{4} \cdot \binom{7}{4} \cdot 4!$ . You may have counted it differently, but check if your answer matches!  $\square$

### 1.2.3 Multinomial Coefficients

Now we'll see if we can generalize our binomial coefficients to solve even more interesting problems. Actually, they can be derived easily from binomial coefficients.

How many ways can you arrange the letters in "MATH"?

$4! = 24$ , since they are distinct objects.

But if we want to rearrange the letters in "POOPOO", we have indistinct letters (two types - P and O). How do we approach this?

One approach is to choose where the 2 P's go, and then the O's have to go in the remaining 4 spots ( $\binom{4}{2} = 1$  way). Or, we can choose where the 4 O's go, and then the remaining P's are set ( $\binom{2}{2} = 1$  way).

Either way, we get,

$$\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$$

Another interpretation of this formula is that we are first arranging the 6 letters as if they were distinct:  $P_1O_1O_2P_2O_3O_4$ . Then, we divide by 4! and 2! to account for 4 duplicate O's and 2 duplicate P's.

What if we got even more complex, let's say three different letters? For example, rearranging the word 'BABYYYBAY'. There are 3 B's, 2 A's, and 4 Y's, for a total of 9 letters. We can choose where the 3 B's should go of the 9 spots:  $\binom{9}{3}$  (order doesn't matter since all the B's are identical). Then out of the remaining 6 spots, we should choose 2 for the A's:  $\binom{6}{2}$ . Finally, out of the 4 remaining spots, we put the 4 Y's there:  $\binom{4}{4} = 1$ . By the product rule, our answer is

$$\binom{9}{3} \binom{6}{2} \binom{4}{4} = \frac{9!}{3!6!} \frac{6!}{2!4!} \frac{4!}{4!0!} = \frac{9!}{3!2!4!}$$

Note that we could have chosen to assign the Y's first instead: Out of 9 positions, we choose 4 to be Y:  $\binom{9}{4}$ . Then from the 5 remaining spots, choose where the 2 A's go:  $\binom{5}{2}$ , and the last three spots must be B's:  $\binom{3}{3} = 1$ . This gives us the equivalent answer

$$\binom{9}{4} \binom{5}{2} \binom{3}{3} = \frac{9!}{4!5!} \frac{5!}{2!3!} \frac{3!}{3!0!} = \frac{9!}{3!2!4!}$$

This shows once again that there are many correct ways to count something. This type of problem also frequently appears, and so we have a special notation (called a **multinomial coefficient**)

$$\binom{9}{3, 2, 4} = \frac{9!}{3!2!4!}$$

Note the order of the bottom three numbers does not matter (since the multiplication in the denominator is commutative), and that the bottom numbers must add up to the top number.

#### Definition 1.2.3: Multinomial Coefficients

If we have  $k$  types of objects ( $n$  total), with  $n_1$  of the first type,  $n_2$  of the second, ..., and  $n_k$  of the  $k$ -th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!n_2!\dots n_k!}$$

This is a **multinomial coefficient**, the generalization of binomial coefficients.

Above, we had  $k = 3$  objects (B, A, Y) with  $n_1 = 3$  (number of B's),  $n_2 = 2$  (number of A's), and  $n_3 = 4$  (number of Y's), for an answer of  $\binom{n}{n_1, n_2, n_3} = \frac{9!}{3!2!4!}$ .

#### Example(s)

How many ways can you arrange the letters in "GODOGGY"?

*Solution* There are  $n = 7$  letters. There are only  $k = 4$  distinct letters -  $\{G, O, D, Y\}$ .

$n_1 = 3$  - there are 3 G's.

$n_2 = 2$  - there are 2 O's.

$n_3 = 1$  - there is 1 D.

$n_4 = 1$  - there is 1 Y.

This gives us the number of possible arrangements:

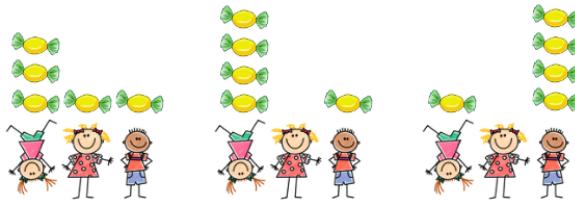
$$\binom{7}{3, 2, 1, 1} = \frac{7!}{3!2!1!1!}$$

It is important to note that even though the 1's are "useless" since  $1! = 1$ , we still must write every number on the bottom since they have to add to the top number.  $\square$

#### 1.2.4 Stars and Bars/Divider Method

Now we tackle another common type of problem, which seems complicated at first. It turns out though that it can be reduced to binomial coefficients!

How many ways can we give 5 (indistinguishable) candies to these 3 (distinguishable) kids? Here are three possible distributions of candy:

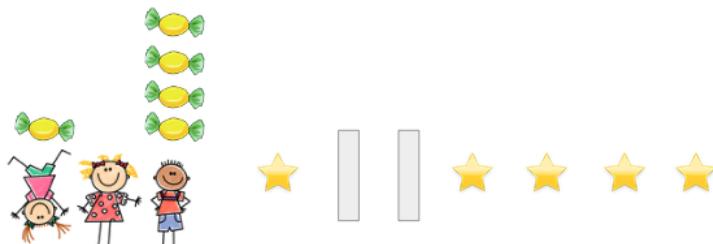


Notice that the second and third pictures show different possible distributions, since the kids are distinguishable (different). Any idea on how we can tackle this problem?

The idea here is that we will count something equivalent. Let's say there are 5 "stars" for the 5 candies and 2 "bars" for the dividers (dividing 3 kids). For instance, this distribution of candies corresponds to this arrangement of 5 stars and 2 bars:



Here is another example of the correspondence between a distribution of candies and the arrangement of stars and bars:



For each candy distribution, there is exactly one corresponding way to arrange the stars and bars. Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Hence, the number of ways to distribute 5 candies to the 3 kids is the number of arrangements of 5 stars and 2 bars.

This is simply

$$\binom{7}{2} = \binom{7}{5} = \frac{7!}{2!5!}$$

Amazing right? We just reduced this candy distribution problem to reordering letters!

**Definition 1.2.4: Stars and Bars/Divider Method**

The number of ways to distribute  $n$  indistinguishable balls into  $k$  distinguishable bins is

$$\binom{n + (k - 1)}{k - 1} = \binom{n + (k - 1)}{n}$$

since we set up  $n$  stars for the  $n$  balls, and  $k - 1$  bars dividing the  $k$  bins.

**Example(s)**

How many ways can we assign 20 students to 4 different professors? Assume the students are indistinguishable to the professors; who only care *how many* students they have, and not which ones.

*Solution* This is actually the perfect setup for stars and bars. We have 20 stars (students) and 3 bars (professors), and so our answer is  $\binom{23}{3} = \binom{23}{20}$ .  $\square$

**1.2.5 Exercises**

1. There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

**Solution:** Again, there may be many correct approaches. We can first choose which 5 out of 10 seats in the front row we want to give, so we have  $\binom{10}{5}$  ways of doing this. Then, assign those 5 students to these seats, to which there are  $5!$  ways. Finally, assign the other 35 students in any way, for  $35!$  ways. By the product rule, there are  $\binom{10}{5} \cdot 5! \cdot 35!$  ways to do so.

2. If we roll a fair 3-sided die 11 times, what is the number of ways that we can get 4 1's, 5 2's, and 2 3's?

**Solution:** We can write the outcomes as a sequence of length 11, each digit of which is 1, 2 or 3. Hence, the number of ways to get 4 1's, 5 2's, and 2 3's, is the number of orderings of 11112222233, which is  $\binom{11}{4,5,2} = \frac{11!}{4!5!2!}$ .

3. These two problems are almost identical, but have drastically different approaches to them. These are both extremely hard/tricky problems, though they may look deceptively simple. These are probably the two coolest problems I've encountered in counting, as they do have elegant solutions!
  - (a) How many 7-digit phone numbers are such that the numbers are strictly increasing (digits must go up)? (e.g., 014-5689, 134-6789, etc.)
  - (b) How many 7-digit phone numbers are such that the numbers are monotone increasing (digits can stay the same or go up)? (e.g., 011-5566, 134-6789, etc.) Hint: Reduce this to stars and bars.

**Solution:**

- (a) We choose 7 out of 10 digits, which has  $\binom{10}{7}$  possibilities, and then once we do, there is only 1 valid ordering (must put them in increasing order). Hence, the answer is simply  $\binom{10}{7}$ . This question has a deceptively simple solution, as many students (including myself at one point), would have

started by choosing the first digit. But the choices for the next digit depend on the first digit. And so on for the third. This leads to a complicated, nearly unsolvable mess!

- (b) This is a very difficult problem to frame in terms of stars and bars. We need to map one phone number to exactly one ordering of stars and bars, and vice versa. Consider letting the 9 bars being an increase from one-digit to the next, and 7 stars for the 7 digits. This is extremely complicated, so we'll give 3 examples of what we mean.
- i. The phone number 011-5566 is represented as  $*| * *| ||| * *| * *| |$ . We start a counter at 0, we see a digit first (a star), so we mark down 0. Then we see a bar, which tells us to increase our counter to 1. Then, two more digits (stars), which say to mark down 2 1's. Then, 4 bars which tell us to increase count from 1 to 5. Then two \*'s for the next two 5's, and a bar to increase to 6. Then, two stars indicate to put down 2 6's. Then, we increment count to 9 but don't put down any more digits.
  - ii. The phone number 134-6789 is represented as  $| * || * | * || * | * | * | *$ . We start a counter at 0, and we see a bar first, so we increase count to 1. Then a star tells us to actually write down 1 as our first digit. The two bars tell us to increase count from 1 to 3. The star says mark a 3 down now. Then, a bar to increase to 4. Then a star to write down 4. Two bars to increase to 6. And so on.
  - iii. The stars and bars ordering  $||| * | * * * | | * || *$  represents the phone number 455-5579. We start a counter at 0. We see 4 bars so we increment to 4. The star says to mark down a 4. Then increment count by 1 to 5 due to the next bar. Then, mark 5 down 4 times (4 stars). Then increment count by 2, put down a 7, and repeat to put down a 9.

Hence there is a bijection between these phone numbers and arrangements of 7 stars and 9 bars. So the number of satisfying phone numbers is  $\binom{16}{7} = \binom{16}{9}$ .

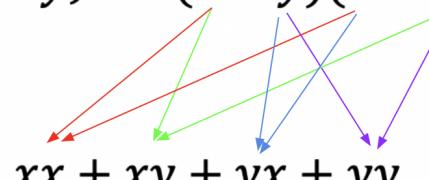
## Chapter 1: Combinatorial Theory

### 1.3: No More Counting Please

In this section, we don't really have a nice successive ordering where one topic leads to the next as we did earlier. This section serves as a place to put all the final miscellaneous but useful concepts in counting.

#### 1.3.1 Binomial Theorem

We talked last time about binomial coefficients of the form  $\binom{n}{k}$ . Today, we'll see how they are used to prove the binomial theorem, which we'll use more later on. For now, we'll see how they can be used to expand (possibly large) exponents below. You may have learned this technique of FOIL (first, outer, inner, last) for expanding  $(x + y)^2$ .

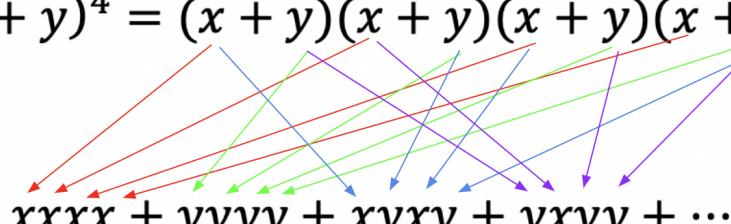
$$(x + y)^2 = (x + y)(x + y) \quad \text{FOIL}$$


xx + xy + yx + yy

We then combine like-terms ( $xy$  and  $yx$ ).

$$\begin{aligned} (x + y)^2 &= (x + y)(x + y) \\ &= xx + xy + yx + yy \\ &= x^2 + 2xy + y^2 \end{aligned} \quad [\text{FOIL}]$$

But, let's say that we wanted to do this for a binomial raised to some higher power, say  $(x + y)^4$ . There would be a lot more terms, but we could use a similar approach.

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$$


xxxx + yyyy + xyxy + yxxy + ...

$$\begin{aligned}
 (x+y)^4 &= (x+y)(x+y)(x+y)(x+y) \\
 &= xxxx + yyyy + xyxy + yxxy + \dots
 \end{aligned}$$

But what are the terms exactly that are included in this expression? And how could we combine the like-terms though?

Notice that each term will be a mixture of  $x$ 's and  $y$ 's. In fact, each term will be in the form  $x^k y^{n-k}$  (in this case  $n = 4$ ). This is because there will be exactly  $n$   $x$ 's or  $y$ 's in each term, so if there are  $k$   $x$ 's, then there must be  $n - k$   $y$ 's. That is, we will have terms of the form  $x^4, x^3y, x^2y^2, xy^3, y^4$ , with most appearing more than once.

For a specific  $k$  though, how many times does  $x^k y^{n-k}$  appear? For example, in the above case, take  $k = 1$ , then note that  $yyyy = yxyy = yyxy = yyyx = xy^3$ , so  $xy^3$  will appear with the coefficient of 4 in the final simplified form (just like for  $(x+y)^2$  the term  $xy$  appears with a coefficient 2). Does this look familiar? It should remind you yet again of rearranging words with duplicate letters!

Now, we can generalize this, as the number of terms will simplify to  $x^k y^{n-k}$  will be equivalent to the number of ways to choose exactly  $k$  of the binomials to give us  $x$  (and let the remaining  $n-k$  give us  $y$ ). Alternatively, we need to arrange  $k$   $x$ 's and  $n-k$   $y$ 's. To think of this in the above example with  $k = 1$  and  $n = 4$ , we were consider which of the four binomials would give us the single  $x$ , the first, second, third, or fourth, for a total of  $\binom{4}{1} = 4$ .

Let's consider  $k = 2$  in the above example. We want to know how many terms are equivalent to  $x^2y^2$ . Well, we then have  $xxyy = yxxy = yyxx = xyxy = yxyx = xyyx = x^2y^2$ , so there are six ways and the coefficient on the simplified term  $x^2y^2$  will be  $\binom{4}{2} = 6$ .

Notice that we are essentially choosing which of the binomials gives us an  $x$  such that  $k$  of the  $n$  binomials do. That is, the coefficient for  $x^k y^{n-k}$  where  $k$  ranges from 0 to  $n$  is simply  $\binom{n}{k}$ . This is why it was also called a binomial coefficient.

That leads us to the binomial theorem:

**Theorem 1.3.1: Binomial Theorem**

Let  $x, y \in \mathbb{R}$  be real numbers and  $n \in \mathbb{N}$  a positive integer. Then:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

This essentially states that in the expansion of the left side, the coefficient of the term with  $x$  raised to the power of  $k$  and  $y$  raised to the power of  $n-k$  will be  $\binom{n}{k}$ , and we know this because we are considering the number of ways to choose  $k$  of the  $n$  binomials in the expression to give us  $x$ .

This can also be proved by induction, but this is left as an exercise for the reader.

**Example(s)**

Calculate the coefficient of  $a^{45}b^{14}$  in the expansion  $(4a^3 - 5b^2)^{22}$ .

*Solution* Let  $x = 4a^3$  and  $y = -5b^2$ . Then, we are looking for the coefficient of  $x^{15}y^7$  (because  $x^{15}$  gives us

$a^{45}$  and  $y^7$  gives us  $b^{14}$ ), which is  $\binom{22}{15}$ . So we have the term

$$\binom{22}{15} x^{15} y^7 = \binom{22}{15} (4a^3)^{15} (-5b^2)^7 = \left( -\binom{22}{15} 4^{15} 5^7 \right) a^{45} b^{14}$$

and our answer is  $-\binom{22}{15} 4^{15} 5^7$ . □

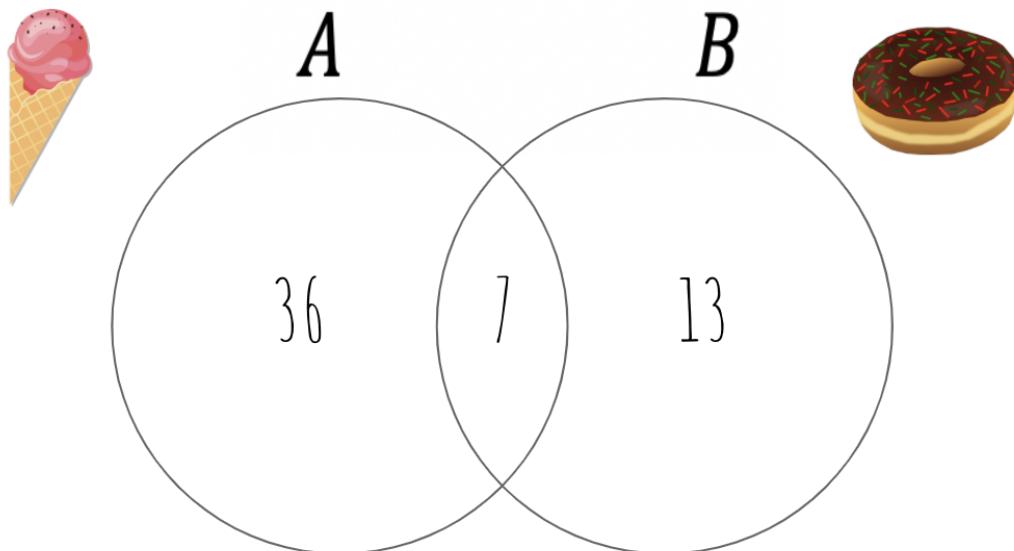
### 1.3.2 Inclusion-Exclusion

Say we did an anonymous survey where we asked whether students in CSE312 like ice cream, and found that 43 people liked ice cream. Then we did another anonymous survey where we asked whether students in CSE312 liked donuts, and found that 20 people liked donuts. With this information can we determine how many people like ice cream or donuts (or both)?

Let  $A$  be the set of people who like ice cream, and  $B$  the set of people who like donuts. The sum rule from 1.1 said that, if  $A, B$  were mutually exclusive (it wasn't possible to like both donuts and ice cream:  $A \cap B = \emptyset$ ), then we could just add them up:  $|A \cup B| = |A| + |B| = 43 + 20 = 63$ . But this is not the case, since it is possible that to like both. We can't quite figure this out yet without knowing how many people overlapped: the size of  $A \cap B$ .

So, we did another anonymous survey in which we asked whether students in CSE312 like both ice cream and donuts, and found that only 7 people like both. Now, do we have enough information to determine how many students like ice cream or donuts?

Yes! Knowing that 43 people like ice cream and 7 people like both ice cream and donuts, we can conclude that 36 people like ice cream but don't like donuts. Similarly, knowing that 20 people like donuts and 7 people like both ice cream and donuts, we can conclude that 13 people like donuts but don't like ice cream. This leaves us with the following picture, where  $A$  is the students who like ice cream.  $B$  is the students who like donuts (this implies  $|A \cap B| = 7$  is the number of students who like both):



So we have the following:

$$\begin{aligned}|A| &= 43 \\ |B| &= 20 \\ |A \cap B| &= 7\end{aligned}$$

Now, to go back to the question of how many students like either ice cream or donuts, we can just add up the 36 people that just like ice cream, the 7 people that like both ice cream and donuts, and the 13 people that just like donuts, and get  $36 + 7 + 13 = 56$ . Alternatively, we could consider this as adding up the 43 people who like ice cream (including both the 36 those who just like ice cream and the 7 who like both) and the 20 people who like donuts (including the 13 who just like donuts and the 7 who like both) and then subtracting the 7 who like both since they were counted twice. That is  $43 + 20 - 7 = 56$ . That leaves us with:

$$|A \cup B| = 36 + 7 + 13 = 56 = 43 + 20 - 7 = |A| + |B| - |A \cap B|$$

Recall that  $|A \cup B|$  is the students who like donuts or ice cream (the union of the two sets).

Theorem 1.3.2: Inclusion-Exclusion

Let  $A, B$  be sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Further, in general, if  $A_1, A_2, \dots, A_n$  are sets, then:

$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \dots \\ &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n|) + \dots\end{aligned}$$

where singles are the sizes of all the single sets ( $\binom{n}{1}$  terms), doubles are the sizes of all the intersections of two sets ( $\binom{n}{2}$  terms), triples are the size of all the intersections of three sets ( $\binom{n}{3}$  terms), quads are all the intersection of four sets, and so forth.

Example(s)

How many numbers in the set  $[360] = \{1, 2, \dots, 360\}$  are divisible by:

1. 4, 6, and 9.
2. 4, 6 or 9.
3. neither 4, 6, nor 9.

*Solution*

1. This is just the multiples of  $\text{lcm}(4, 6, 9) = 36$ , which there are  $\frac{360}{36} = 10$  of.
2. Let  $D_i$  be the number of numbers in  $[360]$  which are divisible by  $i$ , for  $i = 4, 6, 9$ . Hence, the number of numbers which are divisible by 4, 6, or 9 is  $|D_4 \cup D_6 \cup D_9|$ . We can apply inclusion-exclusion (singles

minus doubles plus triples):

$$\begin{aligned}|D_4 \cup D_6 \cup D_9| &= |D_4| + |D_6| + |D_9| - |D_4 \cap D_6| - |D_4 \cap D_9| - |D_6 \cap D_9| + |D_4 \cap D_6 \cap D_9| \\ &= \frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{12} - \frac{360}{36} - \frac{360}{18} + \frac{360}{36}\end{aligned}$$

Notice the denominators for the paired terms are again, dividing by the least common multiple.

3. Complementary counting - this is just 360 minus the answer from the previous part!

□

Many times it may be possible to avoid this ugly mess using complementary counting, but sometimes it isn't.

### 1.3.3 Pigeonhole Principle

The Pigeonhole Principle is a tool that allows us to make guarantees when we tackle problems like: if we want to assign 20 third-grade students to 3 (equivalent) classes, how can we minimize the largest class size? It turns out we can't do any better than having 7 people in the largest class. The reason is because of the pigeonhole principle!

We'll start with a smaller but similar problem. If 11 children have to share 3 beds, how can we minimize the number of children on the most crowded bed? The idea might be just to spread them "uniformly". Maybe number the beds A,B,C, and assign the first child to A, the second to B, the third to C, the fourth to A, and so on. This turns out to be optimal as it spreads the kids out as evenly as possible. The pigeonhole principle tells us the best worst-case scenario: that at least one bed must have at least 4 children.

You might first distribute the children evenly amongst the beds, say put 3 children in each bed to start. That leaves us with 3 times 3 equals 9 children accounted for, and 2 children remaining with a bed. Well, they must be put to bed, so we can put each of them in a separate bed and we finish with the first bed having 4, the second bed having 4, and the third bed having 3. No matter how we move the children around, we can't have an arrangement where at least one bed will contain at least 4 children.

We could also have found this by dividing 11 by 3 and rounding up to account for the remainder (which must go somewhere). Before formally defining the pigeonhole principle, we need to define the floor and ceiling functions.

#### Definition 1.3.1: Floor and Ceiling Functions

The **floor** function  $\lfloor x \rfloor$  returns the largest integer  $\leq x$  (i.e. rounds down).

The **ceiling** function  $\lceil x \rceil$  returns the smallest integer  $\geq x$  (i.e. rounds up). Note the difference is just whether the bracket is on top (ceiling) or bottom (floor).

#### Example(s)

Solve the following:  $\lfloor 2.5 \rfloor$ ,  $\lfloor 16.999999 \rfloor$ ,  $\lceil 5 \rceil$ ,  $\lceil 2.5 \rceil$ ,  $\lceil 9.000301 \rceil$ ,  $\lceil 5 \rceil$ .

#### Solution

$$\lfloor 2.5 \rfloor = 2$$

$$\lceil 2.5 \rceil = 3$$

$$\lfloor 16.999999 \rfloor = 16$$

$$\lceil 9.000301 \rceil = 10$$

$$\lceil 5 \rceil = 5$$

$$\lceil 5 \rceil = 5$$

□

### Theorem 1.3.3: Pigeonhole Principle

If there are  $n$  pigeons we want to put into  $k$  holes (where  $n > k$ ), then at least one pigeonhole must contain at least 2 pigeons.

More generally, if there are  $n$  pigeons we want to put into  $k$  pigeonholes, then at least one pigeonhole must contain at least  $\lceil n/k \rceil$  pigeons.

This fact or rule may seem trivial to you, but the hard part of pigeonhole problems is knowing how to apply it. See the examples below!

#### Example(s)

Show that there exists a number made up of only 1's (e.g., 1111 or 11) which is divisible by 333.

*Solution* Consider the sequence of 334 numbers  $x_1, x_2, x_3, \dots, x_{334}$  where  $x_i$  is the number made of exactly  $i$  1's (e.g.,  $x_2 = 11$ ,  $x_5 = 11, 111$ , etc.). We'll use the notation  $x_i = 1^i$  to mean  $i$  1's concatenated together.

The number of possible remainders when dividing by 333 is 333:  $\{0, 1, 2, \dots, 332\}$ , so by the pigeonhole principle, since  $334 > 333$ , two numbers  $x_i$  and  $x_j$  have the same remainder (suppose  $i < j$  without loss of generality) when divided by 333. The number  $x_j - x_i$  is of the form  $1^{j-i}0^i$ ; that is  $j - i$  1's followed by  $i$  0's (e.g.,  $x_5 - x_2 = 11111 - 11 = 11100 = 1^30^2$ ). This number must be divisible by 333 because  $x_i \equiv x_j \pmod{333} \Rightarrow (x_j - x_i) \equiv 0 \pmod{333}$ .

Now, keep deleting zeros (by dividing by 10) until there aren't any more left - this doesn't affect whether or not 333 goes in since neither 2 nor 5 divides 333. Now we're left with a number divisible by 333 made up of all ones ( $1^{j-i}$  to be exact)!

Note that 333 was not special - we could have used any number that wasn't divisible by 2 nor 5. □

#### Example(s)

Show that in a group of  $n$  people (who may be friends with any number of other people), two must have the same number of friends.

*Solution* We have two cases.

1. Case 1: Everyone has at least one friend. Then, everyone has a number of friends between  $1, 2, \dots, n-1$ . By the pigeonhole principle, since there are  $n$  people and  $n-1$  possibilities, at least two people have the same number of friends.
2. Case 2: At least one person has no friends. Let's take one such person and call them A. Then, the other  $n-1$  people can have number of friends from  $0, \dots, n-2$  since they can't be friends with A. We have two more cases within this case unfortunately.
  - (a) Case 2a: If one of these  $n-1$  people has no friends, we are done since A and this person both have 0 friends.
  - (b) Case 2b: Otherwise, these people all have at least one friend, from  $1, \dots, n-2$ , and since there are  $n-1$  people and  $n-2$  possibilities, at least two people have the same number of friends.

In all cases we are guaranteed that two people have the same number of friends.

□

### 1.3.4 Combinatorial Proofs

You may have taken a discrete mathematics/formal logic class before this if you are a computer science major. If that's the case, you would have learned how to write proofs (e.g., induction, contradiction). Now that we know how to count, we can actually prove some algebraic identities using counting instead!

Suppose we wanted to show that  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  was true for any positive integer  $n \in \mathbb{N}$  and  $0 \leq k \leq n$ .

We could start with an algebraic approach and try something like:

$$\begin{aligned}
 \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)}{(k-1)!(n-k)!} + \frac{(n-1)}{k!(n-1-k)!} && [\text{def of binomial coef}] \\
 &\dots && [\text{lots of algebra}] \\
 &= \frac{n!}{k!(n-k)!} \\
 &= \binom{n}{k}
 \end{aligned}$$

However, those  $\dots$  may be tedious and take a lot of algebra we don't want to do.

So, let's consider another approach. A combinatorial proof is one where you prove two quantities are equal by imagining a situation/something to count. Then, you argue that the left side and right side are two equivalent ways to count the same thing, and hence must be equal. We've seen earlier often how there are multiple approaches to counting!

In this case, let's consider the set of numbers  $[n] = \{1, 2, \dots, n\}$ . We will argue that the LHS and RHS both count the number of subsets of size  $k$ .

1. LHS:  $\binom{n}{k}$  is literally the number of subsets of size  $k$ , since we just want to choose any  $k$  items out of  $n$  (order doesn't matter).
2. RHS: We take a slightly more convoluted approach, splitting on cases depending on whether or not the number 1 was included in the subset.

**Case 1: Our subset of size  $k$  includes the number 1.** Then we need to choose  $k-1$  of the remaining  $n-1$  numbers ( $n$  numbers excluding 1 is  $n-1$  numbers) to make a subset of size  $k$  which includes 1.

**Case 2: Our subset of size  $k$  does not include the number 1.** Then we need to choose  $k$  numbers from the remaining  $n-1$  numbers. There are  $\binom{n-1}{k}$  ways to do this. So, in total we have  $\binom{n-1}{k-1} + \binom{n-1}{k}$  possible subsets of size  $k$ .

Since the left side and right side count the same thing, they must be equal! Note that we dreamed up this situation and you may wonder how we did - this just comes from practicing many types of counting problems. You'll get used to it!

**Definition 1.3.2: Combinatorial Proofs**

To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

**Example(s)**

Prove the following two identities combinatorially (NOT algebraically):

1. Prove that  $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ .
2. Prove that  $2^n = \sum_{k=0}^n \binom{n}{k}$

**Solution**

1. We'll show that both sides count, from a group of  $n$  people, the number of committees of size  $m$ , and within that committee a subcommittee of size  $k$ .

**Left-hand side:** We first choose  $m$  people to be on the committee from  $n$  total; there are  $\binom{n}{m}$  ways to do so. Then, within those  $m$ , we choose  $k$  to be on a specialized subcommittee; there are  $\binom{m}{k}$  ways to do so. By the product rule, the number of ways to assign these is  $\binom{n}{m} \binom{m}{k}$ .

**Right-hand side:** We first choose which  $k$  to be on the subcommittee of size  $k$ ; there are  $\binom{n}{k}$  ways to do so. From the remaining  $n - k$  people, we choose  $m - k$  to be on the committee (but not the subcommittee). By the product rule, the number of ways to assign these is  $\binom{n}{k} \binom{n-k}{m-k}$ .

Since the LHS and RHS both count the same thing, they must be equal.

2. We'll argue that both sides count the number of subsets of the set  $[n] = \{1, 2, \dots, n\}$ .

**Left-hand side:** Each element we can have in our subset or not. For the first element, we have 2 choices (in or out). For the second element, we also have 2 choices (in or out). And so on. So the number of subsets is  $2^n$ .

**Right-hand side:** The subset can be of any size ranging from 0 to  $n$ , so we have a sum. Now how many subsets are there of size exactly  $k$ ? There are  $\binom{n}{k}$  because we choose  $k$  out of  $n$  to have in our set (and order doesn't matter in sets)! Hence, the number of subsets is  $\sum_{k=0}^n \binom{n}{k}$

Since the LHS and RHS both count the same thing, they must be equal.

It's cool to note we can also prove this with the binomial theorem setting  $x = 1$  and  $y = 1$  - try this out! It takes just one line!

□

### 1.3.5 Exercises

1. These problems involve using the pigeonhole principle. How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:

- (a) A single pair? (e.g., AA, 99, JJ)
- (b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)
- (c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)
- (d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?
- (e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)
- (f) A straight flush (5 cards which are both a straight and a flush)?

**Solution:**

- (a) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is 14.
- (b) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is 17.
- (c) The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is 27.
- (d) The worst that could happen is to draw all the A - 4, 6 - 9, and J - K. After drawing these  $11 \cdot 4 = 44$  cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is 45.
- (e) The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is 17.
- (f) Same as straight, 45.

## Chapter 2: Discrete Probability

### 2.1: Intro to Discrete Probability

We're just about to learn about the axioms (rules) of probability, and see how all that counting stuff from chapter 1 was relevant at all. This should align with your current understanding of probability (I only assume you might be able to tell me the probability I roll an even number on a fair six-sided die at this point), and formalize it.

We'll be using a lot of set theory from here on out, so review that in Chapter 0 if you need to!

#### 2.1.1 Definitions

##### Definition 2.1.1: Sample Space

The **sample space** is the set  $\Omega$  of all possible outcomes of an experiment.

##### Example(s)

Find the sample space of...

1. a single coin flip.
2. two coin flips.
3. the roll of a fair 6-sided die.

##### *Solution*

1. The sample space of a single coin flip is:  $\Omega = \{H, T\}$  (heads or tails).
2. The sample space of two coin flips is:  $\Omega = \{HH, HT, TH, TT\}$ .
3. The sample space of the roll of a die is:  $\Omega = \{1, 2, 3, 4, 5, 6\}$ .

□

##### Definition 2.1.2: Event

An **event** is any subset  $E \subseteq \Omega$ .

##### Example(s)

List out the set of outcomes for the following events:

1. Getting at least one head in two coin flips.
2. Rolling an even number on a fair 6-sided die.

*Solution*

1. Getting at least one head in two coin flips:  $E = \{HH, HT, TH\}$
2. Rolling an even number:  $E = \{2, 4, 6\}$

□

### Definition 2.1.3: Mutual Exclusion

Events  $E$  and  $F$  are mutually exclusive if  $E \cap F = \emptyset$ . (i.e. they can't simultaneously happen).

### Example(s)

Say  $E$  is the event of rolling an even number:  $E = \{2, 4, 6\}$ , and  $F$  is the event of rolling an odd number:  $F = \{1, 3, 5\}$ . Are  $E$  and  $F$  mutually exclusive?

*Solution*  $E$  and  $F$  are mutually exclusive because  $E \cap F = \emptyset$ .

□

### Example(s)

Let's consider another example in which our experiment is the rolling of two fair 4-sided dice, one which is blue  $D1$  and one which is red  $D2$  (so they are distinguishable, or effectively, order matters). We can represent each element in the sample set as an ordered pair  $(D1, D2)$  where  $D1, D2 \in \{1, 2, 3, 4\}$  and represent the respective value rolled by the blue and red die.

The sample space  $\Omega$  is the set of all possible ordered pairs of values that could be rolled by the die ( $|\Omega| = 4 \cdot 4 = 16$  by the product rule). Let's consider some events:

1.  $A = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ , the event that the blue die,  $D1$  is a 1.
2.  $B = \{(2, 4), (3, 3), (4, 2)\}$ , the event that the sum of the two rolls is 6 ( $D1 + D2 = 6$ ).
3.  $C = \{(2, 1), (4, 2)\}$ , the event that the value on the blue die is twice the value on the red die ( $D1 = 2 * D2$ ).

All of these events and the sample space are shown below:

		DIE 2 (RED)			
		1	2	3	4
DIE 1 (BLUE)	1	(1, 1) <sup>A</sup>	(1, 2) <sup>A</sup>	(1, 3) <sup>A</sup>	(1, 4) <sup>A</sup>
	2	(2, 1) <sup>B</sup>	(2, 2)	(2, 3)	(2, 4) <sup>B</sup>
	3	(3, 1)	(3, 2)	(3, 3) <sup>B</sup>	(3, 4)
	4	(4, 1)	(4, 2) <sup>B</sup>	(4, 3)	(4, 4)

Are  $A$  and  $B$  mutually exclusive? Are  $B$  and  $C$  mutually exclusive?

*Solution* Now, let's consider whether  $A$  and  $B$  are mutually exclusive. Well, they do not overlap, as we can see that  $A \cap B = \emptyset$ , so yes they are mutually exclusive.

$B$  and  $C$  are not mutually exclusive, since there is a case in which they can happen at the same time  $B \cap C = \{(4, 2)\} \neq \emptyset$ , so they are not mutually exclusive.  $\square$

Again, to summarize, we learned that  $\Omega$  was the sample space (set of all outcomes of an experiment), and  $E \subseteq \Omega$  is just a subset of outcomes we are interested in.

## 2.1.2 Axioms of Probability and their Consequences

### Definition 2.1.4: Axioms of Probability and their Consequences

Let  $\Omega$  denote the sample space and  $E, F \subseteq \Omega$  be events.

#### Axioms:

1. (Nonnegativity)  $\mathbb{P}(E) \geq 0$ ; that is, no event has a negative probability.
2. (Normalization)  $\mathbb{P}(\Omega) = 1$ ; that is, the probability of the entire sample space is always 1 (something is guaranteed to happen)
3. (Countable Additivity) If  $E$  and  $F$  are *mutually exclusive*, then  $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ . This actually holds for any countable (finite or countably infinite) collection of pairwise mutually exclusive events  $E_1, E_2, E_3, \dots$ .

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

The word “axiom” means: things that we take for granted and assume to be true **without proof**.

#### Corollaries:

1. (Complementation)  $\mathbb{P}(E^C) = 1 - \mathbb{P}(E)$ .
2. (Monotonicity) If  $E \subseteq F$ , then  $\mathbb{P}(E) \leq \mathbb{P}(F)$ .
3. (Inclusion-Exclusion)  $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ .

The word “corollary” means: results that follow almost immediately from a previous result (in this case, the axioms).

### Explanation of Axioms

1. Non-negativity is simply because we cannot consider an event to have a negative probability. It just wouldn't make sense. A probability of  $1/6$  would mean that on average, something would happen 1 out of every 6 trials. What about a probability of  $-1/4$ ?
2. Normalization is based on the fact that when we run an experiment, there must be *some* outcome, and all possible outcomes are in the sample space. So, we say the probability of observing some outcome

from the sample space is 1.

3. Countable additivity is because if two events are mutually exclusive, they don't overlap at all; that is, they don't share any outcomes. This means that the union of them will contain the same outcomes of each together, so the probability of their union is the the sum of their individual probabilities. (This is like the sum role of counting).

### Explanation of Corollaries

1. Complementation is based on the fact that the sample space is all the possible outcomes. This means that  $E^C = \Omega \setminus E$ , so  $\mathbb{P}(E^C) = 1 - \mathbb{P}(E)$ . (This is like complementary counting).
2. Monotonocity is because if  $E$  is a subset of  $F$ , then all outcomes in the event  $E$  are in the event  $F$ . This means that all the outcomes that contribute to the probability of  $E$  contribute to the probability of  $F$ , so it's probability is greater than or equal to that of  $E$  (since probabilities are non-negative).
3. Inclusion-Exclusion follows because if  $E$  and  $F$  have some intersection, this would be counted twice by adding their probabilities, so we have to subtract it once to only count it once and not overcount. (This is like inclusion-exclusion for counting).

*Proof of Corollaries.* The proofs of these corollaries only depend on the 3 axioms which we assume to be true.

1. Since  $E$  and  $E^C = \Omega \setminus E$  are mutually exclusive,

$$\begin{aligned} \mathbb{P}(E) + \mathbb{P}(E^C) &= \mathbb{P}(E \cup E^C) && \text{[axiom 3]} \\ &= \mathbb{P}(\Omega) && \text{[}E \cup E^C = \Omega\text{]} \\ &= 1 && \text{[axiom 2]} \end{aligned}$$

Now just subtract  $\mathbb{P}(E)$  from both sides.

2. Since  $E \subseteq F$ , consider the sets  $E$  and  $F \setminus E$ . Then,

$$\begin{aligned} \mathbb{P}(F) &= \mathbb{P}(E \cup (F \setminus E)) && \text{[draw a picture of } E \text{ inside event } F\text{]} \\ &= \mathbb{P}(E) + \mathbb{P}(F \setminus E) && \text{[mutually exclusive, axiom 3]} \\ &\geq \mathbb{P}(E) + 0 && \text{[since } \mathbb{P}(F \setminus E) \geq 0 \text{ by axiom 1]} \end{aligned}$$

3. Left to the reader. Hint: Draw a picture.

□

### 2.1.3 Equally Likely Outcomes

Now we'll see why counting was so useful. We can compute probabilities in the special case where each outcome is equally likely (e.g., rolling a *fair* 6-sided die has each outcome in  $\Omega = \{1, 2, \dots, 6\}$  equally likely). If events are equally likely, then in determining probabilities, we only care about the number of outcomes that are in an event. That let's us conclude the following:

Theorem 2.1.4: Probability in Sample Space with Equally Likely Outcomes

If  $\Omega$  is a sample space such that each of the unique outcome elements in  $\Omega$  **are equally likely**, then for any event  $E \subseteq \Omega$ :

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

*Proof of Equally Likely Outcomes Formula.* If outcomes are equally likely, then for any outcome in the sample space  $\omega \in \Omega$ , we have  $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$  (since there are  $|\Omega|$  total outcomes). Then, if we list the  $|E|$  outcomes that make up event  $E$ , we can write

$$E = \{\omega_1, \omega_2, \dots, \omega_{|E|}\}$$

Every set is the union of the (mutually exclusive) singleton sets containing each element (e.g.,  $\{1, 2, 3\} = \{1\} \cup \{2\} \cup \{3\}$ ), and so by countable additivity, we get

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i=1}^{|E|} \{\omega_i\}\right) &= \sum_{i=1}^{|E|} \mathbb{P}(\{\omega_i\}) && [\text{countable additivity axiom}] \\ &= \sum_{i=1}^{|E|} \frac{1}{|\Omega|} && [\text{equally likely outcomes}] \\ &= \frac{|E|}{|\Omega|} && [\text{sum constant } |E| \text{ times}] \end{aligned}$$

The notation in the first line is like summation or product notation: just union all the sets  $\{\omega_1\} \cup \{\omega_2\} \cup \dots \cup \{\omega_{|E|}\}$ .  $\square$

Example(s)

If we flip two fair coins independently, what is the probability we get at least one head?

*Solution* Since the sample space  $\Omega = \{HH, HT, TH, TT\}$  is such that events are equally likely and the event of getting at least one head is  $E = \{HH, HT, TH\}$ , we can say that

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{3}{4}$$

$\square$

Example(s)

Consider the example of rolling the red and blue fair 4-sided dice again (above), a blue die  $D1$  and a red die  $D2$ . What is the probability that the two die's rolls sum up to 6?

*Solution* We called that event  $B = \{(2, 4), (3, 3), (4, 2)\}$ . What is the probability of the event  $B$  happening?

Well, the 16 possible outcomes that make up all the elements of  $\Omega$  are each equally likely because each die has an equal chance of landing on any of the 4 numbers. So,  $\mathbb{P}(E) = \frac{|B|}{|\Omega|} = \frac{3}{16}$ , so the probability is  $\frac{3}{16}$ .  $\square$

## 2.1.4 Exercises

1. If there are 5 people named A, B, C, D, and E, and they are randomly arranged in a row (with each ordering equally likely), what is the probability that A and B are placed next to each other?

**Solution:** The size of the sample space is the number of ways to organize 5 people randomly, which is  $|\Omega| = 5! = 120$ . The event space is the number of ways to have A and B sit next to each other. We did a similar problem in 1.1, and so the answer was  $2! \cdot 4! = 48$  (why?). Hence, *since the outcomes are equally likely*,  $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{48}{120}$ .

2. Suppose I draw 4 cards from a standard 52-card deck. What is the probability they are all aces (there are exactly 4 aces in a deck)?

**Solution:** There are two ways to define our sample space, one where order matters, and one where it doesn't. These two approaches are equivalent.

- (a) If *order matters*, then  $|\Omega| = P(52, 4) = 52 \cdot 51 \cdot 50 \cdot 49$ , as the number of ways to pick 4 cards out of 52. The event space  $E$  is the number of ways to pick all 4 aces (with order mattering), which is  $P(4, 4) = 4 \cdot 3 \cdot 2 \cdot 1$ . Hence, *since the outcomes are equally likely*,  $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{P(52, 4)}{P(4, 4)} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}$
- (b) If *order does not matter*, then  $|\Omega| = \binom{52}{4}$ , since we just care which 4 out of 52 cards we get. Then, there is only  $\binom{4}{4} = 1$  way to get all 4 aces, and, *since the outcomes are equally likely*,  $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{\binom{52}{4}}{\binom{4}{4}} = \frac{P(52, 4)/4!}{P(4, 4)/4!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}$ .

Notice how it did not matter whether order mattered or not, but we had to be consistent! The 4! accounting for the ordering of the 4 cards gets cancelled out :).

3. Given 3 different spades (S) and 3 different hearts (H), shuffle them. Compute  $\mathbb{P}(E)$ , where  $E$  is the event that the suits of the shuffled cards are in alternating order (e.g., SHSHSH or HSHSHS)

**Solution:** The sample space  $|\Omega|$  is the number of ways to order the 6 (distinct) cards:  $6!$ . The number of ways to organize the three spades is  $3!$  and same for the three hearts. Once we do that, we either lead with spades or hearts, so we get  $2 \cdot 3!^2$  for the size of our event space  $E$ . Hence, *since the outcomes are equally likely*,  $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{2 \cdot 3!^2}{6!}$ .

Note that all of these exercises are just counting two things! We count the size of the sample space, then the event space and divide them. It is very important to acknowledge that we can only do this when the outcomes are *equally likely*.

You can see how we can get even more fun and complicated problems - the three exercises above displayed counting problems on the “easier side”. The reason we didn’t give “harder” problems is because computing probability in the case of equally likely outcomes reduces to doing two counting problems (counting  $|E|$  and  $|\Omega|$ , where computing  $|\Omega|$  is generally easier than computing  $|E|$ ). Just use the techniques from Chapter 1 to do this!

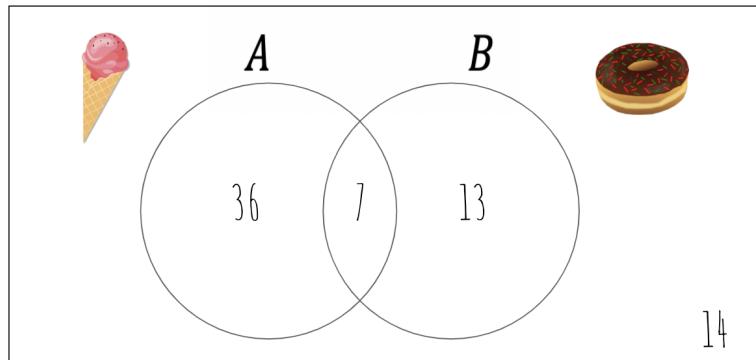
## Chapter 2: Discrete Probability

### 2.2: Conditional Probability

#### 2.2.1 Conditional Probability

Sometimes we would like to incorporate new information into our probability. For example, you may be feeling symptoms of some disease, and so you take a test to see whether you have it or not. Let  $D$  be the event you have a disease, and  $T$  be the event you test positive ( $T^C$  is the event you test negative). It could be that  $\mathbb{P}(D) = 0.01$  (1% chance of having the disease without knowing anything). But how can we update this probability *given* that we tested positive (or negative)? This will be written as  $\mathbb{P}(D | T)$  or  $\mathbb{P}(D | T^C)$  respectively. You would think  $\mathbb{P}(D | T) > \mathbb{P}(D)$  since you're more likely to have the disease once you test positive, and  $\mathbb{P}(D | T^C) < \mathbb{P}(D)$  since you're less likely to have the disease once you test negative. These are called conditional probabilities - they are the probability of an event, given that you know some other event occurred. Is there a formula for updating  $\mathbb{P}(D)$  given new information? Yes!

Let's go back to the example of students in CSE312 liking donuts and ice cream. Recall we defined event  $A$  as liking ice cream and event  $B$  as liking donuts. Then, remember we had 36 students that only like ice cream ( $A \cap B^C$ ), 7 students that like donuts and ice cream ( $A \cap B$ ), and 13 students that only like donuts ( $B \cap A^C$ ). Let's also say that we have 14 students that don't like either ( $A^C \cap B^C$ ). That leaves us with the following picture, which makes up the whole sample space:



Now, what if we asked the question, what's the probability that someone likes ice cream, **given** that we know they like donuts? We can approach this with the knowledge that 20 of the students like donuts (13 who don't like ice cream and 7 who do). What this question is getting at, is: given the knowledge that someone likes donuts, what is the chance that they also like ice cream? Well, 7 of the 20 who like donuts like ice cream, so we are left with the probability  $\frac{7}{20}$ . We write this as  $\mathbb{P}(A | B)$  (read the “probability of  $A$  given  $B$ ”) and in this case we have the following:

$$\begin{aligned}
 \mathbb{P}(A | B) &= \frac{7}{20} \\
 &= \frac{|A \cap B|}{|B|} && [|B| = 20 \text{ people like donuts}, |A \cap B| = 7 \text{ people like both}] \\
 &= \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} && [\text{divide top and bottom by } |\Omega|, \text{ which is equivalent}] \\
 &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} && [\text{if we have equally likely outcomes}]
 \end{aligned}$$

This intuition (which worked only in the special case equally likely outcomes), leads us to the definition of conditional probability:

#### Definition 2.2.1: Conditional Probability

The **conditional probability** of event  $A$  given that event  $B$  happened is:

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

An equivalent and useful formula we can derive (by multiplying both sides by the denominator,  $\mathbb{P}(B)$ , and switching the sides of the equation is:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B) \mathbb{P}(B)$$

Let's consider an important question: does  $\mathbb{P}(A | B) = \mathbb{P}(B | A)$ ? No!

This is a common misconception we can show with some examples. In the above example with ice cream, we showed already  $\mathbb{P}(A | B) = \frac{7}{20}$ , but  $\mathbb{P}(B | A) = \frac{7}{36}$ , and these are not equal.

Consider another example where  $W$  is the event that you are wet and  $S$  is the event you are swimming. Then, the probability you are wet given you are swimming,  $\mathbb{P}(W | S) = 1$ , as if you are swimming you are certainly wet. But, the probability you are swimming given you are wet,  $\mathbb{P}(S | W) \neq 1$ , because there are numerous other reasons you could be wet that don't involve swimming (being in the rain, showering, etc.).

## 2.2.2 Bayes Theorem

This brings us to Bayes Theorem:

Theorem 2.2.5: Bayes Theorem

Let  $A, B$  be events with nonzero probability. Then,

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(B | A) \mathbb{P}(A)}{\mathbb{P}(B)}$$

Note that in the above  $\mathbb{P}(A)$  is called the **prior**, which is our belief without knowing anything about event  $B$ .  $\mathbb{P}(A | B)$  is called the **posterior**, our belief after learning that event  $B$  occurred.

This theorem is important because it allows to “reverse the conditioning”! Notice that both  $\mathbb{P}(A | B)$  and  $\mathbb{P}(B | A)$  appear in this equation on opposite sides. So if we know  $\mathbb{P}(A)$  and  $\mathbb{P}(B)$  and can more easily calculate one of  $\mathbb{P}(A | B)$  or  $\mathbb{P}(B | A)$ , we can use **Bayes Theorem** to derive the other.

*Proof of Bayes Theorem.* Recall the (alternate) definition of conditional probability from above:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B) \mathbb{P}(B) \quad (2.2.6)$$

Swapping the roles of  $A$  and  $B$  we can also get that:

$$\mathbb{P}(B \cap A) = \mathbb{P}(B | A) \mathbb{P}(A) \quad (2.2.7)$$

But, because  $A \cap B = B \cap A$  (since these are the outcomes in both events  $A$  and  $B$ , and the order of intersection does not matter),  $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$ , so (2.2.1) and (2.2.2) are equal and we have (by setting the right-hand sides equal):

$$\mathbb{P}(A | B) \mathbb{P}(B) = \mathbb{P}(B | A) \mathbb{P}(A)$$

We can divide both sides by  $\mathbb{P}(B)$  and get Bayes Theorem:

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(B | A) \mathbb{P}(A)}{\mathbb{P}(B)}$$

Wow, I wish I was alive back then and had this important (and easy to prove) theorem named after me! □

### Example(s)

We'll investigate two slightly different questions whose answers don't seem that they should be different, but are. Suppose a family has two children (whom at birth, were each equally likely to be male or female). Let's say a telemarketer calls home and one of the two children picks up.

1. If the child who responded was male, and says “Let me get my *older* sibling”, what is the probability that both children are male?
2. If the child who responded was male, and says “Let me get my *other* sibling”, what is the probability that both children are male?

*Solution* There are four equally likely outcomes, MM, MF, FM, and FF (where M represents male and F represents female). Let  $A$  be the event both children are male.

1. In this part, we're given that the *younger* sibling is male. So we can rule out 2 of the 4 outcomes above and we're left with MF and MM. Out of these two, in one of these cases we get MM, and so our desired probability is  $1/2$ .

More formally, let this event be  $B$ , which happens with probability  $2/4$  (2 out of 4 equally likely outcomes). Then,  $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$ , since  $P(A \cap B)$  is the probability both children are male, which happens in 1 out of 4 equally likely scenarios. This is because the older sibling's sex is *independent* of the younger sibling's, so knowing the younger sibling is male doesn't change the probability of the older sibling being male (which is what we computed just now).

2. In this part, we're given that *at least one sibling* is male. That is, out of the 4 outcomes, we can only rule out the FF option. Out of the remaining options MM, MF, and FM, only one has both siblings being male. Hence, the probability desired is  $1/3$ . You can do a similar more formal argument like we did above!

See how a slight wording change changed the answer? □

We'll see a disease testing example later, which requires the next section first. If you test positive for a disease, how concerned should you be? The result may surprise you!

### 2.2.3 Law of Total Probability

Let's say you sign up for a chemistry class, but are assigned to one of three teachers randomly. Furthermore, you know the probabilities you fail the class if you were to have each teacher (from historical results, or word-of-mouth from classmates who have taken the class). Can we combine this information to compute the overall probability that you fail chemistry (before you know which teacher you get)? Yes - using the law of total probability below! We first need to define what a partition is.

#### Definition 2.2.2: Partitions

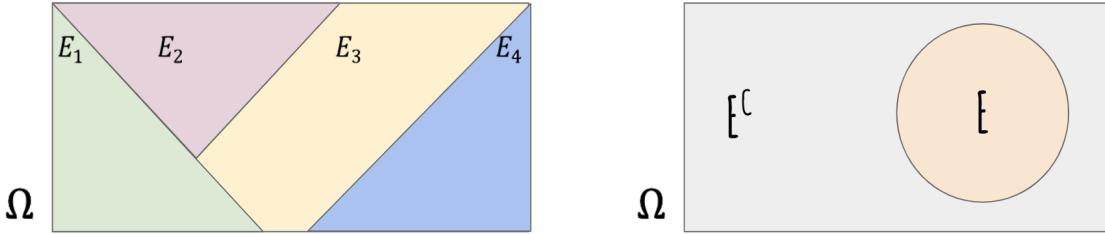
Non-empty events  $E_1, \dots, E_n$  **partition** the sample space  $\Omega$  if they are:

- (**Exhaustive**)  $E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^n E_i = \Omega$ ; that is, they cover the entire sample space.
- (**Pairwise Mutually Exclusive**) For all  $i \neq j$ ,  $E_i \cap E_j = \emptyset$ ; that is, none of them overlap.

Note that for any event  $E$ ,  $E$  and  $E^C$  always form a partition of  $\Omega$ .

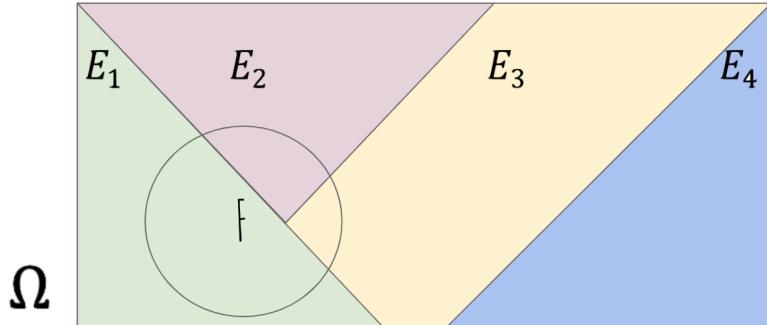
#### Example(s)

Two example partitions can be seen in the image below:



You can see that partition is a very appropriate word here! In the first image, the four events  $E_1, \dots, E_4$  don't overlap and cover the sample space. In the second image, the two events  $E, E^C$  do the same thing! This is useful when you know *exactly* one of a few things will happen. For example, for the chemistry example, there might be only three teachers, and you will be assigned to exactly one of them: at most one because you can't have two teachers (mutually exclusive), and at least one because there aren't other teachers possible (exhaustive).

Now, suppose we have some event  $F$  which intersects with various events that form a partition of  $\Omega$ . This is illustrated by the picture below:



Notice that  $F$  is composed of its intersection with each of  $E_1, E_2$ , and  $E_3$ , and so we can split  $F$  up into smaller pieces. This means that we can write the following (green chunk  $F \cap E_1$ , plus pink chunk  $F \cap E_2$  plus yellow chunk  $F \cap E_3$ ):

$$\mathbb{P}(F) = \mathbb{P}(F \cap E_1) + \mathbb{P}(F \cap E_2) + \mathbb{P}(F \cap E_3)$$

Note that  $F$  and  $E_4$  do not intersect, so  $F \cap E_4 = \emptyset$ . For completion, we can include  $E_4$  in the above equation, because  $\mathbb{P}(F \cap E_4) = 0$ . So, in all we have:

$$\mathbb{P}(F) = \mathbb{P}(F \cap E_1) + \mathbb{P}(F \cap E_2) + \mathbb{P}(F \cap E_3) + \mathbb{P}(F \cap E_4)$$

This leads us to the law of total probability.

## Theorem 2.2.6: Law of Total Probability (LTP)

If events  $E_1, \dots, E_n$  partition  $\Omega$ , then for any event  $F$

$$\mathbb{P}(F) = \mathbb{P}(F \cap E_1) + \dots + \mathbb{P}(F \cap E_n) = \sum_{i=1}^n \mathbb{P}(F \cap E_i)$$

Using the definition of conditional probability,  $\mathbb{P}(F \cap E_i) = \mathbb{P}(F | E_i) \mathbb{P}(E_i)$ , we can replace each of the terms above and get the (typically) more useful formula:

$$\mathbb{P}(F) = \mathbb{P}(F | E_1) \mathbb{P}(E_1) + \dots + \mathbb{P}(F | E_n) \mathbb{P}(E_n) = \sum_{i=1}^n \mathbb{P}(F | E_i) \mathbb{P}(E_i)$$

That is, to compute the probability of an event  $F$  overall; suppose we have  $n$  disjoint cases  $E_1, \dots, E_n$  for which we can (easily) compute the probability of  $F$  in each of these cases ( $\mathbb{P}(F | E_i)$ ). Then, take the weighted average of these probabilities, using the probabilities  $\mathbb{P}(E_i)$  as weights (the probability of being in each case).

## Example(s)

Let's consider an example in which we are trying to determine the probability that we fail chemistry. Let's call the event  $F$  failing, and consider the three events  $E_1$  for getting the Mean Teacher,  $E_2$  for getting the Nice Teacher, and  $E_3$  for getting the Hard Teacher which partition the sample space. The following table gives the relevant probabilities:

	Mean Teacher $E_1$	Nice Teacher $E_2$	Hard Teacher $E_3$
Probability of Teaching You $\mathbb{P}(E_i)$	6/8	1/8	1/8
Probability of Failing You $\mathbb{P}(F   E_i)$	1	0	1/2

Solve for the probability of failing.

*Solution* Before doing anything, how are you liking your chances? There is a high probability (6/8) of getting the Mean Teacher, and she will certainly fail you. Therefore, you should be pretty sad.

Now let's do the computation. Notice that the first row sums to 1, as it must, since events  $E_1, E_2, E_3$  partition the sample space (you have exactly one of the three teachers). Using the Law of Total Probability (LTP), we have the following:

$$\begin{aligned} \mathbb{P}(F) &= \sum_{i=1}^3 \mathbb{P}(F | E_i) \mathbb{P}(E_i) = \mathbb{P}(F | E_1) \mathbb{P}(E_1) + \mathbb{P}(F | E_2) \mathbb{P}(E_2) + \mathbb{P}(F | E_3) \mathbb{P}(E_3) \\ &= 1 \cdot \frac{6}{8} + 0 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \frac{13}{16} \end{aligned}$$

Notice to get the probability of failing, what we did was: consider the probability of failing in each of the 3 cases, and take a weighted average of using the probability of each case. This is exactly what the law of total probability lets us do!

You might consider using the LTP when you know the probability of your desired event in  $\square$

Example(s)

Misfortune struck us and we ended up failing chemistry class. What is the probability that we had the Hard Teacher given that we failed?

*Solution* First, this probability should be low intuitively because if you failed, it was probably due to the Hard Teacher (because you are more likely to get them, AND because they have a high fail rate of 100%).

Start by writing out in a formula what you want to compute; in our case, it is  $\mathbb{P}(E_3 | F)$  (getting the hard teacher **given** that we failed). We know  $\mathbb{P}(F | E_3)$  and we want to solve for  $\mathbb{P}(E_3 | F)$ . This is a hint to use Bayes Theorem since we can reverse the conditioning! Using that with the numbers from the table and the previous question:

$$\begin{aligned}\mathbb{P}(E_3 | F) &= \frac{\mathbb{P}(F | E_3) \mathbb{P}(E_3)}{\mathbb{P}(F)} && [\text{bayes theorem}] \\ &= \frac{\frac{1}{2} \cdot \frac{1}{8}}{\frac{13}{16}} \\ &= \frac{1}{13}\end{aligned}$$

$\square$

## 2.2.4 Bayes Theorem with the Law of Total Probability

Oftentimes, the denominator in Bayes Theorem is hard, so we must compute it using the LTP. Here, we just combine two powerful formulae: Bayes Theorem and the Law of Total Probability:

Theorem 2.2.7: Bayes Theorem with the Law of Total Probability

Let events  $E_1, \dots, E_n$  partition the sample space  $\Omega$ , and let  $F$  be another event. Then:

$$\begin{aligned}\mathbb{P}(E_1 | F) &= \frac{\mathbb{P}(F | E_1) \mathbb{P}(E_1)}{\mathbb{P}(F)} && [\text{by bayes theorem}] \\ &= \frac{\mathbb{P}(F | E_1) \mathbb{P}(E_1)}{\sum_{i=1}^n \mathbb{P}(F | E_i) \mathbb{P}(E_i)} && [\text{by the law of total probability}]\end{aligned}$$

In particular, in the case of a simple partition of  $\Omega$  into  $E$  and  $E^C$ , if  $E$  is an event with nonzero probability, then:

$$\begin{aligned}\mathbb{P}(E | F) &= \frac{\mathbb{P}(F | E) \mathbb{P}(E)}{\mathbb{P}(F)} && [\text{by bayes theorem}] \\ &= \frac{\mathbb{P}(F | E) \mathbb{P}(E)}{\mathbb{P}(F | E) \mathbb{P}(E) + \mathbb{P}(F | E^C) \mathbb{P}(E^C)} && [\text{by the law of total probability}]\end{aligned}$$

## 2.2.5 Exercises

- Suppose the llama flu disease has become increasingly common, and now 0.1% of the population has it (1 in 1000 people). Suppose there is a test for it which is 98% accurate (e.g., 2% of the time it will give the wrong answer). Given that you tested positive, what is the probability you have the disease? Before any computation, think about what you think the answer might be.

**Solution:** Let  $L$  be the event you have the llama flu, and  $T$  be the event you test positive ( $T^C$  is the event you test negative). You are asked for  $\mathbb{P}(L | T)$ . We do know  $\mathbb{P}(T | L) = 0.98$  because if you have the llama flu, the probably you test positive is 98%. This gives us the hint to use Bayes Theorem!

We get that

$$\mathbb{P}(L | T) = \frac{\mathbb{P}(T | L) \mathbb{P}(L)}{\mathbb{P}(T)}$$

We are given  $\mathbb{P}(T | L) = 0.98$  and  $\mathbb{P}(L) = 0.001$ , but how can we get  $\mathbb{P}(T)$ , the probability of testing positive? Well that depends on whether you have the disease or not. When you have two or more cases ( $L$  and  $L^C$ ), that's a hint to use the LTP! So we can write

$$\mathbb{P}(T) = \mathbb{P}(T | L) \mathbb{P}(L) + \mathbb{P}(T | L^C) \mathbb{P}(L^C)$$

Again, interpret this as a weighted average of the probability of testing positive whether you had llama flu  $\mathbb{P}(T | L)$  or not  $\mathbb{P}(T | L^C)$ , weighting by the probability you are in each of these cases  $\mathbb{P}(L)$  and  $\mathbb{P}(L^C)$ . We know  $\mathbb{P}(L^C) = 0.999$  since these  $\mathbb{P}(L^C) = 1 - \mathbb{P}(L)$  (axiom of probability). But what about  $\mathbb{P}(T | L^C)$ ? This is the probability of testing positive given that you don't have llama flu, which is 0.02 or 2% (due to the 98% accuracy). Putting this all together, we get:

$$\begin{aligned} \mathbb{P}(L | T) &= \frac{\mathbb{P}(T | L) \mathbb{P}(L)}{\mathbb{P}(T)} && \text{[bayes theorem]} \\ &= \frac{\mathbb{P}(T | L) \mathbb{P}(L)}{\mathbb{P}(T | L) \mathbb{P}(L) + \mathbb{P}(T | L^C) \mathbb{P}(L^C)} && \text{[LTP]} \\ &= \frac{0.98 \cdot 0.001}{0.98 \cdot 0.001 + 0.02 \cdot 0.999} \\ &\approx 0.046756 \end{aligned}$$

Not even a 5% chance we have the disease, what a relief! But wait, how can that be? The test is so accurate, and it said you were positive? This is because the prior probability of having the disease  $\mathbb{P}(L)$  was so low at 0.1% (actually this is pretty high for a disease rate). If you think about it, the posterior probability we computed  $\mathbb{P}(L | T)$  is 47× larger than the prior probability  $\mathbb{P}(L)$  ( $\mathbb{P}(L | T) / \mathbb{P}(L) \approx 0.047 / 0.001 = 47$ ), so the test did make it a lot more likely we had the disease after all!

- Suppose we have four fair die: one with three sides, one with four sides, one with five sides, and one with six sides (The numbering of an  $n$ -sided die is 1, 2, ...,  $n$ ). We pick one of the four die, each with equal probability, and roll the same die three times. We get all 4's. What is the probability we chose the 5-sided die to begin with?

**Solution:** Let  $D_i$  be the event we rolled the  $i$ -sided die, for  $i = 3, 4, 5, 6$ . Notice that these

$D_3, D_4, D_5, D_6$  partition the sample space.

$$\begin{aligned}
 P(D_5|444) &= \frac{P(444|D_5)P(D_5)}{P(444)} && \text{[by bayes theorem]} \\
 &= \frac{P(444|D_5)P(D_5)}{P(444|D_3)P(D_3) + P(444|D_4)P(D_4) + P(444|D_5)P(D_5) + P(444|D_6)P(D_6)} && \text{[by ltp]} \\
 &= \frac{\frac{1}{5^3} \cdot \frac{1}{4}}{\frac{0}{3^3} \cdot \frac{1}{4} + \frac{1}{4^3} \cdot \frac{1}{4} + \frac{1}{5^3} \cdot \frac{1}{4} + \frac{1}{6^3} \cdot \frac{1}{4}} \\
 &= \frac{1/125}{1/64 + 1/125 + 1/216} \\
 &= \frac{1728}{6103} \approx 0.2831
 \end{aligned}$$

Note that we compute  $P(444|D_i)$  by noting there's only one outcome where we get (4, 4, 4) out of the  $i^3$  equally likely outcomes. This is true except when  $i = 3$ , where it's not possible to roll all 4's.

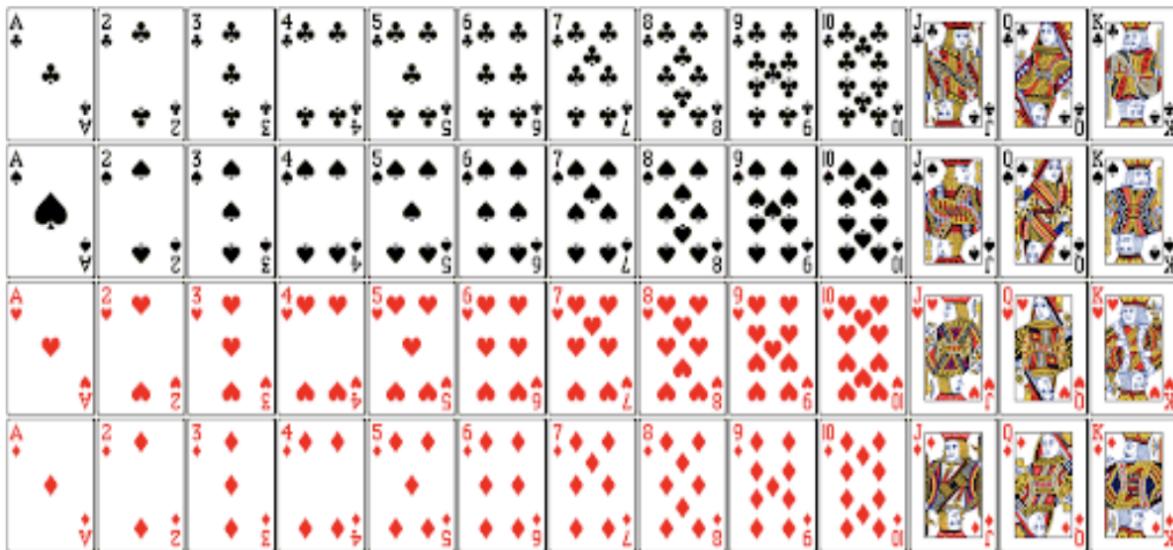
## Chapter 2: Discrete Probability

### 2.3: Independence

#### 2.3.1 Chain Rule

We learned several tools already to compute probabilities (equally likely outcomes, Bayes Theorem, LTP). Now, we will learn how to handle the probability of several events occurring simultaneously: that is,  $\mathbb{P}(A \cap B \cap C \cap D)$  for example. To compute the probability that at least one of several events happens:  $\mathbb{P}(A \cup B \cup C \cup D)$ , you would use inclusion-exclusion! We'll see an example which builds intuition first.

Consider a standard 52 card deck. This has four suits (clubs, spades, hearts, and diamonds). Each of the four suits has 13 cards of different rank (A, 2, 3, 4, 5, 6, 7, 8, 9, 10 J, Q, K).



Now, suppose that we shuffle this deck and draw the top three cards. Let's define:

1.  $A$  to be the event that we get the Ace of spades as our **first** card.
2.  $B$  to be the event that we get the 10 of clubs as our **second** card.
3.  $C$  to be the event that we get the 4 of diamonds as our **third** card.

What is the probability that all three of these events happen? We can write this as  $\mathbb{P}(A, B, C)$  (sometimes we use commas as an alternative to using the intersection symbol, so this is equivalent to  $\mathbb{P}(A \cap B \cap C)$ ). Note that this is equivalent to  $\mathbb{P}(C, B, A)$  or  $\mathbb{P}(B, C, A)$  since order of intersection does not matter.

Intuitively, you might say that this probability is  $\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$ , and you would be correct.

1. The first factor comes from the fact that there are 52 cards that could be drawn, and only one ace of spades. That is, we computed  $\mathbb{P}(A)$ .

2. The second factor comes from the fact that there are 51 cards after we draw the first card and only one 10 of clubs. That is, we computed  $\mathbb{P}(B | A)$ .
3. The final factor comes from the fact that there are 50 cards left after we draw the first two and only one 4 of diamonds. That is, we computed  $\mathbb{P}(C | A, B)$ .

To summarize, we said that

$$\mathbb{P}(A, B, C) = \mathbb{P}(A) \cdot \mathbb{P}(B | A) \cdot \mathbb{P}(C | A, B) = \frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

This brings us to the chain rule:

Theorem 2.3.8: Chain Rule

Let  $A_1, \dots, A_n$  be events with nonzero probabilities. Then:

$$\mathbb{P}(A_1, \dots, A_n) = \mathbb{P}(A_1) \mathbb{P}(A_2 | A_1) \mathbb{P}(A_3 | A_1 A_2) \cdots \mathbb{P}(A_n | A_1, \dots, A_{n-1})$$

In the case of two events,  $A, B$  (this is just the alternate form of the definition of conditional probability from 2.2):

$$\mathbb{P}(A, B) = \mathbb{P}(A) \mathbb{P}(B | A)$$

An easy way to remember this, is if we want to observe  $n$  events, we can observe one event at a time, and condition on those that we've done thus far. And most importantly, since the order of intersection **doesn't matter**, you can actually decompose this into any of  $n!$  orderings. Make sure you "do" one event at a time, conditioning on the intersection of ALL past events like we did above.

*Proof of Chain Rule.* Remember that the definition of conditional probability says  $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B | A)$ . We'll use this repeatedly to break down our  $\mathbb{P}(A_1, \dots, A_n)$ . Sometimes it is easier to use commas, and sometimes it is easier to use the intersection sign  $\cap$ : for this proof, we'll use the intersection sign. We'll prove this for four events, and you'll see how it can be easily extended to any number of events!

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) &= \mathbb{P}((A_1 \cap A_2 \cap A_3) \cap A_4) && [\text{treat } A_1 \cap A_2 \cap A_3 \text{ as one event}] \\ &= \mathbb{P}(A_1 \cap A_2 \cap A_3) \mathbb{P}(A_4 | A_1 \cap A_2 \cap A_3) && [\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B | A)] \\ &= \mathbb{P}((A_1 \cap A_2) \cap A_3) \mathbb{P}(A_4 | A_1 \cap A_2 \cap A_3) && [\text{treat } A_1 \cap A_2 \text{ as one event}] \\ &= \mathbb{P}(A_1 \cap A_2) \mathbb{P}(A_3 | A_1 \cap A_2) \mathbb{P}(A_4 | A_1 \cap A_3 \cap A_3) && [\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B | A)] \\ &= \mathbb{P}(A_1) \mathbb{P}(A_2 | A_1) \mathbb{P}(A_3 | A_1 \cap A_2) \mathbb{P}(A_4 | A_1 \cap A_3 \cap A_3) && [\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B | A)] \end{aligned}$$

Note how we keep "chaining" and applying the definition of conditional probability repeatedly!

□

Example(s)

Consider the 3-stage process. We roll a 6-sided die (numbered 1-6), call the outcome  $X$ . Then, we roll a  $X$ -sided die (numbered 1- $X$ ), call the outcome  $Y$ . Finally, we roll a  $Y$ -sided die (numbered 1- $Y$ ), call the outcome  $Z$ . What is  $P(Z = 5)$ ?

*Solution* There are only three things that could have happened for the triplet  $(X, Y, Z)$  so that  $Z$  takes on the value 5:  $\{(6, 6, 5), (6, 5, 5), (5, 5, 5)\}$ . So

$$\begin{aligned}\mathbb{P}(Z = 5) &= \mathbb{P}(X = 6, Y = 6, Z = 5) + \mathbb{P}(X = 6, Y = 5, Z = 5) + \mathbb{P}(X = 5, Y = 5, Z = 5) && [\text{cases}] \\ &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5} && [\text{chain rule 3x}]\end{aligned}$$

How did we use the chain rule? Let's see for example the last term:

$$\mathbb{P}(X = 5, Y = 5, Z = 5) = P(X = 5)P(Y = 5 | X = 5)P(Z = 5 | X = 5, Y = 5)$$

$P(X = 5) = \frac{1}{6}$  because we rolled a 6-sided die.

$P(Y = 5 | X = 5) = \frac{1}{5}$  since we rolled a  $X = 5$ -sided die.

Finally,  $P(Z = 5 | X = 5, Y = 5) = P(Z = 5 | Y = 5) = \frac{1}{5}$  since we rolled a  $Y = 5$ -sided die. Note we didn't need to know  $X = 5$  once we knew  $Y = 5$ !

□

### 2.3.2 Independence

Let's say we flip a fair coin 3 times *independently* (whatever that means) - what is the probability of getting all heads? You may be inclined to say  $(1/2)^3 = 1/8$  because the probability of getting heads each time is just  $1/2$ . However, we haven't learned such a rule to compute the joint probability  $\mathbb{P}(H_1 \cap H_2 \cap H_3)$  except the chain rule.

Using only what we've learned, we could consider equally likely outcomes. There are  $2^3 = 8$  possible outcomes when flipping a coin three times (by product rule), and only one of those (HHH) makes up the event we care about:  $H_1 \cap H_2 \cap H_3$ . Since the outcomes are equally likely,

$$\mathbb{P}(H_1 \cap H_2 \cap H_3) = \frac{|\{H_1 \cap H_2 \cap H_3\}|}{|\Omega|} = \frac{|\{HHH\}|}{2^3} = \frac{1}{8}$$

We'd love a rule to say  $\mathbb{P}(H_1 \cap H_2 \cap H_3) = \mathbb{P}(H_1) \cdot \mathbb{P}(H_2) \cdot \mathbb{P}(H_3) = 1/2 \cdot 1/2 \cdot 1/2 = 1/8$  - and it turns out this is true when the events are independent!

But first, let's consider the smaller case: does  $\mathbb{P}(A, B) = \mathbb{P}(A)\mathbb{P}(B)$  in general? No! How do we know this though? Well recall that by the chain rule, we know that:

$$\mathbb{P}(A, B) = \mathbb{P}(A)\mathbb{P}(B | A)$$

So, unless  $\mathbb{P}(B | A) = \mathbb{P}(B)$  the equality does not hold. However, when this equality does hold, it is a special case, which brings us to independence.

#### Definition 2.3.1: Independence

Events  $A$  and  $B$  are **independent** if any of the following equivalent statements hold:

1.  $\mathbb{P}(A | B) = \mathbb{P}(A)$
2.  $\mathbb{P}(B | A) = \mathbb{P}(B)$

3.  $\mathbb{P}(A, B) = \mathbb{P}(A) \mathbb{P}(B)$

Intuitively what it means for  $\mathbb{P}(A | B) = \mathbb{P}(A)$  is that: given that we know  $B$  happened, the probability of observing  $A$  is the same as if we didn't know anything. So, event  $B$  has no influence on event  $A$ . The last statement

$$\mathbb{P}(A, B) = \mathbb{P}(A) \mathbb{P}(B)$$

is the most often applied to problems where we are allowed to assume independence.

What about independence of more than just two events? We call this concept “mutual independence” (but most of the time we don't even say the word “mutual”). You might think that for events  $A_1, A_2, A_3, A_4$  to be (mutually) independent, by extension of the definition of two events, we would just need

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \cdot \mathbb{P}(A_4)$$

But it turns out, we need this property to hold for *any* subset of the 4 events. For example, the following must be true (in addition to others):

$$\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_3)$$

$$\mathbb{P}(A_2 \cap A_3 \cap A_4) = \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \cdot \mathbb{P}(A_4)$$

For all  $2^n$  subsets of the 4 events ( $2^4 = 16$  in our case), the probability of the intersection must simply be the product of the individual probabilities.

As you can see, it would be quite annoying to check even if three events were (mutually) independent. Luckily, most of the time we are told to assume that several events are (mutually) independent and we get all of those statements to be true for free. We are rarely asked to demonstrate/prove mutual independence.

**Definition 2.3.2: Mutual Independence**

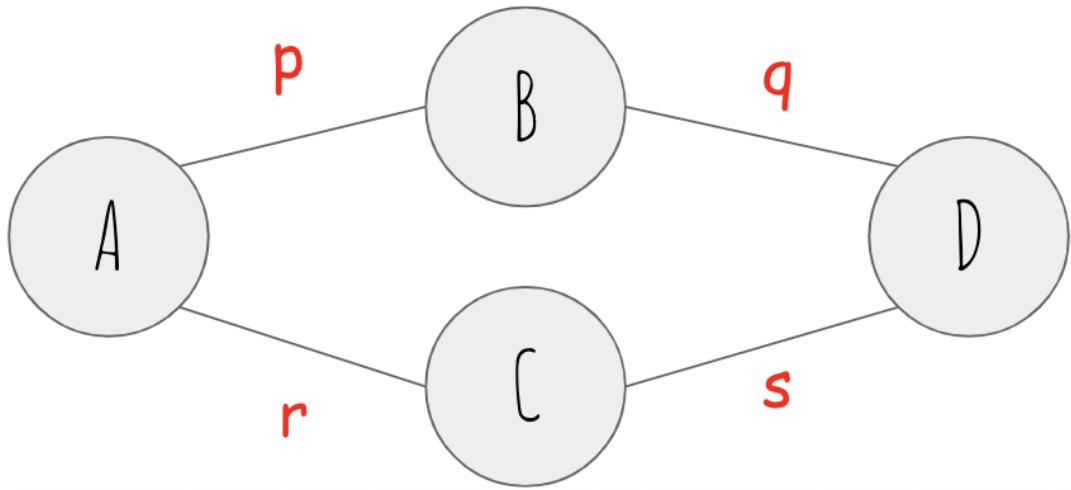
We say  $n$  events  $A_1, A_2, \dots, A_n$  are **(mutually) independent** if, for *any* subset  $I \subseteq [n] = \{1, 2, \dots, n\}$ , we have

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

This is very similar to the last formula  $\mathbb{P}(A, B) = \mathbb{P}(A) \mathbb{P}(B)$  in the definition of independence for two events, just extended to multiple events. It must hold for any subset of the  $n$  events, and so this equation is actually saying  $2^n$  equations are true!

**Example(s)**

Suppose we have the following network, in which circles represents a node in the network ( $A, B, C$ , and  $D$ ) and the links have the probabilities  $p, q, r$  and  $s$  of successfully working. That is, for example, the probability of successful communication from  $A$  to  $B$  is  $p$ . Each link is independent of the others though.



Now, let's consider the question, what is the probability that  $A$  and  $D$  can successfully communicate?

*Solution* There are two ways in which it can communicate: (1) in the top path via  $B$  or (2) in the bottom path via  $C$ . Let's define the event *top* to be successful communication in the top path and the event *bottom* to be successful communication in the bottom path. Let's first consider the probabilities of each of these being successful communication. For the top to be a valid path, *both* links  $AB$  and  $BD$  must work.

$$\begin{aligned}\mathbb{P}(\text{top}) &= \mathbb{P}(AB \cap BD) \\ &= \mathbb{P}(AB)\mathbb{P}(BD) && [\text{by independence}] \\ &= pq\end{aligned}$$

Similarly:

$$\begin{aligned}\mathbb{P}(\text{bottom}) &= \mathbb{P}(AC \cap CD) \\ &= \mathbb{P}(AC)\mathbb{P}(CD) && [\text{by independence}] \\ &= rs\end{aligned}$$

So, to calculate the probability of successful communication between  $A$  and  $D$ , we can take the union of *top* and *bottom* (we just need at least one of the two to work), and so we have:

$$\begin{aligned}\mathbb{P}(\text{top} \cup \text{bottom}) &= \mathbb{P}(\text{top}) + \mathbb{P}(\text{bottom}) - \mathbb{P}(\text{top} \cap \text{bottom}) && [\text{by inclusion-exclusion}] \\ &= \mathbb{P}(\text{top}) + \mathbb{P}(\text{bottom}) - \mathbb{P}(\text{top})\mathbb{P}(\text{bottom}) && [\text{by independence}] \\ &= pq + rs - pqr\end{aligned}$$

□

### 2.3.3 Conditional Independence

In the example above for the chain rule, we made this step:

$$\mathbb{P}(Z = 5 | X = 5, Y = 5) = \mathbb{P}(Z = 5 | Y = 5)$$

This is actually another form of independence, called conditional independence! That is, *given* that  $Y = 5$ , the events  $X = 5$  and  $Z = 5$  are independent (the above equation looks exactly like  $\mathbb{P}(Z = 5 | X = 5) = \mathbb{P}(Z = 5)$  except with extra conditioning on  $Y = 5$  on both sides.

#### Definition 2.3.3: Conditional Independence

Events  $A$  and  $B$  are **conditionally independent given an event  $C$**  if any of the following equivalent statements hold:

1.  $\mathbb{P}(A | B, C) = \mathbb{P}(A | C)$
2.  $\mathbb{P}(B | A, C) = \mathbb{P}(B | C)$
3.  $\mathbb{P}(A, B | C) = \mathbb{P}(A | C)\mathbb{P}(B | C)$

Recall the definition of  $A$  and  $B$  being (unconditionally) independent below:

1.  $\mathbb{P}(A | B) = \mathbb{P}(A)$
2.  $\mathbb{P}(B | A) = \mathbb{P}(B)$
3.  $\mathbb{P}(A, B) = \mathbb{P}(A)\mathbb{P}(B)$

Notice that this is very similar to the definition of independence. There is no difference, except we have just added in conditioning on  $C$  to every probability.

#### Example(s)

Suppose there is a coin  $C_1$  with  $\mathbb{P}(\text{head}) = 0.3$  and a coin  $C_2$  with  $\mathbb{P}(\text{head}) = 0.9$ . We pick one randomly with equal probability and will flip that coin 3 times *independently*. What is the probability we get all heads?

*Solution* Let us call  $HHH$  the event of getting three heads,  $C_1$  the event of picking the first coin, and  $C_2$  the event of getting the second coin. Then we have the following:

$$\begin{aligned} \mathbb{P}(HHH) &= \mathbb{P}(HHH | C_1)\mathbb{P}(C_1) + \mathbb{P}(HHH | C_2)\mathbb{P}(C_2) && \text{[by the law of total probability]} \\ &= (\mathbb{P}(H | C_1))^3\mathbb{P}(C_1) + (\mathbb{P}(H | C_2))^3\mathbb{P}(C_2) && \text{[by conditional independence]} \\ &= (0.3)^3 \frac{1}{2} + (0.9)^3 \frac{1}{2} = 0.378 \end{aligned}$$

It is important to note that getting heads on the first and second flip are NOT independent. The probability of heads on the second, given that we got heads on the first flip, is much higher since we are more likely to have chosen coin  $C_2$ . However, *given which coin we are flipping*, the flips are conditionally independent. Hence, we can write  $\mathbb{P}(HHH | C_1) = \mathbb{P}(H | C_1)^3$ .  $\square$

### 2.3.4 Exercises

- Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.
  - If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?
  - If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?
  - If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?
  - If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judge?

**Solution:**

- Let  $S_i$  be the event a contestant stays in the  $i^{th}$  episode, and  $B$  be the event a contestant is bribing the judges. Then, by the law of total probability,

$$\mathbb{P}(S_1) = \mathbb{P}(S_1 | B)\mathbb{P}(B) + \mathbb{P}(S_1 | B^C)\mathbb{P}(B^C) = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{2}$$

- Again by the law of total probability,

$$\begin{aligned} \mathbb{P}(S_1 \cap S_2) &= \mathbb{P}(S_1 \cap S_2 | B)\mathbb{P}(B) + \mathbb{P}(S_1 \cap S_2 | B^C)\mathbb{P}(B^C) && [\text{LTP}] \\ &= \mathbb{P}(S_1 | B)\mathbb{P}(S_2 | B)\mathbb{P}(B) + \mathbb{P}(S_1 | B^C)\mathbb{P}(S_2 | B^C)\mathbb{P}(B^C) && [\text{conditional independence}] \\ &= 1 \cdot 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{4} \\ &= \frac{1}{3} \end{aligned}$$

Again, it's important to note that staying on the first and second episode are NOT independent. If we know she stayed on the first episode, then it is more likely she stays on the second (since she's more likely to be bribing the judges). However, conditioned on whether or not we are bribing the judges,  $S_1$  and  $S_2$  are independent.

- By Bayes Theorem,

$$\mathbb{P}(S_2^C | S_1) = \frac{\mathbb{P}(S_1 \cap S_2^C)}{\mathbb{P}(S_1)}$$

The denominator is our answer to (a), and the numerator can be computed in the same way as (b).

- By Bayes Theorem,

$$\mathbb{P}(B | S_1) = \frac{\mathbb{P}(S_1 | B)\mathbb{P}(B)}{\mathbb{P}(S_1)}$$

We computed all these quantities in part (a).

2. A parallel system functions whenever at least one of its components works. Consider a parallel system of  $n$  components and suppose that each component works with probability  $p$  independently
- What is the probability the system is functioning?
  - If the system is functioning, what is the probability that component 1 is working?
  - If the system is functioning and component 2 is working, what is the probability that component 1 is working?

**Solution:**

- (a) Let  $C_i$  be the event component  $i$  is functioning, for  $i = 1, \dots, n$ . Let  $F$  be the event the system functions. Then,

$$\begin{aligned}
 \mathbb{P}(F) &= 1 - \mathbb{P}(F^C) \\
 &= 1 - \mathbb{P}\left(\bigcap_{i=1}^n C_i^C\right) && [\text{def of parallel system}] \\
 &= 1 - \prod_{i=1}^n \mathbb{P}(C_i^C) && [\text{independence}] \\
 &= 1 - (1 - p)^n && [\text{prob any fails is } 1 - p]
 \end{aligned}$$

- (b) By Bayes Theorem, and since  $\mathbb{P}(F | C_1) = 1$  (system is guaranteed to function if  $C_1$  is working),

$$P(C_1 | F) = \frac{\mathbb{P}(F | C_1) \mathbb{P}(C_1)}{\mathbb{P}(F)} = \frac{1 \cdot p}{1 - (1 - p)^n}$$

(c)

$$\begin{aligned}
 \mathbb{P}(C_1 | C_2, F) &= \mathbb{P}(C_1 | C_2) && [\text{if given } C_2, \text{ already know } F \text{ is true}] \\
 &= \mathbb{P}(C_1) && [C_1, C_2 \text{ independent}] \\
 &= p
 \end{aligned}$$

## Chapter 3: Discrete Random Variables

### 3.1: Discrete Random Variables Basics

#### 3.1.1 Introduction to Discrete Random Variables

Suppose you flip a fair coin twice. Then the sample space is:

$$\Omega = \{HH, HT, TH, TT\}$$

Sometimes, though, we don't care about the order (*HT* vs *TH*), but just the fact that we got one heads and one tail. So we can define a **random variable** as a numeric function of the outcome.

For example, we can define  $X$  to be the number of heads in the two independent flips of a fair coin. Then  $X$  is a function,  $X : \Omega \rightarrow \mathbb{R}$  which takes outcomes  $\omega \in \Omega$  and maps them to a number. For example, for the outcome *HH*, we have  $X(HH) = 2$  since there are two heads. See the rest below!

$$\begin{aligned} X(HH) &= 2 \\ X(HT) &= 1 \\ X(TH) &= 1 \\ X(TT) &= 0 \end{aligned}$$

$X$  is an example of a random variable, which brings us to the following definition:

##### Definition 3.1.1: Random Variable

Suppose we conduct an experiment with sample space  $\Omega$ . A **random variable (rv)** is a numeric function of the outcome,  $X : \Omega \rightarrow \mathbb{R}$ . That is it maps outcomes  $\omega \in \Omega$  to numbers,  $\omega \rightarrow X(\omega)$ .

The set of possible values  $X$  can take on is its **range/support**, denoted  $\Omega_X$ .

If  $\Omega_X$  is finite or countable infinite (typically integers or a subset),  $X$  is a **discrete random variable (drv)**. Else if  $\Omega_X$  is uncountably large (the size of real numbers),  $X$  is a **continuous random variable**.

##### Example(s)

Below are some examples of random variables, solve for their ranges and classifications as a discrete random variable or continuous random variable.

- $X$ , the number of heads in  $n$  flips of a fair coin.
- $N$ , the number of people born in this year.
- $F$ , the number of flips of a fair coin up to and including the first head.
- $B$ , the amount of time one waits for the next bus in seconds.
- $T$ , the temperature in Celsius of liquid water.

*Solution*

- The range of  $X$  is  $\Omega_X = \{0, 1, \dots, n\}$  because there could be any where from 0 to  $n$  heads flipped. It is a discrete random variable because there are finite  $n + 1$  values that it takes on.
- The range of  $N$  is  $\Omega_N = \{0, 1, 2, \dots\}$  because there is no upper bound on the number of people that can be born. This is countably infinite as it is a subset of all the integers, so it is a discrete random variable.
- The range of  $F$  is  $\Omega_F = \{1, 2, \dots\}$  because it will take at least 1 flip to flip a head or it could always be tails and never flip a head (although the chance is low). This is still countable as a subset of all the integers, so it is a discrete random variable.
- The range of  $B$  is  $\Omega_B = [0, \infty)$ , as there could be partial seconds waited, and it could be anywhere from 0 seconds to a bus never coming. This is a continuous random variable because there are uncountably many values in this range.
- The range of  $C$  is  $\Omega_C = (0, 100)$  because the temperature can be any real number in this range. It cannot be 0 or below because that would be frozen (ice), nor can it be 100 or above because this would be boiling (steam). This is a continuous random variable.

□

### 3.1.2 Probability Mass Functions

Let's return to  $X$  which we defined to be the number of heads in the flip of two fair coins. We already determined that  $\Omega = \Omega = \{HH, HT, TH, TT\}$  and  $X(HH) = 2, X(HT) = 1, X(TH) = 1$  and  $X(TT) = 0$ . The range,  $\Omega_X$ , is  $\{0, 1, 2\}$ .

We can define the **probability mass function (pmf)** of  $X$ , as  $p_X : \Omega_X \rightarrow [0, 1]$ :

$$p_X(k) = \mathbb{P}(X = k)$$

to calculate the probabilities that  $X$  takes on each of these values.

In this case we have the following:

$$p_X(k) = \begin{cases} \frac{1}{4} & k = 0 \\ \frac{1}{2} & k = 1 \\ \frac{1}{4} & k = 2 \end{cases}$$

this is because the number of outcomes for  $X = 0$  is 1 of the 4, the number of outcomes for  $X = 1$  is 2 of the 4, and the number of outcomes for  $X = 2$  is 1 of the 4.

This brings us to the formal definition of a probability mass function:

**Definition 3.1.2: Probability Mass Function (pmf)**

The **probability mass function (pmf)** of a discrete random variable  $X$  assigns probabilities to the possible values of the random variable. That is  $p_X : \Omega_X \rightarrow [0, 1]$  where:

$$p_X(k) = \mathbb{P}(X = k)$$

Note that  $\{X = a\}$  for  $a \in \Omega$  form a partition of  $\Omega$ , since each outcome  $a \in \Omega$  is mapped to exactly one number. Hence,

$$\sum_{z \in \Omega_X} p_X(z) = 1$$

Notice here the only thing consistent is  $p_X$ , as it's the PMF of  $X$ . The value inside is a dummy variable - just like we can write  $f(x) = x^2$  or  $f(t) = t^2$ . To reinforce this, I will constantly use different letters for dummy variables.

### 3.1.3 Expectation

We have this idea of a random variable, which is actually neither random nor a variable (it's a deterministic function  $X : \Omega \rightarrow \Omega_X$ .) However, the way I like to think about it is: it a random quantity which we do not know the value of yet. You might want to know what you might expect it to equal on average. For example,  $X$  could be the random variable which represents the number of babies born in Seattle per day. On average,  $X$  might be equal to 250, and we would write that its average/mean/expectation/expected value is  $\mathbb{E}[X] = 250$ .

Let's go back to the coin example though to define expectation. Your intuition might tell you that the expected number of heads in 2 flips of a fair coin would be 1 (you would be correct).

Since  $X$  was the random variable defined to be the number of heads in 2 flips of a fair coin, we denote this  $\mathbb{E}[X]$ . Think of this as the average value of  $X$ .

More specifically, imagine if we repeated the two coin flip experiment 4 times. Then we would “expect” to get  $HH$ ,  $HT$ ,  $TH$ , and  $TT$  each once. Then, we can divide by the number of times (4) to get 1.

$$\frac{2 + 1 + 1 + 0}{4} = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 1$$

Notice that:

$$\begin{aligned} 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} &= X(HH)\mathbb{P}(HH) + X(HT)\mathbb{P}(HT) + X(TH)\mathbb{P}(TH) + X(TT)\mathbb{P}(TT) \\ &= \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega) \end{aligned}$$

This is the the sum of the random variable's value for each outcome multiplied by the probability of that outcome (a weighted average).

Another way of writing this is by multiplying every value that  $X$  takes on (in its range) with the probability of that value occurring (the pmf). Notice that below is the same exact sum, but groups the common values together (since  $X(HT) = X(TH) = 1$ ). That is:

$$2 \cdot \frac{1}{4} + 1 \cdot \left(\frac{1}{4} + \frac{1}{4}\right) + 0 \cdot \frac{1}{4} = 2 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 0 \cdot \frac{1}{4} = \sum_{k \in \Omega_X} k \cdot p_X(k)$$

This brings us to the definition of expectation.

### Definition 3.1.3: Expectation

The **expectation/expected value/average** of a discrete random variable  $X$  is:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega)$$

or equivalently,

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$$

The interpretation is that we take an average of the possible values, but weighted by their probabilities.

### 3.1.4 Exercises

1. Let  $X$  be the value of single roll of a fair six-sided dice. What is the range  $\Omega_X$ , the PMF  $p_X(k)$ , and the expectation  $\mathbb{E}[X]$ ?

**Solution:** The range is  $\Omega_X = \{1, 2, 3, 4, 5, 6\}$ . The pmf is

$$p_X(k) = \frac{1}{6}, k \in \Omega_X$$

The expectation is

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1}{6}(1 + 2 + \cdots + 6) = 3.5$$

This kind of makes sense right? You expect the “middle number” between 1 and 6, which is 3.5.

2. Suppose at time  $t = 0$ , a frog starts on a 1-dimensional number line at the origin 0. At each step, the frog moves *independently*: left with probability  $1/10$ , and right (with probability  $9/10$ ). Let  $X$  be the position of the frog after 2 time steps. What is the range  $\Omega_X$ , the PMF  $p_X(k)$ , and the expectation  $\mathbb{E}[X]$ ?

**Solution:** The range is  $\Omega_X = \{-2, 0, 2\}$ . To find the pmf, we find the probabilities of being each of those three values.

- For  $X$  to equal  $-2$ , we have to move left both times, which happens with probability  $\frac{1}{10} \cdot \frac{1}{10}$  by independence of the moves.
- For  $X$  to equal  $2$ , we have to move right both times, which happens with probability  $\frac{9}{10} \cdot \frac{9}{10}$  by independence of the moves.
- Finally, for  $X$  to equal  $0$ , we have to take opposite moves. So either LR or RL, which happens with probability  $2 \cdot \frac{1}{10} \cdot \frac{9}{10} = \frac{18}{100}$ . Alternatively, the easier way is to note that these three values sum to 1, so  $\mathbb{P}(X = 0) = 1 - \mathbb{P}(X = 2) - \mathbb{P}(X = -2) = 1 - \frac{81}{100} - \frac{1}{100} = \frac{18}{100}$

So our PMF is:

$$p_X(k) = \begin{cases} 1/100 & k = -2 \\ 18/100 & k = 0 \\ 81/100 & k = 2 \end{cases}$$

The expectation is

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = -2 \cdot \frac{1}{100} + 0 \cdot \frac{18}{100} + 2 \cdot \frac{81}{100} = 1.6$$

You **might** have been able to guess this, but how? At each time step you “expect” to move to the right by  $\frac{9}{10} - \frac{1}{10}$  which is 0.8. So after two steps, you would expect to be at 1.6. We’ll formalize this approach more in the next chapter!

3. Let  $X$  be the number of independent coin flips up to and including our first head, where  $\mathbb{P}(\text{head}) = p$ . What is the range  $\Omega_X$ , the PMF  $p_X(k)$ , and the expectation  $\mathbb{E}[X]$ ?

**Solution:** The range is  $\Omega_X = \{1, 2, 3, \dots\}$ , since it could theoretically take any number of flips. The pmf is

$$p_X(k) = (1-p)^{k-1}p, k \in \Omega_X$$

Why? We can start slowly.

- (a)  $\mathbb{P}(X = 1)$  is the probability we get heads (for the first time) on our first try, which is just  $p$ .
- (b)  $\mathbb{P}(X = 2)$  is the probability we get heads (for the first time) on our second try, which is  $(1-p)p$  since we had to get a tails first.
- (c)  $\mathbb{P}(X = k)$  is the probability we get heads (for the first time) on our  $k^{\text{th}}$  try, which is  $(1-p)^{k-1}p$ , since we had to get all tails on the first  $k-1$  tries (otherwise, our first head would have been earlier).

The expectation is pretty complicated and uses a calculus trick, so don’t worry about it too much. Just understand the first two lines, which are the setup! But before that, what do you think it should be? For example, if  $p = 1/10$ , how many flips do you think it would take until our first head? Possibly 10? And if  $p = 1/7$ , maybe 7? So seems like our guess will be  $\mathbb{E}[X] = \frac{1}{p}$ . It turns out this intuition is actually correct!

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k \in \Omega_X} k \cdot p_X(k) && [\text{def of expectation}] \\ &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\ &= p \sum_{k=1}^{\infty} k(1-p)^{k-1} && [p \text{ is a constant with respect to } k] \\ &= p \sum_{k=1}^{\infty} \frac{d}{dp}(-(1-p)^k) && \left[ \frac{d}{dy} y^k = ky^{k-1} \right] \\ &= -p \left( \frac{d}{dp} \sum_{k=1}^{\infty} (1-p)^{k-1} \right) && [\text{swap sum and integral}] \\ &= -p \left( \frac{d}{dp} \frac{1}{1-(1-p)} \right) && \left[ \text{geometric series formula: } \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \right] \\ &= -p \left( \frac{d}{dp} \frac{1}{p} \right) \\ &= -p \left( -\frac{1}{p^2} \right) \\ &= \frac{1}{p} \end{aligned}$$

## Chapter 3: Discrete Random Variables

### 3.2: More on Expectation

#### 3.2.1 Linearity of Expectation

Right now, the only way you've learned to compute expectation is by first computing the PMF of a random variable  $p_X(k)$  and using the formula  $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$  which is just a weighted sum of the possible values of  $X$ . If you had two random variables  $X$  and  $Y$ , then to compute the expectation of their sum  $Z = X + Y$ , you could compute the PMF of  $Z$  and apply the same formula. But actually, if you knew both  $\mathbb{E}[X]$  and  $\mathbb{E}[Y]$ , you might be inclined to just say  $\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ , and we'll see that this is true! Linearity of expectation is one of the most fundamental and important concepts in probability theory, that you will use almost everywhere! We'll explain it in a simple example, prove it, and then use it to tackle hard problems.

Let's say that you and your friend sell fish for a living. Every day, you catch  $X$  fish, with  $\mathbb{E}[X] = 3$  and your friend catches  $Y$  fish, with  $\mathbb{E}[Y] = 7$ . How many fish do the two of you bring in ( $Z = X + Y$ ) on an average day? You might guess  $3 + 7 = 10$ . This is the formula you just guessed:

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

This property turns out to be true! Furthermore, let's say that you can sell each fish for \$5 at a store, but you need to pay \$20 in rent for the storefront. How much profit do you expect to make? The profit formula would be  $5Z - 20$ : \$5 times the number of total fish, minus \$20. You might guess  $5 \cdot 10 - 20 = 30$  and you would be right once again! This is the formula you just guessed:

$$\mathbb{E}[5Z - 20] = 5\mathbb{E}[Z] - 20 = 5 \cdot 10 - 20 = 30$$

#### Theorem 3.2.9: Linearity of Expectation (LoE)

Let  $\Omega$  be the sample space of an experiment,  $X, Y : \Omega \rightarrow \mathbb{R}$  be (possibly "dependent") random variables both defined on  $\Omega$ , and  $a, b, c \in \mathbb{R}$  be scalars. Then,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

and

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Combining them gives,

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

*Proof of Linearity of Expectation.* Note that  $X$  and  $Y$  are functions (since random variables are functions), so  $X + Y$  is function that is the sum of the outputs of each of the functions. We have the following (in the first equation,  $(X + Y)(\omega)$  is the function  $(X + Y)$  applied to  $\omega$  which is equal to  $X(\omega) + Y(\omega)$ , it is not a

product):

$$\begin{aligned}
 \mathbb{E}[X + Y] &= \sum_{\omega \in \Omega} (X + Y)(\omega) \cdot \mathbb{P}(\omega) && [\text{def of expectation for the rv } X + Y] \\
 &= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) \cdot \mathbb{P}(\omega) && [\text{def of sum of functions}] \\
 &= \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega) + \sum_{\omega \in \Omega} Y(\omega) \cdot \mathbb{P}(\omega) && [\text{property of summation}] \\
 &= \mathbb{E}[X] + \mathbb{E}[Y] && [\text{def of expectation of } X \text{ and } Y]
 \end{aligned}$$

For the second property, note that  $aX + b$  is also a random variable and hence a function (e.g., if  $f(x) = \sin(1/x)$ , then  $(2f - 5)(x) = 2f(x) - 5 = 2\sin(1/x) - 5$ .)

$$\begin{aligned}
 \mathbb{E}[aX + b] &= \sum_{\omega \in \Omega} (aX + b)(\omega) \cdot \mathbb{P}(\omega) && [\text{def of expectation}] \\
 &= \sum_{\omega \in \Omega} (aX(\omega) + b) \cdot \mathbb{P}(\omega) && [\text{def of the function } aX + b] \\
 &= \sum_{\omega \in \Omega} aX(\omega) \cdot \mathbb{P}(\omega) + \sum_{\omega \in \Omega} b \cdot \mathbb{P}(\omega) && [\text{property of summation}] \\
 &= a \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega) + b \sum_{\omega \in \Omega} \mathbb{P}(\omega) && [\text{property of summation}] \\
 &= a\mathbb{E}[X] + b && [\text{def of } \mathbb{E}[X] \text{ and } \sum_{\omega} \mathbb{P}(\omega) = 1]
 \end{aligned}$$

For the last property, we get to assume the first two that we proved already:

$$\begin{aligned}
 \mathbb{E}[aX + bY + c] &= \mathbb{E}[aX] + \mathbb{E}[bY] + \mathbb{E}[c] && [\text{property 1}] \\
 &= a\mathbb{E}[X] + b\mathbb{E}[Y] + c && [\text{property 2}]
 \end{aligned}$$

□

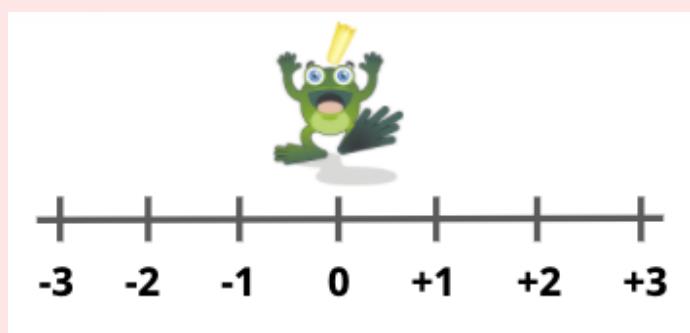
Again, you may think a result like this is “trivial” or “obvious”, but we’ll see the true power of linearity of expectation through examples. It is one of the most important ideas that you will continue to use (and probably take for granted), even when studying some of the most complex topics in probability theory.

### Example(s)

A frog starts on a 1-dimensional number line at 0. At each time step, it moves

- left with probability  $p_L$
- right with probability  $p_R$
- stays with probability  $p_s$

where  $p_L + p_R + p_s = 1$ . Let  $X$  be the position of the frog after 2 (independent) time steps. What is  $\mathbb{E}[X]$ ?



**Brute Force Solution:** When dealing with any random variable, the first thing you should do is identify its range. The frog must end up in one of these positions, since it can move at most 1 to the left and 1 to the right at each step:

$$\Omega_X = \{-2, -1, 0, +1, +2\}$$

So we need to compute 5 values: the probability of each of these. Let's start with the easier ones. The only way to end up at  $-2$  is if the frog moves left at both steps, which happens with probability  $p_L \cdot p_L = p_L^2 = \mathbb{P}(X = -2) = p_X(-2)$ . The only reason we can multiply them is because of our independence assumption. Similarly,  $p_X(2) = p_R \cdot p_R = p_R^2$ .

To get to  $-1$ , there are two possibilities: first going left and staying ( $p_L \cdot p_S$ ), or first staying and then going left ( $p_S \cdot p_L$ ). Adding these disjoint cases gives  $p_X(-1) = 2p_L p_S$ . Again, we can only multiply due to independence. Similarly,  $p_X(1) = 2p_R p_S$ .

Finally, to compute  $p_X(0)$ , we have two options. One is considering all the possibilities (there are three: left right, right left, or stay stay) and adding them up, and you get  $2p_L p_R + p_S^2$ . Alternatively and equivalently, since you know the probabilities of 4 of the values ( $p_X(-2), p_X(2), p_X(-1), p_X(1)$ ), the last one  $p_X(0)$  must be 1 minus the other four since probabilities have to sum to 1! This is a often useful and clever trick - solving for all but one of the probabilities actually gives you the last one!

In summary, we would write the PMF as:

$$p_X(k) = \begin{cases} p_L^2 & k = -2 \quad : \text{Left left} \\ 2p_L p_S & k = -1 \quad : \text{Left and stay, or stay and left} \\ 2p_L p_R + p_S^2 & k = 0 \quad : \text{Right left, or left right, or stay stay} \\ 2p_R p_S & k = 1 \quad : \text{Right and stay, or stay and right} \\ p_R^2 & k = 2 \quad : \text{Right right} \end{cases}$$

Then to solve for the expectation we just multiply the value and probability mass function and take the sum and have the following:

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k \in \Omega_X} k \cdot p_X(k) && [\text{def of expectation}] \\ &= (-2) \cdot p_L^2 + (-1) \cdot 2p_L p_S + (0) \cdot (2p_L p_R + p_S^2) + (1) \cdot 2p_R p_S + (2) \cdot p_R^2 && [\text{plug in our values}] \\ &= 2(p_R - p_L) && [\text{lots of messy algebra}] \end{aligned}$$

The last step of algebra is not important - once you get to more advanced mathematics (like this text), getting the second-to-last formula is sufficient. Everything else is algebra which you could do, or use a computer to do, and so we will omit the useless calculations.

This was quite tedious already; what if instead you were to find the expected location after 100 steps? Then, this method would be completely ridiculous: finding  $\Omega_X = \{-100, -99, \dots, +99, +100\}$  and their 201 probabilities. Since you know the frog always moves with the same probabilities though, maybe we can do something more clever!

**Linearity Solution:**

Let  $X_1, X_2$  be the distance the frog travels at time steps 1,2 respectively.

Important Observation:  $X = X_1 + X_2$ , since your location after 2 time steps is the sum of the displacement of the first time step and the second time step. Therefore,  $\Omega_{X_1} = \Omega_{X_2} = \{-1, 0, +1\}$ . They have the same simple PMF of:

$$p_{X_i}(k) = \begin{cases} p_L & k = -1 \\ p_S & k = 0 \\ p_R & k = 1 \end{cases}$$

So:  $\mathbb{E}[X_i] = -1 \cdot p_L + 0 \cdot p_S + 1 \cdot p_R = p_R - p_L$ , for both  $i = 1$  and  $i = 2$ .

By linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 2(p_R - p_L)$$

Which method is easier? Maybe in this case it is debatable, but if we change the time steps from 2 to 100 or 1000, the brute force solution is entirely infeasible, and the linearity solution will basically be the same amount of work! You could say that  $X_1, \dots, X_{100}$  is the displacement at each of 100 time steps, and hence by linearity:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{100} X_i\right] = \sum_{i=1}^{100} \mathbb{E}[X_i] = \sum_{i=1}^{100} (p_R - p_L) = 100(p_R - p_L)$$

Hopefully now you can come to appreciate more how powerful LoE truly is! We'll see more examples in the next section as well as at the end of this section.

### 3.2.2 Law of the Unconscious Statistician (LOTUS)

Recall the fish example at the beginning of this section regarding profit. Since the expected number of fish was  $\mathbb{E}[Z] = 10$  and the profit was a function of the number of fish  $g(Z) = 5Z - 20$ , we were able to use linearity to say  $\mathbb{E}[5Z - 20] = 5\mathbb{E}[Z] - 20$ . But this formula only holds for nice linear functions (hence the name “**linearity** of expectation”). What if the profit function was instead something weird/non-linear like  $h(Z) = Z^2$  or  $h(Z) = \log(5^Z)$ ? It turns out we can't just say  $\mathbb{E}[Z^2] = \mathbb{E}[Z]^2$  or  $\mathbb{E}[\log(5^Z)] = \log(5^{\mathbb{E}[Z]})$  - this is actually almost never true! Let's see if we can't derive a nice formula for  $\mathbb{E}[g(X)]$  for *any* function  $g$ , linear or not.



Consider we are flipping 2 coins again. Let  $X$  be the number of heads in two independent flips of a fair coin. Recall the range, PMF, and expectation (again, I'm using the dummy letter  $d$  to emphasize that  $p_X$  is the PMF for  $X$ , and the inner variable doesn't matter):

$$\Omega_X = \{0, 1, 2\}$$

,

$$p_X(d) = \begin{cases} \frac{1}{4} & d = 0 \\ \frac{1}{2} & d = 1 \\ \frac{1}{4} & d = 2 \end{cases}$$

$$\mathbb{E}[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

Let  $g$  be the cubing function; i.e.,  $g(t) = t^3$ . Let  $Y = g(X) = X^3$ ; what does this mean? It literally means the cubed number of heads! Let's try to compute  $\mathbb{E}[Y] = \mathbb{E}[X^3]$ , the expected cubed number of heads. We first find its range and PMF. Based on the range of  $X$ , we can calculate the range of  $Y$  to be:

$$\Omega_Y = \{0, 1, 8\}$$

since if we get 0 heads, the cubed number of heads is  $0^3 = 0$ ; if we get 1 head, the cubed number of heads is  $1^3 = 1$ ; and if we get 2 heads, the cubed number of heads is  $2^3 = 8$ .

Now to find the PMF of  $Y = X^3$ . (Again, below I use the notation  $p_Y$  to denote the probability mass function of  $Y = X^3$ ;  $z$  is a dummy variable which could be any letter.)

$$p_Y(z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 8 \end{cases}$$

since there is a  $1/4$  chance of getting 0 cubed heads (the outcome TT),  $1/2$  chance of getting 1 cubed heads (the outcomes HT or TH), and a  $1/4$  chance of getting 8 cubed heads (the outcome HH).

$$\mathbb{E}[X^3] = \mathbb{E}[Y] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4} = 2.5$$

**Is there an easier way to compute  $\mathbb{E}[X^3] = \mathbb{E}[Y]$  without going through the trouble of writing out  $p_Y$ ?** Yes! Since we know  $X$ 's PMF already, why should we have to find the PMF of  $Y = g(X)$ ?

Note this formula below is the same formula as above, rewritten so you can observe something:

$$\mathbb{E}[X^3] = 0^3 \cdot \frac{1}{4} + 1^3 \cdot \frac{1}{2} + 2^3 \cdot \frac{1}{4} = 2.5$$

In fact:

$$\mathbb{E}[X^3] = \sum_{b \in \Omega_X} b^3 p_X(b)$$

That is, we can apply the function to each value in  $\Omega_X$ , and then take the weighted average! We can generalize such that for any function  $g : \Omega_X \rightarrow \mathbb{R}$ , we have:

$$\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) p_X(b)$$

**Caveat:** It is worth noting that  $2.5 = \mathbb{E}[X^3] \neq (\mathbb{E}[X])^3 = 1$ . You cannot just say  $\mathbb{E}[g(X)] = g(\mathbb{E}[X])$  as we just showed!

Theorem 3.2.10: Law of the Unconscious Statistician (LOTUS)

Let  $X$  be a discrete random variable with range  $\Omega_X$  and  $g : D \rightarrow \mathbb{R}$  be a function defined at least over  $\Omega_X$ , ( $\Omega_X \subseteq D$ ). Then

$$\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b)p_X(b)$$

Note that in general,  $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$ . For example,  $\mathbb{E}[X^2] \neq (\mathbb{E}[X])^2$ , and  $\mathbb{E}[\log(X)] \neq \log(\mathbb{E}[X])$ .

Before we formally prove this, it will help if we have some intuition for each step. As an example, let  $X$  have range  $\Omega_X = \{-1, 0, 1\}$  and PMF

$$p_X(k) = \begin{cases} \frac{3}{12} & k = -1 \\ \frac{5}{12} & k = 0 \\ \frac{4}{12} & k = 1 \end{cases}$$

Notice that  $Y = X^2$  has range  $\Omega_Y = \{g(x) : x \in \Omega_X\} = \{(-1)^2, 0^2, 1^2\} = \{0, 1\}$  and the following PMF:

$$p_Y(k) = \begin{cases} \frac{3}{12} + \frac{4}{12} & k = 1 \\ \frac{5}{12} & k = 0 \end{cases}$$

Note that  $p_Y(1) = \mathbb{P}(X = -1) + \mathbb{P}(X = 1)$  because  $\{-1, 1\} = \{x : x^2 = 1\}$ . The crux of the LOTUS proof depends on this fact. We just group things together and sum!

*Proof of LOTUS.* The proof isn't too complicated, but the notation is pretty tricky and may be an impediment to your understanding, so focus on understanding the setup in the next few lines.

Let  $Y = g(X)$ . Note that

$$p_Y(y) = \sum_{x \in \Omega_X : g(x)=y} p_X(x)$$

That is, the total probability that  $Y = y$  is the sum of the probabilities over all  $x \in \Omega_X$  where  $g(x) = y$  (this is like saying  $\mathbb{P}(Y = 1) = \mathbb{P}(X = -1) + \mathbb{P}(X = 1)$  because  $\{x \in \Omega_X : x^2 = 1\} = \{-1, 1\}$ .)

$$\begin{aligned} \mathbb{E}[g(X)] &= \mathbb{E}[Y] && [Y = g(X)] \\ &= \sum_{y \in \Omega_Y} y p_Y(y) && [\text{def of expectation}] \\ &= \sum_{y \in \Omega_Y} y \sum_{x \in \Omega_X : g(x)=y} p_X(x) && [\text{above substitution}] \\ &= \sum_{y \in \Omega_Y} \sum_{x \in \Omega_X : g(x)=y} y p_X(x) && [\text{move } y \text{ into a sum}] \\ &= \sum_{y \in \Omega_Y} \sum_{x \in \Omega_X : g(x)=y} g(x) p_X(x) && [y = g(x) \text{ in the inner sum}] \\ &= \sum_{x \in \Omega_X} g(x) p_X(x) && [\text{the double sum is the same as summing over all } x] \end{aligned}$$

□

### 3.2.3 Exercises

1. Let  $S$  be the sum of three rolls of a fair 6-sided die. What is  $\mathbb{E}[S]$ ?

**Solution:** Let  $X, Y, Z$  be the first, second, and third roll respectively. Then,  $S = X + Y + Z$ . We showed in the first exercise of 3.1 that  $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z] = 3.5$ , so by LoE,

$$\mathbb{E}[S] = \mathbb{E}[X + Y + Z] = \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[Z] = 3.5 + 3.5 + 3.5 = 10.5$$

Alternatively, imagine if we didn't have this theorem. We would find the range of  $S$ , which is  $\Omega_S = \{3, 4, \dots, 18\}$  and find its PMF. What a nightmare!

2. Blind LOTUS Practice: This will all seem useless, but I promise we'll need this in the future. Let  $X$  have PMF

$$p_X(k) = \begin{cases} \frac{3}{12} & k = 5 \\ \frac{5}{12} & k = 2 \\ \frac{4}{12} & k = 1 \end{cases}$$

- (a) Compute  $\mathbb{E}[X^2]$ .
- (b) Compute  $\mathbb{E}[\log(X)]$
- (c) Compute  $\mathbb{E}[e^{\sin(X)}]$ .

**Solution:** LOTUS says that  $\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k)p_X(k)$ . That is,

(a)

$$\mathbb{E}[X^2] = \sum_{k \in \Omega_X} k^2 p_X(k) = 5^2 \cdot \frac{3}{12} + 2^2 \cdot \frac{5}{12} + 1^2 \cdot \frac{4}{12}$$

(b)

$$\mathbb{E}[\log X] = \sum_{k \in \Omega_X} \log(k) \cdot p_X(k) = \log(5) \cdot \frac{3}{12} + \log(2) \cdot \frac{5}{12} + \log(1) \cdot \frac{4}{12}$$

(c)

$$\mathbb{E}[e^{\sin(X)}] = \sum_{k \in \Omega_X} e^{\sin(k)} p_X(k) = e^{\sin(5)} \cdot \frac{3}{12} + e^{\sin(2)} \cdot \frac{5}{12} + e^{\sin(1)} \cdot \frac{4}{12}$$

## Chapter 3: Discrete Random Variables

### 3.3: Variance

#### 3.3.1 Linearity of Expectation with Indicator RVs

We've seen how useful and important linearity of expectation was (e.g., with the frog example). We'll now see how to apply it in a very clever way that is very commonly used to solve seemingly difficult problems.

Suppose there are 7 mermaids in the sea. Below is a table that represents these mermaids and the colors of their hair.

Mermaid	1	2	3	4	5	6	7
Color	RED	BLUE	PURPLE	RED	BLACK	YELLOW	RED
1 / 0	1	0	0	1	0	0	1

Each column in the third row of the table is a variable,  $X_i$ , that is 1 if the  $i$ -th mermaid has red hair and 0 otherwise. We call these sorts of variables *indicator variables* because they are either 1 or 0, and their values indicate the truth of a boolean (red hair or not).

Let the variable  $X$  represent how many of the 7 mermaids have red hair. If I only gave you this third row ( $X_1, X_2, \dots, X_7$  of 1's and 0's), how could you compute  $X$ ?

Well, you would add them all up!  $X = X_1 + X_2 + \dots + X_7 = 3$ . So, there are 3 mermaids in the sea that have red hair. This might seem like a trivial result, but let's go over a more complicated example to illustrate the usefulness of indicator random variables!

#### Example(s)

Suppose  $n$  people go to a party and leave their hat with the hat-check person. At the end of the party, she returns hats randomly and uniformly because she does not care about her job. Let  $X$  be the number of people who get their original hat back. What is  $\mathbb{E}[X]$ ?

*Solution* Your first instinct might be to approach this problem with brute force. Such an approach would involve enumerating the range,  $\Omega_X = \{0, 1, 2, \dots, n-2, n\}$  (all the integers from 0 to  $n$ , except  $n-1$ ), and computing the probability mass function for each of its elements. However, this approach will get very complicated (give it a shot). So, let's use our new friend, linearity of expectation.

**Quick Observation:** Does it matter where you are in line?

If we are first in line,  $\mathbb{P}(\text{gets hat back}) = \frac{1}{n}$ , because there are  $n$  in total and each is equally likely.

If we are last in line,  $\mathbb{P}(\text{gets hat back}) = \frac{1}{n}$ , because there is one left and its just as likely to be yours as any other hat after giving away  $n-1$ .

(Similar logic applies to the other positions in between as well). So actually, no, the probability that someone

will get their original hat back does NOT depend on where they are in line. Each person gets their hat back with probability  $\frac{1}{n}$ .

Let's use linearity with indicator random variables! For  $i = 1, \dots, n$ , let

$$X_i = \begin{cases} 1 & \text{if } i\text{-th person got their hat back} \\ 0 & \text{otherwise} \end{cases}.$$

Then the total number of people who get their hat back is  $X = \sum_{i=1}^n X_i$ . (Why?)

The expected value of each individual indicator random variable can be found as follows, since it can only take on the values 0 and 1:

$$\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \mathbb{P}(i\text{th person got their hat back}) = \frac{1}{n}$$

From here, we will use linearity of expectation:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\ &= \sum_{i=1}^n \mathbb{E}[X_i] && [\text{linearity of expectation}] \\ &= \sum_{i=1}^n \frac{1}{n} \\ &= n \cdot \frac{1}{n} \\ &= 1 \end{aligned}$$

So, the expected number of people to get their hats back is 1 (doesn't even depend on  $n$ )! It is worth noting that these indicator random variables are *not* "independent" (we'll define this formally later). One of the reasons why is because if we know that a particular person did not get their own hat back, then the original owner of that hat will have a probability of 0 that they get that hat back.  $\square$

#### Theorem 3.3.11: Linearity of Expectation with Indicators

If asked only about the expectation of a random variable  $X$  (and not its PMF), then you may be able to write  $X$  as the sum of possibly dependent indicator random variables, and apply linearity of expectation. This technique is used when  $X$  is counting something (the number of people who get their hat back). Finding the PMF for this random variable is extremely complicated, and linearity makes computing the expectation easy (or at least easier than directly finding the PMF).

For an indicator random variable  $X_i$ ,

$$\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1)$$

#### Example(s)

Suppose we flip a coin  $n = 100$  times independently, where the probability of getting a head on each flip is  $p = 0.23$ . What is the expected number of heads we get? Before doing any computation, what

do you think it might be?

*Solution* You might expect  $np = 100 \cdot 0.23 = 23$  heads, and you would be absolutely correct! But we do need to prove/show this.

Let  $X$  be the number of heads total, so  $\Omega_X = \{0, 1, 2, \dots, 100\}$ . The “normal” approach might be to try to find this PMF, which could be a bit complicated (we’ll actually see this in the next section)! But let’s try to use what we just learned instead, and define indicators.

For  $i = 1, 2, \dots, 100$ , let  $X_i = 1$  if the  $i$ -th flip is heads, and  $X_i = 0$  otherwise. Then,  $X = \sum_{i=1}^{100} X_i$  is the total number of heads (why?). To use linearity, we need to find  $\mathbb{E}[X_i]$ .

We showed earlier that

$$\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = p = 0.23$$

and so

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^{100} X_i\right] && [\text{def of } X] \\ &= \sum_{i=1}^{100} \mathbb{E}[X_i] && [\text{linearity of expectation}] \\ &= \sum_{i=1}^{100} 0.23 \\ &= 100 \cdot 0.23 = 23 \end{aligned}$$

□

### 3.3.2 Variance

We’ve talked about the expectation (average/mean) of a random variable, and some approaches to computing this quantity. This provides a nice “summarization” of a random variable, as something we often want to know about it (sometimes even in place of its PMF). But we might want to know another summary quantity: how “variable” the random variable is, or how much it deviates from its mean. This is called the *variance* of a random variable, and we’ll start with a motivating example below!

Consider the following two games. In both games we flip a fair coin. In Game 1, if a heads is flipped you pay me \$1, and if a tails is flipped I pay you \$1. In Game 2, if a heads is flipped you pay me \$1000, and if a tails is flipped I pay you \$1000.

Both games are fair, in the sense that the expected values of playing both games is 0.

$$\mathbb{E}[G_1] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0 = -1000 \cdot \frac{1}{2} + 1000 \cdot \frac{1}{2} = \mathbb{E}[G_2]$$

Which game would you rather play? Maybe the adrenaline junkies among us would be willing to risk it all on Game 2, but I think most of us would feel better playing Game 1. As shown above, there is no difference in the expected value of playing these two games, so we need another metric to explain why Game 1 feels safer than Game 2.

We can measure this by calculating how far away a random variable is from its mean, on average. The quantity  $X - \mathbb{E}[X]$  is the difference between a rv and its mean, but we want a distance, a positive value.

So we will look at the squared difference  $(X - \mathbb{E}[X])^2$  instead (another option would have been the absolute difference  $|X - \mathbb{E}[X]|$ , but someone chose the squared one instead). This is still a random variable (a nonnegative one, since it is squared), and so to get a number (the average distance from the mean), we take the *expectation* of this new rv,  $\mathbb{E}[(X - \mathbb{E}[X])^2]$ . This is called the variance of the original random variable. The definition goes as follows:

### Definition 3.3.1: Variance

The variance of a random variable  $X$  is defined to be

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

The variance is always nonnegative since we take the expectation of a nonnegative random variable  $(X - \mathbb{E}[X])^2$ . The first equality is the *definition* of variance, and the second equality is a more useful identity for doing computation that we show below.

*Proof of Variance Identity.*

Let  $\mu = \mathbb{E}[X]$  as a shorthand. Then,

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[(X - \mu)^2] && [\text{def of variance}] \\ &= \mathbb{E}[X^2 - 2\mu X + \mu^2] && [\text{algebra}] \\ &= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 && [\text{linearity of expectation}] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 && [\mu = \mathbb{E}[X]] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \end{aligned}$$

Notice that  $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$  - this is a perfect time to point this out again. If these were equal, variance would always be zero and this would be a useless construct.  $\square$

The reason that someone chose the squared definition instead of the absolute value definition is because it has this nice splitting property  $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$  (the absolute value definition wouldn't have something this nice), and because the squaring function  $g(t) = t^2$  is differentiable but the absolute value function  $g(t) = |t|$  is not.

There is one problem though - if  $X$  is the height of someone in feet for example, then the average  $\mathbb{E}[X]$  is also in units of feet, but the variance is in terms of square feet (since we square  $X$ ). We'd like to say something like: the height of adults is generally 5.5 feet plus or minus 0.3 feet. To correct for this, we define the standard deviation to be the square root of the variance, which "undoes" the squaring.

### Definition 3.3.2: Standard Deviation

Another measure of a random variable  $X$ 's spread is the **standard deviation**, which is

$$\sigma_X = \sqrt{\text{Var}(X)}$$

This measure is also useful, because the units of variance are squared in terms of the original variable  $X$ , and this essentially "undoes" our squaring, returning our units to the same as  $X$ .

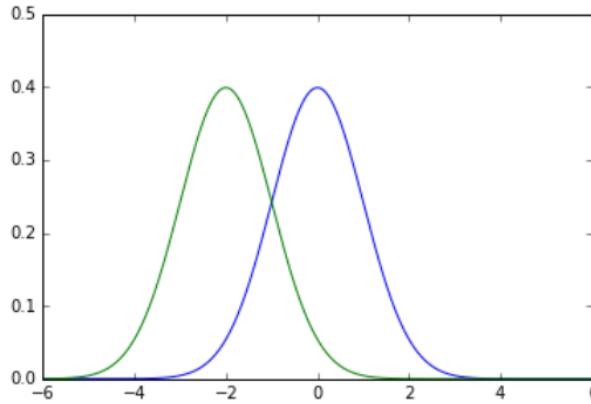
We had something nice happen for the random variable  $aX + b$  when computing its expectation:  $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ , called linearity of expectation. Is there a similar nice property for the variance as well?

## Theorem 3.3.12: Property of Variance

We can also show that for any scalar  $a, b \in \mathbb{R}$ ,

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Before proving this, let's think about and try to understand why  $a$  came out squared, and what happened to the  $b$ . The reason  $a$  is squared is because variance involved squaring the random variable, so the  $a$  had to come out squared. It might not be a great intuitive reason, but we'll prove it below algebraically. The second ( $b$  disappearing) has a nice intuition behind it. Which of the two distributions (random variables) below do you think should have higher variance?



You might agree with me that they have the same variance! Why?

The idea behind variance is that it measures the “spread” of the values that a random variable can take on. The two graphs of random variables (distributions) above have the same “spread”, but one is shifted slightly to the right. Since these graphs have the same “spread”, we want their variance to reflect this similarity. Thus, shifting a random variable by some constant does not change the variance of that random variable. That is,  $\text{Var}(X + b) = \text{Var}(X)$ : that's why the  $b$  got lost!

*Proof of Variance Property:*  $\text{Var}(aX + b) = a^2\text{Var}(X)$

First, we show variance is unaffected by shifts; that is,  $\text{Var}(X + b) = \text{Var}(X)$  for any scalar  $b$ . We use the original definition that  $\text{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}[Y])^2]$ , with  $Y = X + b$ .

$$\begin{aligned} \text{Var}(X + b) &= \mathbb{E}[(X + b - \mathbb{E}[X + b])^2] && [\text{def of variance}] \\ &= \mathbb{E}[(X + b - \mathbb{E}[X] - b)^2] && [\text{linearity of expectation}] \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \text{Var}(X) && [\text{def of variance}] \end{aligned}$$

Then, we use this result to get the final one:

$$\begin{aligned}
 \text{Var}(aX + b) &= \text{Var}(aX) && [\text{shifts don't matter}] \\
 &= \mathbb{E}[(aX)^2] - (\mathbb{E}[aX])^2 && [\text{property of variance}] \\
 &= \mathbb{E}[a^2X^2] - (a\mathbb{E}[X])^2 && [\text{linearity of expectation}] \\
 &= a^2\mathbb{E}[X^2] - a^2\mathbb{E}[X]^2 && [\text{linearity of expectation}] \\
 &= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) \\
 &= a^2\text{Var}(X) && [\text{def of variance}]
 \end{aligned}$$

□

### Example(s)

Let  $X$  be the outcome of a fair 6-sided die roll. Recall that  $\mathbb{E}[X] = 3.5$ . What is  $\text{Var}(X)$ ? Let's say you play a casino game, where you must pay \$10 to roll this die once, but earn twice the value of the roll. What are the expected value and variance of your earnings?

*Solution* Recall that one of the equations for variance is

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Computing the expected value of  $X$  we get

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} xp_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = 3.5$$

And using LOTUS, we can compute the expected value of  $X^2$  to be

$$\mathbb{E}[X^2] = \sum_{x \in \Omega_X} x^2 p_X(x) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \dots + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$

Putting these together with our definition above gives

$$\text{Var}(X) = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

Now, let  $Y$  denote our earnings; then  $Y = 2X - 10$ . So by linearity of expectation,

$$\mathbb{E}[2X - 10] = 2\mathbb{E}[X] - 10 = 2 \cdot 3.5 - 10 = -3$$

By the property of variance, we get

$$\text{Var}(2X - 10) = 2^2\text{Var}(X) = 2^2 \cdot \frac{35}{12} = \frac{35}{3}$$

□

Now you might wonder, what about the variance of a sum  $\text{Var}(X + Y)$ ? You might hope that  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ , but this unfortunately is only true when the random variables are independent (we'll define this in the next section, but you can kind of guess what it means)! It is so important to remember that we made no independence assumptions for linearity of expectation - it's always true!

### 3.3.3 Exercises

1. Suppose you studied hard for a 100-question multiple-choice exam (with 4 choices per question) so that you believe you know the answer to about 80% of the questions, and you guess the answer to the remaining 20%. What is the expected number of questions you answer correctly?

**Solution:** For  $i = 1, \dots, 100$ , let  $X_i$  be the indicator rv which is 1 if you got the  $i^{th}$  question correct, and 0 otherwise. Then, the total number of questions correct is  $X = \sum_{i=1}^{100} X_i$ . To compute  $\mathbb{E}[X]$  we need  $\mathbb{E}[X_i]$  for each  $i = 1, \dots, 100$ .

$$\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \mathbb{P}(\text{correct on question } i) = 1 \cdot 0.8 + 0.25 \cdot 0.2 = 0.85$$

where the second last step was using the law of total probability, conditioning on whether we know the answer to a question or not. Hence,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{100} X_i\right] = \sum_{i=1}^{100} \mathbb{E}[X_i] = \sum_{i=1}^{100} 0.85 = 85$$

This kind of makes sense - I should be guaranteed 80 out of 100, and if I guess on the other 20, I would get about 5 (a quarter of them) right, for a total of 85.

2. Recall exercise 2 from 3.1, where we had a random variable  $X$  with PMF

$$p_X(k) = \begin{cases} 1/100 & k = -2 \\ 18/100 & k = 0 \\ 81/100 & k = 2 \end{cases}$$

The expectation was

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = -2 \cdot \frac{1}{100} + 0 \cdot \frac{18}{100} + 2 \cdot \frac{81}{100} = 1.6$$

Compute  $\text{Var}(X)$ .

**Solution:** Since  $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ , we need to use LOTUS to compute the first part.

$$\mathbb{E}[X^2] = \sum_{k \in \Omega_X} k^2 \cdot p_X(k) = (-2)^2 \cdot \frac{1}{100} + 0^2 \cdot \frac{18}{100} + 2^2 \cdot \frac{81}{100} = 3.28$$

Hence,

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3.28 - 1.6^2 = 0.72$$

## Chapter 3: Discrete Random Variables

### 3.4: Zoo of Discrete Random Variables Part I

In this section, we'll define formally what it means for random variables to be independent. Then, for the rest of the chapter (3.4, 3.5, 3.6), we'll discuss commonly appearing random variables for which we can just cite its properties like its PMF, mean, and variance without doing any work! These situations are so common that we name them, and can refer to them and related quantities easily!

#### 3.4.1 Independence of Random Variables

##### Definition 3.4.1: Independence

Random variables  $X$  and  $Y$  are independent, denoted  $X \perp Y$ , if for all  $x \in \Omega_X$  and all  $y \in \Omega_Y$ , any of the following three equivalent properties holds:

1.  $\mathbb{P}(X = x | Y = y) = \mathbb{P}(X = x)$
2.  $\mathbb{P}(Y = y | X = x) = \mathbb{P}(Y = y)$
3.  $\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$

Note, that this is the same as the event definition of independence, but it must hold for all events  $\{X = x\}$  and  $\{Y = y\}$ .

##### Theorem 3.4.13: Variance Adds for Independent RVs

If  $X \perp Y$ , then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

This will be proved a bit later, but we can start using this fact now! It is important to remember that you *cannot* use this formula if the random variables are not independent (unlike linearity).

A common misconception is that  $\text{Var}(X - Y) = \text{Var}(X) - \text{Var}(Y)$ , but this actually isn't true, otherwise we could get a negative number. In fact, if  $X \perp Y$ , then

$$\text{Var}(X - Y) = \text{Var}(X + (-Y)) = \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X) + (-1)^2 \text{Var}(Y) = \text{Var}(X) + \text{Var}(Y)$$

#### 3.4.2 The Bernoulli Process and Bernoulli Random Variable

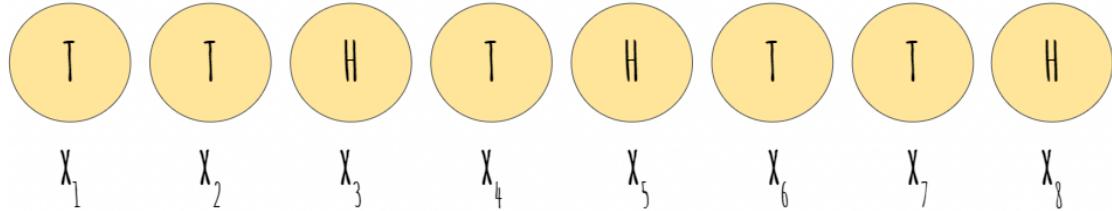
There are several random variables that occur naturally and frequently! It is often useful to be able to recognize these random variables by their characterization, so we can take advantage of relevant properties such as probability mass functions, expected values, and variance. In the rest of this section and chapter 3, we will explore some fundamental discrete random variables, while finding those aforementioned properties, so that we can just cite them instead of doing all the work again!

Before diving into the random variables themselves, let's look at a situation that arises often...

**Definition 3.4.2: Bernoulli Process**

A Bernoulli process with parameter  $p$  is a sequence of independent coin flips  $X_1, X_2, X_3, \dots$  where  $\mathbb{P}(\text{head}) = p$ . If flip  $i$  is heads, then we encode  $X_i = 1$ ; otherwise,  $X_i = 0$ . From this process we can measure many interesting things.

Let's illustrate how this might be useful with an example. Suppose we independently flip 8 coins that land heads with probability  $p$ , and get the following sequence of coin flips



This series of flips is a Bernoulli process. We call each of these coin flips a Bernoulli random variable (or indicator rv).

**Definition 3.4.3: Bernoulli/Indicator Random Variable**

A random variable  $X$  is Bernoulli (or indicator), denoted  $X \sim \text{Ber}(p)$ , if and only if  $X$  has the following PMF:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

Each  $X_i$  in the Bernoulli process with parameter  $p$  is Bernoulli/indicator random variable with parameter  $p$ . It simply represents a binary outcome, like a coin flip.

Additionally,

$$\mathbb{E}[X] = p \text{ and } \text{Var}(X) = p(1 - p)$$

*Proof of Expectation and Variance of Bernoulli.*

Suppose  $X \sim \text{Ber}(p)$ . Then

$$\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

Now for the variance, we compute  $\mathbb{E}[X^2]$  first by LOTUS:

$$\mathbb{E}[X^2] = 1^2 \cdot p + 0^2 \cdot (1 - p) = p$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$$

□

Notice how we found a situation whose general form comes up quite often, and derived a random variable that models that situation well. Now, anytime we need a Bernoulli/indicator random variable we can denote it as follows:  $X \sim \text{Ber}(p)$ .

### 3.4.3 The Binomial Random Variable

If you recall, one of the main reasons that indicator random variables are useful is because they can compose more complicated random variables. For example, in the sequence of coin flips above, we might be interested in modeling the probability of getting a certain number of heads. In order to do this, it would be useful to use a random variable that is equal to the sum of the Bernoulli's that each represent a single flip. These types of random variables also come up frequently, so we have a special name for them, binomial random variables.

$$X \sim \text{Bin}(n, p) = \sum_{i=1}^n X_i, \text{ where } X_i \text{'s are independent Bernoulli random variables}$$

That is, we write  $X \sim \text{Bin}(n, p)$  to be the number of heads in  $n$  *independent* flips of a coin with  $\mathbb{P}(\text{head}) = p$ . Why is it true that if  $X$  is the number of heads in  $n$  flips, that  $X = \sum_{i=1}^n X_i$  (recall the mermaids in 3.3)?

Let's try to derive the PMF of a binomial rv. Its range is  $\Omega_X = \{0, 1, \dots, n\}$  since we can get anywhere from 0 heads to  $n$  heads. Let's consider the case of  $n = 5$  flips, and figure out the probability we get exactly 4 heads,  $\mathbb{P}(X = 4)$ .

Here's one sample sequence of heads and tails with exactly four heads, HTHHH, and its probability is (by independence):

$$\mathbb{P}(\text{HTHHH}) = p \cdot (1-p) \cdot p \cdot p \cdot p = p^4(1-p)^{5-4}$$

But this is not the only sequence of flips which gives exactly 4 heads! How many such sequences are there? There are  $\binom{5}{4}$  since we choose 4 out of the 5 positions to put the heads, and the remaining must be tails. Haha, so our counting knowledge is finally being applied! So we must sum these  $\binom{5}{4}$  disjoint cases, and we get

$$\mathbb{P}(X = 4) = \binom{5}{4} p^4(1-p)^{5-4}$$

We can generalize this as follows to get the PMF of a binomial random variable:

$$p_X(k) = \mathbb{P}(X = k) = \mathbb{P}(\text{exactly } k \text{ heads in } n \text{ Bernoulli trials}) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \Omega_X$$

This hopefully sheds some light on why  $\binom{n}{k}$  is called a binomial coefficient and  $X$  a binomial random variable. Before computing its expectation, let's make sure we didn't make a mistake, and check that our probabilities sum to 1. This will use the binomial *theorem* we learned in chapter 1 finally:  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ .

$$\begin{aligned} \sum_{k=0}^n p_X(k) &= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} && [\text{PMF of Binomial RV}] \\ &= (p + (1-p))^n && [\text{binomial theorem}] \\ &= 1^n = 1 \end{aligned}$$

#### Definition 3.4.4: Binomial Random Variable

A random variable  $X$  has a Binomial distribution, denoted  $X \sim \text{Bin}(n, p)$ , if and only if  $X$  has the

following PMF for  $k \in \Omega_X = \{0, 1, 2, \dots, n\}$ :

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$X$  is the sum of  $n$  independent  $\text{Ber}(p)$  random variables, and represents the number of heads in  $n$  independent coin flips where  $\mathbb{P}(\text{head}) = p$ .

Additionally,

$$\mathbb{E}[X] = np \text{ and } \text{Var}(X) = np(1-p)$$

*Proof of Expectation and Variance of Binomial.* We can use linearity of expectation to compute the expected value of a particular binomial variable (i.e. the expected number of successes in  $n$  Bernoulli trials). Let  $X \sim \text{Bin}(n, p)$ .

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\ &= \sum_{i=1}^n \mathbb{E}[X_i] && [\text{linearity of expectation}] \\ &= \sum_{i=1}^n p && [\text{expectation of Bernoulli}] \\ &= np \end{aligned}$$

This makes sense! If  $X \sim \text{Bin}(100, 0.5)$  (number of heads in 100 independent flips of a fair coin), you expect 50 heads, which is just  $np = 100 \cdot 0.5 = 50$ . Variance can be found in a similar manner

$$\begin{aligned} \text{Var}(X) &= \text{Var}\left(\sum_{i=1}^n X_i\right) \\ &= \sum_{i=1}^n \text{Var}(X_i) && [\text{variance adds if independent rvs}] \\ &= \sum_{i=1}^n p(1-p) && [\text{variance of Bernoulli}] \\ &= np(1-p) \end{aligned}$$

Like Bernoulli rvs, Binomial random variables have a special place in our zoo. Arguably, Binomial rvs are probably the **most important** discrete random variable, so make sure to understand everything above and be ready to use it!

It is important to note for the hat check example in 3.3 that we had the sum of  $n$  Bernoulli/indicator rvs BUT that they were NOT independent. This is because if we know one person gets their hat back, someone else is more likely to (since there are  $n-1$  possibilities instead of  $n$ ). However, linearity of expectation works **regardless of independence**, so we were able to still add their expectations like so

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = n \cdot \frac{1}{n} = 1$$

It would be **incorrect** to say that  $X \sim \text{Bin}(n, \frac{1}{n})$  because the indicator rvs were NOT independent.  $\square$

## Example(s)

A factory produces 100 cars per day, but a car is defective with probability 0.02. What's the probability that the factory produces 2 or more defective cars on a given day?

*Solution* Let  $X$  be the number of defective cars that the factory produces.  $X \sim \text{Bin}(100, 0.02)$ , so

$$\begin{aligned}\mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) && [\text{complement}] \\ &= 1 - \binom{100}{0} (0.02)^0 (1 - 0.02)^{100} - \binom{100}{1} (0.02)^1 (1 - 0.02)^{99} && [\text{plug in binomial PMF}] \\ &\approx 0.5967\end{aligned}$$

So, there is about a 60% chance that 2 or more cars produced on a given day will be defective.  $\square$

### 3.4.4 Exercises

1. An elementary school wants to keep track of how many of their 200 students have acceptable attendance. Each student shows up to school on a particular day with probability 0.85, independently of other days and students.
  - (a) A student has acceptable attendance if they show up to class at least 4 out of 5 times in a school week. What is the probability a student has acceptable attendance?
  - (b) What is the probability that at least 170 out of the 200 students have acceptable attendance? Assume students' attendance are independent since they live separately.
  - (c) What is the expected number of students with acceptable attendance?

**Solution:** Actually, this is a great question because it has nested binomials!

- (a) Let  $X$  be the number of school days a student shows up in a school week. Then,  $X \sim \text{Bin}(n = 5, p = 0.85)$  since a students' attendance on different days is independent as mentioned earlier. We want  $X \geq 4$ ,

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) = \binom{5}{4} 0.85^4 0.15^1 + \binom{5}{5} 0.85^5 0.15^0 = 0.83521$$

- (b) Let  $Y$  be the number of students who have acceptable attendance. Then,  $Y \sim \text{Bin}(n = 200, p = 0.83521)$  since each students' attendance is independent of the rest. So,

$$\mathbb{P}(Y \geq 170) = \sum_{k=170}^{200} \binom{200}{k} 0.83521^k (1 - 0.83521)^{200-k} \approx 0.3258$$

- (c) We have  $\mathbb{E}[Y] = np = 200 \cdot 0.83521 = 167.04$  as the expected number of students! We can just cite it now that we've identified  $Y$  as being Binomial!
2. [From Stanford CS109] When sending binary data to satellites (or really over any noisy channel) the bits can be flipped with high probabilities. In 1947 Richard Hamming developed a system to more reliably send data. By using Error Correcting Hamming Codes, you can send a stream of 4 bits with 3 (additional) redundant bits. If zero or one of the seven bits are corrupted, using error correcting codes,

a receiver can identify the original 4 bits. Let's consider the case of sending a signal to a satellite where each bit is independently flipped with probability  $p = 0.1$ . (Hamming codes are super interesting. It's worth looking up if you haven't seen them before! All these problems could be approached using a binomial distribution (or from first principles).)

- (a) If you send 4 bits, what is the probability that the correct message was received (i.e. none of the bits are flipped).
- (b) If you send 4 bits, with 3 (additional) Hamming error correcting bits, what is the probability that a correctable message was received?
- (c) Instead of using Hamming codes, you decide to send 100 copies of each of the four bits. If for every single bit, more than 50 of the copies are not flipped, the signal will be correctable. What is the probability that a correctable message was received?

**Solution:**

- (a) We have  $X \sim \text{Bin}(n = 4, p = 0.9)$  to be the number of correct (unflipped) bits. So the binomial PMF says:

$$\mathbb{P}(X = 4) = \binom{4}{4} 0.9^4 (0.1)^{4-0} = 0.9^4 = 0.656$$

Note we could have also approached this by letting  $Y \sim \text{Bin}(4, 0.1)$  be the number of corrupted (flipped) bits, and computing  $\mathbb{P}(Y = 0)$ . This is the same result!

- (b) Let  $Z$  be the number of corrupted bits, then  $Z \sim \text{Bin}(n = 7, p = 0.1)$ , so we can use its PMF. A message is correctable if  $Z = 0$  or  $Z = 1$  (mentioned above), so

$$\mathbb{P}(Z = 0) + \mathbb{P}(Z = 1) = \binom{7}{0} 0.1^7 0.9^0 + \binom{7}{1} 0.1^6 0.9^1 = 0.850$$

This is a 30% (relative) improvement compared to above by just using 3 extra bits!

- (c) For  $i = 1, \dots, 4$ , let  $X_i \sim \text{Bin}(n = 100, p = 0.9)$ . We need  $X_1 > 50$ ,  $X_2 > 50$ ,  $X_3 > 50$ , and  $X_4 > 50$  for us to get a correctable message. For  $X_i > 50$ , we just sum the binomial PMF from 51 to 100:

$$\mathbb{P}(X_i > 50) = \sum_{k=51}^{100} \binom{100}{k} 0.9^k (1-p)^{100-k}$$

Then, since we need all 4 to work, by independence, we get

$$\begin{aligned} \mathbb{P}(X_1 > 50, X_2 > 50, X_3 > 50, X_4 > 50) &= \mathbb{P}(X_1 > 50) \mathbb{P}(X_2 > 50) \mathbb{P}(X_3 > 50) \mathbb{P}(X_4 > 50) \\ &= \left( \sum_{k=51}^{100} \binom{100}{k} 0.9^k (1-p)^{100-k} \right)^4 \\ &> 0.999 \end{aligned}$$

But this required 400 bits instead of just the 7 required by Hamming codes! This is well worth the tradeoff.

- 3. Suppose  $A$  and  $B$  are random, independent (possibly empty) subsets of  $\{1, 2, \dots, n\}$ , where each subset is equally likely to be chosen. Consider  $A \cap B$ , i.e., the set containing elements that are in both  $A$  and  $B$ . Let  $X$  be the random variable that is the size of  $A \cap B$ . What is  $\mathbb{E}[X]$ ?

**Solution:** Then,  $X \sim \text{Bin}(n, \frac{1}{4})$ , so  $\mathbb{E}[X] = \frac{n}{4}$  (since we know the expected value of  $\text{Bin}(n, p)$  is  $np$ ). How did we do that??

Choosing a random subset of  $\{1, \dots, n\}$  can be thought of as follows: for each element  $i = 1, \dots, n$ , with probability  $1/2$  take the element (and with probability  $1/2$  don't take it), independently of other elements. This is a crucial observation.

For each element  $i = 1, \dots, n$ , the element is either in  $A \cap B$  or not. So let  $X_i$  be the indicator/Bernoulli rv of whether  $i \in A \cap B$  or not. Then,  $\mathbb{P}(X_i = 1) = \mathbb{P}(i \in A, i \in B) = \mathbb{P}(i \in A)\mathbb{P}(i \in B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$  because  $A, B$  are chosen independently, and each element is in  $A$  or  $B$  with probability  $1/2$ . Note that these  $X_i$ 's are independent because one element being in the set does not affect another element being in the set. Hence,  $X = \sum_{i=1}^n X_i$  the number of elements in our intersection, so  $X \sim \text{Bin}(n, \frac{1}{4})$  and  $\mathbb{E}[X] = np = \frac{n}{4}$ .

Note that it was not necessary that these variables were independent; we could have still applied linearity of expectation anyway to get  $\frac{n}{4}$ . We just wouldn't have been able to say  $X \sim \text{Bin}(n, \frac{1}{4})$ .

## Chapter 3: Discrete Random Variables

### 3.5: Zoo of Discrete Random Variables Part II

#### 3.5.1 The Uniform (Discrete) Random Variable

In the this lecture we will continue to expand our zoo of discrete random variables. The next one we will discuss is the uniform random variable. This models situations where the probability of each value in the range is equally likely, like the roll of a fair die.

##### Definition 3.5.1: Uniform Random Variable

$X$  is a uniform random variable, denoted  $X \sim \text{Unif}(a, b)$ , where  $a < b$  are integers, if and only if  $X$  has the following probability mass function

$$p_X(k) = \begin{cases} \frac{1}{b-a+1}, & k \in \{a, a+1, \dots, b\} \\ 0, & \text{otherwise} \end{cases}$$

$X$  is equally likely to take on any value in  $\Omega_X = \{a, a+1, \dots, b\}$ . This set contains  $b - a + 1$  integers, which is why  $\mathbb{P}(X = k)$  is always  $\frac{1}{b-a+1}$ .

Additionally,

$$\mathbb{E}[X] = \frac{a+b}{2} \text{ and } \text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

As you might expect, the expected value is just the average of the endpoints that the uniform random variable is defined over.

*Proof of Expectation and Variance of Uniform.*

Suppose  $X \sim \text{Unif}(a, b)$ . We need to use the fact that  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$  and  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$  to compute some quantities. I will skip some steps as it is pretty tedious and just algebra, but just focus on the setup.

$$\mathbb{E}[X] = \sum_{k=a}^b k \cdot p_X(k) = \sum_{k=a}^b k \cdot \frac{1}{b-a+1} = \frac{1}{b-a+1} \sum_{k=a}^b k = \dots = \frac{a+b}{2}$$

$$\mathbb{E}[X^2] = \sum_{k=a}^b k^2 \cdot p_X(k) = \sum_{k=a}^b k^2 \cdot \frac{1}{b-a+1} = \frac{1}{b-a+1} \sum_{k=a}^b k^2 = \dots$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{(b-a)(b-a+2)}{12}$$

□

This variable models situations like rolling a fair six sided die. Let  $X$  be the random variable whose value is the number face up on a die roll. Since the die is fair each outcome is equally likely, which means that  $X \sim \text{Unif}(1, 6)$  so

$$p_X(k) = \begin{cases} \frac{1}{6}, & k \in \{1, 2, \dots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

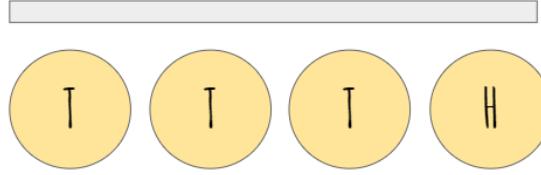
This is fairly intuitive, but is nice to have these formulas in our zoo so we can make computations quickly, and think about random processes in an organized fashion. Using the equations above we can find that

$$\mathbb{E}[X] = \frac{1+6}{2} = 3.5 \text{ and } \text{Var}(X) = \frac{(6-1)(6-1+2)}{12} = \frac{35}{12}$$

### 3.5.2 The Geometric Random Variable

Another random variable that arises from the Bernoulli process is the Geometric random variable. It models situations that can be thought of as the number of trials up to and including the first success.

For example, suppose we are betting on how many **independent** flips it will take for a coin to land heads for the first time. The coin lands heads with a probability  $p$ , and you feel confident that it will take four flips to get your first head. The only way that this can occur is with the following sequence of flips (since our first head must have been on the fourth trial, we know everything before must be tails):



Let  $X$  be the random variable that represents the number of **independent** coin flips up to and including your first head. Lets compute  $\mathbb{P}(X = 4)$ .  $X = 4$  occurs exactly when there are 3 tails followed by a head. So,

$$\mathbb{P}(X = 4) = \mathbb{P}(TTTH) = (1-p)(1-p)(1-p)p = (1-p)^3p$$

In general,

$$p_X(k) = (1-p)^{k-1} p$$

This is because there must be  $k-1$  tails in a row followed by a head occurring on the  $k$ th trial.

Let's also verify that the probabilities sum to 1.

$$\begin{aligned} \sum_{k=1}^{\infty} p_X(k) &= \sum_{k=1}^{\infty} (1-p)^{k-1} p && [\text{Geometric PMF}] \\ &= p \sum_{k=1}^{\infty} (1-p)^{k-1} && [\text{take out constant}] \\ &= p \sum_{k=0}^{\infty} (1-p)^k && [\text{reindex to 0}] \\ &= p \left( \frac{1}{1-(1-p)} \right) && \left[ \text{geometric series formula: } \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \right] \\ &= p \cdot \frac{1}{p} = 1 \end{aligned}$$

The second last step used the geometric series formula - this may be why this random variable is called Geometric!

**Definition 3.5.2: Geometric Random Variable**

$X$  is a Geometric random variable, denoted  $X \sim \text{Geo}(p)$ , if and only if  $X$  has the following probability mass function (and range  $\Omega_X = \{1, 2, \dots\}$ ):

$$p_X(k) = (1 - p)^{k-1}p, \quad k = 1, 2, 3, \dots$$

Additionally,

$$\mathbb{E}[X] = \frac{1}{p} \quad \text{and} \quad \text{Var}(X) = \frac{1-p}{p^2}$$

*Proof of Expectation and Variance of Geometric.*

Suppose  $X \sim \text{Geo}(p)$ . The expectation is pretty complicated and uses a calculus trick, so don't worry about it too much. Just understand the first two lines, which are the setup! But before that, what do you think it should be? For example, if  $p = 1/10$ , how many flips do you think it would take until our first head? Possibly 10? And if  $p = 1/7$ , maybe 7? So seems like our guess will be  $\mathbb{E}[X] = \frac{1}{p}$ . It turns out this intuition is actually correct!

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k \in \Omega_X} k \cdot p_X(k) && [\text{def of expectation}] \\ &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\ &= p \sum_{k=1}^{\infty} k(1-p)^{k-1} && [p \text{ is a constant with respect to } k] \\ &= p \sum_{k=1}^{\infty} \frac{d}{dp}(-(1-p)^k) && \left[ \frac{d}{dy} y^k = ky^{k-1}, \text{ and chain rule of calculus} \right] \\ &= -p \left( \frac{d}{dp} \sum_{k=1}^{\infty} (1-p)^{k-1} \right) && [\text{swap sum and integral}] \\ &= -p \left( \frac{d}{dp} \frac{1}{1-(1-p)} \right) && \left[ \text{geometric series formula: } \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \right] \\ &= -p \left( \frac{d}{dp} \frac{1}{p} \right) \\ &= -p \left( -\frac{1}{p^2} \right) \\ &= \frac{1}{p} \end{aligned}$$

We'll actually have a much nicer proof of this fact in 5.3 using the law of total expectation, so look forward to that! I hope you'll take my word that  $\mathbb{E}[X^2]$  is even worse, so I will not provide that proof. But the expectation and variance are here for you to cite!  $\square$

**Example(s)**

Let's say you buy lottery tickets every day, and the probability you win on a given day is 0.01, independently of other days. What is the probability that after a year (365 days), you still haven't won? What is the expected number of days until you win your first lottery?

*Solution* If  $X$  is the number of days until the first win, then  $X \sim \text{Geo}(p = 0.01)$ . Hence, the probability we don't win after a year is (using the PMF)

$$\mathbb{P}(X \geq 365) = 1 - \mathbb{P}(X < 365) = 1 - \sum_{k=1}^{364} \mathbb{P}(X = k) = 1 - \sum_{k=1}^{364} (1 - 0.01)^{k-1} 0.01$$

This is great, but for the geometric, we can actually get a closed-form formula by thinking of what it means that  $X \geq 365$  in English.  $X \geq 365$  happens if and only if we lose for the first 365 days, which happens with probability  $0.99^{365}$ . If you evaluated that nasty sum above and this quantity, you would find that they are equal!

Finally, we can just cite the expectation of the Geometric RV:

$$\mathbb{E}[X] = \frac{1}{p} = \frac{1}{0.01} = 100$$

This is the point of the zoo! We do all these generic calculations so we can use them later anytime.  $\square$

### Example(s)

You gamble by flipping a fair coin independently up to and including the first head. If it takes  $k$  tries, you earn  $\$2^k$  (i.e., if your first head was the flip, you would earn  $\$8$ ). How much would you pay to play this game?

*Solution* Let  $X$  be the number of flips to the first head. Then,  $X \sim \text{Geo}(\frac{1}{2})$  because its a fair coin, and

$$p_X(k) = \left(1 - \frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right) = \frac{1}{2^k} \quad k = 1, 2, 3, \dots$$

It is usually unwise to gamble, especially if your expected earnings are lower than the price to play. So, let  $Y$  be your expected earnings. Note that  $Y = 2^X$  because the amount you win depends the number of flips it takes to get a heads. We will use LOTUS to compute  $\mathbb{E}[Y] = \mathbb{E}[2^X]$ . Recall  $\mathbb{E}[2^X] \neq 2^{\mathbb{E}[X]} = 2^2 = 4$  as we've seen many times now.

$$\mathbb{E}[Y] = \mathbb{E}[2^X] = \sum_{k=1}^{\infty} 2^k p_X(k) = \sum_{k=1}^{\infty} 2^k \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

So, you are expected to win an infinite amount of money!

Some might say they would be willing to pay any finite amount of money to play this game. Think about why that would be unwise, and what this means regarding the modeling tools we have provided you so far.  $\square$

### 3.5.3 The Negative Binomial Random Variable

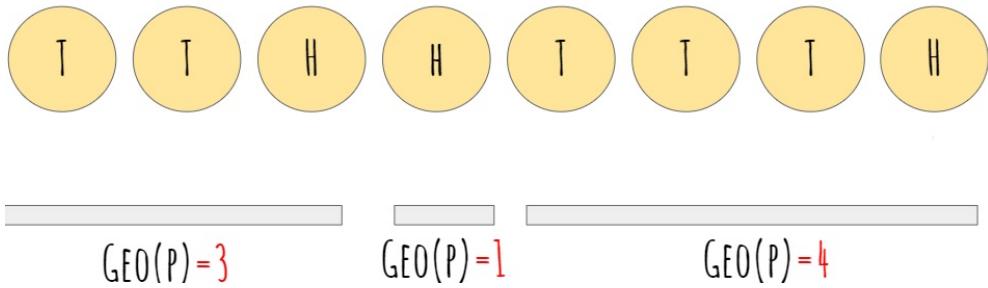
Consider the situation where we are not just betting on the first head, but on the first  $r$  heads. How could we use a random variables to model this scenario?

If you'll recall from the last lecture, multiple Bernoulli random variables sum together to produce a more complicated random variable, the binomial. We might try to do something similar with geometric random variables.

Let  $X$  be a random variable that represents the number of coin flips it takes to get our  $r^{th}$  head.

$$X = \sum_{i=1}^r X_i$$

where  $X_i$  is a geometric random variable that represents the number of flips it takes to get the  $i^{th}$  head after  $i-1$  heads have already occurred. Since all the flips are independent, so are the rvs  $X_1, \dots, X_r$ . For example, if  $r = 3$  we might observe the following sequence of flips



In this case,  $X_1 = 3$  and represents the number of trials between the 0th to the 1st head;  $X_2 = 1$  and represents the number of trials between the 1st to the 2nd head;  $X_3 = 4$  and represents the number of trials between the 2nd and the 3rd head. Remember this fact for later!

How do we find  $\mathbb{P}(X = 8)$ ? There must be exactly 5 heads and 3 tails, so it is reasonable to expect  $(1-p)^5 p^3$  to come up somewhere in our final formula, but how many ways can we get a valid sequence of flips? Note that the last coin flip *must* be a heads, otherwise we would've gotten our  $r$  heads earlier than our 8th flip. From here, any 2 of the first 7 flips can be heads, and 5 of must be tails. Thus, there are  $\binom{7}{2}$  valid sequences of coin flips.

Each of these 7 flip sub-sequences (of the 8 total flips) occurs with probability  $(1-p)^5 p^2$  and there is no overlap. However, we need to include the probability that the last coin flip is a heads. So,

$$p_X(8) = \mathbb{P}(X = 8) = \binom{7}{2} (1-p)^5 p^2 \cdot p = \binom{7}{2} (1-p)^5 p^3$$

We can generalize as follows:

$$p_X(k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r$$

Again, the interpretation is that our  $r^{th}$  head must come at the  $k^{th}$  trial exactly; so in the first  $k-1$  we can get  $r-1$  heads anywhere (hence the binomial coefficient), and overall we have  $r$  heads and  $k-r$  tails.

If we are interested in finding the expected value of  $X$  we might try the brute force approach directly from the definition of expected value

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k p_X(k) = \sum_{k \in \Omega_X} k \binom{k-1}{r-1} (1-p)^{k-r} p^r$$

but this approach is overly complicated, and there is a much simpler way using linearity of expectation! Suppose  $X_1, \dots, X_r \sim \text{Geo}(p)$  are independent. As we showed earlier,  $X = \sum_{i=1}^r X_i$ , and we showed that each  $\mathbb{E}[X_i] = 1/p$ . Using linearity of expectation, we can derive the following:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^r X_i\right] = \sum_{i=1}^r \mathbb{E}[X_i] = \sum_{i=1}^r \frac{1}{p} = \frac{r}{p}$$

Using a similar technique and the (yet unproven) fact that  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ , we can find the variance of  $X$  from the sum of the variances of multiple geometric random variables

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^r X_i\right) = \sum_{i=1}^r \text{Var}(X_i) = \sum_{i=1}^r \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$$

This random variable is called the negative binomial random variable. It is quite common so it too deserves a special place in our zoo.

#### Definition 3.5.3: Negative Binomial Random Variable

$X$  is a negative binomial random variable, denoted  $X \sim \text{NegBin}(r, p)$ , if and only if  $X$  has the following probability mass function (and range  $\Omega_X = \{r, r+1, \dots\}$ ):

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots$$

$X$  is the sum of  $r$  independent  $\text{Geo}(p)$  random variables.

Additionally,

$$\mathbb{E}[X] = \frac{r}{p} \quad \text{and} \quad \text{Var}(X) = \frac{r(1-p)}{p^2}$$

Also, note that  $\text{Geo}(p) \equiv \text{NegBin}(1, p)$ , and that if  $X, Y$  are independent such that  $X \sim \text{NegBin}(r, p)$  and  $Y \sim \text{NegBin}(s, p)$ , then  $X + Y \sim \text{NegBin}(r+s, p)$  (waiting for  $r+s$  heads).

### 3.5.4 Exercises

1. You are a hardworking boxer. Your coach tells you that the probability of your winning a boxing match is 0.25, independently of every other match.
  - How many matches do you expect to fight until you win once?
  - How many matches do you expect to fight until you win ten times?
  - You only get to play 12 matches every year. To win a spot in the Annual Boxing Championship, a boxer needs to win at least 10 matches in a year. What is the probability that you will go to the Championship this year?
  - Let  $q$  be your answer from the previous part. How many times can you expect to go to the Championship in your 20 year career?

**Solution:**

- (a) Let  $X$  be the matches you have to fight until you win once. Then,  $X \sim \text{Geo}(p = 0.25)$ , so  $\mathbb{E}[X] = \frac{1}{p} = \frac{1}{0.25} = 4$ .
- (b) Let  $Y$  be the matches you have to fight until you win ten times. Then,  $Y \sim \text{NegBin}(r = 10, p = 0.25)$ , so  $\mathbb{E}[Y] = \frac{r}{p} = \frac{10}{0.25} = 40$ .
- (c) Let  $Z$  be the number of matches you win out of 12. Then,  $Z \sim \text{Bin}(n = 12, p = 0.25)$ , and we want

$$\mathbb{P}(Z \geq 10) = \sum_{k=10}^{12} \binom{12}{k} 0.2^k (1 - 0.2)^{12-k}$$

- (d) Let  $W$  be the number of times we make it to the Championship in 20 years. Then,  $W \sim \text{Bin}(n = 20, p = q)$ , and

$$\mathbb{E}[W] = np = 20q$$

2. You are in music class, and your cruel teacher says you cannot leave until you play the 1000-note song Fur Elise correctly 5 times. You start playing the song, and if you play an incorrect note, you immediately start the song over from scratch. You play each note correctly independently with probability 0.999.
- What is the probability you play the 1000-note song Fur Elise correctly immediately? (i.e., the first 1000 notes are all correct).
  - What is the probability you take exactly 20 attempts to correctly play the song 5 times?
  - What is the probability you take at least 20 attempts to correctly play the song 5 times?
  - (Challenge) What is the expected number of **notes** you play until you finish playing Fur Elise correctly 5 times?

**Solution:**

- Let  $X$  be the number of correct notes we play in Fur Elise in one attempt, so  $X \sim \text{Bin}(1000, 0.999)$ . We need  $\mathbb{P}(X = 1000) = 0.999^{1000} \approx 0.3677$ .
- If  $Y$  is the number of attempts until we play the song correctly 5 times, then  $Y \sim \text{NegBin}(5, 0.3677)$ , and so

$$\mathbb{P}(Y = 20) = \binom{20-1}{5-1} 0.3677^5 (1 - 0.3677)^{15} \approx 0.0269$$

- We can actually take two approaches to this. We can either take our  $Y$  from earlier, and compute

$$\mathbb{P}(Y \geq 20) = 1 - \mathbb{P}(Y < 20) = 1 - \sum_{k=5}^{19} \binom{k-1}{4} 0.3677^5 (1 - 0.3677)^{k-5} \approx 0.1161$$

Notice the sum starts at 5 since that's the lowest possible value of  $Y$ . This would be exactly the probability of the statement asked. We could alternatively rephrase the question as: what is the probability we play the song correctly at most 4 times correctly in the first 19 times? Check that these questions are equivalent! Then, we can let  $Z \sim \text{Bin}(19, 0.3677)$  and instead compute

$$\mathbb{P}(Z \leq 4) = \sum_{k=0}^4 \binom{19}{k} 0.3677^k (1 - 0.3677)^{19-k} \approx 0.1161$$

- We will have to revisit this question later in the course! Note that we could have computed the expected number of **attempts** to finish playing Fur Elise though, as it would follow a  $\text{NegBin}(5, 0.3677)$  distribution with expectation  $\frac{5}{0.3677} \approx 13.598$ .

## Chapter 3: Discrete Random Variables

### 3.6: Zoo of Discrete RV's Part III

#### 3.6.1 The Poisson Random Variable

So far, none of the random variables can measure the number of events in a unit time. For example:

- How many babies born in the next minute?
- How many car crashes happen per hour?

If we wanted a count like this, we might try to use the binomial random variable. But what should we choose for  $n$ ? Ideally, we wouldn't have an upper bound or maximum number of events. We'll actually derive a new random variable called a Poisson rv, which is derived from the Binomial rv as follows.

##### The Poisson RV (Idea)

Let's say we want to model babies born in the next minute, and the historical average is 2 babies/min. Our strategy will be to take a minute interval and split it into infinitely many small chunks (think milliseconds, then nanoseconds, etc.).

We start by breaking one unit of time into 5 parts, and we say at each of the five chunks, either a baby is born or not. That means we'll be using a binomial rv with  $n = 5$ . The choice of  $p$  that will keep our average to be 2 is  $\frac{2}{5}$ , because the expected value of binomial RV is  $np = 2$ .

Similarly, if we break the time into even smaller chunks such as  $n = 10$  or  $n = 70$ , we can get the corresponding  $p$  to be  $\frac{2}{10}$  or  $\frac{2}{70}$  respectively (either a baby is born or not in 1/70 of a second).

One Unit of Time

—————  $Bin(n = 5, p = 2/5)$   
—————  $Bin(n = 10, p = 2/10)$   
—————  $Bin(n = 70, p = 2/70)$

And we keep increasing  $n$  so that it gets down to the smallest fraction of a second; we have  $n \rightarrow \infty$  and  $p \rightarrow 0$  in this fashion while maintaining the condition that  $np = 2$ .

Let  $\lambda$  be the historical average number of events per unit of time. Send  $n \rightarrow \infty$  and  $p \rightarrow 0$  in such a way that  $np = \lambda$  is fixed (i.e.,  $p = \frac{\lambda}{n}$ ).

Let  $X_n \sim Bin(n, \frac{\lambda}{n})$  and  $Y \sim \lim_{n \rightarrow \infty} X_n$  be the limit of this sequence of Binomial rvs. Then, we say  $Y \sim Poi(\lambda)$  and measures the number of events in a unit time, where the historical average is  $\lambda$ . We'll derive its PMF by taking the limit of the binomial PMF.

We'll need to recall how we defined the base of the natural logarithm  $e$ . There are two equivalent formulations.

$$\begin{aligned}
 e^x &= \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n & & \\
 e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} & & \\
 p_Y(k) &= \lim_{n \rightarrow \infty} p_{X_n}(k) & & [\text{def of } Y \sim \text{Poi}(\lambda)] \\
 &= \lim_{n \rightarrow \infty} \binom{n}{k} p^k (1-p)^{n-k} & & [\text{binomial PMF}] \\
 &= \lim_{n \rightarrow \infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} & & [p = \lambda/n, \text{ expand binomial coefficient}] \\
 &= \lim_{n \rightarrow \infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k} & & [\text{algebra}] \\
 &= \frac{\lambda^k}{k!} \cdot \lim_{n \rightarrow \infty} \frac{n!}{(n-k)!} \frac{1}{n^k} \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k} & & [\text{algebra}] \\
 &= \frac{\lambda^k}{k!} \cdot \lim_{n \rightarrow \infty} \left( \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-k+1}{n} \right) \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k} & & \left[ \frac{n!}{(n-k)!} = P(n, k) = n(n-1) \dots (n-k+1) \right] \\
 &= \frac{\lambda^k}{k!} \cdot \lim_{n \rightarrow \infty} \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k} & & \left[ \lim_{n \rightarrow \infty} \left( \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-k+1}{n} \right) = 1 \right] \\
 &= \frac{\lambda^k}{k!} \cdot \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n & & \left[ \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = 1 \text{ since } k \text{ is finite} \right] \\
 &= \frac{\lambda^k}{k!} e^{-\lambda} & & \left[ e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \right]
 \end{aligned}$$

We'll now verify that the Poisson PMF does sum to 1, and is valid.

Recall the Taylor series for  $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ , so

$$\sum_{k=0}^{\infty} p_Y(k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

### Definition 3.6.1: The Poisson RV

$X \sim \text{Poi}(\lambda)$  if and only if  $X$  has the following probability mass function (and range  $\Omega_X = \{0, 1, 2, \dots\}$ ):

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

If  $\lambda$  is the historical average of events per unit of time, then  $X$  is the number of events that occur in a unit of time.

Additionally,

$$\mathbb{E}[X] = \lambda, \quad \text{Var}(X) = \lambda$$

*Proof of Expectation and Variance of Poisson.* Let  $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$  and  $Y \sim \lim_{n \rightarrow \infty} X_n = \text{Poi}(\lambda)$ . By the properties of the binomial random variable, the mean and variance are as follows for any  $n$  (plug in  $\lambda = np$  or equivalently,  $p = \lambda/n$ ):

$$\mathbb{E}[X_n] = np = \lambda$$

$$\text{Var}(X_n) = np(1-p) = \lambda \left(1 - \frac{\lambda}{n}\right)$$

Therefore:

$$\mathbb{E}[Y] = \mathbb{E}\left[\lim_{n \rightarrow \infty} X_n\right] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \lim_{n \rightarrow \infty} \lambda = \lambda$$

$$\text{Var}(Y) = \text{Var}\left(\lim_{n \rightarrow \infty} X_n\right) = \lim_{n \rightarrow \infty} \text{Var}(X_n) = \lim_{n \rightarrow \infty} \lambda \left(1 - \frac{\lambda}{n}\right) = \lambda$$

□

### Example(s)

Suppose the average number of babies born in Seattle historically is 2 babies every 15 minutes.

1. What is the probability no babies are born in the next *hour* in Seattle?
2. What is the expected number of babies born in the next hour?
3. What is the probability no babies are born in the next *5 minutes* in Seattle?

### Solution

1. Since  $\text{Poi}(\lambda)$  is the number of events in a *single* unit of time (matching units as  $\lambda$ ), we must convert our rate to hours (since we are interested in one hour). So the number of babies born in the next hour can be modelled as  $X \sim \text{Poi}(\lambda = 8/\text{hr})$ , and so the probability no babies are born is

$$\mathbb{P}(X = 0) = e^{-8} \frac{8^0}{0!} = e^{-8}$$

2. Since  $X \sim \text{Poi}(8)$ , we can look up its expectation to be just  $\lambda = 8$  babies in an hour.
3. Since our units of interest now are 5 minutes, we now have  $Y \sim \text{Poi}(\lambda = 2/3)$  because the average rate is  $2/3$  babies per 5 minutes. Now,

$$\mathbb{P}(Y = 0) = e^{-2/3} \frac{(2/3)^0}{0!} = e^{-2/3}$$

□

Before doing the next example, let's talk about the sum of two independent Poisson rvs. Almost by definition, if  $X, Y$  are *independent* with  $X \sim \text{Poi}(\lambda)$  and  $Y \sim \text{Poi}(\mu)$ , then  $X + Y \sim \text{Poi}(\lambda + \mu)$  (if the average number of babies born per minute in the USA is 5 and in Canada is 2, then the total babies in the next minute combined is  $\text{Poi}(5+2)$  since the average combined rate is 7. We'll prove this fact that the sum of independent Poisson rvs is Poisson with the sum of their rates in a future chapter!)

## Example(s)

Suppose Lookbook gets on average 120 new users per hour, and Quickgram gets 180 new users per hour, independently. What is the probability that, combined, less than 2 users sign up in the next minute?

*Solution* Convert  $\lambda$ 's to the same unit of interest. For us, it's a minute. We can always change the rate  $\lambda$  (e.g., 120 per hour is the same as 2 per minute), but we can't change the unit of time we're interested in.

$$X \sim \text{Poi}(\lambda = 2 \text{ users/min}), Y \sim \text{Poi}(\lambda = 3 \text{ users/min})$$

Then their total is Poisson:

$$Z = X + Y \sim \text{Poi}(2 + 3) = \text{Poi}(5)$$

$$\mathbb{P}(Z < 2) = p_Z(0) + p_Z(1) = e^{-5} \frac{5^0}{0!} + e^{-5} \frac{5^1}{1!} = 6e^{-5} \approx 0.04$$

□

### 3.6.2 The Poisson Process

Now we'll define the Poisson process, which is related to the Poisson random variable, among ones in our future (Exponential and Gamma). We'll discuss this a bit more after seeing the definition.

#### Definition 3.6.2: The Poisson Process

A **Poisson process** with rate  $\lambda > 0$  per unit of time, is a continuous-time stochastic process indexed by  $t \in [0, \infty)$ , so that  $X(t)$  is the number of events that happens in the interval  $[0, t]$ . Notice that if  $t_1 < t_2$ , then  $X(t_2) - X(t_1)$  is the number of events in  $(t_1, t_2]$ . The process has three properties:



- $X(0) = 0$ . That is, we initially start with an empty counter at time 0.
- The number of events happening in any two disjoint intervals  $[a, b]$  and  $[c, d]$  are independent.
- The number of events in any time interval  $[t_1, t_2]$  is  $\text{Poi}(\lambda(t_2 - t_1))$ . This is because on average  $\lambda$  events happen per unit time, so in  $t_2 - t_1$  units of time, the average rate is  $\lambda(t_2 - t_1)$ . Again, we can scale our rate but not our period of interest.

All this formality is saying is that, events happen independently at a constant rate.

### 3.6.3 The Hypergeometric Random Variable

This will be our last one in our Zoo of discrete random variables!

Suppose there is a candy bag of  $N = 9$  total candies,  $K = 4$  of which are lollipops. Our parents allow us grab  $n = 3$  of them. Let  $X$  be the number of lollipops we grab. What is the probability that we get exactly 2 lollipops?

The number of ways to grab three candies is just  $\binom{9}{3}$ , and we need to get exactly 2 lollipops out of 4, which is  $\binom{4}{2}$ . Out of the other 5 candies, we only need one of them, which yields  $\binom{5}{1}$  ways.

$$p_X(2) = \mathbb{P}(X = 2) = \frac{\binom{4}{2} \binom{5}{1}}{\binom{9}{3}}$$

We say the number of successes we draw is  $X \sim \text{HypGeo}(N, K, n)$ , where  $K$  out of  $N$  items in a bag are successes, and we draw  $n$  *without* replacement.

**Definition 3.6.3: The Hypergeometric RV**

$X \sim \text{HypGeo}(N, K, n)$  if and only if  $X$  has the following probability mass function:

$$p_X(k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}, k = \max\{0, n+K-N\}, \dots, \min\{K, n\}$$

$X$  is the number of successes when drawing  $n$  items *without* replacement from a bag containing  $N$  items,  $K$  of which are successes (hence  $N - K$  failures).

$$\mathbb{E}[X] = n \frac{K}{N} \quad \text{Var}(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)}$$

Note that if we drew *with* replacement, then we would model this situation using  $\text{Bin}(n, \frac{K}{N})$  as each draw would be an independent trial.

*Proof of Expectation and Variance of Hypergeometric.*

Suppose  $X \sim \text{HypGeo}(N, K, n)$ , and let  $X_1, \dots, X_n$  be indicator RV's (NOT independent) so that  $X_i = 1$  if we got a lollipop on the  $i^{\text{th}}$  draw, and 0 otherwise. So  $X = \sum_{i=1}^n X_i$ .

Then, each  $X_i$  is Bernoulli, but with what parameter? The probability of getting a lollipop on the first draw ( $X_1$  being equal to 1) is just  $K/N$ .

$$\mathbb{P}(X_1 = 1) = \frac{K}{N}$$

What about  $\mathbb{P}(X_2 = 1)$ , the probability we get a lollipop on our second draw? Well, it depends on whether or not we got one the first draw! So we can use the LTP conditioning on whether we got one ( $X_1 = 1$ ) or we didn't ( $X_1 = 0$ ).

$$\begin{aligned} \mathbb{P}(X_2 = 1) &= \mathbb{P}(X_2 = 1 | X_1 = 1) \mathbb{P}(X_1 = 1) + \mathbb{P}(X_2 = 1 | X_1 = 0) \mathbb{P}(X_1 = 0) \quad [\text{LTP}] \\ &= \frac{K-1}{N-1} \cdot \frac{K}{N} + \frac{K}{N-1} \cdot \frac{N-K}{N} = \frac{K(N-1)}{N(N-1)} = \frac{K}{N} \end{aligned}$$

Actually, each  $X_i \sim \text{Ber}(K/N)$ , at every draw  $i$ ! You could continue the above logic for  $X_3$  and so on. This

makes sense, because, if you just think about the  $i$ -th draw and you didn't know anything about the first  $i-1$ , the probability you get a lollipop would just be  $K/N$ .

$$\mathbb{E}[X_i] = \frac{K}{N}$$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \frac{K}{N} = n \frac{K}{N}$$

Note again it would be wrong to say  $X \sim \text{Bin}(n, K/N)$  because the trials are NOT independent, but we are still able to use linearity of expectation. If we did this experiment *with* replacement though (take one and put it back), then the draws would be independent, and modelled as  $\text{Bin}(n, K/N)$ . Note the expectation with or without replacement is the same because linearity of expectation doesn't care about independence! The variance is a nightmare, and will be proven in 5.4 when figure out how to compute the variance of the sum of these dependent indicator variables.  $\square$

**The Zoo of Discrete RV's:** Here are all the distributions in our zoo of discrete random variables!

- The Bernoulli RV
- The Binomial RV
- The Uniform (Discrete) RV
- The Geometric RV
- The Negative Binomial RV
- The Poisson RV
- The Hypergeometric RV

Congratulations on making it through this chapter on all these wonderful discrete random variables! There are several practice problems below which require using a lot of these zoo elements. It will definitely take some time to get used to all of these - you'll need to do practice! See our handy reference sheet for one place to see all of them while doing problems.

### 3.6.4 Exercises

1. Suppose that on average, 40 babies are born per hour in Seattle.
  - (a) What is the probability that over 1000 babies are born in a single day in Seattle?
  - (b) What is the probability that in a 365-day year, over 1000 babies are born on exactly 200 days?

**Solution:**

- (a) The number of babies born in a single average day is  $40 \cdot 24 = 960$ , so  $X \sim \text{Poi}(\lambda = 960)$ . Then,

$$\mathbb{P}(X > 1000) = 1 - \mathbb{P}(X \leq 1000) = 1 - \sum_{k=0}^{1000} e^{-960} \frac{960^k}{k!}$$

- (b) Let  $q$  be the answer from part (a). The number of days where over 1000 babies are born is  $Y \sim \text{Bin}(n = 365, p = q)$ , so

$$\mathbb{P}(Y = 200) = \binom{365}{200} q^{200} (1 - q)^{165}$$

2. Suppose the Senate consists of 53 Republicans and 47 Democrats. Suppose we were to create a bipartisan committee of 20 senators by randomly choosing from the 100 total.
- What is the probability we end up with exactly 9 Republicans and 11 Democrats?
  - What is the expected number of Democrats on the committee?

**Solution:**

- (a) Let  $X$  be the number of Republican senators chosen. Then  $X \sim \text{HypGeo}(N = 100, K = 53, n = 20)$ , and the desired probability is

$$\mathbb{P}(X = 9) = \frac{\binom{53}{9} \binom{47}{11}}{\binom{100}{20}}$$

since choosing 9 out of 20 Republicans also implies immediately we have 11 out of 20 Democrats. Note we could have flipped the roles of Democrats and Republicans. If  $Y$  is the number of Democratic senators chosen, then  $Y \sim \text{HypGeo}(N = 100, K = 47, n = 20)$ , and

$$\mathbb{P}(Y = 11) = \frac{\binom{47}{11} \binom{53}{9}}{\binom{100}{20}}$$

- (b) The number of Democrats as mentioned earlier is  $Y \sim \text{HypGeo}(N = 100, K = 47, n = 20)$ , and so

$$\mathbb{E}[Y] = n \frac{K}{N} = 20 \cdot \frac{47}{100} = 9.4$$

3. **(Poisson Approximation to Binomial)** Suppose the famous chip company “Bayes” produces  $n = 10000$  bags per day. They need to do a quality check, and they know that 0.1% of their bags independently have “bad” chips in them.

- What is the exact probability that at most 5 bags contain “bad” chips?
- Recall the Poisson was derived from the Binomial with  $n \rightarrow \infty$  and  $p \rightarrow 0$ , so it suggests that a Poisson distribution would be a good approximation to a Binomial with large  $n$  and small  $p$ . Use a Poisson rv instead to compute the same probability as in part (a). How close are the answers?

**Note:** The reason we use a Poisson approximation sometimes is because the binomial PMF is hard to compute. Imagine  $X \sim \text{Bin}(10000, 0.256)$ , computing  $\mathbb{P}(X = 2000) = \binom{10000}{2000} 0.256^{2000} (1 - 0.256)^{8000}$  has at least 10000 multiplication operations for the probabilities. Furthermore,  $\binom{10000}{2000} = \frac{10000!}{2000! 8000!}$  - good luck avoiding overflow on your computer!

**Solution:**

- (a) If  $X$  is the number of bags with “bad” chips, then  $X \sim \text{Bin}(n = 10000, p = 0.001)$ , so

$$\mathbb{P}(X \leq 5) = \sum_{k=0}^5 \binom{10000}{k} 0.001^k (1 - 0.001)^{10000-k} \approx 0.06699$$

- (b) Since  $n$  is large and  $p$  is small, we might approximate  $X$  as Poisson rv, with  $\lambda = np = 10000 \cdot 0.001 = 10$ . Then, since  $X \approx \text{Poi}(10)$ , we have

$$\mathbb{P}(X \leq 5) = \sum_{k=0}^5 e^{-10} \frac{10^k}{k!} \approx 0.06709$$

This approximation is not bad at all!

4. You are writing a 250-page book, but you make an average of one typo every two pages. For a lot of these questions, if you cite the correct distribution, the answer follows immediately.
- What is the probability that a particular page contains (at least) one typo?
  - What is the expected number of typos in total?
  - What is the probability that your book contains at most 50 pages with (at least) one typo on them?
  - What is the expected “page number” which contains your first typo?
  - Suppose your book has exactly 50 pages with a typo (and 200 without). If I look at 20 different pages randomly, what is the probability that exactly 5 contain (at least) one typo?

**Solution:**

- (a) The average rate of typos is one per two pages, or equivalently,  $1/2$  per one page. Hence, if  $X$  is the number of typos on a page, then  $X \sim \text{Poi}(\lambda = 1/2)$ , and

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - e^{-1/2} \frac{(1/2)^0}{0!} = 1 - e^{-1/2} \approx 0.39347$$

- (b) Since we are interested in a 250 page “time period”, the average rate of typos is 125 per 250 pages. If  $Y$  is the number of typos in total, then  $Y \sim \text{Poi}(\lambda = 125)$ , and  $\mathbb{E}[Y] = \lambda = 125$ .

- (c) We can consider each page as  $\text{Poi}(1/2)$  like in part (a). Let  $Z$  be the number of pages with at least one typo. Then,  $Z \sim \text{Bin}(n = 250, p = 0.39347)$ , and

$$\mathbb{P}(Z \leq 50) = \sum_{k=0}^{50} \binom{250}{k} 0.39347^k (1 - 0.39347)^{250-k}$$

- (d) Let  $V$  be the first page that contains (at least) one typo. Then,  $V \sim \text{Geo}(0.39347)$ , so

$$\mathbb{E}[V] = \frac{1}{0.39347} = 2.5415$$

- (e) If  $W$  is the number of pages out of 20 that have a typo, then  $W \sim \text{HypGeo}(N = 250, K = 50, n = 20)$ , and

$$\mathbb{P}(W = 5) = \frac{\binom{50}{5} \binom{200}{15}}{\binom{250}{20}}$$

## Chapter 4: Continuous Random Variables

### 4.1: Continuous Random Variables Basics

Up to this point, we have only been talking about *discrete* random variables - ones that only take values in a countable (finite or countably infinite) set like the integers or a subset. What if we wanted to model quantities that were continuous - that could take on *uncountably infinitely* many values? If you haven't studied or seen cardinality (or types of infinities) before, you can think of this as being intervals of the real line, which take decimal values. Our tools from the previous chapter were not suitable to modelling these situations, and so we need a new type of random variable.

#### Definition 4.1.1: Continuous Random Variables

A **continuous random variable** is a random variable that takes values from an uncountably infinite set, such as the set of real numbers or an interval. For e.g., height (5.6312435 feet, 6.1123 feet, etc.), weight (121.33567 lbs, 153.4642 lbs, etc.) and time (2.5644 seconds, 9321.23403 seconds, etc.) are continuous random variables that take on values in a continuum.

Why do we need continuous random variables?

Suppose we want a random number in the interval  $[0, 10]$ , with each possibility being "equally likely".

- What is  $\mathbb{P}(X = 3.141592)$  for such a random variable  $X$ ? That is, if I chose a random decimal number (with infinite precision/decimal places), what is the probability you guess it exactly right (matching infinitely many decimal places)? The probability is actually 0, it's not even a tiny positive number!
- What is  $\mathbb{P}(5 \leq X \leq 8)$  for such a random variable  $X$ ? That is, what if you were allowed to guess a range instead of a single number? As you might expect,  $\frac{\text{size of the required interval}}{\text{size of the total interval}} = \frac{3}{10}$  since the random number is uniformly distributed.

Suppose we want to study the set of possible heights (in feet) a person can have, supposing that the range of possible heights is the interval  $[1, 8]$ .

- What is the probability that someone has a height of 5.2311333 feet? This is again 0, since you have to be exactly precise!
- What is the probability that someone has a height between 5 and 6 feet? This is non-zero, since we are studying an interval. It isn't necessarily  $\frac{6-5}{8-1} = \frac{1}{7}$  though since heights aren't necessarily uniformly distributed! More people will have heights in the interval  $[4, 6]$  feet than say  $[1, 3]$  feet.

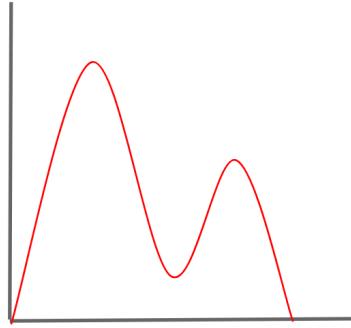
Notice, that since these values can have infinite precision, the probability that a variable has a specific value is 0, in contrast to discrete random variables.

#### 4.1.1 Probability Density Functions (PDFs)

Every continuous random variable has a probability density function (PDF), *instead* of a probability mass function (PMF), that defines the relative likelihood that a random variable  $X$  has a particular value. Why

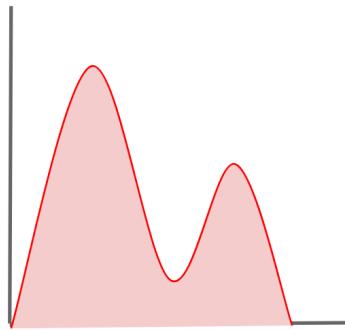
do we need this new construct? We already said that  $\mathbb{P}(X = a) = 0$  for any value of  $a$ , and so a “PMF” for a continuous random variable would equal 0 for any input and be useless. It wouldn’t satisfy the constraint that the sum of the probabilities is 1 (assuming we could even sum over uncountably many values; we can’t). Instead, we have the idea of a probability density function where the  $x$ -axis has values in the random variable’s range (usually an interval), and the  $y$ -axis has the probability *density* (not mass), which is explained below.

A PDF may look something like this:

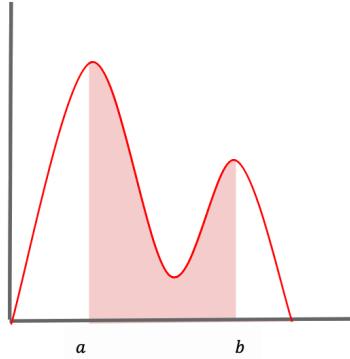


The probability density function  $f_X$  has some characteristic properties (denoted with  $f_X$  to distinguish from PMFs  $p_X$ ). Notice again I will use different dummy variables inside the function like  $f_X(z)$  or  $f_X(t)$  to ensure you get the idea that the density is  $f_X$  (subscript indicates for rv  $X$ ) and the dummy variable can be anything.

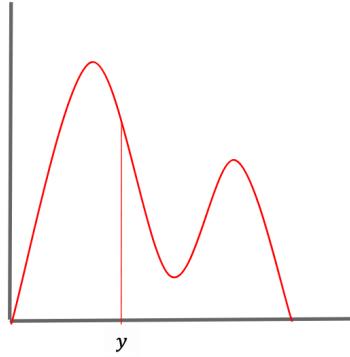
- $f_X(z) \geq 0$  for all  $z \in \mathbb{R}$ ; i.e., it is always non-negative, just like a probability mass function.
- $\int_{-\infty}^{\infty} f_X(t)dt = 1$ ; i.e., the area under the entire curve is equal to 1, just like the sum of all the probabilities of a discrete random variable equals 1.



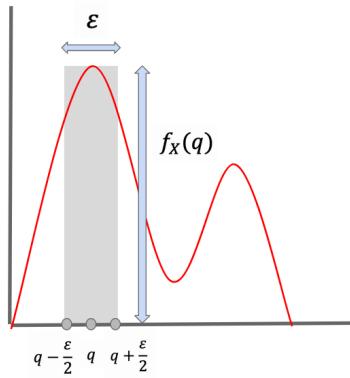
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(w)dw$ ; i.e., the probability that  $X$  lies in the interval  $a$  to  $b$  is the area under the curve from  $a$  to  $b$ . This is key - **integrating  $f_X$  gives us probabilities**.



- $\mathbb{P}(X = y) = \mathbb{P}(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$ . The probability of being a particular value is 0, and NOT equal to the density  $f_X(y)$  which is nonzero. This is particularly confusing at first.

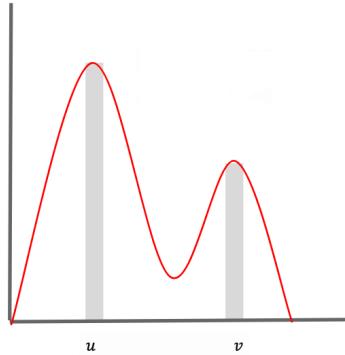


- $\mathbb{P}(X \approx q) = \mathbb{P}(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}) \approx \varepsilon f_X(q)$ ; i.e., with a small epsilon value, we can obtain a good rectangle approximation of the area under the curve. The width of the rectangle is  $\varepsilon$  (from the difference between  $q + \frac{\varepsilon}{2}$  and  $q - \frac{\varepsilon}{2}$ ). The height of the rectangle is  $f_X(q)$ , the value of the probability density function  $f_X$  at  $q$ . So, the area of the rectangle is  $\varepsilon f_X(q)$ . This is similar to the idea of Riemann integration.



- $\frac{\mathbb{P}(X \approx u)}{\mathbb{P}(X \approx v)} \approx \frac{\varepsilon f_X(u)}{\varepsilon f_X(v)} = \frac{f_X(u)}{f_X(v)}$ ; i.e., the PDF tells us ratios of probabilities of being “near” a point. From the previous point, we know the probabilities of  $X$  being approximately  $u$  and  $v$ , and through algebra,

we see their ratios. Since the density is twice as high at  $u$  as it is at  $v$ , it means we are twice as likely to get a point “near”  $u$  as we are to get one “near”  $v$ .



#### Definition 4.1.2: Probability Density Function (PDF)

Let  $X$  be a continuous random variable (one whose range is typically an interval or union of intervals). The probability density function (PDF) of  $X$  is the function  $f_X : \mathbb{R} \rightarrow \mathbb{R}$ , such that the following properties hold:

- $f_X(z) \geq 0$  for all  $z \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(t) dt = 1$
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(w) dw$
- $\mathbb{P}(X = y) = 0$  for any  $y \in \mathbb{R}$
- The probability that  $X$  is close to  $q$  is proportional to its density  $f_X(q)$ ;

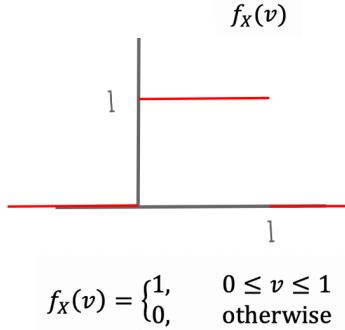
$$\mathbb{P}(X \approx q) = \mathbb{P}\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$$

- Ratios of probabilities of being “near points” are maintained;

$$\frac{\mathbb{P}(X \approx u)}{\mathbb{P}(X \approx v)} \approx \frac{\varepsilon f_X(u)}{\varepsilon f_X(v)} = \frac{f_X(u)}{f_X(v)}$$

#### 4.1.2 Cumulative Distribution Functions (CDFs)

Here is the density function of a “uniform” random variable on the interval  $[0, 1]$ :



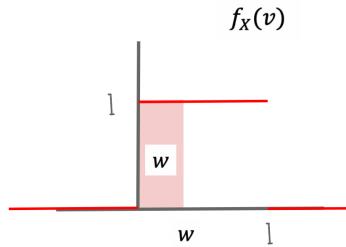
We know this is valid, because the area under the curve is the area of a square with side lengths 1, which is  $1 \cdot 1 = 1$ .

We define the cumulative distribution function (CDF) of  $X$  to be  $F_X(w) = \mathbb{P}(X \leq w)$ . That is, the all the area to the left of  $w$  in the density function. Note we also have CDFs for discrete random variables, they are defined exactly the same way (the probability of being less than or equal to a certain value)! They just don't usually have a nice closed form like they do for continuous RVs. Note for continuous random variables, the CDF at  $w$  is just the cumulative area to the left of  $w$ , which can be found by an integral (the dummy variable of integration should be different than the input variable  $w$ )

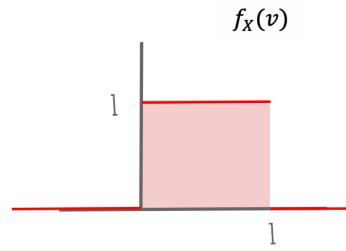
$$F_X(w) = \mathbb{P}(X \leq w) = \int_{-\infty}^w f_X(y) dy$$

Let's try to compute the CDF of this uniform random variable on  $[0, 1]$ . There are three cases to consider here.

- If  $w < 0$ ,  $F_X(w) = 0$  since  $\Omega_X = [0, 1]$ . For example, if  $w = -1$ , then  $F_X(w) = \mathbb{P}(X \leq -1) = 0$  since there is no chance that  $X \leq -1$ . Formally, there is also no area to the left of  $w = -1$  as you can see from the PDF above, so the integral evaluates to 0!
- If  $0 \leq w \leq 1$ , the area up to  $w$  is a rectangle of height 1 and width  $w$  (see below), so  $F_X(w) = w$ . That is,  $\mathbb{P}(X \leq w) = w$ . For example, if  $w = 0.5$ , then the probability  $X \leq 0.5$  is actually just 0.5 since  $X$  is just equally likely to be anywhere in  $\Omega_X = [0, 1]$ ! Note here we didn't do an integral since there are nice shapes, and we sometimes don't have to! We just looked at the area to the left of  $w$ .



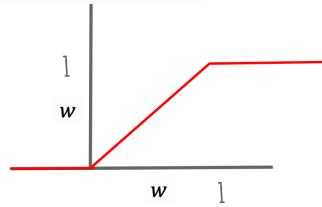
- If  $w > 1$ , all the area is up to the left of  $w$ , so  $F_X(w) = 1$ . Again, since  $\Omega_X = [0, 1]$  and suppose  $w = 2$ , then  $F_X(w) = \mathbb{P}(X \leq 2) = 1$  since  $X$  is always between 0 and 1 ( $X$  must be less than or equal to 2). Formally, the cumulative area to the left of  $w = 2$  is 1 (just the area of the square)!



We can put these conclusions together to show:

$$F_X(w) = \begin{cases} 0 & \text{if } w < 0 \\ w & \text{if } 0 \leq w \leq 1 \\ 1 & \text{if } w > 1 \end{cases}$$

On a graph,  $F_X(w)$  looks like this:



The cumulative distribution function has some characteristic properties:

- $F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(w)dw$  for all  $t \in \mathbb{R}$  - i.e. the probability that  $X \leq t$  is the area to the left of  $t$  of the density curve.
- You have a function that is defined to be the integral up to a point of a function, so by the Fundamental Theorem of Calculus, the derivative of the CDF is actually the PDF - i.e.  $\frac{d}{dw}F_X(u) = f_X(u)$ . **This is probably the most important observation** that explains the relationship between PDF and CDF.
- The probability that  $X$  is between  $a$  and  $b$  is the probability that  $X \leq b$  minus the probability that  $X \leq a$ ; i.e.,  $\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a)$ . For those with an eagle eye, you might have noticed I lied a little - it should be  $\mathbb{P}(a < X \leq b)$ . But since  $\mathbb{P}(X = a) = 0$ , it doesn't matter!
- $F_X$  is always monotone increasing because we are integrating a non-negative function ( $f_X \geq 0$ ). That is, if  $c \leq d$ , then  $F_X(c) \leq F_X(d)$ . For example, if  $c = 2$  and  $d = 5$ , then  $\mathbb{P}(X \leq 2) \leq \mathbb{P}(X \leq 5)$  because  $X \leq 2$  implies that  $X \leq 5$  automatically.
- As  $v \rightarrow -\infty$ , the CDF at  $v$  is the probability that  $X$  is less than negative infinity which is 0; so the left-hand limit is 0, i.e.  $\lim_{v \rightarrow -\infty} F_X(v) = \mathbb{P}(X \leq -\infty) = 0$ .
- With similar logic to the previous point,  $\lim_{v \rightarrow +\infty} F_X(v) = \mathbb{P}(X \leq +\infty) = 1$ .

**Definition 4.1.3: Cumulative Distribution Function (CDF)**

Let  $X$  be a continuous random variable (one whose range is typically an interval or union of intervals). The cumulative distribution function (CDF) of  $X$  is the function  $F_X : \mathbb{R} \rightarrow \mathbb{R}$  such that:

- $F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(w) dw$  for all  $t \in \mathbb{R}$
- $\frac{d}{du} F_X(u) = f_X(u)$
- $\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a)$
- $F_X$  is monotone increasing, since  $f_X \geq 0$ . That is,  $F_X(c) \leq F_X(d)$  for  $c \leq d$ .
- $\lim_{v \rightarrow -\infty} F_X(v) = \mathbb{P}(X \leq -\infty) = 0$
- $\lim_{v \rightarrow +\infty} F_X(v) = \mathbb{P}(X \leq +\infty) = 1$

### 4.1.3 From Discrete to Continuous

Here is a nice summary chart of how similar the formulae for continuous RVs and discrete RVs are! Note that to compute the expected value of a discrete random variable, we took a weighted sum of each value multiplied by its probability. For continuous random variables though, we take an integral of each value multiplied by its density function! We'll see some examples below.

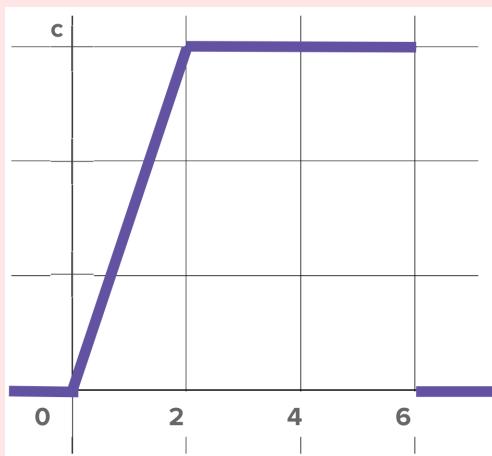
	Discrete	Continuous
<b>PMF/PDF</b>	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
<b>CDF</b>	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
<b>Normalization</b>	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
<b>Expectation/LOTUS</b>	$\mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$

#### Example(s)

Suppose the number of hours that a package gets delivered past noon is modelled by the following PDF:

$$f_X(x) = \begin{cases} x/10 & 0 \leq x \leq 2 \\ c & 2 < x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Here is a graph of the PDF as described above:



1. What is the range  $\Omega_X$ ?
2. What is the value of  $c$  that makes  $f_X$  a valid density function?
3. Find the cumulative distribution function (CDF) of  $X$ ,  $F_X(x)$ , and make sure to define it piecewise for any real number  $x$ .
4. What is the probability that the delivery arrives between 2pm and 6pm?
5. What is the expected time that the package arrives at?

*Solution*

1. The range is all values where the density is nonzero; in our case, that is  $\Omega_X = [0, 6]$  (or  $(0, 6)$ ), but we don't care about single points or endpoints because the probability of being exactly that value is 0.
2. Formally, we need the density function to integrate to 1; that is,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

But, the density function is split into three parts, we can split our integral into three. However, anywhere the density is zero, we will get an integral of zero, so we'll only set up the two integrals that are nontrivial:

$$\int_0^2 x/10 dx + \int_2^6 c dx = 1$$

Solving this equation for  $c$  would definitely work. But let's try to use geometry instead, as we do know how to compute the area of a triangle and rectangle. So the left integral is the area of the triangle with base from 0 to 2 and height  $c$ , so that area is  $2c/2 = c$  (the area of a triangle is  $b \cdot h/2$ ). The area of the rectangle with base from 2 to 6 is  $4c$ . We need the total area of  $c + 4c = 1$ , so  $c = 1/5$ .

3. Our CDF needs four cases: when  $x < 0$ , when  $0 \leq x \leq 2$ , when  $2 < x \leq 6$ , and when  $x > 6$ .
  - (a) The outer cases are usually the easiest ones: if  $x < 0$ , then  $F_X(x) = \mathbb{P}(X \leq x) = 0$  since  $X$  cannot be less than zero.
  - (b) If  $x > 6$ , then  $F_X(x) = \mathbb{P}(X \leq x) = 1$  since  $X$  is guaranteed to be at most 6.

- (c) For  $0 \leq x \leq 2$ , we need the cumulative area to the left of  $x$ , which happens to be a triangle with base  $x$  and height  $x/10$ , so the area is  $x^2/20$ . Alternatively, evaluate the integral

$$F_X(x) = \int_{-\infty}^x f_X(t)dt = \int_0^x t/10dt = t^2/20$$

- (d) For  $2 < x \leq 6$ , we have the entire triangle of area  $2 \cdot 1/5 \cdot 0.5 = 1/5$ , but also a rectangle of base  $x - 2$  and height  $1/5$ , for a total area of  $1/5 + 1/5(x - 2) = x/5 - 1/5$ . Alternatively, the integral would be

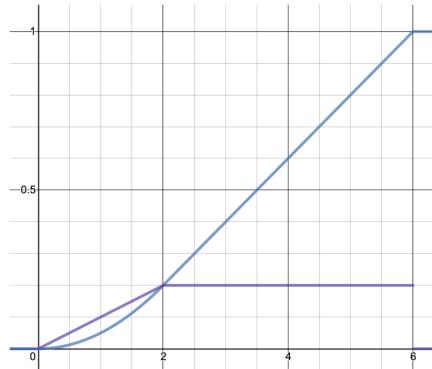
$$F_X(x) = \int_{-\infty}^x f_X(t)dt = \int_0^2 t/10dt + \int_2^x 1/5dt = x/5 - 1/5$$

Again, I skipped all the integral evaluation steps as they are purely computational, but feel free to verify!

Finally, putting this together gives

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/20 & 0 \leq x \leq 2 \\ x/5 - 1/5 & 2 < x \leq 6 \\ 1 & x > 6 \end{cases}$$

Here is the plot of the CDF alongside the PDF:



4. Using the formula, we find the area between 2 and 6 to get  $\mathbb{P}(2 \leq X \leq 6) = \int_2^6 f_X(t)dt = \int_2^6 1/5dt = 4/5$ . Alternatively, we can just see the area from 2 to 6 is just a rectangle with base 4 and height  $1/5$ , so the probability is just  $4/5$ .

We could also use the CDF we so painstakingly computed.

$$\mathbb{P}(2 \leq X \leq 6) = F_X(6) - F_X(2) = (6/5 - 1/5) - (2^2/20) = 1 - 1/5 = 4/5$$

This is just the area to the left of 6, minus the area to the left of 2, which gives us the area between 2 and 6.

5. We'll use the formula for expectation of a continuous RV, but split into three integrals again due to the piecewise definition of our density. However, the integral outside the range  $[0, 6]$  will evaluate to zero, so we won't include it.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^2 xf_X(x)dx + \int_2^6 xf_X(x)dx = \int_0^2 x \cdot (x/10)dx + \int_2^6 x \cdot (1/5)dx$$

We won't do the computation because it's not important, but hopefully you get the idea of how similar this is to the discrete version!

□

#### 4.1.4 Exercises

1. Suppose  $X$  is continuous with density

$$f_X(x) = \begin{cases} cx^2 & 0 \leq x \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Write an expression for the value of  $c$  that makes  $X$  a valid pdf, and set up expressions (integrals) for its mean and variance. Also, find the cdf of  $X$ ,  $F_X$ .

**Solution:** We need the total area under the curve to be 1, so

$$1 = \int_{-\infty}^{\infty} f_X(y)dy = \int_0^9 cy^2 dy = c \left[ \frac{1}{3}y^3 \right]_0^9 = c \frac{729}{3} = 243c$$

Hence,  $c = \frac{1}{243}$ . The expected value is the weighted average of each point weighted by its density, so

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} z f_X(z) dz = \int_0^9 z \frac{1}{243} z^2 dz = \frac{1}{243} \int_0^9 z^3 dz$$

Similarly, by LOTUS,

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f_X(z) dz = \int_0^9 z^2 \frac{1}{243} z^2 dz = \frac{1}{243} \int_0^9 z^4 dz$$

Finally, we can set

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

For the CDF, we know that

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(y) dy$$

We actually have three cases, similar to the example earlier. If  $t < 0$ ,  $F_X(t) = 0$  since there's no way to get a negative number (the range is  $\Omega_X = [0, 9]$ ). If  $t > 9$ ,  $F_X(t) = 1$  since we are guaranteed to get a number less than  $t$ . And for  $0 \leq t \leq 9$ , we just do a normal integral to get that

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(s) ds = \int_{-\infty}^0 f_X(s) ds + \int_0^t f_X(s) ds = 0 + \int_0^t cs^2 ds = \frac{c}{3}t^3$$

Putting this together gives:

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{c}{3}t^3 & 0 \leq t \leq 9 \\ 1 & t > 9 \end{cases}$$

2. Suppose  $X$  is continuous with pdf

$$f_X(x) = \begin{cases} \frac{c}{x^2} & 1 \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Write an expression for the value of  $c$  that makes  $X$  a valid pdf, and set up expressions (integrals) for its mean and variance. Also, find the cdf of  $X$ ,  $F_X$ .

**Solution:** We need the total area under the curve to be 1, so

$$1 = \int_{-\infty}^{\infty} f_X(y)dy = \int_1^{\infty} \frac{c}{y^2} dy = -c \left[ \frac{1}{y} \right]_1^{\infty} = -c(0 - 1) = c$$

Hence,  $c = 1$ . The expected value is the weighted average of each point weighted by its density, so

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} z f_X(z) dz = \int_1^{\infty} z \cdot \frac{1}{z^2} dz = \int_1^{\infty} \frac{1}{z} dz = [\ln(z)]_1^{\infty} = \infty$$

Actually, the mean and variance are undefined (since they are infinite)! If the integral for  $\mathbb{E}[X]$  did not converge, then the integral for  $\mathbb{E}[X^2]$  had no chance either (try it)! For the cdf, we know that

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(y)dy$$

We actually have two cases. If  $t < 1$ ,  $F_X(t) = 0$  since there's no way to get a number less than 1 (the range is  $\Omega_X = [1, \infty)$ ). For  $t > 1$ , we just do a normal integral to get that

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(s)ds = \int_{-\infty}^1 f_X(s)ds + \int_1^t f_X(s)ds = \int_1^t \frac{1}{s^2} ds = - \left[ \frac{1}{s} \right]_1^t = - \left( \frac{1}{t} - 1 \right) = 1 - \frac{1}{t}$$

Putting this together gives:

$$F_X(t) = \begin{cases} 0 & t < 1 \\ 1 - \frac{1}{t} & t \geq 1 \end{cases}$$

## Chapter 4: Continuous Random Variables

### 4.2: Zoo of Continuous RVs

Now that we've learned about the properties of continuous random variables, we'll discover some frequently used RVs just like we did for discrete RVs! In this section, we'll learn the continuous Uniform distribution, the Exponential distribution, and Gamma distribution. In the next section, we'll finally learn about the Normal/Gaussian (bell-shaped) distribution which you all may have heard of before!

#### 4.2.1 The (Continuous) Uniform RV

The continuous uniform random variable models a situation where there is no preference for any particular value over a bounded interval. This is very similar to the discrete uniform random variable (e.g., roll of a fair die), except extended to include decimal values. The probability of equalling any particular value is again 0 since we are dealing with a continuous RV.

##### Definition 4.2.1: Uniform (Continuous) RV

$X \sim \text{Unif}(a, b)$  (continuous) where  $a < b$  are real numbers, if and only if  $X$  has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

$X$  is equally likely to be take on any value in  $[a, b]$ . Note the similarities and differences it has with the discrete uniform! The value of the density function is constant at  $\frac{1}{b-a}$ , for any input  $x \in [a, b]$ , and makes it a rectangle whose area integrates to 1.

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}$$

The cdf is

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

*Proof of Expectation and Variance of Uniform.* I'm setting up the integrals but omitting the steps that are not relevant to your understanding of probability theory (computing integrals):

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{a+b}{2}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{a^2 + ab + b^2}{3}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

□

## Example(s)

Suppose we think that a Hollywood movie's overall rating is equally likely to be any decimal value in the interval  $[1, 5]$  (this may not be realistic). You may be able to do these questions "in your head", but I encourage you to formalize the questions and solutions to practice the notation and concepts we've learned. (You probably wouldn't be able to do them "in your head" if the movie rating wasn't uniformly distributed!)

1. A movie is considered average if its overall rating is between 1.5 and 4.5. What is the probability that it is average?
2. A movie is considered a huge success if its overall rating is at least 4.5. What is the probability that it is a huge success?
3. A movie is considered legendary if its overall rating is at least 4.95. *Given that* a movie is a huge success, what is the probability it is legendary?

*Solution* Before starting, we can write that the overall rating of a movie is  $X \sim \text{Unif}(1, 5)$ . Hence, its density function is  $f_X(x) = \frac{1}{5-1} = \frac{1}{4}$  for  $x \in [1, 5]$  (and 0 otherwise).

1. We know the probability of being in the range  $[1.5, 4.5]$  is the area under the density function from 1.5 to 4.5, so

$$\mathbb{P}(1.5 \leq X \leq 4.5) = \int_{1.5}^{4.5} f_X(x) dx = \int_{1.5}^{4.5} \frac{1}{4} dx = \frac{3}{4}$$

You could have also drawn a picture of this density function (which is flat at 1/4), and exploited geometry to figure that the base of the rectangle is 3 and the height is 1/4.

2. Similarly,

$$\mathbb{P}(X \geq 4.5) = \int_{4.5}^{\infty} f_X(x) dx = \int_{4.5}^5 \frac{1}{4} dx = \frac{1}{8}$$

Note that the density function for values  $x \geq 5$  is zero, so that's why the integral changed its upper bound from  $\infty$  to 5 when replacing the density!

3. We'll use Bayes Theorem:

$$\mathbb{P}(X \geq 4.95 \mid X \geq 4.5) = \frac{\mathbb{P}(X \geq 4.5 \mid X \geq 4.95) \mathbb{P}(X \geq 4.95)}{\mathbb{P}(X \geq 4.5)}$$

Note that  $\mathbb{P}(X \geq 4.5 \mid X \geq 4.95) = 1$  (why?) and  $\mathbb{P}(X \geq 4.95) = \frac{1}{80}$  (do a similar integral again or use geometry), so plugging in these numbers gives

$$= \frac{1 \cdot \frac{1}{80}}{\frac{1}{8}} = \frac{1}{10}$$

Think about why this also might make sense intuitively!

□

### 4.2.2 The Exponential RV

Now we'll learn a distribution which is typically used to model waiting time until an event, like a server failure or the bus arriving. This is a *continuous* RV since the time taken has decimal places, like 3.5341109 minutes or 9.9324 seconds. This is like the continuous extension of the Geometric (discrete) RV which is the number of trials until a success occurs.

Recall the Poisson Process with parameter  $\lambda > 0$  has events happening at average rate of  $\lambda$  per unit time forever. The exponential RV measures the *time* (e.g., 4.33212 seconds, 9.382 hours, etc.) until the first occurrence of an event, so is a continuous RV with range  $[0, \infty)$  (unlike the Poisson RV, which counts the *number of occurrences* in a unit of time, with range  $\{0, 1, 2, \dots\}$  and is a discrete RV).

Let  $Y \sim \text{Exp}(\lambda)$  be the time until the first event. We'll first compute its CDF  $F_Y(t)$  and then differentiate it to find its PDF  $f_Y(t)$ .

Let  $X(t) \sim \text{Poi}(\lambda t)$  be the number of events in the first  $t$  units of time, for  $t \geq 0$  (if average is  $\lambda$  per unit of time, then it is  $\lambda t$  per  $t$  units of time). Then,  $Y > t$  (wait longer than  $t$  units of time until the first event) *if and only if*  $X(t) = 0$  (no events happened in the first  $t$  units of time). This allows us to relate the Exponential CDF to the Poisson PMF.

$$\mathbb{P}(Y > t) = \mathbb{P}(\text{no events in the first } t \text{ units}) = \mathbb{P}(X(t) = 0) = e^{-\lambda t} \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

Note that we plugged in the  $\text{Poi}(\lambda t)$  PMF at 0 in the second last equality. Now, the CDF is just the complement of the probability we computed:

$$F_Y(t) = \mathbb{P}(Y \leq t) = 1 - \mathbb{P}(Y > t) = 1 - e^{-\lambda t}$$

Remember since the CDF was the integral of the PDF, the PDF is the derivative of the CDF by the fundamental theorem of calculus:

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \lambda e^{-\lambda t}$$

#### Definition 4.2.2: The Exponential RV

$X \sim \text{Exp}(\lambda)$ , if and only if  $X$  has the following pdf (and range  $\Omega_X = [0, \infty)$ ):

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$X$  is the waiting time until the first occurrence of an event in a Poisson Process with parameter  $\lambda$ .

$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

The cdf is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

*Proof of Expectation and Variance of Exponential.* You can use integration by parts if you want to solve these integrals, or you can use WolframAlpha. Again, I'm omitting the steps that are not relevant to your understanding of probability theory (computing integrals):

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

□

If you usually skip examples, please don't skip the next two. The first example here highlights the relationship between the Poisson and Exponential RVs, and the second highlights the memoryless property!

### Example(s)

Suppose that, on average, 13 car crashes occur each *day* on Highway 101. What is the probability that no car crashes occur in the next *hour*? Be careful of units of time!

*Solution* We will solve this problem with three equivalent approaches! Take the time to understand why each of them work.

1. Then, on average there are  $\frac{13}{24}$  car crashes per *hour*. So the number of crashes in the next hour is  $X \sim \text{Poi}(\lambda = \frac{13}{24})$ .

$$\mathbb{P}(X = 0) = e^{-13/24} \frac{(13/24)^0}{0!} = e^{-13/24}$$

2. Similar to above, the time (in *hours*) until the first car crash is  $Y \sim \text{Exp}(\lambda = \frac{13}{24})$ , since on average  $13/24$  car crashes happen per *hour*. Then, the probability no car crashes happen in the next hour is

$$\mathbb{P}(Y > 1 \text{ (hour)}) = 1 - \mathbb{P}(Y \leq 1) = 1 - F_Y(1) = 1 - (1 - e^{-13/24 \cdot 1}) = e^{-13/24}$$

3. If we don't want to change the units, then we can say the waiting time until the next car crash (in *days*) is  $Z \sim \text{Exp}(\lambda = 13)$ . since on average 13 car crashes happen per *day*. Then, the probability no car crashes occur in the next hour ( $1/24$  of a day) is the probability that we wait longer than  $1/24$  day:

$$\mathbb{P}(Z > 1/24) = 1 - \mathbb{P}(Z \leq 1/24) = 1 - F_Z(1/24) = 1 - (1 - e^{-13 \cdot 1/24}) = e^{-13/24}$$

Hopefully the first and second solutions show you the relationship between the Poisson and Exponential RVs (they both come from the Poisson process), and the second and third solution show you how to be careful with units and that you'll get the same answer as long as you are consistent. □

### Example(s)

Suppose the average battery life of an AAA battery is approximately 50 hours.

1. What is the probability the battery lasts more than 60 hours?
2. What is the probability the battery lasts more than 40 hours?
3. What is the probability the battery lasts more than 100 hours, *given* that the battery has

already lasted 60 hours? That is, what is the probability it can last 40 additional hours? Relate this to your answer from the previous part!

*Solution* Since we want to model battery life, we should use an Exponential distribution. Since we know the average battery life is 50 hours, and that the expected value of an exponential RV is  $1/\lambda$  (see above), we should say that the battery life is  $X \sim \text{Exp}(\lambda = \frac{1}{50} = 0.02)$ .

1. If we want the probability the battery lasts more than 60 hours, then we want

$$\mathbb{P}(X \geq 60) = \int_{60}^{\infty} f_X(t) dt = \int_{60}^{\infty} 0.02e^{-0.02t} dt = e^{-1.2}$$

But continuous distributions have a CDF which we can and should take advantage of! We can look up the CDF above as well:

$$\mathbb{P}(X \geq 60) = 1 - \mathbb{P}(X < 60) = 1 - F_X(60) = 1 - (1 - e^{-0.02 \cdot 60}) = e^{-1.2}$$

We made a step above that said  $\mathbb{P}(X < 60) = F_X(60)$ , but  $F_X(60) = \mathbb{P}(X \leq 60)$ . It turns out they are the same for continuous RVs, since the probability  $X = 60$  exactly is zero!

2. Similarly,

$$\mathbb{P}(X \geq 40) = 1 - \mathbb{P}(X < 40) = 1 - F_X(40) = 1 - (1 - e^{-0.02 \cdot 40}) = e^{-0.8}$$

3. By Bayes Theorem,

$$\mathbb{P}(X \geq 100 \mid X \geq 60) = \frac{\mathbb{P}(X \geq 60 \mid X \geq 100) \mathbb{P}(X \geq 100)}{\mathbb{P}(X \geq 60)}$$

Note that  $\mathbb{P}(X \geq 60 \mid X \geq 100) = 1$  (why?) and  $\mathbb{P}(X \geq 100) = e^{-0.02 \cdot 100} = e^{-2}$  (same process as above), so plugging in these numbers gives

$$= \frac{1 \cdot e^{-2}}{e^{-1.2}} = e^{-0.8}$$

Note that this is exactly the same as  $\mathbb{P}(X \geq 40)$  above, the probability we the battery lasted at least 40 hours. This says that the previous 60 hours don't matter -  $\mathbb{P}(X \geq 40 + 60 \mid X \geq 60) = \mathbb{P}(X \geq 40)$ . This property is called memorylessness, since the battery essentially forgets that it was alive for 60 hours! We'll discuss this more formally below and prove it.

□

### 4.2.3 Memorylessness

#### Definition 4.2.3: Memorylessness

A random variable  $X$  is **memoryless** is for all  $s, t \geq 0$ ,

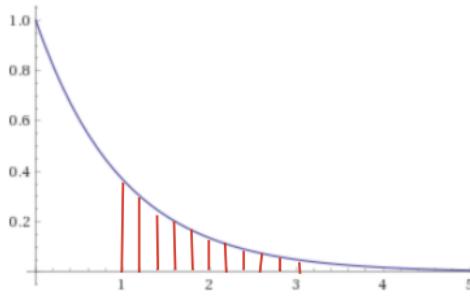
$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

We just saw a concrete example above, but let's see another. Let  $s = 7, t = 2$ . So  $\mathbb{P}(X > 9 \mid X > 7) = \mathbb{P}(X > 2)$ .

This memoryless property says that, given we've waited (at least) 7 minutes, the probability we wait (at least) 2 more minutes, is the same as the probability we waited (at least 2) more from the beginning. That is, the random variable “forgot” how long we've already been waiting.

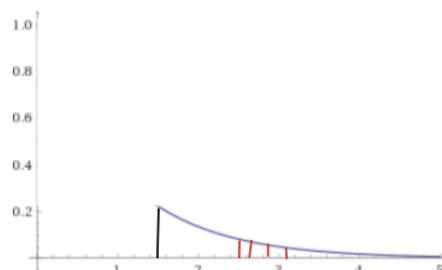
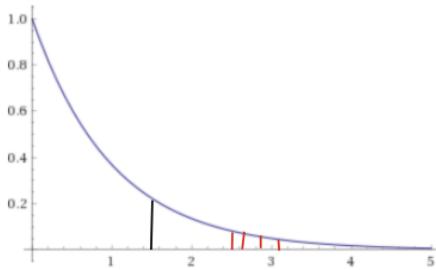
The only memoryless RVs are the **Geometric** (discrete) and **Exponential** (Continuous)! This is because events happen *independently* over time/trials, and so the past doesn't matter.

We've seen it algebraically and intuitively, but let's see it pictorially as well. Here is a picture of the probability is greater than 1 for an exponential RV. It is the area to the right of 1 of the density function  $\lambda e^{-\lambda x}$  for  $x \geq 0$  (shaded in red).



$$P(X > 1)$$

Below is a picture of the probability  $X > 2.5$  given  $X > 1.5$ . (shaded in red; note that the red area is to the right of the black line where  $X = 1.5$ ). If you hide the area to the left of the black line, you can see the ratio of the red area to the right of the black line is the same as  $P(X > 1)$  above. So this exponential density function has memoryless built in!



$$P(X > 2.5 | X > 1.5)$$

$$P(X > 2.5 | X > 1.5)$$

Theorem 4.2.14: Memorylessness of Exponential

If  $X \sim \text{Exp}(\lambda)$ , then  $X$  has the memoryless property.

*Proof of Memorylessness of Exponential.*

If  $X \sim \text{Exp}(\lambda)$  and  $x \geq 0$ , then recall

$$\mathbb{P}(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$\begin{aligned}\mathbb{P}(X > s + t \mid X > s) &= \frac{\mathbb{P}(X > s \mid X > s + t) \mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} && [\text{Bayes Theorem}] \\ &= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} && [\mathbb{P}(X > s \mid X > s + t) = 1] \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} && [\text{plug in formula above}] \\ &= e^{-\lambda t} \\ &= \mathbb{P}(X > t)\end{aligned}$$

□

Theorem 4.2.15: Memorylessness of Geometric

If  $X \sim \text{Geo}(p)$ , then  $X$  has the memoryless property.

*Proof of Memorylessness of Geometric.*

If  $X \sim \text{Geo}(p)$ , then for  $k \in \Omega_X = \{1, 2, \dots\}$ , then by independence of the trials,

$$\mathbb{P}(X > k) = \mathbb{P}(\text{no successes in first } k \text{ trials}) = (1 - p)^k$$

Then, I'll leave it to you to do the same computation as above (using Bayes Theorem). You'll see it work out almost exactly the same way! □

#### 4.2.4 The Gamma RV

Just like the Exponential RV is the continuous extension of the Geometric RV (from discrete trials to continuous time), we have a Gamma RV which models the time until the  $r$ -th event. This should remind you of the Negative Binomial RV, which modelled the number of trials until the  $r$ -th success, and so was the sum of  $r$  independent and identically distributed (iid)  $\text{Geo}(p)$  RVs.

**Definition 4.2.4: Gamma RV**

$X \sim \text{Gamma}(r, \lambda)$  if and only if  $X$  has the following pdf:

$$f_X(x) = \begin{cases} \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$X$  is the sum of  $r$  independent  $\text{Exp}(\lambda)$  random variables.

Gamma is to Exponential as Negative Binomial to Geometric. It is the waiting time until the  $r$ -th event, rather than just the first event. So you can write it as a sum of  $r$  independent exponential random variables.

$$\mathbb{E}[X] = \frac{r}{\lambda}, \quad \text{Var}(X) = \frac{r}{\lambda^2}$$

$X$  is the waiting time until the  $r^{th}$  occurrence of an event in a Poisson Process with parameter  $\lambda$ . Notice that  $\text{Gamma}(1, \lambda) \equiv \text{Exp}(\lambda)$ . By definition, if  $X, Y$  are independent with  $X \sim \text{Gamma}(r, \lambda)$  and  $Y \sim \text{Gamma}(s, \lambda)$ , then  $X + Y \sim \text{Gamma}(r + s, \lambda)$ .

*Proof of Expectation and Variance of Gamma.* The PDF of the Gamma looks very ugly and hard to deal with, so let's use our favorite trick: Linearity of Expectation! As mentioned earlier, if  $X \sim \text{Gamma}(r, \lambda)$ , then  $X = \sum_{i=1}^r X_i$  where each  $X_i \sim \text{Exp}(\lambda)$  is independent with  $\mathbb{E}[X_i] = \frac{1}{\lambda}$  and  $\text{Var}(X_i) = \frac{1}{\lambda^2}$ . So by LoE,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^r X_i\right] = \sum_{i=1}^r \mathbb{E}[X_i] = \sum_{i=1}^r \frac{1}{\lambda} = \frac{r}{\lambda}$$

Now, we can use the fact that the variance of a sum of *independent* rvs is the sum of the variances (we have yet to prove this fact).

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^r X_i\right) = \sum_{i=1}^r \text{Var}(X_i) = \sum_{i=1}^r \frac{1}{\lambda^2} = \frac{r}{\lambda^2}$$

□

### 4.2.5 Exercises

- Suppose that on average, 40 babies are born every hour in Seattle.
  - What is the probability that no babies are born in the next minute? Try solving this in two different but equivalent ways - using a Poisson and Exponential RV.
  - What is the probability that it takes more than 20 minutes for the first 10 babies to be born? Again, try solving this in two different but equivalent ways - using a Poisson and Gamma RV.
  - What is the expected time until the 5th baby is born?

**Solution:**

- The number of babies born in the next minute is  $X \sim \text{Poi}(40/60)$ , so  $\mathbb{P}(X = 0) = e^{-40/60} \approx 0.5134$ . Alternatively, the time in minutes until the next baby is born is  $Y \sim \text{Exp}(40/60)$ , and we want the probability that no babies are born in the next minute; i.e., it takes at least one minute for the first baby to be born. Hence,

$$\mathbb{P}(Y > 1) = 1 - F_Y(1) = 1 - (1 - e^{-2/3 \cdot 1}) = e^{-2/3}$$

We got the same answer with two different approaches!

- The number of babies born in the next 20 minutes is  $W \sim \text{Poi}(40/3)$ , so

$$\mathbb{P}(W \leq 10) = \sum_{k=0}^{10} e^{-40/3} \frac{(40/3)^k}{k!}$$

Alternatively, the time in minutes until the tenth baby is born is  $Z \sim \text{Gamma}(10, 40/60)$ , and we are asking what's the probability this is over 20 minutes, is

$$\mathbb{P}(Z > 20) = 1 - F_Z(20) = 1 - \int_0^{20} \frac{(40/60)^{10}}{(10-1)!} x^{10-1} e^{-(40/60)x} dx$$

Unfortunately, there isn't a nice closed form for the Gamma CDF, but this would evaluate to the same result!

- (c) The time in minutes until the 5th baby is born is  $V \sim \text{Gamma}(5, 40/60)$ , so  $\mathbb{E}[V] = \frac{r}{\lambda} = \frac{5}{40/60} = 7.5$  minutes.

## Chapter 4: Continuous Random Variables

### 4.3: The Normal/Gaussian Random Variable

#### 4.3.1 Standardizing RVs

Let's say you took two tests. You got 90% on history, and 50% on math. In which test did you do "better"? You might think it's obviously history, but actually your performance depends on the mean and standard deviation of scores in the class! We need to compare them on a fair playing ground then - this process is called standardizing. Let's see which test you truly did better on, given some extra information.

1. On your history test, you got a 90% when the mean was 70% and the standard deviation was 10%.
2. On your math test, you got a 50% when the mean was 35% and the standard deviation was 5%.

You scored higher in history, but how many standard deviations above the mean?

$$\frac{\text{your history score} - \text{mean history score}}{\text{standard deviation of history scores}} = \frac{90 - 70}{10} = 2$$

On your math test,

$$\frac{\text{your math score} - \text{mean math score}}{\text{standard deviation of math scores}} = \frac{50 - 35}{5} = 3$$

Then, in terms of standard deviations above the mean, you actually did better in math! What we just computed here was

$$\frac{X - \mu}{\sigma}$$

in order to calculate the number of standard deviations above the mean a random variable's value is. (Note how we are using standard deviation instead of variance here so the units are the same!)

Recall that in general, if  $X$  is any random variable (discrete or continuous) with  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ , and  $a, b \in \mathbb{R}$ . Then,

$$\begin{aligned}\mathbb{E}[aX + b] &= a\mathbb{E}[X] + b = a\mu + b \\ \text{Var}(aX + b) &= a^2\text{Var}(X) = a^2\sigma^2\end{aligned}$$

In particular, we call  $\frac{X - \mu}{\sigma}$  a standardized version of  $X$ , as it measures how many standard deviations above the mean a point is. We standardize random variables for fair comparison. Applying linearity of expectation and variance of random variables to standardized random variables, we get the expectation and variance of standardized random variables:

$$\mathbb{E}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}(\mathbb{E}[X] - \mu) = 0$$

$$\text{Var}\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}\text{Var}(X - \mu) = \frac{1}{\sigma^2}\text{Var}(X) = \frac{1}{\sigma^2}\sigma^2 = 1 \implies \sigma_X = \sqrt{\text{Var}(X)} = \sqrt{1} = 1$$

It turns out the mean is 0 and the standard deviation (and variance) is 1! This makes sense because on average, someone is average (0 standard deviations above the mean), and the standard deviation is 1.

### 4.3.2 The Normal/Gaussian Random Variable

**Definition 4.3.1:** Normal (Gaussian, "bell curve") distribution

$X \sim \mathcal{N}(\mu, \sigma^2)$  if and only if  $X$  has the following PDF (and range  $\Omega_X = (-\infty, +\infty)$ ):

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where  $\exp(y) = e^y$ . This Normal random variable actually has as parameters its mean and variance, and hence:

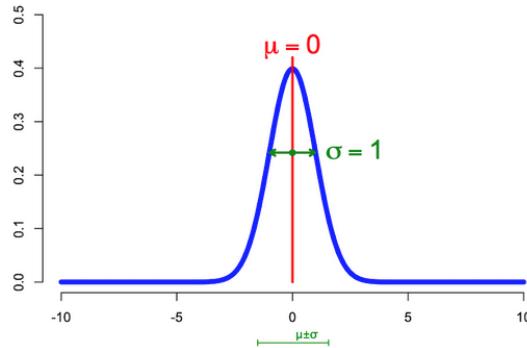
$$\mathbb{E}[X] = \mu$$

$$\text{Var}(X) = \sigma^2$$

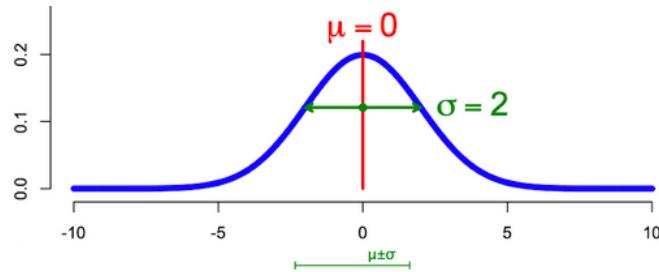
Unfortunately, there is no closed form formula for the CDF (there wasn't one for the Gamma RV) either. We'll see how to compute these probabilities anyway though soon using a lookup table!

Normal distributions produce bell-shaped curves. Here are some visualizations of the density function for varying  $\mu$  and  $\sigma^2$ .

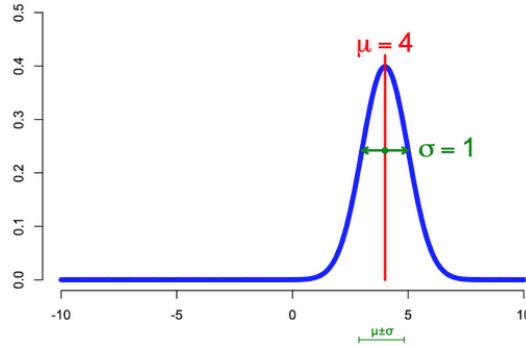
For instance, a normal distribution with  $\mu = 0$  and  $\sigma = 1$  produces the following bell curve:



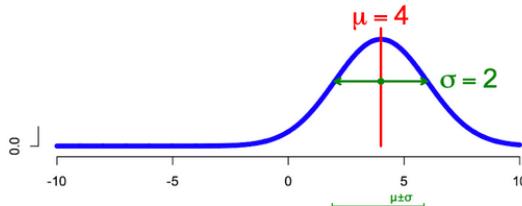
If the standard deviation increases, it becomes more likely for the variable to be farther away from the mean, so the distribution becomes flatter. For instance, a curve with the same  $\mu = 0$  but higher  $\sigma = 2$  ( $\sigma^2 = 4$ ) looks like this:



If you change the mean, the distribution will shift left or right. For instance, increasing the mean so  $\mu = 4$  shifts the distribution 4 to the right. The shape of the curve remains unchanged:



If you change the mean AND standard deviation, the curves shape changes and shifts. For instance, changing the mean so  $\mu = 4$  and standard deviation so  $\sigma = 2$  gives us a flatter, shifted curve:



### 4.3.3 Closure Properties of the Normal Random Variable

Occasionally, when we sum two independent random variables of the same type, we get the same type. For example, if  $X \sim \text{Bin}(n, p)$  and  $Y \sim \text{Bin}(m, p)$  are independent, then  $X + Y \sim \text{Bin}(n + m, p)$  because  $X$  is the number of successes in  $n$  trials, and  $Y$  is the number of successes in  $m$  trials. It also turns out similar properties hold for the Poisson, Negative Binomial, and Gamma random variables when you think of their English meaning. We'll formally prove some of these results in 5.5 though.

However, scaling and shifting a random variable often does not keep it in the same family. Continuous uniform rvs are the only ones we learned so far that do: if  $X \sim \text{Unif}(0, 1)$ , then  $3X + 2 \sim \text{Unif}(2, 5)$ : we'll learn how to prove this in the next section! However, this is not true for the others; for example, the range of a  $\text{Poi}(\lambda)$  is  $\{0, 1, 2, \dots\}$  as it is the number of events in a unit of time, and  $2X$  has range  $\{0, 2, 4, 6, \dots\}$  so cannot be Poisson (cannot be an odd number)! We'll see that Normal random variables have these closure properties.

Recall that in general, if  $X$  is any random variable (discrete or continuous) with  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ , and  $a, b \in \mathbb{R}$ . Then,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b = a\mu + b$$

$$\text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$$

#### Definition 4.3.2: Closure of the Normal Under Scale and Shift

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

In particular,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

We will prove this theorem later in section 5.6 using Moment Generating Functions! This is really amazing - the mean and variance are no surprise. The fact that scaling and shifting a Normal random variable results in another Normal random variable is very interesting!

Let  $X, Y$  be ANY *independent* random variables (discrete or continuous) with  $\mathbb{E}[X] = \mu_X$ ,  $\mathbb{E}[Y] = \mu_Y$ ,  $\text{Var}(X) = \sigma_X^2$ ,  $\text{Var}(Y) = \sigma_Y^2$  and  $a, b, c \in \mathbb{R}$ . Recall,

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c = a\mu_X + b\mu_Y + c$$

$$\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

#### Definition 4.3.3: Closure of the Normal Under Addition

If  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$  and  $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$  (both independent normal random variables), then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

Again, this is really amazing. The mean and variance aren't a surprise again, but the fact that adding two independent Normals results in another Normal distribution is not trivial, and we will prove this later as well!

#### Example(s)

Suppose you believe temperatures in the Vancouver, Canada each day are approximately normally distributed with mean 25 degrees Celsius and standard deviation 5 degrees Celsius. However, your American friend only understands Fahrenheit.

1. What is the distribution of temperatures each day in Vancouver in Fahrenheit? To convert Celsius ( $C$ ) to Fahrenheit ( $F$ ), the formula is  $F = \frac{9}{5}C + 32$ .
2. What is the distribution of the average temperature over a week in the Vancouver, in Fahrenheit? That is, if you were to sample a random week's average temperature, what is its distribution? Assume the temperature each day is independent of the rest (this may not be a realistic assumption).

#### Solution

1. The degrees in Celsius are  $\mathcal{N}(\mu_C = 25, \sigma_C^2 = 5^2)$ . Since  $F = \frac{9}{5}C + 32$ , we know by linearity of expectation and properties of variance:

$$\mu_F = \mathbb{E}[F] = \mathbb{E}\left[\frac{9}{5}C + 32\right] = \frac{9}{5}\mathbb{E}[C] + 32 = \frac{9}{5}25 + 32 = 77$$

$$\sigma_F^2 = \text{Var}(F) = \text{Var}\left(\frac{9}{5}C + 32\right) = \left(\frac{9}{5}\right)^2 \text{Var}(C) = \left(\frac{9}{5}\right)^2 5^2 = 81$$

These values are no surprise, but by closure of the Normal distribution, we can say that  $F \sim \mathcal{N}(\mu_F = 77, \sigma_F^2 = 9^2)$ .

2. Let  $F_1, F_2, \dots, F_7$  be independent temperatures over a week, so each  $F_i \sim \mathcal{N}(\mu_F = 77, \sigma_F^2 = 81)$ . Let  $\bar{F} = \frac{1}{7} \sum_{i=1}^7 F_i$  denote the average temperature over this week. Then, by linearity of expectation and properties of variance (requiring independence),

$$\mathbb{E} \left[ \frac{1}{7} \sum_{i=1}^7 F_i \right] = \frac{1}{7} \sum_{i=1}^7 \mathbb{E}[F_i] = \frac{1}{7} \cdot 7 \cdot 77 = 77$$

$$\text{Var} \left( \frac{1}{7} \sum_{i=1}^7 F_i \right) = \frac{1}{7^2} \sum_{i=1}^7 \text{Var}(F_i) = \frac{1}{7^2} \cdot 7 \cdot 81 = \frac{81}{7}$$

Note that the mean is the same, but the variance is smaller. This might make sense because we expect the average temperature over a week should match that of a single day, but it is more stable (has lower variance). By closure properties of the Normal distribution, since we take a sum of independent Normal RVs and then divide it by 7,  $\bar{F} = \frac{1}{7} \sum_{i=1}^7 F_i \sim \mathcal{N}(\mu = 77, \sigma^2 = 81/7)$ .

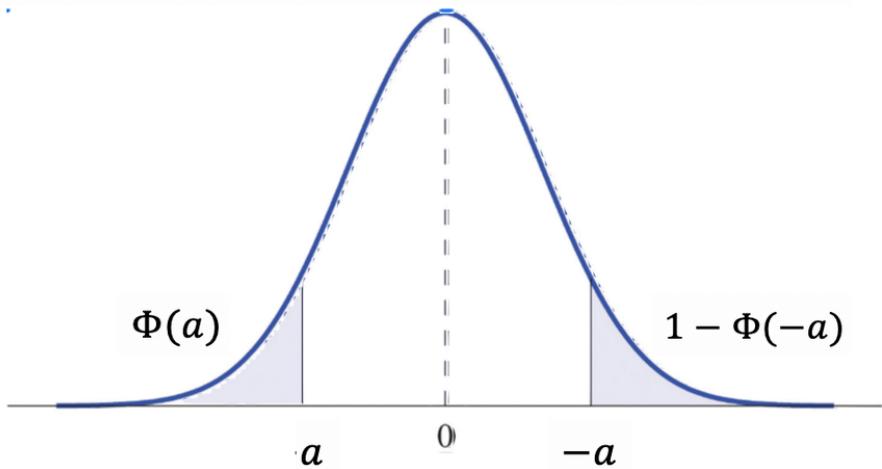
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#### 4.3.4 The Standard Normal CDF

We'll finally learn how to use to calculate probabilities like  $\mathbb{P}(X \leq 55)$  if  $X$  has a Normal distribution!

If  $Z \sim \mathcal{N}(0, 1)$  is the **standard normal** (the normal RV with mean 0 and variance/standard deviation 1), we denote the CDF  $\Phi(a) = F_Z(a) = \mathbb{P}(Z \leq a)$ , since it is so commonly used. There is no closed-form formula, so this CDF is stored in a  $\Phi$  table (read a "Phi Table"). Remember,  $\Phi(a)$  is just the area to the left of  $a$ .

Since the normal distribution curve is symmetric, the area to the left of  $a$  is the same as the area to the right of  $-a$ . This picture below shows that  $\Phi(a) = 1 - \Phi(-a)$ .



To get the CDF  $\Phi(1.09) = \mathbb{P}(Z \leq 1.09)$  from the  $\Phi$  table, we look at the row with a value of 1.0, and column with value 0.09, as marked here:

<i>z</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

From this, we see that  $\mathbb{P}(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$ .

This table usually only has positive numbers, so if you want to look up negative numbers, it's necessary to use the fact that  $\Phi(a) = 1 - \Phi(-a)$ . For example, if we want  $\mathbb{P}(Z \leq -2.13) = \Phi(-2.13)$ , we need to do  $1 - \Phi(2.13) = 1 - 0.9834 = 0.0166$  (try to find  $\Phi(2.13)$  yourself above).

How does this help though when  $X$  is Normal but not the standard normal? In general, for a  $X \sim \mathcal{N}(\mu, \sigma^2)$ , we can calculate the CDF of  $X$  by standardizing it to be standard normal,

$$\begin{aligned}
 F_X(y) &= \mathbb{P}(X \leq y) && [\text{def of CDF}] \\
 &= \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{y - \mu}{\sigma}\right) && [\text{standardizing both sides}] \\
 &= \mathbb{P}\left(Z \leq \frac{y - \mu}{\sigma}\right) && [\text{since } Z = \frac{X - \mu}{\sigma} \text{ is the standardized normal, } Z \sim \mathcal{N}(0, 1)] \\
 &= \Phi\left(\frac{y - \mu}{\sigma}\right) && [\text{def of } \Phi]
 \end{aligned}$$

We can also find  $\mathbb{P}(a \leq X \leq b)$ ,

$$\begin{aligned}\mathbb{P}(a \leq X \leq b) &= F_X(b) - F_X(a) && [\text{def of CDF}] \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) && [\text{def of } \Phi]\end{aligned}$$

#### Definition 4.3.4: Standard Normal Random Variable

The “standard normal” random variable is typically denoted  $Z$  and has mean 0 and variance 1. By the closure property of normals, if  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$ . The CDF has no closed form, but we denote the CDF of the standard normal by  $\Phi(a) = F_Z(a) = \mathbb{P}(Z \leq a)$ . Note that by symmetry of the density about 0,  $\Phi(-a) = 1 - \Phi(a)$ .

See some examples below of how we can use the  $\Phi$  table to calculate probabilities associated with the Normal distributions! Again, the  $\Phi$  table gives the CDF of the standard Normal since it doesn’t have a closed form like Uniform/Exponential. Also, *any* Normal RV can be standardized so we can look up probabilities in the  $\Phi$  table!

#### Example(s)

Suppose the age of a random adult in the United States is (approximately) normally distributed with mean 50 and standard deviation 15.

1. What is the probability that a randomly selected adult in the US is over 70 years old?
2. What is the probability that a randomly selected adult in the US is under 25 years old?
3. What is the probability that a randomly selected adult in the US is between 40 and 45 years old?

#### Solution

1. The height of a random adult is  $X \sim \mathcal{N}(\mu = 50, \sigma^2 = 15^2)$ , so remember we standardize to use the standard Gaussian:

$$\begin{aligned}\mathbb{P}(X > 70) &= \mathbb{P}\left(\frac{X-50}{15} > \frac{70-50}{15}\right) && [\text{standardize}] \\ &= \mathbb{P}(Z > 1.33) && \left[\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0, 1)\right] \\ &= 1 - \mathbb{P}(Z \leq 1.33) && [\text{complement}] \\ &= 1 - \Phi(1.33) && [\text{def of } \Phi] \\ &= 1 - 0.9082 && [\text{look up } \Phi \text{ table from earlier}] \\ &= 0.0918\end{aligned}$$

2. We do a similar calculation:

$$\begin{aligned}
 \mathbb{P}(X < 25) &= \mathbb{P}\left(\frac{X - 50}{15} > \frac{25 - 50}{15}\right) && \text{[standardize]} \\
 &= \mathbb{P}(Z < -2/3) && \left[\frac{X - \mu}{\sigma} = Z \sim \mathcal{N}(0, 1)\right] \\
 &= \Phi(-0.67) && \text{[recall since continuous rv, identical to less than or equal]} \\
 &= 1 - \Phi(0.67) && \text{[symmetry trick to make positive]} \\
 &= 1 - 0.7486 && \text{[look up } \Phi \text{ table from earlier]} \\
 &= 0.2514
 \end{aligned}$$

3. We do a similar calculation:

$$\begin{aligned}
 \mathbb{P}(40 < X < 45) &= \mathbb{P}\left(\frac{40 - 50}{15} < \frac{X - 50}{15} < \frac{45 - 50}{15}\right) && \text{[standardize]} \\
 &= \mathbb{P}(-2/3 < Z < -1/3) && \left[\frac{X - \mu}{\sigma} = Z \sim \mathcal{N}(0, 1)\right] \\
 &= \Phi(-0.33) - \Phi(-0.67) && [\mathbb{P}(a < X < b) = F_X(b) - F_X(a)] \\
 &= (1 - \Phi(0.33)) - (1 - \Phi(0.67)) && \text{[symmetry trick to make positive]} \\
 &= \Phi(0.67) - \Phi(0.33) \\
 &= 0.7486 - 0.6293 && \text{[look up } \Phi \text{ table from earlier]} \\
 &= 0.1193
 \end{aligned}$$

□

### 4.3.5 Exercises

- Suppose the time (in hours) it takes for you to finish pset  $i$  is approximately  $X_i \sim \mathcal{N}(\mu = 10, \sigma^2 = 9)$  (for  $i = 1, \dots, 5$ ) and the time (in hours) it takes for you to finish a project is approximately  $Y \sim \mathcal{N}(\mu = 20, \sigma^2 = 10)$ . Let  $W = X_1 + X_2 + X_3 + X_4 + X_5 + Y$  be the time it takes to complete all 5 psets and the project.
  - What are the mean and variance of  $W$ ?
  - What is the distribution of  $W$  and what are its parameter(s)?
  - What is the probability that you complete all the homework in under 60 hours?

**Solution:**

- The mean by linearity of expectation is  $\mathbb{E}[W] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_5] + \mathbb{E}[Y] = 50 + 20 = 70$ . Variance adds for independent RVs, so  $\text{Var}(W) = \text{Var}(X_1) + \dots + \text{Var}(X_5) + \text{Var}(Y) = 45 + 10 = 55$ .
- Since  $W$  is the sum of independent Normal random variables,  $W$  is also normal with the parameters we calculated above. So  $W \sim \mathcal{N}(\mu = 70, \sigma^2 = 55)$ .
- 

(c)

$$\mathbb{P}(W < 60) = \mathbb{P}\left(\frac{W - 70}{\sqrt{55}} < \frac{60 - 70}{\sqrt{55}}\right) \approx \mathbb{P}(Z < -1.35) = \Phi(-1.35) = 1 - \Phi(1.35) = 1 - 0.9115 = 0.0885$$

## Chapter 4: Continuous Random Variables

### 4.4: Transforming Continuous RVs

Suppose the amount of gold a company can mine is  $X$  tons per year, and you have some (continuous) distribution to model this. However, your earning is not simply  $X$  - it is actually a function of the amount of product, some  $Y = g(X)$ . What is the distribution of  $Y$ ?

Since we know the distribution of  $X$ , this will help us model the distribution of  $Y$  by *transforming random variables*.

#### 4.4.1 Transforming 1-D (Continuous) RVs via CDF

When we are dealing with discrete random variables, this process wasn't too bad. Let's say  $X$  had range  $\{-1, 0, 1\}$  and PMF

$$p_X(x) = \begin{cases} 0.3 & x = -1 \\ 0.2 & x = 0 \\ 0.5 & x = 1 \end{cases}$$

and  $Y = g(X) = X^2$ . Then,  $\Omega_Y = \{0, 1\}$ , and we could say

$$p_Y(y) = \begin{cases} p_X(-1) + p_X(1) = 0.3 + 0.5 = 0.8 & y = 1 \\ p_X(0) = 0.2 & y = 0 \end{cases}$$

This is because  $Y = 1$  if and only if  $X \in \{-1, 1\}$ , so to find  $\mathbb{P}(Y = 1)$ , we sum over all values  $x$  such that  $x^2 = 1$  of its probability. That's all this formula below says (the “:” means “such that”):

$$p_Y(y) = \sum_{x \in \Omega_X : g(x) = y} p_X(x)$$

But for continuous random variables, we have density functions instead of mass functions. That means  $f_X$  is not actually a probability and so we can't do this same technique. We want to work with the CDF  $F_X(x) = \mathbb{P}(X \leq x)$  instead because it actually does represent a probability! It's best to see this idea through an example.

#### Example(s)

Suppose you know  $X \sim \text{Unif}(0, 9)$  (continuous). What is the PDF of  $Y = \sqrt{X}$ ?

*Solution* We know the range of  $X$ ,

$$\Omega_X = [0, 9]$$

We also know the PDF of  $X$ , which is uniform from 0 to 9, and 0 elsewhere.

$$f_X(x) = \begin{cases} \frac{1}{9} & \text{if } 0 \leq x \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

The CDF of  $X$  is derived by taking the integral of the PDF, giving us (can also cite this),

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{9} & \text{if } 0 \leq x \leq 9 \\ 1 & \text{if } x > 9 \end{cases}$$

Now, we determine the range of  $Y$ . The smallest value that  $Y$  can take is  $\sqrt{0} = 0$ , and the largest value that  $Y$  can take is  $\sqrt{9} = 3$ , from the range of  $X$ . Since the square root function is monotone increasing, this gives us,

$$\Omega_Y = [0, 3]$$

But can we assume that, because  $X$  has a uniform distribution,  $Y$  does too?

This is not the case! Notice that values of  $X$  in the range  $[0, 1]$  will map to  $Y$  values in the range  $[0, 1]$ . But,  $X$  values in the range  $[1, 4]$  map to  $Y$  values in the range  $[1, 2]$  and  $X$  values in the range  $[4, 9]$  map to  $Y$  values in the range  $[2, 3]$ .

So, there is a much larger range of values of  $X$  that map to  $[2, 3]$  than to  $[0, 1]$  (since  $[4, 9]$  is a larger range than  $[0, 1]$ ). Therefore,  $Y$ 's distribution shouldn't be uniform. So, we cannot define the PDF of  $Y$  using the assumption that  $Y$  is uniform.

Instead, we will first compute the CDF  $F_Y$  and then, differentiate that to get the PDF  $f_Y$  for  $y \in [0, 3]$ .

To compute  $F_Y$  for any  $y$  in  $[0, 3]$ , we first take the CDF at  $y$ :

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) && [\text{def of CDF}] \\ &= \mathbb{P}(\sqrt{X} \leq y) && [\text{def of } Y] \\ &= \mathbb{P}(X \leq y^2) && [\text{squaring both sides}] \\ &= F_X(y^2) && [\text{def of CDF of } X \text{ evaluated at } y^2] \\ &= \frac{y^2}{9} && [\text{plug in CDF of } X, \text{ since } y^2 \in [0, 9]] \end{aligned}$$

Be very careful when squaring both sides of an equation - it may not keep the inequality true. In this case we didn't have to worry since  $X$  and  $Y$  were both guaranteed positive.

Differentiating the CDF to get the PDF  $f_Y$ , for  $y \in [0, 3]$ ,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2y}{9}$$

□

This is the general strategy for transforming continuous RVs! We'll summarize the steps below.

**Definition 4.4.1: Steps to get PDF of  $Y = g(X)$  from  $X$  (via CDF)**

1. Write down the range  $\Omega_X$ , PDF  $f_X$ , and CDF  $F_X$ .
2. Compute the range  $\Omega_Y = \{g(x) : x \in \Omega_X\}$ .

3. Start computing the CDF of  $Y$  on  $\Omega_Y$ ,  $F_Y(y) = \mathbb{P}(g(X) \leq y)$ , in terms of  $F_X$ .
4. Differentiate the CDF  $F_Y(y)$  to get the PDF  $f_Y(y)$  on  $\Omega_Y$ .  $f_Y$  is 0 outside  $\Omega_Y$ .

### Example(s)

Let  $X$  be continuous with range  $\Omega_X = [-1, +1]$  have density function

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose  $Y = X^4$ . Find the density function  $f_Y(y)$ .

*Solution* We'll follow the 4-step procedure as outlined above.

1. First, we list out the range, PDF, and CDF of the original variable  $X$ . We were given the range and PDF, but not the CDF, so let's compute it. For  $x \in [-1, +1]$  (note the use of the dummy variable  $t$  since  $x$  is already taken),

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(t) dt = \int_{-1}^x \frac{3}{4}(1-t^2) dt = \frac{1}{4}(2+3x-x^3)$$

So the complete CDF is:

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{4}(2+3x-x^3) & -1 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

2. The range of  $Y = X^4$  is  $\Omega_Y = \{x^4 : x \in [-1, +1]\} = [0, 1]$ , since  $x^4$  is always positive and between 0 and 1 for  $x \in [-1, +1]$ .
3. Be careful in the third equation below to include *both* lower and upper bounds (draw the function  $y = x^4$  to see why). For  $y \in \Omega_Y = [0, 1]$ , we will compute the CDF:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) && [\text{def of CDF}] \\ &= \mathbb{P}(X^4 \leq y) && [\text{def of } Y] \\ &= \mathbb{P}(-\sqrt[4]{y} \leq X \leq \sqrt[4]{y}) && [\text{don't forget the negative side}] \\ &= \mathbb{P}(X \leq \sqrt[4]{y}) - \mathbb{P}(X \leq -\sqrt[4]{y}) \\ &= F_X(\sqrt[4]{y}) - F_X(-\sqrt[4]{y}) && [\text{def of CDF of } X] \\ &= \frac{1}{4}(2+3\sqrt[4]{y}-\sqrt[4]{y^3}) - \frac{1}{4}(2+3(-\sqrt[4]{y})-(-\sqrt[4]{y})^3) && [\text{plug in CDF}] \end{aligned}$$

4. The last step is to differentiate the CDF to get the PDF, which is just computational, so I'll skip it!

□

### 4.4.2 Transforming 1-D RVs via Explicit Formula

Now, it turns out actually that in some special cases, there is an explicit formula for the density function of  $Y = g(X)$ , and we don't have to go through all the same steps above. It's important to note that the CDF

method *can always be applied*, but this next method has restrictions.

Theorem 4.4.16: Formula to get PDF of  $Y = g(X)$  from  $X$

If  $Y = g(X)$  and  $g : \Omega_X \rightarrow \Omega_Y$  is **strictly monotone** and **invertible** with inverse  $X = g^{-1}(Y) = h(Y)$ , then

$$f_Y(y) = \begin{cases} f_X(h(y)) \cdot |h'(y)| & \text{if } y \in \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

That is, the PDF of  $Y$  at  $y$  is the PDF of  $X$  evaluated at  $h(y)$  (the value of  $x$  that maps to  $y$ ) multiplied by the absolute value of the derivative of  $h(y)$ .

Note that the formula method is not as general as the previous method (using CDF), since  $g$  must satisfy monotonicity and invertibility. So transforming via CDF always works, but transforming may not work with this explicit formula all the time.

*Proof of Formula to get PDF of  $Y = g(X)$  from  $X$ .*

Suppose  $Y = g(X)$  and  $g$  is strictly monotone and invertible with inverse  $X = g^{-1}(Y) = h(Y)$ . We'll assume  $g$  is strictly monotone *increasing* and leave it to you to prove it for the case when  $g$  is strictly monotone *decreasing* (it's very similar).

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) && [\text{def of CDF}] \\ &= \mathbb{P}(g(X) \leq y) && [\text{def of } Y] \\ &= \mathbb{P}(X \leq g^{-1}(y)) && [\text{invertibility, AND monotone increasing keeps the sign}] \\ &= F_X(g^{-1}(y)) && [\text{def of CDF of } X \text{ evaluated at } g^{-1}(y)] \\ &= F_X(h(y)) && [h(y) = g^{-1}(y)] \end{aligned}$$

Hence, by the chain rule (of calculus),

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(h(y)) \cdot h'(y)$$

A similar proof would hold if  $g$  were monotone decreasing, except in the third line we would flip the sign of the inequality and make the  $h'(y)$  become an absolute value:  $|h'(y)|$ .

□ Now let's try the same example as we did earlier, but using this new method instead.

Example(s)

Suppose you know  $X \sim \text{Unif}(0, 9)$  (continuous). What is the PDF of  $Y = \sqrt{X}$ ?

*Solution* Recall, we know the range of  $X$ ,

$$\Omega_X = [0, 9]$$

We also know the PDF of  $X$ , which is uniform from 0 to 9 and 0 elsewhere.

$$f_X(x) = \begin{cases} \frac{1}{9} & \text{if } 0 \leq x \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Our goal is to use the formula given  $f_Y(y) = f_X(h(y)) \cdot |h'(y)|$ , after verifying some conditions on  $g$ .

Let  $g(t) = \sqrt{t}$ . This is strictly monotone increasing on  $\Omega_X = [0, 9]$ . This means that as  $t$  increases,  $\sqrt{t}$  also increases - therefore,  $g(t)$  is an increasing function.

What is the inverse of this function  $g$ ? The inverse of the square root function is just the squaring function:

$$h(y) = g^{-1}(y) = y^2$$

Then, we find it's derivative:

$$h'(y) = 2y$$

Now, we can use the explicit formula to find the PDF of  $Y$ .

For  $y \in [0, 3]$ ,

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| = \frac{1}{9} |2y| = \frac{2}{9}y$$

Note that we dropped the absolute value because we already assume  $y \in [0, 3]$  and hence  $2y$  is always positive. This gives the same formula as earlier, as it should!  $\square$

### 4.4.3 Transforming Multidimensional RVs via Formula

For completion, we've cited a formula to transform  $n$  random variables to  $n$  other random variables. For example, this might be useful if you have a system of two equations. For example,  $(R, \Theta)$  (polar) coordinates which are random variables, and wanting to convert to Cartesian coordinates to the two random variables  $(X, Y)$  where  $X = R \cos(\Theta)$  and  $Y = R \sin(\Theta)$ . This extends the formula we just learned to multi-dimensional random variables!

Theorem 4.4.17: Formula to get PDF of  $Y = g(X)$  from  $X$  (Multidimensional Case)

Let  $\mathbf{X} = (X_1, \dots, X_n)$ ,  $\mathbf{Y} = (Y_1, \dots, Y_n)$  be continuous random vectors (each component is a continuous rv) with the same dimension  $n$  (so  $\Omega_{\mathbf{X}}, \Omega_{\mathbf{Y}} \subseteq \mathbb{R}^n$ ), and  $\mathbf{Y} = g(\mathbf{X})$  where  $g : \Omega_{\mathbf{X}} \rightarrow \Omega_{\mathbf{Y}}$  is invertible and differentiable, with differentiable inverse  $\mathbf{X} = g^{-1}(\mathbf{y}) = h(\mathbf{y})$ . Then,

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h(\mathbf{y})) \left| \det \left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right) \right|$$

where  $\left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right) \in \mathbb{R}^{n \times n}$  is the Jacobian matrix of partial derivatives of  $h$ , with

$$\left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right)_{ij} = \frac{\partial (h(\mathbf{y}))_i}{\partial \mathbf{y}_j}$$

Hopefully this formula looks very similar to the one for the single-dimensional case! This formula is just for your information and you'll never have to use it in this class.

### 4.4.4 Exercises

1. Suppose  $X$  has range  $\Omega_X = (1, \infty)$  and density function

$$f_X(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

For reference, the CDF is also given

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^2} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Let  $Y = \frac{e^X - 1}{2}$ .

- (a) Compute the density function of  $Y$  via the CDF transformation method.
- (b) Compute the density function of  $Y$  using the formula, but explicitly verify the monotonicity and invertibility conditions.

**Solution:**

- (a) The range of  $Y$  is  $\Omega_Y = \left(\frac{e-1}{2}, \infty\right)$ . For  $y \in \Omega_Y$ ,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) && [\text{def of CDF}] \\ &= \mathbb{P}\left(\frac{e^X - 1}{2} \leq y\right) && [\text{def of } Y] \\ &= \mathbb{P}(e^X \leq 2y + 1) \\ &= \mathbb{P}(X \leq \ln(2y + 1)) \\ &= F_X(\ln(2y + 1)) && [\text{def of CDF}] \\ &= 1 - \frac{1}{[\ln(2y + 1)]^2} && \left[F_X(x) = 1 - \frac{1}{x^2}\right] \end{aligned}$$

The derivative is (don't forget the chain rule)

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{[\ln(2y + 1)]^3} \cdot \frac{1}{2y + 1} \cdot 2 = \frac{4}{(2y + 1)[\ln(2y + 1)]^3}$$

This density is valid for  $y \in \Omega_Y$ , and 0 everywhere else.

- (b) The function  $g(t) = \frac{e^t - 1}{2}$  is monotone increasing (since  $e^t$  is, and we shift and scale it by a positive constant), and has inverse  $h(y) = g^{-1}(y) = \ln(2y + 1)$ . We have  $h'(y) = \frac{2}{2y + 1}$ . By the formula, we get

$$\begin{aligned} f_Y(y) &= f_X(h(y))|h'(y)| && [\text{formula}] \\ &= \frac{2}{[\ln(2y + 1)]^3} \cdot \frac{2}{2y + 1} && \left[f_X(x) = \frac{2}{x^3}\right] \\ &= \frac{4}{(2y + 1)[\ln(2y + 1)]^3} \end{aligned}$$

This gives the same answer as part (a)!

## Chapter 5: Multiple Random Variables

### 5.1: Joint Discrete Distributions

This chapter, especially Sections 5.1-5.6, are arguably the most difficult in this entire text. They might take more time to fully absorb, but you'll get it, so don't give up!

We are finally going to talk about what happens when we want the probability distribution of more than one random variable. This will be called the joint distribution of two or more random variables. In this section, we'll focus on joint discrete distributions, and in the next, joint continuous distributions. We'll also finally prove that variance the variance of the sum of independent RVs is the sum of the variances, an important fact that we've been using without proof! But first, we need to review what a Cartesian product of sets is.

#### 5.1.1 Cartesian Products of Sets

##### Definition 5.1.1: Cartesian Product of Sets

Let  $A, B$  be sets. The Cartesian product of  $A$  and  $B$  is denoted:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Further if  $A, B$  are finite sets, then  $|A \times B| = |A| \cdot |B|$  by the product rule of counting.

##### Example(s)

Write each of the following in a notation that does not involve a Cartesian product:

1.  $\{1, 2, 3\} \times \{4, 5\}$
2.  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

##### *Solution*

1. Here, we have:

$$\{1, 2, 3\} \times \{4, 5\} = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

We have each of the elements of the first set paired with each of the elements of the second set. Note that  $|\{1, 2, 3\}| = 3$ ,  $|\{4, 5\}| = 2$ , and  $|\{1, 2, 3\} \times \{4, 5\}| = 6$ .

2. This is the  $xy$ -plane (2D space), which is denoted:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$$

□

### 5.1.2 Joint PMFs and Expectation

We will now talk about how we can model the distribution of two or more random variables, using an example to start.

Suppose we roll two fair 4-sided die independently, one blue and one red. Let  $X$  be the value of the blue die and  $Y$  be the value of the red die. Note:

$$\begin{aligned}\Omega_X &= \{1, 2, 3, 4\} \\ \Omega_Y &= \{1, 2, 3, 4\}\end{aligned}$$

Then we can also consider  $\Omega_{X,Y}$ , the joint range of  $X$  and  $Y$ . The joint range happens to be any combination of  $\{1, 2, 3, 4\}$  for both rolls. This can be written as:

$$\Omega_{X,Y} = \Omega_X \times \Omega_Y$$

Further each of these will be equally likely (as shown in the table below):

$X \setminus Y$	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

Above is a suitable way to write the joint probability mass function of  $X$  and  $Y$ , as it enumerates every probability of every pair of values. If we wanted to write it as a formula,  $p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$  for  $x, y \in \Omega_{X,Y}$  we have:

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{16}, & x, y \in \Omega_{X,Y} \\ 0, & \text{otherwise} \end{cases}$$

Note that either this piecewise function or the table above are valid ways to express the joint PMF.

#### Definition 5.1.2: Joint PMFs

Let  $X, Y$  be discrete random variables. The joint PMF of  $X$  and  $Y$  is:

$$p_{X,Y}(a,b) = \mathbb{P}(X = a, Y = b)$$

The joint range is the set of pairs  $(c, d)$  that have nonzero probability:

$$\Omega_{X,Y} = \{(c, d) : p_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that the probabilities in the table must sum to 1:

$$\sum_{(s,t) \in \Omega_{X,Y}} p_{X,Y}(s, t) = 1$$

Further, note that if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function, then LOTUS extends to the multidimensional case:

$$\mathbb{E}[g(X, Y)] = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} g(x, y) p_{X,Y}(x, y)$$

A lot of things are just the same as what we learned in Chapter 3, but extended! Note that the joint range above  $\Omega_{X,Y}$  was always a subset of  $\Omega_X \times \Omega_Y$ , and they're not necessarily equal. Let's see an example of this.

Back to our example of the blue and red die rolls. Again, let  $X$  be the value of the blue die and  $Y$  be the value of the red die. Now, let  $U = \min\{X, Y\}$  (the smaller of the two die rolls) and  $V = \max\{X, Y\}$  (the larger of the two die rolls). Then:

$$\Omega_U = \{1, 2, 3, 4\}$$

$$\Omega_V = \{1, 2, 3, 4\}$$

because both random variables can take on any of the four values that appear on the dice (e.g., it is possible that the minimum is 4 if we roll (4, 4) and the maximum to be 1 if we roll (1, 1)).

However, there is the constraint that the minimum value  $U$  is always at most the maximum value  $V$ . That is, the joint range would not include the pair  $(u, v) = (4, 1)$  for example, since the probability that the minimum is 4 and the maximum is 1 is zero. We can write this formally as the subset of the Cartesian product subject to  $u \leq v$ :

$$\Omega_{U,V} = \{(u, v) \in \Omega_U \times \Omega_V : u \leq v\} \neq \Omega_U \times \Omega_V$$

This will just be all the ordered pairs of the values that can appear as  $U$  and  $V$ . Now, however these are not equally likely, as shown in the table below. Notice that any pair  $(u, v)$  with  $u > v$  has zero probability, as promised. We'll explain how we got the other numbers under the table.

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

As discussed earlier, we can't have the case where  $U > V$ , so these are all 0. The cases where  $U = V$  occurs when the blue and red die have the same value, each which occurs with probability of  $\frac{1}{16}$  as shown earlier. For example,  $p_{U,V}(2,2) = \mathbb{P}(U = 2, V = 2) = 1/16$  since only one of the 16 equally likely outcomes  $(2,2)$  gives this result. The others in which  $U < V$  each occur with probability  $\frac{2}{16}$  because it could be the red die with the max and the blue die with the min, or the reverse. For example,  $p_{U,V}(1,3) = \mathbb{P}(U = 1, V = 3) = 2/16$  because two of the 16 outcomes  $(1,3)$  and  $(3,1)$  would result in the min being 1 and the max being 3.

So for the joint PMF as a formula  $p_{U,V}(u,v) = \mathbb{P}(U = u, V = v)$  for  $u, v \in \Omega_{U,V}$  we have:

$$p_{U,V}(u,v) = \begin{cases} \frac{2}{16}, & u, v \in \Omega_U \times \Omega_V, \quad v > u \\ \frac{1}{16}, & u, v \in \Omega_U \times \Omega_V, \quad v = u \\ 0, & \text{otherwise} \end{cases}$$

Again, the piecewise function and the table are both valid ways to express the joint PMF, and you may choose whichever is easier for you. When the joint range is larger, it might be infeasible to use a table though!

### 5.1.3 Marginal PMFs

Now suppose we didn't care about both  $U$  and  $V$ , just  $U$  (the minimum value). That is, we wanted to solve for the PMF  $p_U(u) = \mathbb{P}(U = u)$  for  $u \in \Omega_U$ . Intuitively, how would you do it? Take a look at the table version of their joint PMF above.

You might think the answer is  $7/16$ , but how did you get that? Well,  $\mathbb{P}(U = 1)$  would be the sum of the first row, since that is all the cases where  $U = 1$ . You computed

$$\mathbb{P}(U = 1) = \mathbb{P}(U = 1, V = 1) + \mathbb{P}(U = 1, V = 2) + \mathbb{P}(U = 1, V = 3) + \mathbb{P}(U = 1, V = 4) = \frac{1}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{7}{16}$$

Mathematically, we have

$$\mathbb{P}(U = u) = \sum_{v \in \Omega_V} \mathbb{P}(U = u, V = v)$$

Does this look like anything we learned before? It's just the law of total probability (intersection version) that we derived in 2.2, as the events  $\{V = v\}_{v \in \Omega_V}$  partition the sample space ( $V$  takes on exactly one value)! We can refer to the table above sum each row (which corresponds to a value of  $u$  to find the probability of that value of  $u$  occurring). That gives us the following:

$$p_U(u) = \begin{cases} \frac{7}{16}, & u = 1 \\ \frac{5}{16}, & u = 2 \\ \frac{3}{16}, & u = 3 \\ \frac{1}{16}, & u = 4 \end{cases}$$

One more example with  $u = 4$  is:

$$\mathbb{P}(U = 4) = \mathbb{P}(U = 4, V = 1) + \mathbb{P}(U = 4, V = 2) + \mathbb{P}(U = 4, V = 3) + \mathbb{P}(U = 4, V = 4) = 0 + 0 + 0 + \frac{1}{16} = \frac{1}{16}$$

This brings us to the definition of marginal PMFs. The idea of these is: given a joint probability distribution, what is the distribution of just one of them (or a subset)? We get this by *marginalizing* (summing) out the other variables.

**Definition 5.1.3: Marginal PMFs**

Let  $X, Y$  be discrete random variables. The marginal PMF of  $X$  is:

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a, b)$$

Similarly, the marginal PMF of  $Y$  is:

$$p_Y(d) = \sum_{c \in \Omega_X} p_{X,Y}(c, d)$$

(Extension) If  $Z$  is also a discrete random variable, then the marginal PMF of  $z$  is:

$$p_Z(z) = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} p_{X,Y,Z}(x, y, z)$$

This follows from the law of total probability, and is just like taking the sum of a row in the example above.

Now if asked for  $\mathbb{E}[U]$  for example, we actually don't need the joint PMF anymore. We've extracted the pertinent information in the form of  $p_U(u)$ , and compute  $\mathbb{E}[U] = \sum_u u p_U(u)$  normally.

We'll do more examples right after the next section!

## 5.1.4 Independence

We'll now redefine independence of RVs in terms of the joint PMF. This is completely the same as the definition we gave earlier, just with the new notation we learned.

**Definition 5.1.4: Independence (DRVs)**

Discrete random variables  $X, Y$  are independent, written  $X \perp Y$ , if for all  $x \in \Omega_X$  and  $y \in \Omega_Y$ :

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Again, this just says that  $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$  for every  $x, y$ .

## Theorem 5.1.18: Check for Independence (DRVs)

Recall the joint range  $\Omega_{X,Y} = \{(x,y) : p_{X,Y}(x,y) > 0\} \subseteq \Omega_X \times \Omega_Y$  is always a subset of the Cartesian product of the individual ranges. A necessary but not sufficient condition for independence is that  $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ . That is, if  $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ , then  $X$  and  $Y$  cannot be independent, but if  $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ , then we have to check the condition above.

This is because if there is some  $(a,b) \in \Omega_X \times \Omega_Y$  but not in  $\Omega_{X,Y}$ , then  $p_{X,Y}(a,b) = 0$  but  $p_X(a) > 0$  and  $p_Y(b) > 0$ , violating independence. For example, suppose the joint PMF looks like:

$X \setminus Y$	8	9	Row Total $p_X(x)$
3	1/3	1/2	5/6
7	1/6	0	1/6
Col Total $p_Y(y)$	1/2	1/2	1

Also side note that the marginal distributions are named what they are, since we often write the row and column totals in the margins. The joint range  $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$  since one of the entries is 0, and so  $(7,9) \notin \Omega_{X,Y}$  but  $(7,9) \in \Omega_X \times \Omega_Y$ . This immediately tells us they cannot be independent -  $p_X(7) > 0$  and  $p_Y(9) > 0$ , yet  $p_{X,Y}(7,9) = 0$ .

## Example(s)

Suppose  $X, Y$  are jointly distributed with joint PMF:

$X \setminus Y$	6	9	Row Total
0	3/12	5/12	?
2	1/12	2/12	?
3	0	1/12	?
Col Total	?	?	1

1. Find the marginal probability mass functions  $p_X(x)$  and  $p_Y(y)$ .
2. Find  $\mathbb{E}[Y]$ .
3. Are  $X$  and  $Y$  independent?
4. Find  $\mathbb{E}[X^Y]$ .

## Solution

1. Actually these can be found by filling in the row and column totals, since

$$p_X(x) = \sum_y p_{X,Y}(x,y), \quad p_Y(y) = \sum_x p_{X,Y}(x,y)$$

For example,  $\mathbb{P}(X = 0) = p_X(0) = \sum_y p_{X,Y}(0,y) = p_{X,Y}(0,6) + p_{X,Y}(0,9) = 3/12 + 5/12 = 8/12$  is the sum of the first row.

$X \setminus Y$	6	9	Row Total $p_X(x)$
0	3/12	5/12	8/12
2	1/12	2/12	3/12
3	0	1/12	1/12
Col Total $p_Y(y)$	4/12	8/12	1

Hence,

$$p_X(x) = \begin{cases} 8/12 & x = 0 \\ 3/12 & x = 2 \\ 1/12 & x = 3 \end{cases}$$

$$p_Y(y) = \begin{cases} 4/12 & y = 6 \\ 8/12 & y = 9 \end{cases}$$

2. We can actually compute  $\mathbb{E}[Y]$  just using  $p_Y$  now that we've eliminated/marginalized out  $X$  - we don't need the joint PMF anymore. We go back to the definition:

$$\mathbb{E}[Y] = \sum_y y p_Y(y) = 6 \cdot \frac{4}{12} + 9 \cdot \frac{8}{12} = 8$$

3.  $X, Y$  are independent, if for every table entry  $(x, y)$ , we have  $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ . However, notice  $p_{X,Y}(3, 6) = 0$  but  $p_X(3) > 0$  and  $p_Y(6) > 0$ . Hence we found an entry where this condition isn't true, so they cannot be independent. This is like the comment mentioned earlier: if  $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ , they have no chance of being independent.

4. We use the LOTUS formula:

$$\mathbb{E}[X^Y] = \sum_x \sum_y x^y p_{X,Y}(x, y) = 0^6 \cdot \frac{3}{12} + 0^9 \cdot \frac{5}{12} + 2^6 \cdot \frac{1}{12} + 2^9 \cdot \frac{2}{12} + 3^6 \cdot 0 + 3^9 \cdot \frac{1}{12}$$

This just sums over all the entries in the table  $(x, y)$  and takes a weighted average of all values  $x^y$  weighted by  $p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$ .

□

### 5.1.5 Variance Adds for Independent Random Variables

We will finally prove that variance adds for independent RVs. You are highly encouraged to read them because they give practice with expectations with joint distributions and LOTUS!

#### Lemma 5.1.1: Variance Adds for Independent RVs

If  $X, Y$  are independent random variables, denoted  $X \perp Y$ , then:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

If  $a, b, c \in R$  are scalars, then:

$$\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

Note this property relies on the fact that they are independent, whereas linearity of expectation always holds, regardless.

To prove this, we must first prove the following lemma:

**Lemma 5.1.2: Expected Value of the Product of Independent Random Variables**

If  $X \perp Y$ , then  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ .

*Proof of Lemma.*

$$\begin{aligned}
 \mathbb{E}[XY] &= \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} x y p_{X,Y}(x,y) && [\text{LOTUS}] \\
 &= \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} x y p_X(x) p_Y(y) && [X \perp Y, \text{ so } p_{X,Y}(x,y) = p_X(x)p_Y(y)] \\
 &= \sum_{x \in \Omega_X} x p_X(x) \sum_{y \in \Omega_Y} y p_Y(y) \\
 &= \mathbb{E}[X]\mathbb{E}[Y]
 \end{aligned}$$

□

*Proof of Variance Adds for Independent RVs.* Now we have the following:

$$\begin{aligned}
 \text{Var}(X+Y) &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 && [\text{def of variance}] \\
 &= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 && [\text{linearity of expectation}] \\
 &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - (\mathbb{E}[Y])^2 && [\text{linearity of expectation}] \\
 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 + \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 + 2\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] && [\text{rearranging}] \\
 &= (\mathbb{E}[X^2] - (\mathbb{E}[X])^2) + (\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2) + 2(\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y]) && [\text{lemma, since } X \perp Y] \\
 &= \text{Var}(X) + \text{Var}(Y) + 0 && [\text{def of variance}]
 \end{aligned}$$

□

### 5.1.6 Reproving Linearity of Expectation

*Proof of Linearity of Expectation.* Let  $X, Y$  be (possibly dependent) random variables. We'll prove that  $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ .

$$\begin{aligned}
 \mathbb{E}[X+Y] &= \sum_x \sum_y (x+y) p_{X,Y}(x,y) && [\text{LOTUS}] \\
 &= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y) && [\text{split sum}] \\
 &= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y) && [\text{algebra}] \\
 &= \sum_x x p_X(x) + \sum_y y p_Y(y) && [\text{def of marginal PMF}] \\
 &= \mathbb{E}[X] + \mathbb{E}[Y] && [\text{def of expectation}]
 \end{aligned}$$

□

### 5.1.7 Exercises

- Suppose we flip a fair coin three times independently. Let  $X$  be the number of heads in the first two flips, and  $Y$  be the number of heads in the last two flips (there is overlap).
  - What distribution do  $X$  and  $Y$  have marginally, and what are their ranges?
  - What is  $p_{X,Y}(x,y)$ ? Fill in this table below. You may want to fill in the marginal distributions first!
  - What is  $\Omega_{X,Y}$ , using your answer to (b)?
  - Write a formula for  $\mathbb{E}[\cos(XY)]$ .
  - Are  $X, Y$  independent?

**Solution:**

- Since  $X$  counts the number of heads in two independent flips of a fair coin, then  $X \sim \text{Bin}(n = 2, p = 0.5)$ .  $Y$  also has this distribution! Their ranges are  $\Omega_X = \Omega_Y = \{0, 1, 2\}$ .
- First, fill in the marginal distributions, which should be  $1/4, 1/2, 1/4$  for the probability that  $X = 0, X = 1$ , and  $X = 2$  respectively (same for  $Y$ ).  
First let's start with  $p_{X,Y}(2,2) = \mathbb{P}(X = 2, Y = 2)$ . If  $X = 2$ , that means the first two flips must've been heads. If  $Y = 2$ , that means the last two flips must've been heads. So the probability that  $X = 2, Y = 2$  is the probability of the single outcome HHH, which is  $1/8$ . Apply similar logic for  $p_{X,Y}(0,0) = \mathbb{P}(X = 0, Y = 0)$  which is the probability of TTT.

Then,  $p_{X,Y}(0,2) = \mathbb{P}(X = 0, Y = 2)$ . If  $X = 0$  then the first two flips are tails. If  $Y = 2$ , the last two flips are heads. This is impossible, so  $\mathbb{P}(X = 0, Y = 2) = 0$ . Similarly,  $\mathbb{P}(X = 2, Y = 0) = 0$  as well. Now use the constraints (the row totals and col totals) to fill in the rest! For example, the first row must sum to  $1/4$ , and we have two out of three of the entries  $p_{X,Y}(0,0)$  and  $p_{X,Y}(0,2)$ , so  $p_{X,Y}(0,1) = 1/4 - 1/8 - 0 = 1/8$ .

$X \setminus Y$	0	1	2	Row Total $p_X(x)$
0	1/8	1/8	0	1/4
1	1/8	1/4	1/8	1/2
2	0	1/8	1/8	1/4
Col Total $p_Y(y)$	1/4	1/2	1/4	1

- From the previous part, we can see that the joint range is everything in the Cartesian product except  $(0, 2)$  and  $(2, 0)$ , so  $\Omega_{X,Y} = (\Omega_X \times \Omega_Y) \setminus \{(0, 2), (2, 0)\}$ .
- By LOTUS extended to multiple variables,

$$\mathbb{E}[\cos(XY)] = \sum_x \sum_y \cos(xy) p_{X,Y}(x,y)$$

- No, the joint range is not equal to the Cartesian product. This immediately makes independence impossible. The intuitive reason is that, since  $(0, 2) \notin \Omega_{X,Y}$  for example, if we know  $X = 0$ , then  $Y$  cannot be 2. Formally, there exists a pair  $(x, y) \in \Omega_X \times \Omega_Y$  (namely  $(x, y) = (0, 2)$ ) such that  $p_{X,Y}(0, 2) = 0$  but  $p_X(0) > 0$  and  $p_Y(2) > 0$ . Hence,  $p_{X,Y}(0, 2) \neq p_X(0)p_Y(2)$ , which violates independence.
- Suppose radioactive particles at Area 51 are emitted at an average rate of  $\lambda$  per second. You want to measure how many particles are emitted, but your geiger-counter (device that measures radioactivity) fails to record each particle independently with some small probability  $p$ . Let  $X$  be the number of particles emitted, and  $Y$  be the number of particles observed (by your geiger-counter).

- (a) Describe the joint range  $\Omega_{X,Y}$  using set notation.
- (b) Write a formula (not a table) for  $p_{X,Y}(x,y)$ .
- (c) Write a formula for  $p_Y(y)$ .

**Solution:**

- (a)  $X \sim \text{Poi}(\lambda)$  can be any nonnegative integer  $\{0, 1, 2, \dots\}$ , and  $Y$  must be between 0 and  $X$ . Hence, the joint range is  $\Omega_{X,Y} = \{(x,y) \in \mathbb{Z}^2 : 0 \leq y \leq x\}$ , where  $\mathbb{Z}$  denotes the set of integers.
- (b) We know the Poisson PMF is

$$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

and that the distribution of  $Y$  given  $X = x$  is binomial  $(Y \mid X = x) \sim \text{Bin}(x, 1 - p)$ . This is because, given that  $X = x$  particles were emitted, we observe each one independently with probability  $1 - p$ . Hence,

$$\mathbb{P}(Y = y \mid X = x) = \binom{x}{y} (1 - p)^y p^{x-y}$$

By the chain rule (or definition of conditional probability),

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \mathbb{P}(Y = y \mid X = x) = e^{-\lambda} \frac{\lambda^x}{x!} \cdot \binom{x}{y} (1 - p)^y p^{x-y}$$

for  $(x,y) \in \Omega_{X,Y}$ .

- (c) We are asked to find the probability that we observe  $Y = y$  particles. To make this concrete, let's say we want  $p_Y(5) = \mathbb{P}(Y = 5)$ . Then there is some chance of this if  $X = 5$  (observing 100% of particles), or  $X = 6$ , or  $X = 7$ , etc. Hence, for any  $y \in \Omega_Y$ ,

$$p_Y(y) = \sum_{x \in \Omega_X} p_{X,Y}(x,y) = \sum_{x=y}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \cdot \binom{x}{y} (1 - p)^y p^{x-y}$$

That is, we sum over all cases where  $x \geq y$ .

- 3. Suppose there are  $N$  marbles in a bag, composed of  $r$  different colors. Suppose there are  $K_1$  of color 1,  $K_2$  of color 2, ...,  $K_r$  of color  $r$ , where  $\sum_{i=1}^r K_i = N$ . We reach in and draw  $n$  *without replacement*. Let  $(X_1, \dots, X_r)$  be a random vector where  $X_i$  is the count of how many marbles of color  $i$  we drew. What is  $p_{X_1, \dots, X_r}(k_1, \dots, k_r)$  for valid values of  $k_1, \dots, k_r$ ? We say the random vector  $(X_1, \dots, X_r) \sim MVHG(N, K_1, \dots, K_r, n)$  has a multivariate hypergeometric distribution!

**Solution:**

$$p_{X_1, \dots, X_r}(k_1, \dots, k_r) = \frac{\binom{K_1}{k_1} \cdots \binom{K_r}{k_r}}{\binom{N}{n}} = \frac{\prod_{i=1}^r \binom{K_i}{k_i}}{\binom{N}{n}}$$

## Chapter 5: Multiple Random Variables

### 5.2: Joint Continuous Distributions

#### 5.2.1 Joint PDFs and Expectation

The joint continuous distribution is the continuous counterpart of a joint discrete distribution. Therefore, conceptual ideas and formulas will be roughly similar to that of discrete ones, and the transition will be much like how we went from single variable discrete RVs to continuous ones.

To think intuitively about joint continuous distributions, consider throwing darts at a dart board. A dart board is two-dimensional and a certain 2D position on the dart board is  $(x, y)$ . Because  $x$  and  $y$  positions are continuous, we want to think about the joint distribution between two continuous random variables  $X$  and  $Y$  representing the location of the dart. What is the joint density function describing this scenario?

##### Definition 5.2.1: Joint PDFs

Let  $X, Y$  be continuous random variables. The joint PDF of  $X$  and  $Y$  is:

$$f_{X,Y}(a, b) \geq 0$$

The joint range is the set of pairs  $(c, d)$  that have nonzero density:

$$\Omega_{X,Y} = \{(c, d) : f_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that the double integral over all values must be 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u, v) dudv = 1$$

Further, note that if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function, then LOTUS extends to the multidimensional case:

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s, t) f_{X,Y}(s, t) dsdt$$

The joint PDF must satisfy the following (similar to univariate PDFs):

$$\mathbb{P}(a \leq X < b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dydx$$

##### Example(s)

Let  $X$  and  $Y$  be two jointly continuous random variables with the following joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

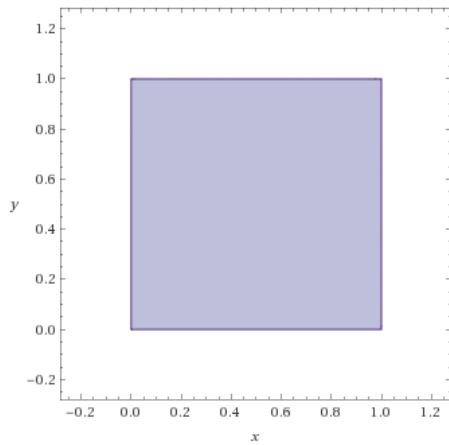
- Find and sketch the joint range  $\Omega_{X,Y}$ .
- Find the constant  $c$  that makes  $f_{X,Y}$  a valid joint PDF.

(c) Find  $\mathbb{P}(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$ .

*Solution*

(a)

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$



(b) To find  $c$ , the following condition has to be satisfied:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$$

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy \\ &= \int_0^1 \int_0^1 (x + cy^2) dx dy \\ &= \int_0^1 \left[ \frac{1}{2}x^2 + cy^2 x \right]_{x=0}^{x=1} dy \\ &= \int_0^1 \left( \frac{1}{2} + cy^2 \right) dy \\ &= \left[ \frac{1}{2}y + \frac{1}{3}cy^3 \right]_{y=0}^{y=1} \\ &= \frac{1}{2} + \frac{1}{3}c \end{aligned}$$

Thus,  $c = \frac{3}{2}$ .

(c)

$$\begin{aligned}
\mathbb{P}\left(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}\right) &= \int_0^{1/2} \int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx dy \\
&= \int_0^{1/2} \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_{x=0}^{x=1/2} dy \\
&= \int_0^{1/2} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy \\
&= \frac{3}{32}
\end{aligned}$$

□

Example(s)

Let  $X$  and  $Y$  be two jointly continuous random variables with the following PDF:

$$f_{X,Y}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find  $\mathbb{E}[XY^2]$ .

*Solution* By LOTUS,

$$\begin{aligned}
\mathbb{E}[XY^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy^2) f_{X,Y}(x, y) dx dy \\
&= \int_0^1 \int_0^1 xy^2(x + y) dx dy \\
&= \int_0^1 \left(\frac{1}{3}y^2 + \frac{1}{2}y^3\right) dy \\
&= \frac{17}{72}
\end{aligned}$$

□

### 5.2.2 Marginal PDFs

**Definition 5.2.2: Marginal PDFs**

Suppose that  $X$  and  $Y$  are jointly distributed continuous random variables with joint PDF  $f_{X,Y}(x, y)$ . The marginal PDFs of  $X$  and  $Y$  are respectively given by the following:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

Note this is exactly like for joint discrete random variables, with integrals instead of sums.

(Extension): If  $Z$  is also a continuous random variable, then the marginal PDF of  $Z$  is:

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z) dx dy$$

*Solution*

Example(s)

Find the marginal PDFs  $f_X(x)$  and  $f_Y(y)$  given the joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, compute  $\mathbb{E}[X]$ . (This is the same joint density as the first example, plugging in  $c = 3/2$ ).

For  $0 \leq x \leq 1$ :

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \\ &= \int_0^1 \left( x + \frac{3}{2}y^2 \right) dy \\ &= \left[ xy + \frac{1}{2}y^3 \right]_{y=0}^{y=1} \\ &= x + \frac{1}{2} \end{aligned}$$

Thus, the marginal PDF  $f_X(x)$  is:

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

For  $0 \leq y \leq 1$ :

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx \\ &= \int_0^1 \left( x + \frac{3}{2}y^2 \right) dx \\ &= \left[ \frac{1}{2}x^2 + \frac{3}{2}y^2 x \right]_{x=0}^{x=1} \\ &= \frac{3}{2}y^2 + \frac{1}{2} \end{aligned}$$

Thus, the marginal PDF  $f_Y(y)$  is:

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that to compute  $\mathbb{E}[X]$  for example, we can either use LOTUS, or just the marginal PDF  $f_X(x)$ . These methods are equivalent. By LOTUS (taking  $g(X, Y) = X$ ),

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dx dy = \int_0^1 \int_0^1 x \left( x + \frac{3}{2} y^2 \right) dx dy$$

Alternatively, by definition of expectation for a single RV,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \left( x + \frac{1}{2} \right) dx$$

It only takes two lines or so of algebra to show they are equal!  $\square$

### 5.2.3 Independence of Continuous Random Variables

#### Definition 5.2.3: Independence of Continuous Random Variables

Continuous random variables  $X, Y$  are independent, written  $X \perp Y$ , if for all  $x \in \Omega_X$  and  $y \in \Omega_Y$ ,

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Recall  $\Omega_{X,Y} = \{(x, y) : f_{X,Y}(x, y) > 0\} \subseteq \Omega_X \times \Omega_Y$ . A necessary but not sufficient condition for independence is that  $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ . That is, if  $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ , then we have to check the condition, but if not, then we know they are not independent.

This is because if there is some  $(a, b) \in \Omega_X \times \Omega_Y$  but not in  $\Omega_{X,Y}$ , then  $f_{X,Y}(a, b) = 0$  but  $f_X(a) > 0$  and  $f_Y(b) > 0$ , which violates independence. (This is very similar to independence for discrete RVs).

### 5.2.4 Multivariate: From Discrete to Continuous

The following table tells us the relationships between discrete and continuous joint distributions.

	Discrete	Continuous
<b>Joint PMF/PDF</b>	$p_{X,Y}(x, y) = P(X = x, Y = y)$	$f_{X,Y}(x, y) \neq P(X = x, Y = y)$
<b>Joint CDF</b>	$F_{X,Y}(x, y) = \sum_{t \leq x} \sum_{s \leq y} p_{X,Y}(t, s)$	$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(t, s) ds dt$
<b>Normalization</b>	$\sum_x \sum_y p_{X,Y}(x, y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$
<b>Marginal PMF/PDF</b>	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$
<b>Expectation</b>	$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$	$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$
<b>Conditional PMF/PDF</b>	$p_{X Y}(x   y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$	$f_{X Y}(x   y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$
<b>Conditional Expectation</b>	$E[X   Y = y] = \sum_x x p_{X Y}(x   y)$	$E[X   Y = y] = \int_{-\infty}^{\infty} x f_{X Y}(x   y) dx$
<b>Independence</b>	$\forall x, y, p_{X,Y}(x, y) = p_X(x)p_Y(y)$	$\forall x, y, f_{X,Y}(x, y) = f_X(x)f_Y(y)$

We'll explore the two conditional rows (second and third last rows) in the next section more, but you can guess that  $p_{X|Y}(x | y) = \mathbb{P}(X = x | Y = y)$ , and use the definition of conditional probability to see that it is  $\mathbb{P}(X = x, Y = y) / \mathbb{P}(Y = y)$ , as stated!

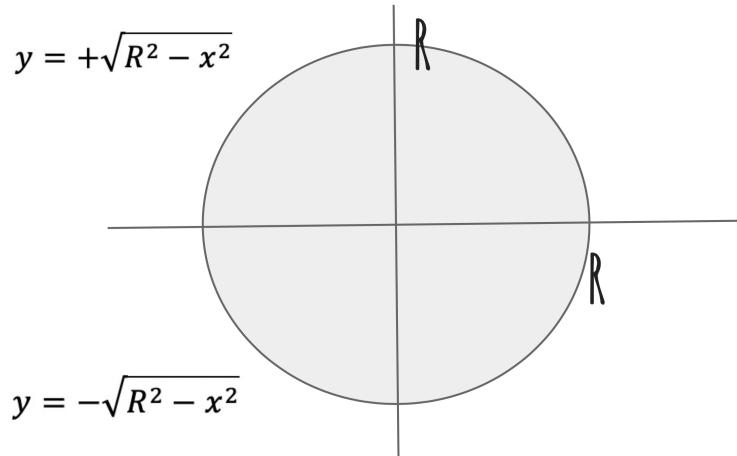
### Example(s)

Let's return to our dart example. Suppose  $(X, Y)$  are jointly and uniformly distributed on the circle of radius  $R$  centered at the origin (example a dart throw).

1. First find and sketch the joint range  $\Omega_{X,Y}$ .
2. Now, write an expression for the joint PDF  $f_{X,Y}(x, y)$  and carefully define it for all  $x, y \in \mathbb{R}$ .
3. Now, solve for the range of  $X$  and write an expression we can evaluate to find  $f_X(x)$ , the marginal PDF for  $X$ .
4. Now, let  $Z$  be the distance from the center that the dart falls. Find  $\Omega_Z$  and write an expression for  $\mathbb{E}[Z]$ .
5. Finally, determine using the definition of independence whether  $X$  and  $Y$  are independent.

### Solution

1. The joint range is  $\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$  since the values must be within the circle of radius  $R$ . We can sketch the range as follows, with the semi-circles below and above the  $y$ -axis labeled with their respective equations.



2. The height of the density function is constant, say  $h$ , since it is uniform. The double integral over all  $x$  and  $y$  must equal one ( $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$ ), meaning the volume of this cylinder must be 1. The volume is base times height, which is  $\pi R^2 \cdot h$ , and setting it equal to 1 gives  $h = \frac{1}{\pi R^2}$ . This gives us the following joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

3. Well,  $X$  can range from  $-R$  to  $R$ , since there are points on the circle with  $x$  values in this range. So the range of  $X$  is:

$$\Omega_X = [-R, R]$$

**Setting up this integral will be trickier than in the earlier examples**, because when finding  $f_X(x)$  and integrating out the  $y$ , the limits of integration actually depend on  $x$ . Imagine making a tick mark at some  $x \in [-R, R]$  (on the  $x$ -axis) and drawing a vertical line through  $x$ : where does  $y$  enter and leave (like summing a column in a joint PMF)? Based on the equations we had earlier for  $y$  in terms of  $x$  (see the sketch above), this give us:

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f_{X,Y}(x, y) dy$$

Again, this is different from the previous examples, and you MUST sketch/plot the joint range to figure this out. If you learned how to do double integrals, this is exactly the same idea.

4. Well, the distance will be given by  $Z = \sqrt{X^2 + Y^2}$ , which is the definition of distance. We can further see that  $Z$  will take on any value from 0 to  $R$ , since the point could be at the origin and as far as  $R$ . This gives,  $\Omega_Z = [0, R]$ .

Then, to solve for the expected value of  $Z$ , we can use LOTUS, and only integrate over the joint range of  $X$  and  $Y$  (since the joint PDF is 0 elsewhere). We have to be careful in setting up the bounds of our integral.  $X$  will range from  $-R$  to  $R$  as we discussed earlier. But as  $X$  ranges across these values,  $Y$  will range from  $-\sqrt{R^2 - x^2}$  to  $\sqrt{R^2 - x^2}$ . We had  $Z = \sqrt{X^2 + Y^2}$ , so for the expected value we have:

$$\mathbb{E}[Z] = \mathbb{E}[\sqrt{X^2 + Y^2}] = \int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dy dx$$

Note that we could've set up this integral  $dxdy$  instead - what would the limits of integration have been? It would've been

$$\mathbb{E}[Z] = \mathbb{E}[\sqrt{X^2 + Y^2}] = \int_{-R}^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dx dy$$

Your outer limits must be just the range of  $Y$  (both constants), and your inner limits may depend on the outer variable of integration.

5. No, they are not independent. We can see this with the test:  $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ . This is because  $X$  and  $Y$  both have marginal range from  $-R$  to  $R$ , but the joint range is not a rectangle of this region (it is a circle). More explicitly, take a point  $(0.99R, 0.99R)$  which is basically the top right of the square  $(R, R)$ . We get  $0 = f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R)f_Y(0.99R) > 0$ . This is because the joint PDF is defined to be 0 at  $(0.99R, 0.99R)$  (not in the circle), but the marginal PDFs of both  $X$  and  $Y$  are nonzero at  $0.99R$  (since  $0.99R$  is in the marginal range of both).

□

### Example(s)

Now let's consider another example where we have a continuous joint distribution  $(X, Y)$ , where  $X \in [0, 1]$  is the proportion of the time until the midterm that you actually spend studying for it and  $Y \in [0, 1]$  is your percentage score on the exam.

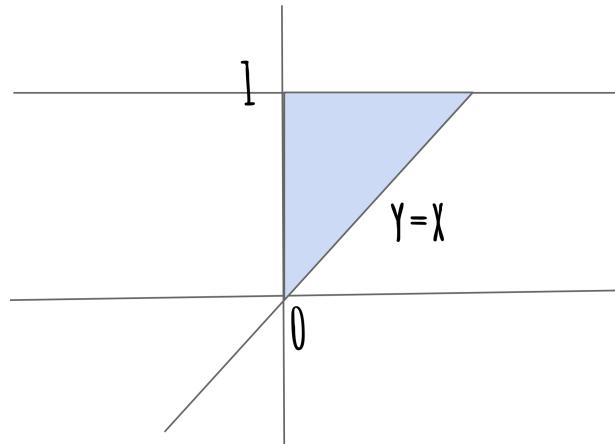
Suppose the joint PDF is:

$$f_{X,Y}(x, y) = \begin{cases} ce^{-(y-x)} & x, y \in [0, 1] \text{ and } y \geq x \\ 0 & \text{otherwise} \end{cases}$$

1. First, consider the joint range and sketch it. Then, interpret it in English in the context of the problem.
2. Now, write an expression for  $c$  in the PDF above.
3. Now, find  $\Omega_Y$  and write an expression that we could evaluate to find  $f_Y(y)$ .
4. Now, write an expression that we could evaluate to find  $\mathbb{P}(Y \geq 0.9)$ .
5. Now, write an expression that we can evaluate to find  $\mathbb{E}[Y]$ , the expected score on the exam.
6. Finally, consider whether  $X$  and  $Y$  are independent.

*Solution*

1.  $X$  can range from any value in  $[0, 1]$  without conditions. Then  $Y$  will only be bounded in that it must be less than or equal to  $X$ . We can first draw the line  $y = x$ , and then the region above this line for which  $x, y$  are less than 1 will be our range. That gives us the following:



In English, this means that your score is at least the percentage of time that you studied, as your score will be that proportion or more.

2. To solve for  $c$ , we should find the volume above this triangle on the  $x$ - $y$  plane and invert it, since  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$ . To find the area we can integrate in terms of  $x$  or  $y$  first, which will give us the following two equivalent expressions:

$$c = \frac{1}{\int_0^1 \int_x^1 e^{-(y-x)} dy dx} = \frac{1}{\int_0^1 \int_0^y e^{-(y-x)} dx dy}$$

We'll explain the first equality using the  $dydx$  ordering. Since  $dx$  is the outer integral, the limits must be just the range of  $X$ , which is  $[0, 1]$ . For each value of  $x$  (draw a vertical line through  $x$  on the

$x$ -axis),  $y$  goes between  $x$  and 1, so those are the inner limits of integration.

Now, for the second equality using  $dxdy$  ordering, the outer integral is  $dy$ , so the limits are the range of  $Y$ , also  $[0, 1]$ . Then, for each value of  $y$  (draw a horizontal line through  $y$  on the  $y$ -axis),  $x$  goes between 0 and  $y$ , and so those are the inner limits of integration.

3. Well,  $\Omega_Y = [0, 1]$  as we can see in our graph above that  $Y$  takes on values in this range. For the marginal PDF we have to integrate in respect to  $X$ , which will take on values in the range 0 to  $y$  based on our graph. So, we have:

$$f_Y(y) = \int_0^y ce^{-(y-x)} dx$$

4. We can integrate from 0.9 to 1 to solve for this, using the marginal PDF that we solved for above. This takes us back to the univariate case essentially, and gives us the following:

$$\mathbb{P}(Y \geq 0.9) = \int_{0.9}^1 f_Y(y) dy = \int_{0.9}^1 \int_0^y ce^{-(y-x)} dx dy$$

5. By definition of expectation (univariate), or LOTUS, we have:

$$\mathbb{E}[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 \int_0^y cye^{-(y-x)} dx dy$$

6.  $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$  since the sketch of the range is not a rectangle. The joint range is not equal to the cartesian product of the marginal ranges. To be concrete, consider the point  $(x = 0.99, y = 0.01)$  (basically the corner  $(1, 0)$ ). I chose this point because it was in the Cartesian product  $\Omega_X \times \Omega_Y = [0, 1] \times [0, 1]$ , but not in the joint range (see the picture from the first part). Since it's not in the joint range (shaded region), we have  $f_{X,Y}(0.99, 0.01) = 0$ , but since  $0.99 \in \Omega_X$  and  $0.01 \in \Omega_Y$ ,  $f_X(0.99) > 0$  and  $f_Y(0.01) > 0$ . Hence, I've found a pair of points  $(x, y)$  where the joint density isn't equal to the product of the marginal densities, violating independence.

□

## Chapter 5: Multiple Random Variables

### 5.3: Conditional Distributions

#### 5.3.1 Conditional PMFs/PDFs

Now that we've finished talking about joint distributions (whew), we can move on to conditional distributions and conditional expectation. This is actually just applying the concepts from 2.2 about conditional probability, generalizing to random variables (instead of events)!

##### Definition 5.3.1: Conditional PMFs and PDFs

If  $X, Y$  are discrete random variables, then the **conditional PMF** of  $X$  given  $Y$  is:

$$p_{X|Y}(a | b) = \mathbb{P}(X = a | Y = b) = \frac{p_{X,Y}(a, b)}{p_Y(b)} = \frac{p_{Y|X}(b | a)p_X(a)}{p_Y(b)}$$

Note that this should remind you of Bayes Theorem (because that's what it is)!

If  $X, Y$  are continuous random variables, then the **conditional PDF** of  $X$  given  $Y$  is:

$$f_{X|Y}(u | v) = \frac{f_{X,Y}(u, v)}{f_Y(v)} = \frac{f_{Y|X}(v | u)f_X(u)}{f_Y(v)}$$

Again, this is just a generalization from discrete to continuous as we've been doing!

It's important to note that, for each *fixed value* of  $b$ , the probabilities that  $X = a$  must sum to 1:

$$\sum_{a \in \Omega_X} p_{X|Y}(a | b) = 1$$

If  $X$  and  $Y$  are mixed (one discrete, one continuous), then a similar extension can be made where any discrete random variable has a  $p$  (a probability mass function) any continuous random variable has an  $f$  (a probability density function).

##### Example(s)

Back to our example of the blue and red die rolls from 5.1. Suppose we roll a fair blue 4-sided die and a fair red 4-sided die independently. Recall that  $U = \min\{X, Y\}$  (the smaller of the two die rolls) and  $V = \max\{X, Y\}$  (the larger of the two die rolls). Then, their joint PMF was:

$U \setminus V$	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

What is  $p_{U|V}(u | 3) = \mathbb{P}(U = u | V = 3)$  for each value of  $u \in \Omega_U$  (these should sum to 1)!

*Solution* Well, we know by the definition of conditional probability that

$$p_{U|V} = \frac{p_{U,V}(u, 3)}{p_V(3)}$$

We need to compute the denominator which is the marginal PMF of  $V$  (the sum of the third column):

$$p_V(3) = \sum_{a \in \Omega_U} p_{U,V}(a, 3) = 2/16 + 2/16 + 1/16 + 0 = 5/16$$

Hence, our conditional PMF is

$$p_{U|V}(u | 3) = \begin{cases} \frac{2/16}{5/16} = \frac{2}{5} & u = 1 \\ \frac{2/16}{5/16} = \frac{2}{5} & u = 2 \\ \frac{1/16}{5/16} = \frac{1}{5} & u = 3 \\ \frac{0}{5/16} = 0 & u = 4 \end{cases}$$

□

### 5.3.2 Conditional Expectation

Just like conditional probabilities helped us compute “normal” (unconditional) probabilities in Chapter 2 (using LTP), we will learn about conditional expectation which will help us compute “normal” expectations!

Let’s try to find out how we might define this idea of conditional expectation of a random variable  $X$ , given that we know some other RV  $Y$  takes on a particular value  $y$ . Well since  $\mathbb{E}[X]$  (for discrete RVs) is defined to be:

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \mathbb{P}(X = x) = \sum_{x \in \Omega_X} x p_X(x)$$

it’s only fair that the conditional expectation of  $X$ , given knowledge that some other RV  $Y$  is equal to  $y$  is the same exact thing, EXCEPT the probabilities should be conditioned on  $Y = y$  now:

$$\mathbb{E}[X | Y = y] = \sum_{x \in \Omega_X} x \mathbb{P}(X = x | Y = y) = \sum_{x \in \Omega_X} x p_{X,Y}(x | y)$$

Most notably, we are still summing over  $x$  and NOT  $y$ , since this expression should depend on  $y$  right? Given that  $Y = y$ , what is the expectation of  $X$ ?

### Definition 5.3.2: Conditional Expectation

Let  $X, Y$  be jointly distributed random variables.

If  $X$  is discrete (and  $Y$  is either discrete or continuous), then we define the **conditional expectation** of  $g(X)$  given (the event that)  $Y = y$  as:

$$\mathbb{E}[g(X) | Y = y] = \sum_{x \in \Omega_X} g(x)p_{X|Y}(x | y)$$

If  $X$  is continuous (and  $Y$  is either discrete or continuous), then we define the conditional expectation of  $g(X)$  given (the event that)  $Y = y$  as:

$$\mathbb{E}[g(X) | Y = y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x | y)dx$$

Notice that these sums and integrals are **over  $x$**  (not  $y$ ), since  $\mathbb{E}[g(X) | Y = y]$  is a function of  $y$ . These formulas are exactly the same as  $\mathbb{E}[g(X)]$ , except the PMF/PDF of  $X$  is replaced with the conditional PMF/PDF of  $X | Y = y$ .

### Example(s)

Suppose  $X \sim \text{Unif}(0, 1)$  (continuous). We repeatedly draw independent  $Y_1, Y_2, Y_3, \dots \sim \text{Unif}(0, 1)$  (continuous) until the first random time  $T$  such that  $Y_T < X$ . What is  $\mathbb{E}[T]$ ?

The question is basically asking the following: we get some uniformly random decimal number  $X$  from  $[0, 1]$ . We keep drawing uniform random numbers until we get a value less than our initial value. What is the expected number of draws until this happens?

*Solution* We'll do this problem in a "bad" way (the only way we know how to know), and then learn the Law of Total Expectation next to see how this solution could be much simpler!

To find  $\mathbb{E}[T]$ , since  $T$  is discrete with range  $\Omega_T = \{1, 2, 3, \dots\}$ , we can find its PMF  $p_T(t) = \mathbb{P}(T = t)$  for any value  $t$  and use the usual formula for expectation. However,  $T$  depends on the value of the initial number  $X$  right? If  $X = 0.1$  it would take longer to get a number less than this than if  $X = 0.99$ . Let's try to find the probability  $T = t$  given that  $X = x$  first:

$$\mathbb{P}(T = t | X = x) = (1 - x)^{t-1}x$$

because the probability we get a number smaller than  $x$  is just  $x$  (Uniform CDF), and so we need to get  $t - 1$  failures first before our first success. Actually,  $(T|X = x) \sim \text{Geo}(x)$  so that's another way we could've computed this conditional PMF. Then, let's use the LTP to find  $\mathbb{P}(T = t)$  (we need to *integrate* over all values of  $t$  because  $T$  is continuous, not discrete):

$$\mathbb{P}(T = t) = \int_0^1 \mathbb{P}(T = t | X = x) f_X(x)dx = \int_0^1 (1 - x)^{t-1}x \cdot 1dx = \dots = \frac{1}{t(t+1)}$$

after skipping some purely computational steps. Finally, since we have the PMF of  $T$ , we can compute

expectation in the normal way:

$$\mathbb{E}[T] = \sum_{t=1}^{\infty} tp_T(t) = \sum_{t=1}^{\infty} t \frac{1}{t(t+1)} = \sum_{t=1}^{\infty} \frac{1}{t+1} = \infty$$

The reason this is  $\infty$  is because this is like the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  which is known to diverge to  $\infty$ . This is surprising right? The expected time until you get a number smaller than your first is infinite!

□

### 5.3.3 Law of Total Expectation (LTE)

Now we'll see how the Law of Total Expectation can make our lives easier! We'll also see an extremely cool application, which is to *elegantly* prove the expected value of a  $\text{Geo}(p)$  RV is  $1/p$  (we did this algebraically in 3.5, but this was messy).

Theorem 5.3.19: Law of Total Expectation

Let  $X, Y$  be jointly distributed random variables.

If  $Y$  is discrete (and  $X$  is either discrete or continuous), then:

$$\mathbb{E}[g(X)] = \sum_{y \in \Omega_Y} \mathbb{E}[g(X) | Y = y] p_Y(y)$$

If  $Y$  is continuous (and  $X$  is either discrete or continuous), then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} \mathbb{E}[g(X) | Y = y] f_Y(y) dy$$

This looks exactly like the law of total probability we are used to. Basically to solve for  $\mathbb{E}[g(X)]$ , we need to take a weighted average of  $\mathbb{E}[g(X) | Y = y]$  over all possible values of  $y$ .

*Proof of LTE.* Now we will prove the law of total expectation.

Suppose that  $X, Y$  are discrete (note that the same proof holds for any combination of  $X, Y$  being discrete or continuous, but swapping sums to integrals as necessary). Then:

$$\begin{aligned} \sum_{y \in \Omega_Y} \mathbb{E}[g(X) | Y = y] p_Y(y) &= \sum_{y \in \Omega_Y} \left( \sum_{x \in \Omega_X} g(x) p_{X|Y}(x | y) \right) p_Y(y) && [\text{def of conditional expectation}] \\ &= \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} g(x) p_{X|Y}(x | y) p_Y(y) && [\text{swap sums}] \\ &= \sum_{x \in \Omega_X} g(x) \sum_{y \in \Omega_Y} p_{X,Y}(x, y) && [\text{def of conditional pmf}] \\ &= \sum_{x \in \Omega_X} g(x) p_X(x) && [\text{def of marginal pmf}] \\ &= \mathbb{E}[g(X)] && [\text{def of expectation/LOTUS}] \end{aligned}$$

□

## Example(s)

(This is the same example as earlier): Suppose  $X \sim \text{Unif}(0, 1)$  (continuous). We repeatedly draw independent  $Y_1, Y_2, Y_3, \dots \sim \text{Unif}(0, 1)$  (continuous) until the first random time  $T$  such that  $Y_T < X$ . What is  $\mathbb{E}[T]$ ?

*Solution* Using the LTE now, we can solve this in a much simpler fashion. We know that  $(T \mid X = x) \sim \text{Geo}(p)$  as stated earlier. By citing the expectation of a Geometric RV, we know that  $\mathbb{E}[T \mid X = x] = \frac{1}{x}$ . By the LTE, conditioning on  $x$ :

$$\mathbb{E}[T] = \int_0^1 \mathbb{E}[T \mid X = x] f_X(x) dx = \int_0^1 \frac{1}{x} 1 dx = [\ln(x)]_0^1 = \infty$$

This was a much faster way to getting to the answer than before!  $\square$

## Example(s)

Let's finally prove that if  $X \sim \text{Geo}(p)$ , then  $\mu = \mathbb{E}[X] = \frac{1}{p}$ . Recall that the Geometric random variable is the number of independent Bernoulli trials with parameter  $p$  up to and including the first success.

*Solution* First, let's condition on whether our first flip was heads or tails (these events partition the sample space):

$$\mathbb{E}[X] = \mathbb{E}[X \mid H]\mathbb{P}(H) + \mathbb{E}[X \mid T]\mathbb{P}(T) \text{[Law of Total Expectation]}$$

What are those four values on the right though? We know  $\mathbb{P}(H) = p$  and  $\mathbb{P}(T) = 1 - p$ , so that's out of the way.

What is  $\mathbb{E}[X \mid H]$ ? If we got heads on the first try, then  $\mathbb{E}[X \mid H] = 1$  since we are immediately done (i.e., the number of trials it took to get our first heads, given we got heads on the first trial, is 1).

What is  $\mathbb{E}[X \mid T]$ ? This is a bit trickier: because the trials are independent, and we got a tail on the first try, we basically have to restart (memorylessness), and so our conditional expectation is just  $\mathbb{E}[1 + X]$ , since we are back to square one except with one additional trial!

Plugging these four values in gives a recursive formula ( $\mathbb{E}[X]$  appears on both sides):

$$\mathbb{E}[X] = p + (1 + \mathbb{E}[X]) \cdot (1 - p)$$

We can solve this, using  $\mu = \mathbb{E}[X]$  (for notational convenience):

$$\begin{aligned} \mu &= p + (1 + \mu)(1 - p) \\ \mu &= p + 1 - p + \mu - \mu p \\ \mu &= 1 + \mu - \mu p \\ 0 &= 1 - \mu p \\ \mu p &= 1 \\ \mu &= \frac{1}{p} \end{aligned}$$

This is a really “cute” proof of the expectation of a Geometric RV! See the notes in 3.5 to see the “ugly” calculus proof.  $\square$

### 5.3.4 Exercises

1. What happens to linearity of expectation when you sum a *random* number of random variables? We know it holds for fixed values of  $n$ , but let’s see what happens if we sum a random number  $N$  of them. It turns out, you get something very nice!

Let  $X_1, X_2, X_3, \dots$  be a sequence of independent and identically distributed (iid) RVs, with common mean  $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \dots$ . Let  $N$  be a random variable which has range  $\Omega_N \subseteq \{0, 1, 2, \dots\}$  (nonnegative integers), independent of all the  $X_i$ ’s. Show that  $\mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}[X_1]\mathbb{E}[N]$ . That is, the expected sum of a random number of random variables is the expected number of random variables times the expected value of each (which you might think is intuitively true, but we have to prove it!).

**Solution:** We have the following:

$$\begin{aligned}
 \mathbb{E}\left[\sum_{i=1}^N X_i\right] &= \sum_{n \in \Omega_N} \mathbb{E}\left[\sum_{i=1}^N X_i \mid N = n\right] p_N(n) && \text{[Law of Total Expectation]} \\
 &= \sum_{n \in \Omega_N} \mathbb{E}\left[\sum_{i=1}^n X_i \mid N = n\right] p_N(n) && \text{[given } N = n : \text{ substitute in the upper limit]} \\
 &= \sum_{n \in \Omega_N} \mathbb{E}\left[\sum_{i=1}^n X_i\right] p_N(n) && \text{[} N \text{ independent of } X'_i \text{'s]} \\
 &= \sum_{n \in \Omega_N} n\mathbb{E}[X_1] p_N(n) && \text{[Linearity of Expectation]} \\
 &= \mathbb{E}[X_1] \sum_{n \in \Omega_N} np_N(n) \\
 &= \mathbb{E}[X_1]\mathbb{E}[N] && \text{[def of } \mathbb{E}[N] \text{]}
 \end{aligned}$$

## Chapter 5: Multiple Random Variables

### 5.4: Covariance and Correlation

In this section, we'll learn about covariance; which as you might guess, is related to variance. It is a function of two random variables, and tells us whether they have a positive or negative linear relationship. It also helps us finally compute the variance of a sum of *dependent* random variables, which we have not yet been able to do.

#### 5.4.1 Covariance and Properties

We will start with the definition of covariance:  $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ . By LOTUS, we know this is equal to (where  $\mu_X = \mathbb{E}[X]$  and  $\mu_Y = \mathbb{E}[Y]$ )

$$\sum_x \sum_y (x - \mu_X)(y - \mu_Y) p_{X,Y}(x, y)$$

Intuitively, we can see the following possibilities:

- $x > \mu_X, y > \mu_Y \Rightarrow (x - \mu_X)(y - \mu_Y) > 0$  ( $X, Y$  both above their means)
- $x < \mu_X, y < \mu_Y \Rightarrow (x - \mu_X)(y - \mu_Y) > 0$  ( $X, Y$  both below their means)
- $x < \mu_X, y > \mu_Y \Rightarrow (x - \mu_X)(y - \mu_Y) < 0$  ( $X$  below its mean,  $Y$  above its mean)
- $x > \mu_X, y < \mu_Y \Rightarrow (x - \mu_X)(y - \mu_Y) < 0$  ( $X$  above its mean,  $Y$  below its mean)

So we get a weighted average (by  $p_{X,Y}$ ) of these positive or negative quantities. Just with this brief intuition, we can say that covariance is positive when  $X, Y$  are usually both above/below their means, and negative if they are opposite. That is, covariance is positive in general when increasing one variable leads to an increase in the other, and negative when increasing one variable leads to a decrease in the other.

##### Definition 5.4.1: Covariance

Let  $X, Y$  be random variables. The **covariance** of  $X$  and  $Y$  is:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

This should remind you of the definition of variance - think of replacing  $Y$  with  $X$  and you'll see it!  
Note: Covariance can be negative, unlike variance.

Covariance satisfies the following properties:

1. If  $X \perp Y$ , then  $\text{Cov}(X, Y) = 0$  (but not necessarily vice versa, because the covariance could be zero but  $X$  and  $Y$  could not be independent).
2.  $\text{Cov}(X, X) = \text{Var}(X)$ . (Just plug in  $Y = X$ ).
3.  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ . (Multiplication is commutative).

4.  $\text{Cov}(X + c, Y) = \text{Cov}(X, Y)$ . (Shifting doesn't and shouldn't affect the covariance).
5.  $\text{Cov}(aX + bY, Z) = a \cdot \text{Cov}(X, Z) + b \cdot \text{Cov}(Y, Z)$ . This can be easily remembered like the distributive property of scalars  $(aX + bY)Z = a(XZ) + b(YZ)$ .
6.  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$ , and hence if  $X \perp Y$ , then  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$  (as we discussed earlier).
7.  $\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j)$ . That is covariance works like FOIL (first, outer, inner, last) for multiplication of sums  $((a + b + c)(d + e) = ad + ae + bd + be + cd + ce)$ .

*Proof of Covariance Alternate Formula.* We will prove that  $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ .

$$\begin{aligned}
 \text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] && [\text{def of covariance}] \\
 &= \mathbb{E}[XY - \mathbb{E}[X]Y - X\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]] && [\text{algebra}] \\
 &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] && [\text{Linearity of Expectation}] \\
 &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] && [\text{algebra}]
 \end{aligned}$$

□

*Proof of Property 1: Covariance of Independent RVs is 0.*

We actually proved in 5.1 already that  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$  when  $X, Y$  are independent. Hence,

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

□

*Proof of Property 6: Variance of Sum of RVs.*

We will show that in general, for any RVs  $X$  and  $Y$ , that

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

$$\begin{aligned}
 \text{Var}(X + Y) &= \text{Cov}(X + Y, X + Y) && [\text{covariance with self} = \text{variance}] \\
 &= \text{Cov}(X, X) + \text{Cov}(X, Y) + \text{Cov}(Y, X) + \text{Cov}(Y, Y) && [\text{covariance like FOIL}] \\
 &= \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y) && [\text{covariance with self, and symmetry}]
 \end{aligned}$$

□

Example(s)

Let  $X$  and  $Y$  be two independent  $\mathcal{N}(0, 1)$  random variables and:

$$Z = 1 + X + XY^2$$

$$W = 1 + X$$

Find  $\text{Cov}(Z, W)$ .

*Solution* First note that  $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 1 + 0^2 = 1$  (rearrange variance formula and solve for  $\mathbb{E}[X^2]$ ). Similarly,  $\mathbb{E}[Y^2] = 1$ .

$$\begin{aligned}
 \text{Cov}(Z, W) &= \text{Cov}(1 + X + XY^2, 1 + X) \\
 &= \text{Cov}(X + XY^2, X) && \text{[Property 4]} \\
 &= \text{Cov}(X, X) + \text{Cov}(XY^2, X) && \text{[Property 7]} \\
 &= \text{Var}(X) + \mathbb{E}[X^2Y^2] - \mathbb{E}[XY^2]\mathbb{E}[X] && \text{[Property 2 and def of covariance]} \\
 &= 1 + \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y^2] && \text{[Because } X \text{ and } Y \text{ are independent]} \\
 &= 1 + 1 - 0 = 2
 \end{aligned}$$

□

### 5.4.2 (Pearson) Correlation

Covariance has a “problem” in measuring linear relationships, in that  $\text{Cov}(X, Y)$  will be positive when there is a positive linear relationship and negative when there is a negative linear relationship, but  $\text{Cov}(2X, Y) = 2\text{Cov}(X, Y)$ . Scaling one of the random variables should not affect the *strength* of their relationship, which it seems to do. It would be great if we defined some metric that was normalized (had a maximum and minimum), and was invariant to scale. This metric will be called correlation!

**Definition 5.4.2: (Pearson) Correlation**

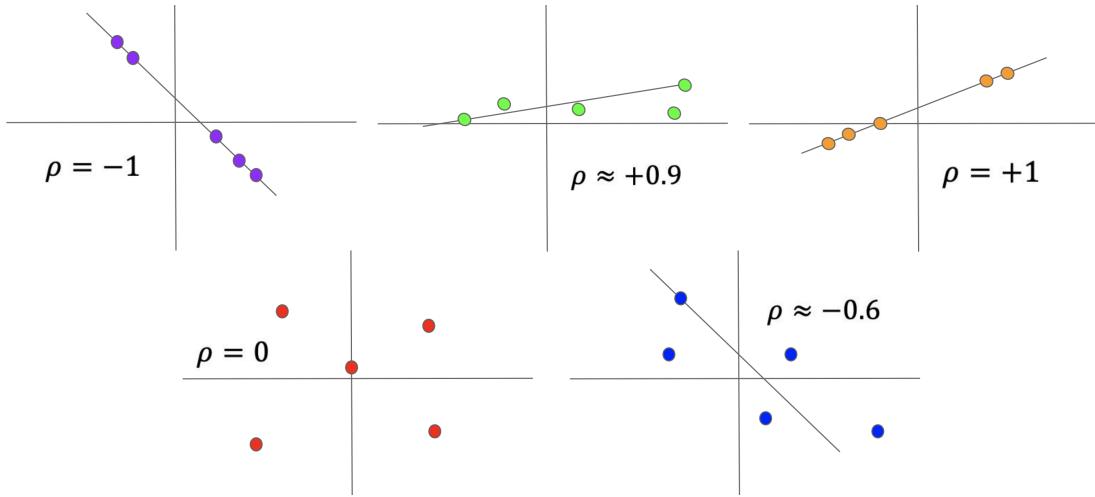
Let  $X, Y$  be random variables. The (Pearson) correlation of  $X$  and  $Y$  is:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

We can prove by the Cauchy-Schwarz inequality (from linear algebra),  $-1 \leq \rho(X, Y) \leq 1$ . That is, correlation is just a normalized version of covariance. Most notably,  $\rho(X, Y) = \pm 1$  if and only if  $Y = aX + b$  for some constants  $a, b \in \mathbb{R}$ , and then the sign of  $\rho$  is the same as that of  $a$ .

In linear regression (“line-fitting”) from high school science class, you may have calculated some  $R^2$ ,  $0 \leq R^2 \leq 1$ , and this is actually  $\rho^2$ , and measure how well a linear relationship exists between  $X$  and  $Y$ .  $R^2$  is the percentage of variance in  $Y$  which can be explained by  $X$ .

Let’s take a look at some example graphs which shows a sample of data and their (Pearson) correlations, to get some intuition.



The 1st (purple) plot has a perfect negative linear relationship and so the correlation is  $-1$ .

The 2nd (green) plot has a positive relationship, but it is not perfect, so the correlation is around  $+0.9$ .

The 3rd (orange) plot is a perfectly linear positive relationship, so the correlation is  $+1$ .

The 4th (red) plot appears to have data that is independent, so the correlation is  $0$ .

The 5th (blue) plot has a negative trend that isn't strongly linear, so the correlation is around  $-0.6$ .

### Example(s)

Suppose  $X$  and  $Y$  are random variables, where  $Y = -5X + 2$ . Show that, since there is a perfect negative linear relationship,  $\rho(X, Y) = -1$ .

*Solution* To find the correlation, we need the covariance and the two individual variances. Let's write them in terms of  $\text{Var}(X)$ .

$$\text{Var}(Y) = \text{Var}(-5X + 2) = (-5)^2 \text{Var}(X) = 25 \text{Var}(X)$$

By properties of covariance (shifting by 2 doesn't matter),

$$\text{Cov}(X, Y) = \text{Cov}(X, -5X + 2) = -5\text{Cov}(X, X) = -5\text{Var}(X)$$

Finally,

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}} = \frac{-5\text{Var}(X)}{\sqrt{\text{Var}(X)} \sqrt{25\text{Var}(X)}} = \frac{-5\text{Var}(X)}{5\text{Var}(X)} = -1$$

Note that the  $-5$  and  $2$  did not matter at all (except that  $-5$  was negative and made the correlation negative)!

□

### 5.4.3 Variance of Sums of Random Variables

Perhaps the most useful application of covariance is in finding the variance of a sum of *dependent* random variables. We'll extend the case of  $\text{Var}(X + Y)$  to more than two random variables.

## Theorem 5.4.20: Variance of Sums of RVs

If  $X_1, X_2, \dots, X_n$  are random variables, then

$$\text{Var} \left( \sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)$$

*Proof of Variance of Sums of RVs.* We'll first do something unintuitive - making our expression more complicated. The variance of the sum  $X_1 + X_2 + \dots + X_n$  is the covariance with itself! We'll use  $i$  to index one of the sums  $\sum_{i=1}^n X_i$  and  $j$  for the other  $\sum_{j=1}^n X_i$ . Keep in mind these both represent the same quantity; you'll see why we used different dummy variables soon!

$$\begin{aligned} \text{Var} \left( \sum_{i=1}^n X_i \right) &= \text{Cov} \left( \sum_{i=1}^n X_i, \sum_{j=1}^n X_j \right) && [\text{covariance with self} = \text{variance}] \\ &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) && [\text{by FOIL}] \\ &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) && [\text{by symmetry (see image below)}] \end{aligned}$$

The final step comes from the definition of covariance of a variable with itself and the symmetry of the covariance. It is illustrated below where the red diagonal is the covariance of a variable with itself (which is its variance), and the green off-diagonal are the symmetric pairs of covariance. We used the fact that  $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$  to require us to only sum the lower triangle (where  $i < j$ ), and multiply by 2 to account for the upper triangle.

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$
$X_1$	Red				
$X_2$		Red			
$X_3$			Red		
$X_4$				Red	
$X_5$					Red

It is important to remember than if all the RVs were independent, all the  $\text{Cov}(X_i, X_j)$  terms (for  $i \neq j$ ) would be zero, and so we would just be left with the sum of the variances as we showed earlier!  $\square$

## Example(s)

Recall in the hat check problem in 3.3, we had  $n$  people who go to a party and leave their hats with a hat check person. At the end of the party, the hats are returned randomly though.

We let  $X$  be the number of people who get their original hat back. We solved for  $\mathbb{E}[X]$  with indicator random variables  $X_1, \dots, X_n$  for whether the  $i$ -th person got their hat back.

We showed that:

$$\begin{aligned}\mathbb{E}[X_i] &= \mathbb{P}(X_i = 1) \\ &= \mathbb{P}(i^{\text{th}} \text{ person get their hat back}) \\ &= \frac{1}{n}\end{aligned}$$

So,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\ &= \sum_{i=1}^n \mathbb{E}[X_i] \\ &= \sum_{i=1}^n \frac{1}{n} \\ &= n \cdot \frac{1}{n} \\ &= 1\end{aligned}$$

Above was all review: now compute  $\text{Var}(X)$ .

*Solution* Recall that each  $X_i \sim \text{Ber}\left(\frac{1}{n}\right)$  (1 with probability  $\frac{1}{n}$ , and 0 otherwise). (Remember these were NOT independent RVs, but we still could apply linearity of expectation.) In our previous proof, we showed that

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)$$

Recall that  $X_i, X_j$  are indicator random variables which are in  $\{0, 1\}$ , so their product  $X_i X_j \in \{0, 1\}$  as well.

This allows us to calculate:

$$\begin{aligned}\mathbb{E}[X_i X_j] &= \mathbb{P}(X_i X_j = 1) && [\text{since indicator, is just probability of being 1}] \\ &= \mathbb{P}(X_i = 1, X_j = 1) && [\text{product is 1 if and only if both are 1}] \\ &= \mathbb{P}(X_i = 1) \mathbb{P}(X_j = 1 \mid X_i = 1) && [\text{chain rule}] \\ &= \frac{1}{n} \left( \frac{1}{n-1} \right)\end{aligned}$$

This is because we need both person  $i$  and person  $j$  to get their hat back: person  $i$  gets theirs back with probability  $\frac{1}{n}$ , and *given* this is true, person  $j$  gets theirs back with probability  $\frac{1}{n-1}$

So, by definition of covariance (recall each  $\mathbb{E}[X_i] = \frac{1}{n}$ ):

$$\begin{aligned}\text{Cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] \\ &= \frac{1}{n} \left( \frac{1}{n-1} \right) - \frac{1}{n} \cdot \frac{1}{n} && \text{[plug in]} \\ &= \frac{n}{n^2(n-1)} - \frac{n-1}{n^2(n-1)} && \text{[algebra]} \\ &= \frac{1}{n^2(n-1)} && \text{[algebra]}\end{aligned}$$

Further, since  $X_i$  is a Bernoulli (indicator) random variable:

$$\text{Var}(X_i) = p(1-p) = \left( \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right)$$

Finally, we have

$$\begin{aligned}\text{Var}(X) &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) && \text{[formula for variance of sum]} \\ &= \sum_{i=1}^n \frac{1}{n} \left( 1 - \frac{1}{n} \right) + 2 \sum_{i < j} \frac{1}{n^2(n-1)} && \text{[plug in]} \\ &= n \left( \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right) + 2 \binom{n}{2} \left( \frac{1}{n^2(n-1)} \right) && \text{[there are } \binom{n}{2} \text{ pairs with } i < j] \\ &= \left( 1 - \frac{1}{n} \right) + 2 \frac{n(n-1)}{2} \left( \frac{1}{n^2(n-1)} \right) \\ &= \left( 1 - \frac{1}{n} \right) + \frac{1}{n} \\ &= 1\end{aligned}$$

How many pairs are there with  $i < j$ ? This is just  $\binom{n}{2} = \frac{n(n-1)}{2}$  since we just choose two different elements. Another way to see this is that there was an  $n \times n$  square, and we removed the diagonal of  $n$  elements, so we are left with  $n^2 - n = n(n-1)$ . Divide by two to get just the lower half.

This is very surprising and interesting! When returning  $n$  hats randomly and uniformly, the expected number of people who get their hat back is 1, and so is the variance! These don't even depend on  $n$  at all!  $\square$  It takes practice to get used to these formulae, so let's do one more problem.

### Example(s)

Suppose we throw 12 balls independently and uniformly into 7 bins. What are the mean and variance of the number of empty bins after this process? (Hint: Indicators).

*Solution* Let  $X$  be the total number of empty bins, and  $X_1, \dots, X_7$  be the indicator of whether or not bin  $i$  is empty so that  $X = \sum_{i=1}^7 X_i$ . Then,

$$\mathbb{P}(X_i = 1) = \left( \frac{6}{7} \right)^{12}$$

since we need to avoid this bin (with probability  $6/7$ ) 12 times independently. That is,

$$X_i \sim \text{Ber} \left( p = \left( \frac{6}{7} \right)^{12} \right)$$

Hence,  $\mathbb{E}[X_i] = p \approx 0.1573$  and  $\text{Var}(X_i) = p(1-p) \approx 0.1325$ . These random variables are surely dependent, since knowing one bin is empty means the 12 balls had to go to the other 6 bins, making it less likely that another bin is empty.

However, dependence doesn't bother us for computing the expectation; by linearity of expectation, we get

$$\mathbb{E}[X] = \mathbb{E} \left[ \sum_{i=1}^7 X_i \right] = \sum_{i=1}^7 \mathbb{E}[X_i] = \sum_{i=1}^7 \left( \frac{6}{7} \right)^{12} = 7 \left( \frac{6}{7} \right)^{12} \approx 1.1009$$

Now for the variance, we need to find  $\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j]$  for  $i \neq j$ . Well,  $X_i X_j \in \{0, 1\}$  since both  $X_i, X_j \in \{0, 1\}$ , so  $X_i X_j$  is indicator/Bernoulli as well with

$$\mathbb{E}[X_i X_j] = \mathbb{P}(X_i X_j = 1) = \mathbb{P}(X_i = 1, X_j = 1) = \mathbb{P}(\text{both bin } i \text{ and } j \text{ are empty}) = \left( \frac{5}{7} \right)^{12}$$

since all the balls must go into the other 5 bins during each of the 12 independent throws. Finally,

$$\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] = \left( \frac{5}{7} \right)^{12} - \left( \frac{6}{7} \right)^{12} \left( \frac{6}{7} \right)^{12} \approx -0.0071$$

Recall that  $\text{Var}(X_i) = p(1-p) \approx 0.1325$ , and so putting this all together gives:

$$\begin{aligned} \text{Var}(X) &= \sum_{i=1}^7 \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) && \text{[formula for variance of sum]} \\ &\approx \sum_{i=1}^7 0.1325 + 2 \sum_{i < j} (-0.0071) && \text{[plug in approximate decimal values]} \\ &= 7 \cdot 0.1325 + 2 \binom{7}{2} (-0.0071) \\ &\approx 0.62954 \end{aligned}$$

□

Recall the hypergeometric RV  $X \sim \text{HypGeo}(N, K, n)$  which was the number of lollipops we get when we draw  $n$  candies from a bag of  $N$  total candies ( $K \leq N$  are lollipops). We stated without proof that  $\text{Var}(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)}$ . You have the tools now to prove this if you like using indicators and covariances, but we'll prove this later in 5.8 as well!

## Chapter 5: Multiple Random Variables

### 5.5: Convolution

In section 4.4, we explained how to transform random variables (finding the density function of  $g(X)$ ). In this section, we'll talk about how to find the distribution of the sum of two independent random variables,  $X + Y$ , using a technique called convolution. It will allow us to prove some statements we made earlier without proof (like sums of independent Binomials are Binomial, sums of independent Poissons are Poisson), and also derive the density function of the Gamma distribution which we just stated.

#### 5.5.1 Law of Total Probability for Random Variables

We did secretly use this in some previous examples, but let's formally define this!

##### Definition 5.5.1: Law of Total Probability for Random Variables

Discrete version: If  $X, Y$  are discrete random variables:

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \sum_y p_{X|Y}(x|y)p_Y(y)$$

Continuous version: If  $X, Y$  are continuous random variables:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy$$

This should just remind of you of the LTP we learned in section 2.2, or the definition of marginal PMF/PDFs from earlier in the chapter! We'll use this LTP to help us derive the formulae for convolution.

#### 5.5.2 Convolution

Convolution is a mathematical operation that allows to derive the distribution of a sum of two independent random variables. For example, suppose the amount of gold a company can mine is  $X$  tons per year in country A, and the amount of gold the company can mine is  $Y$  tons per year in country B, independently. You have some distribution to model each. What is the distribution of the total amount of gold you mine,  $Z = X + Y$ ? Combining this with 4.4, if you know your profit is some function of  $g(Z) = \sqrt{X + Y}$  of the total amount of gold, you can now find the density function of your profit!

I think this is best learned through examples:

##### Example(s)

Let  $X, Y \sim \text{Unif}(1, 4)$  be independent rolls of a fair 4-sided die. What is the PMF of  $Z = X + Y$ ?

*Solution* We know that for the range of  $Z$  we have the following, since it is the sum of two values each in the range  $\{1, 2, 3, 4\}$ :

$$\Omega_Z = \{2, 3, 4, 5, 6, 7, 8\}$$

Should the probabilities be uniform? That is, would you be equally likely to roll a 2 as a 5? No, because there is only one way to get a 2 (rolling  $(1, 1)$ ), but many ways to get a 5.

If I wanted to compute the probability that  $Z = 3$  for example, I could just sum over all possible values of  $X$  in  $\Omega_X = \{1, 2, 3, 4\}$  to get:

$$\begin{aligned} \mathbb{P}(Z = 3) &= \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 1) + \mathbb{P}(X = 3, Y = 0) + \mathbb{P}(X = 4, Y = -1) \\ &= \mathbb{P}(X = 1)\mathbb{P}(Y = 2) + \mathbb{P}(X = 2)\mathbb{P}(Y = 1) + \mathbb{P}(X = 3)\mathbb{P}(Y = 0) + \mathbb{P}(X = 4)\mathbb{P}(Y = -1) \\ &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 \\ &= \frac{2}{16} \end{aligned}$$

where the first line is all ways to get a 3, and the second line uses independence. Note that is is not possible that  $Y = 0$  or  $Y = -1$ , but we write this for completion. More generally, to find  $p_Z(z) = \mathbb{P}(Z = z)$  for any value of  $z$ , we just write

$$\begin{aligned} p_Z(z) &= \mathbb{P}(Z = z) \\ &= \sum_{x \in \Omega_X} \mathbb{P}(X = x, Y = z - x) \\ &= \sum_{x \in \Omega_X} \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\ &= \sum_{x \in \Omega_X} p_X(x)p_Y(z - x) \end{aligned}$$

The intuition is that if we want  $Z = z$ , we sum over all possibilities of  $X = x$  but require that  $Y = z - x$  so that we get the desired sum of  $z$ . It is very possible that  $p_Y(z - x) = 0$  as we saw above.  $\square$

It turns out that formula at the bottom was extremely general, and works for any sum of two independent discrete RVs. Now let's consider the continuous case. What if  $X$  and  $Y$  are continuous RVs and we define  $Z = X + Y$ ; how can we solve for the probability density function for  $Z$ ,  $f_Z(z)$ ? It turns out the formula is extremely similar, just replacing  $p$  with  $f$ !

Theorem 5.5.21: Convolution

Let  $X, Y$  be *independent* RVs, and  $Z = X + Y$ .

Discrete version: If  $X, Y$  are discrete:

$$p_Z(z) = \sum_{x \in \Omega_X} p_X(x)p_Y(z - x)$$

Continuous version: If  $X, Y$  are continuous:

$$f_Z(z) = \int_{x \in \Omega_X} f_X(x)f_Y(z - x)dx$$

Note: You can swap the roles of  $X$  and  $Y$ . Note the similarity between the cases!

*Proof of Convolution.:*

- Discrete case: Even though we proved this earlier, we'll do it again a different way (using the LTP/def of marginal):

$$\begin{aligned}
 p_Z(z) &= \mathbb{P}(Z = z) \\
 &= \sum_{x \in \Omega_X} \mathbb{P}(X = x, Z = z) && [\text{LTP/marginal}] \\
 &= \sum_{x \in \Omega_X} \mathbb{P}(X = x, Y = z - x) && [(X = x, Z = z) \text{ equivalent to } (X = x, Y = z - x)] \\
 &= \sum_{x \in \Omega_X} \mathbb{P}(X = x) \mathbb{P}(Y = z - x) && [X \text{ and } Y \text{ are independent}] \\
 &= \sum_{x \in \Omega_X} p_X(x) p_Y(z - x)
 \end{aligned}$$

- Continuous case: Since we should never work with densities as probabilities, let's start with the CDF and differentiate:

$$\begin{aligned}
 F_Z(z) &= \mathbb{P}(Z \leq z) \\
 &= \mathbb{P}(X + Y \leq z) && [\text{def of } Z] \\
 &= \int_{x \in \Omega_X} \mathbb{P}(X + Y \leq z \mid X = x) f_X(x) dx && [\text{LTP, conditioning on } X] \\
 &= \int_{x \in \Omega_X} \mathbb{P}(x + Y \leq z \mid X = x) f_X(x) dx && [\text{given } X = x] \\
 &= \int_{x \in \Omega_X} \mathbb{P}(Y \leq z - x \mid X = x) f_X(x) dx && [\text{algebra}] \\
 &= \int_{x \in \Omega_X} \mathbb{P}(Y \leq z - x) f_X(x) dx && [X \text{ and } Y \text{ are independent}] \\
 &= \int_{x \in \Omega_X} F_Y(z - x) f_X(x) dx && [\text{def of CDF of } Y]
 \end{aligned}$$

Now we can take the derivative (with respect to  $z$ ) of the CDF to get the density ( $F_Y$  becomes  $f_Y$ ):

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{x \in \Omega_X} f_X(x) f_Y(z - x) dx$$

Note the striking similarity in the formulae!

□

Example(s)

Suppose  $X$  and  $Y$  are two independent random variables such that  $X \sim \text{Poi}(\lambda_1)$  and  $Y \sim \text{Poi}(\lambda_2)$ , and let  $Z = X + Y$ . Prove that  $Z \sim \text{Poi}(\lambda_1 + \lambda_2)$ .

The range of  $X, Y$  are  $\Omega_X = \Omega_Y = \{0, 1, 2, \dots\}$ , and so  $\Omega_Z = \{0, 1, 2, \dots\}$  as well. For  $n \in \Omega_Z$ : Note that the convolution formula says:

$$p_Z(n) = \sum_{k \in \Omega_X} p_X(k) p_Y(n - k) = \sum_{k=0}^{\infty} p_X(k) p_Y(n - k)$$

However, if you blindly plug in the PMFs  $p_X$  and  $p_Y$ , you will get the wrong answer, and here's why. We only want to sum things that are non-zero (otherwise what's the point?), and if we want  $p_X(k)p_Y(n-k) > 0$ , we need BOTH to be nonzero. That means,  $k$  must be in the range of  $X$  AND  $n - k$  must be in the range of  $Y$ . Remember the dice example (we had  $p_Y(-1)$  at some point, which would be 0 and not  $1/4$ ). We are guaranteed  $p_X(k) > 0$  because we are only summing over valid  $k \in \Omega_X$ , but we must have  $n - k$  be a nonnegative integer (in the range  $\Omega_Y = \{0, 1, 2, \dots\}$ , so actually, we must have  $k \leq n$ ). Now, we can just plug and chug:

$$\begin{aligned}
 p_Z(n) &= \sum_{k=0}^n p_X(k)p_Y(n-k) && \text{[convolution formula]} \\
 &= \sum_{k=0}^n e^{-\lambda_1} \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} && \text{[plug in Poisson PMFs]} \\
 &= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^n \frac{1}{k!(n-k)!} \lambda_1^k (1-\lambda_2)^{n-k} && \text{[algebra]} \\
 &= e^{-(\lambda_1+\lambda_2)} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \lambda_1^k (1-\lambda_2)^{n-k} && \text{[multiply and divide by } n! \text{]} \\
 &= e^{-(\lambda_1+\lambda_2)} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \lambda_1^k (1-\lambda_2)^{n-k} && \left[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \right] \\
 &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!} && \text{[binomial theorem]}
 \end{aligned}$$

Thus,  $Z \sim \text{Poi}(\lambda_1 + \lambda_2)$ , as its PMF matches that of a Poisson distribution! Note we wouldn't have been able to do that last step if our sum was still  $k = 0$  to  $n$ . You MUST watch out for this at the beginning, and after that, it's just algebra.

### Example(s)

Suppose  $X, Y$  are independent and identically distributed (iid) continuous  $\text{Unif}(0, 1)$  random variables. Let  $Z = X + Y$ . What is  $f_Z(z)$ ?

*Solution* We always begin by calculating the range: we have  $\Omega_Z = [0, 2]$ . Again, we shouldn't expect  $Z$  to be uniform, since we should expect a number around 1, but not 0 or 2.

For a  $U \sim \text{Unif}(0, 1)$  (continuous) random variable, we know  $\Omega_U = [0, 1]$ , and that

$$f_U(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{x \in \Omega_X} f_X(x)f_Y(z-x)dx = \int_0^1 f_X(x)f_Y(z-x)dx = \int_0^1 f_Y(z-x)dx$$

where the last formula holds since  $f_X(x) = 1$  for all  $0 \leq x \leq 1$  as we saw above. Remember, we need to make sure  $z - x \in \Omega_Y = [0, 1]$ , otherwise the density will be 0.

For  $f_Y(z-x) > 0$ , we need  $0 \leq z - x \leq 1$ . We'll split into two cases depending on whether  $z \in [0, 1]$  or  $z \in [0, 2]$ , which compose its range  $\Omega_Z = [0, 2]$ .

- If  $z \in [0, 1]$ , we already have  $z - x \leq 1$  since  $z \leq 1$  (and  $x \in [0, 1]$ ). We also need  $z - x \geq 0$  for the density to be nonzero:  $x \leq z$ . Hence, our integral becomes:

$$\begin{aligned} f_Z(z) &= \int_0^z f_Y(z - x)dx + \int_z^1 f_Y(z - x)dx \\ &= \int_0^z 1dx + 0 = [x]_0^z = z \end{aligned}$$

- If  $z \in [1, 2]$ , we already have  $z - x \geq 0$  since  $z \geq 1$  (and  $x \in [0, 1]$ ). We now need the other condition  $z - x \leq 1$  for the density to be nonzero:  $x \geq z - 1$ . Hence, our integral becomes:

$$\begin{aligned} f_Z(z) &= \int_0^{z-1} f_Y(z - x)dx + \int_{z-1}^1 f_Y(z - x)dx \\ &= 0 + \int_{z-1}^1 1dx = [x]_{z-1}^1 = 2 - z \end{aligned}$$

Thus, putting these two cases together gives:

$$f_Z(z) = \begin{cases} z & 0 \leq z \leq 1 \\ 2 - z & 1 \leq z \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

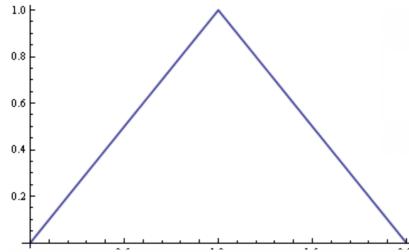


Figure 5.5.1: Triangular distribution for  $Z$

This makes sense because there are “more ways” to get a value of 1 for example than any other point. Whereas to get a value of 2, there’s only one way - we need both  $X, Y$  to be equal to 1.  $\square$

### Example(s)

Mitchell and Alex are competing together in a 2-mile relay race. The time Mitchell takes to finish (in hours) is  $X \sim \text{Exp}(2)$  and the time Alex takes to finish his mile (in hours) is continuous  $Y \sim \text{Unif}(0, 1)$ . Alex starts immediately after Mitchell finishes his mile, and their performances are independent. What is the distribution of  $Z = X + Y$ , the total time they take to finish the race?

*Solution* First, we know that  $\Omega_X = [0, \infty)$  and  $\Omega_Y = [0, 1]$ , so  $\Omega_Z = [0, \infty)$ . We know from our distribution chart that

$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0 \quad \text{and} \quad f_Y(y) = 1, 0 \leq y \leq 1$$

Let  $z \in \Omega_Z$ . We’ll use the convolution formula, but this time over the range of  $Y$  (you could also do over  $X$  too!). We can do this because  $X + Y = Y + X$ , and there was no reason why we had to condition on  $X$  first.

$$f_Z(z) = \int_{\Omega_Y} f_Y(y) f_X(z - y) dy = \int_0^1 f_Y(y) f_X(z - y) dy$$

Since we are integrating over  $y$ , we don't need to worry about  $f_Y(y)$  being 0, but we do need to make sure  $f_X(z-y) > 0$ . There are two cases again:

- If  $z \in [0, 1]$ , then since we need  $z - y \geq 0$ , we need  $y \leq z$ :

$$f_Z(z) = \int_0^z f_Y(y) f_X(z-y) dy = \int_0^z 1 \cdot \lambda e^{-\lambda(z-y)} dy = 1 - e^{-\lambda z}$$

- if  $z \in (1, \infty)$ , then  $y \leq z$  always (since  $y \in [0, 1]$ ), so

$$f_Z(z) = \int_0^1 f_Y(y) f_X(z-y) dy = (e^\lambda - 1) e^{-\lambda z}$$

Note this tiny difference in the upper limit of the integral made a huge difference! Our final result is

$$f_Z(z) = \begin{cases} 1 - e^{-\lambda z} & z \in [0, 1] \\ (e^\lambda - 1) e^{-\lambda z} & z \in (1, \infty) \\ 0 & \text{otherwise} \end{cases}$$

□

The moral of the story is: always watch out for the ranges, otherwise you might not get what you expect! The range of the random variable exists for a reason, so be careful!

## Chapter 5: Multiple Random Variables

### 5.6: Moment Generating Functions

Last time, we talked about how to find the distribution of the sum of two independent random variables. Some of the most important use cases are to prove the results we've been using for so long: the sum of independent Binomials is Binomial, the sum of independent Poissons is Poisson (we proved this in 5.5 using convolution), etc. We'll now talk about Moment Generating Functions, which allow us to do these in a different (and arguably easier) way. These will also be used to prove the Central Limit Theorem (next section), probably the most important result in all of statistics! Also, to derive the Chernoff bound (6.2). The point is, these are used to *prove* a lot of important results. They might not be as direct applicable to problems though.

#### 5.6.1 Moments

First, we need to define what a moment is.

##### Definition 5.6.1: Moments

Let  $X$  be a random variable and  $c \in \mathbb{R}$  a scalar. Then: The  $k^{\text{th}}$  moment of  $X$  is:

$$\mathbb{E}[X^k]$$

and the  $k^{\text{th}}$  moment of  $X$  (about  $c$ ) is:

$$\mathbb{E}[(X - c)^k]$$

The first four moments of a distribution/RV are commonly used, though we have only talked about the first two of them. I'll briefly explain each but we won't talk about the latter two much.

1. The first moment of  $X$  is the mean of the distribution  $\mu = \mathbb{E}[X]$ . This describes the center or average value.
2. The second moment of  $X$  about  $\mu$  is the variance of the distribution  $\sigma^2 = \text{Var}(X) = \mathbb{E}[(X - \mu)^2]$ . This describes the spread of a distribution (how much it varies).
3. The third standardized moment is called skewness  $\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$  and typically tells us about the asymmetry of a distribution about its peak. If skewness is positive, then the mean is larger than the median and there are a lot of extreme high values. If skewness is negative, then the median is larger than the mean and there are a lot of extreme low values.
4. The fourth standardized moment is called kurtosis  $\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mathbb{E}[X^4]}{\sigma^4}$ , which measures how peaked a distribution is. If the kurtosis is positive, then the distribution is thin and pointy, and if the kurtosis is negative, the distribution is flat and wide.

## 5.6.2 Moment Generating Functions (MGFs)

We'll first define the MGF of a random variable  $X$ , and then explain its use cases and importance.

### Definition 5.6.2: Moment Generating Functions (MGFs)

Let  $X$  be a random variable. The **moment generating function (MGF)** of  $X$  is a function of a dummy variable  $t$ :

$$M_X(t) = \mathbb{E}[e^{tX}]$$

If  $X$  is discrete, by LOTUS:

$$M_X(t) = \sum_{x \in \Omega_X} e^{tx} p_X(x)$$

If  $X$  is continuous, by LOTUS:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

We say that the MGF of  $X$  exists, if there is a  $\varepsilon > 0$  such that the MGF is finite for all  $t \in (-\varepsilon, \varepsilon)$ , since it is possible that the sum or integral diverges.

Let's do some example computations before discussing why it might be useful.

### Example(s)

Find the MGF of the following random variables:

(a)  $X$  is a discrete random variable with PMF:

$$p_X(k) = \begin{cases} 1/3 & k = 1 \\ 2/3 & k = 2 \end{cases}$$

(b)  $Y$  is a  $\text{Unif}(0, 1)$  continuous random variable.

*Solution*

(a)

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_x e^{tx} p_X(x) && [\text{LOTUS}] \\ &= \frac{1}{3}e^t + \frac{2}{3}e^{2t} \end{aligned}$$

(b)

$$\begin{aligned}
M_Y(t) &= \mathbb{E}[e^{tY}] \\
&= \int_0^1 e^{ty} f_Y(y) dy && \text{[LOTUS]} \\
&= \int_0^1 e^{ty} \cdot 1 dy && [f_Y(y) = 1, 0 \leq y \leq 1] \\
&= \frac{e^t - 1}{t}
\end{aligned}$$

□

### 5.6.3 Properties and Uniqueness of MGFs

There are some useful properties of MGFs that we will discuss. Let  $X, Y$  be *independent* random variables, and  $a, b \in \mathbb{R}$  be scalars. Then, recall that the moment generating function of  $X$  is:  $M_X(t) = \mathbb{E}[e^{tX}]$ .

**1. Computing MGF of Linear Transformations:** We'll first see how we can compute the MGF of  $aX + b$  if we know the MGF of  $X$ :

$$M_{aX+b}(t) = \mathbb{E}[e^{t(aX+b)}] = e^{tb} \mathbb{E}[e^{(at)X}] = e^{tb} M_X(at)$$

**2. Computing MGF of Sums:** We can also compute the MGF of the sum of independent RVs  $X$  and  $Y$  given their individual MGFs: (the third step is due to independence):

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX} e^{tY}] = \mathbb{E}[e^{tX}] \mathbb{E}[e^{tY}] = M_X(t) M_Y(t)$$

**3. Generating Moments with MGFs:** The reason why MGFs are named they way they are, is because they *generate moments* of  $X$ . That means, they can be used to compute  $\mathbb{E}[X]$ ,  $\mathbb{E}[X^2]$ ,  $\mathbb{E}[X^3]$ , and so on. How? Let's take the derivative of an MGF (with respect to  $t$ ):

$$M'_X(t) = \frac{d}{dt} \mathbb{E}[e^{tX}] = \frac{d}{dt} \sum_{x \in \Omega_X} e^{tx} p_X(x) = \sum_{x \in \Omega_X} \frac{d}{dt} (e^{tx} p_X(x)) = \sum_{x \in \Omega_X} x e^{tx} p_X(x)$$

note in the last step that  $x$  is a constant with respect to  $t$  and so  $\frac{d}{dt} e^{tx} = x e^{tx}$ .

Note that if evaluate the derivative at  $t = 0$ , we get  $\mathbb{E}[X]$  since  $e^0 = 1$ :

$$M'_X(0) = \sum_{x \in \Omega_X} x e^{0x} p_X(x) = \sum_{x \in \Omega_X} x p_X(x) = \mathbb{E}[X]$$

Now, let's consider the second derivative:

$$M''_X(t) = \frac{d}{dt} M'_X(t) = \frac{d}{dt} \sum_{x \in \Omega_X} x e^{tx} p_X(x) = \sum_{x \in \Omega_X} \frac{d}{dt} (x e^{tx} p_X(x)) = \sum_{x \in \Omega_X} x^2 e^{tx} p_X(x)$$

If we evaluate the second derivative at  $t = 0$ , we get  $\mathbb{E}[X^2]$ :

$$M_X''(0) = \sum_{x \in \Omega_X} x^2 e^{0x} p_X(x) = \sum_{x \in \Omega_X} x^2 p_X(x) = \mathbb{E}[X^2]$$

Seems like there's a pattern - if we take the  $n$ -th derivative of  $M_X(t)$ , then we will generate the  $n$ -th moment  $\mathbb{E}[X^n]$ !

Theorem 5.6.22: Properties and Uniqueness of Moment Generating Functions

For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we will denote  $f^{(n)}(x)$  to be the  $n^{\text{th}}$  derivative of  $f(x)$ . Let  $X, Y$  be independent random variables, and  $a, b \in \mathbb{R}$  be scalars. Then MGFs satisfy the following properties:

1.  $M_X'(0) = \mathbb{E}[X]$ ,  $M_X''(0) = \mathbb{E}[X^2]$ , and in general  $M_X^{(n)} = \mathbb{E}[X^n]$ . This is why we call  $M_X$  a *moment generating* function, as we can use it to generate the moments of  $X$ .
2.  $M_{aX+b}(t) = e^{tb} M_X(at)$ .
3. If  $X \perp Y$ , then  $M_{X+Y}(t) = M_X(t)M_Y(t)$ .
4. (**Uniqueness**) The following are equivalent:
  - (a)  $X$  and  $Y$  have the same distribution.
  - (b)  $f_X(z) = f_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (c)  $F_X(z) = F_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (d) There is an  $\varepsilon > 0$  such that  $M_X(t) = M_Y(t)$  for all  $t \in (-\varepsilon, \varepsilon)$  (they match on a small interval around  $t = 0$ ).

That is  $M_X$  uniquely identifies a distribution, just like PDFs or CDFs do.

We proved the first three properties before stating all the theorems, so all that's left is property 4. This is a very complex proof (out of the scope of this course), but we can prove it for a special case.

*Proof of Property 4 for a Special Case.* We'll prove that, if  $X, Y$  are discrete rvs with range  $\Omega = \{0, 1, 2, \dots, m\}$  and whose MGFs are equal everywhere, that  $p_X(k) = p_Y(k)$  for all  $k \in \Omega$ . That is, if two distributions have the same MGF, they have the same distribution (PMF).

For any  $t$ , we have

$$M_X(t) = M_Y(t)$$

By definition of MGF, we get

$$\sum_{k=0}^m e^{tk} p_X(k) = \sum_{k=0}^m e^{tk} p_Y(k)$$

Subtracting the right-hand side from both sides gives:

$$\sum_{k=0}^m e^{tk} (p_X(k) - p_Y(k)) = 0$$

Let  $a_k = p_X(k) - p_Y(k)$  for  $k = 0, \dots, m$  and write  $e^{tk}$  as  $(e^t)^k$ . Then, we get

$$\sum_{k=0}^m a_k (e^t)^k = 0$$

Note that this is an  $m$ -th degree polynomial in  $e^t$ , and remember that this equation holds for (uncountably) infinitely many  $t$ . An  $m^{th}$  degree polynomial can only have  $m$  roots, unless all the coefficients are 0. Hence  $a_k = 0$  for all  $k$ , and so  $p_X(k) = p_Y(k)$  for all  $k$ .  $\square$

Now we'll see how to use MGFs to prove some results we've been using.

Example(s)

Suppose  $X \sim \text{Poi}(\lambda)$ , meaning  $X$  has range  $\Omega_X = \{0, 1, 2, \dots\}$  and PMF:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Compute  $M_X(t)$ .

*Solution* First, let's recall the Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} p_X(k) = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} (e^t)^k \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} \quad [\text{Taylor series with } x = \lambda e^t] \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

$\square$

We can use MGFs in our proofs of certain facts about RVs.

Example(s)

If  $X \sim \text{Poi}(\lambda)$ , compute  $\mathbb{E}[X]$  using its MGF we computed earlier  $M_X(t) = e^{\lambda(e^t - 1)}$ .

*Solution* We can prove that  $\mathbb{E}[X] = \lambda$  as follows.

First we take the derivative of the moment generating function (don't forget the chain rule of calculus) and see that:

$$M'_X(t) = e^{\lambda(e^t - 1)} \cdot \lambda e^t$$

Then, we know that:

$$\mathbb{E}[X] = M'_X(0) = e^{\lambda(e^0 - 1)} \cdot \lambda e^0 = \lambda$$

$\square$

## Example(s)

If  $Y \sim \text{Poi}(\gamma)$  and  $Z \sim \text{Poi}(\mu)$  and  $Y \perp Z$ , show that  $Y + Z \sim \text{Poi}(\gamma + \mu)$  using the uniqueness property of MGFs. (Recall we did this exact problem using convolution in 5.5).

*Solution* First note that a  $\text{Poi}(\gamma + \mu)$  RV has MGF  $e^{(\gamma+\mu)(e^t-1)}$  (just plugging in  $\gamma + \mu$  as the parameter). Since  $Y$  and  $Z$  are independent, by property 3,

$$M_{Y+Z}(t) = M_Y(t)M_Z(t) = e^{\gamma(e^t-1)}e^{\mu(e^t-1)} = e^{(\gamma+\mu)(e^t-1)}$$

The MGF of  $Y + Z$  which we computed is the same as that of  $\text{Poi}(\gamma + \mu)$ . So, by the uniqueness of MGFs (which implies that an MGF can uniquely describe a distribution),  $Y + Z \sim \text{Poi}(\gamma + \mu)$ .

Which way was easier for you - this approach or using convolution? MGF's have limitations though whereas convolution doesn't (besides independence) - we need to compute the MGF of  $Y, Z$  but we also need to know the MGF of what distribution we are trying to "get".  $\square$

## Example(s)

Now, use MGFs to prove the closure properties of Gaussian RVs (which we've been using without proof).

- If  $V \sim \mathcal{N}(\mu, \sigma^2)$  and  $W \sim \mathcal{N}(\nu, \gamma^2)$  are independent, that  $V + W \sim \mathcal{N}(\mu + \nu, \sigma^2 + \gamma^2)$ .
- If  $a, b \in \mathbb{R}$  are constants and  $X \sim \mathcal{N}(\mu, \sigma^2)$ , show that  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

You may use the fact that if  $Y \sim \mathcal{N}(\mu, \sigma^2)$ , that

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

*Solution*

- If  $V \sim \mathcal{N}(\mu, \sigma^2)$  and  $W \sim \mathcal{N}(\nu, \gamma^2)$  are independent, we have the following:

$$M_{V+W}(t) = M_V(t)M_W(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} e^{\nu t + \frac{\gamma^2 t^2}{2}} = e^{(\mu+\nu)t + \frac{(\sigma^2 + \gamma^2)t^2}{2}}$$

This is the MGF of a Normal distribution with mean  $\mu + \nu$  and variance  $\sigma^2 + \gamma^2$ . So, by uniqueness of MGFs,  $Y + Z \sim \mathcal{N}(\mu + \nu, \sigma^2 + \gamma^2)$ .

- Let us examine the moment generating function for  $aX + b$ . (We'll use the notation  $\exp(z) = e^z$  so that we can actually see what's in the exponent clearly):

$$M_{aX+b}(t) = e^{bt} M_X(at) = \exp(bt) \exp\left(\mu(at) + \frac{\sigma^2(at)^2}{2}\right) = \exp\left((a\mu + b)t + \frac{(a^2\sigma^2)t^2}{2}\right)$$

Since this is the moment generating function for a RV that is  $\mathcal{N}(a\mu + b, a^2\sigma^2)$ , we have shown that by the uniqueness of MGFs that  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .  $\square$

## Chapter 5: Multiple Random Variables

### 5.7: Limit Theorems

This is definitely one of the most important sections in the entire text! The Central Limit Theorem is used everywhere in statistics (hypothesis testing), and it also has its applications in computing probabilities. We'll see three results here, each getting more powerful and surprising.

If  $X_1, \dots, X_n$  are iid random variables with mean  $\mu$  and variance  $\sigma^2$ , then we define the sample mean to be  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . We'll see the following results:

- The expectation of the sample mean  $\mathbb{E}[\bar{X}_n]$  is exactly the true mean  $\mu$ , and the variance  $\text{Var}(\bar{X}_n) = \sigma^2/n$  goes to 0 as you get more samples.
- (Law of Large Numbers) As  $n \rightarrow \infty$ , the sample mean  $\bar{X}_n$  converges (in probability) to the true mean  $\mu$ . That is, as you get more samples, you will be able to get an excellent estimate of  $\mu$ .
- (Central Limit Theorem) In fact,  $\bar{X}_n$  follows a *Normal distribution* as  $n \rightarrow \infty$  (in practice  $n$  as low as 30 is good enough for this to be true). When we talk about the distribution of  $\bar{X}_n$ , this means: if we take  $n$  samples and take the sample mean, another  $n$  samples and take the sample mean, and so on, how will these sample means look in a histogram? This is crazy - regardless of what the distribution of  $X_i$ 's were (discrete, continuous), their average will be approximately Normal! We'll see pictures and describe this more soon!

#### 5.7.1 The Sample Mean

Before we start, we will define the sample mean of  $n$  random variables, and compute its mean and variance.

##### Definition 5.7.1: The Sample Mean + Properties

Let  $X_1, X_2, \dots, X_n$  be a sequence of iid (independent and identically distributed) random variables with mean  $\mu$  and variance  $\sigma^2$ . The **sample mean** is:

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Further:

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n\mu = \mu$$

Also, since the  $X_i$ 's are independent:

$$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

Again, none of this is “mind-blowing” to prove: we just used linearity of expectation and properties of

variance to show this.

What is this saying? Basically, if you wanted to estimate the mean height of the U.S. population by sampling  $n$  people uniformly at random:

- In expectation, your sample average will be “on point” at  $\mathbb{E}[\bar{X}_n] = \mu$ . This even includes the case  $n = 1$ : if you just sample one person, on average, you will be correct. However, the variance is high.
- The variance of your estimate (the sample mean) for the true mean goes down ( $\sigma^2/n$ ) as your sample size  $n$  gets larger. This makes sense right? If you have more samples, you have more confidence in your estimate because you are more “sure” (less variance).

In fact, as  $n \rightarrow \infty$ , the variance of the sample mean approaches 0. A distribution with mean  $\mu$  and variance 0 is essentially the degenerate random variable that takes on  $\mu$  with probability 1. We’ll actually see that the Law of Large Numbers argues exactly that!

### 5.7.2 The Law of Large Numbers (LLN)

Using the fact that the variance is approaching 0 as  $n \rightarrow \infty$ , we can argue that, by averaging more and more samples ( $n \rightarrow \infty$ ), we get a really good estimate of the true mean  $\mu$  since the variance of the sample mean is  $\sigma^2/n \rightarrow 0$  (as we showed earlier). Here is the formal mathematical statement:

Theorem 5.7.23: The Law of Large Numbers

**Weak Law of Large Numbers (WLLN):** Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables with mean  $\mu$ . Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  be the sample mean. Then,  $\bar{X}_n$  converges **in probability** to  $\mu$ . That is for any  $\epsilon > 0$ :

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\bar{X}_n - \mu| > \epsilon) = 0$$

**Strong Law of Large Numbers (SLLN):** Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables with mean  $\mu$ . Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  be the sample mean. Then,  $\bar{X}_n$  converges **almost surely** to  $\mu$ . That is:

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \bar{X}_n = \mu\right) = 1$$

The SLLN implies the WLLN, but not vice versa. The difference is subtle and is basically swapping the limit and probability operations.

The proof the WLLN will be given in 6.1 when we prove Chebyshev’s inequality, but the proof of the SLLN is out of the scope of this class and much harder to prove.

### 5.7.3 The Central Limit Theorem (CLT)

#### Theorem 5.7.24: The Central Limit Theorem (CLT)

Let  $X_1, \dots, X_n$  be a sequence of independent and identically distributed random variables with mean  $\mu$  and (finite) variance  $\sigma^2$ . We've seen that the sample mean  $\bar{X}_n$  has mean  $\mu$  and variance  $\frac{\sigma^2}{n}$ . Then as  $n \rightarrow \infty$ , the following equivalent statements hold:

1.  $\bar{X}_n \rightarrow \mathcal{N}(\mu, \frac{\sigma^2}{n})$ .
2.  $\frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} \rightarrow \mathcal{N}(0, 1)$
3.  $\sum_{i=1}^n X_i \sim \mathcal{N}(n\mu, n\sigma^2)$ . This is not "technically" correct, but is useful for applications.
4.  $\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n\sigma^2}} \rightarrow \mathcal{N}(0, 1)$

The mean or variance are not a surprise (we computed these at the beginning of these notes for any sample mean); the importance of the CLT is, regardless of the distribution of  $X_i$ 's, the sample mean approaches a Normal distribution as  $n \rightarrow \infty$ .

We will prove the central limit theorem in 5.11 using MGFs, but take a second to appreciate this crazy result! The LLN say that as  $n \rightarrow \infty$ , the sample mean of iid variables  $\bar{X}_n$  converges to  $\mu$ . The CLT says that, as  $n \rightarrow \infty$ , the sample mean actually converges to a Normal distribution! For **any** original distribution of the  $X_i$ 's (discrete or continuous), the average/sum will become approximately normally distributed.

If you're still having trouble with figuring out what "the distribution of the sample mean" means, that's completely normal (double pun!). Let's consider  $n = 2$ , so we just take the average of  $X_1 + X_2$ , which is  $\frac{X_1 + X_2}{2}$ . The distribution of  $X_1 + X_2$  means: if we repeatedly sample  $X_1, X_2$  and add them, what might the density look like? For example, if  $X_1, X_2 \sim \text{Unif}(0, 1)$  (continuous), we showed the density of  $X_1 + X_2$  looked like a triangle. We figured out how to compute the PMF/PDF of the sum using convolution in 5.5, and the average is just dividing this by 2:  $\frac{X_1 + X_2}{2}$ , which you can find the PMF/PDF by transforming RVs in 4.4. On the next page, you'll see exactly the CLT applied to these Uniform distributions. With  $n = 1$ , it looks (and is) Uniform. When  $n = 2$ , you get the triangular shape. And as  $n$  gets larger, it starts looking more and more like a Normal!

You'll see some examples below of how we start with some arbitrary distributions and how the density function of their mean becomes shaped like a Gaussian (you know how to compute the pdf of the mean now using convolution in 5.5 and transforming RV's in 4.4)!

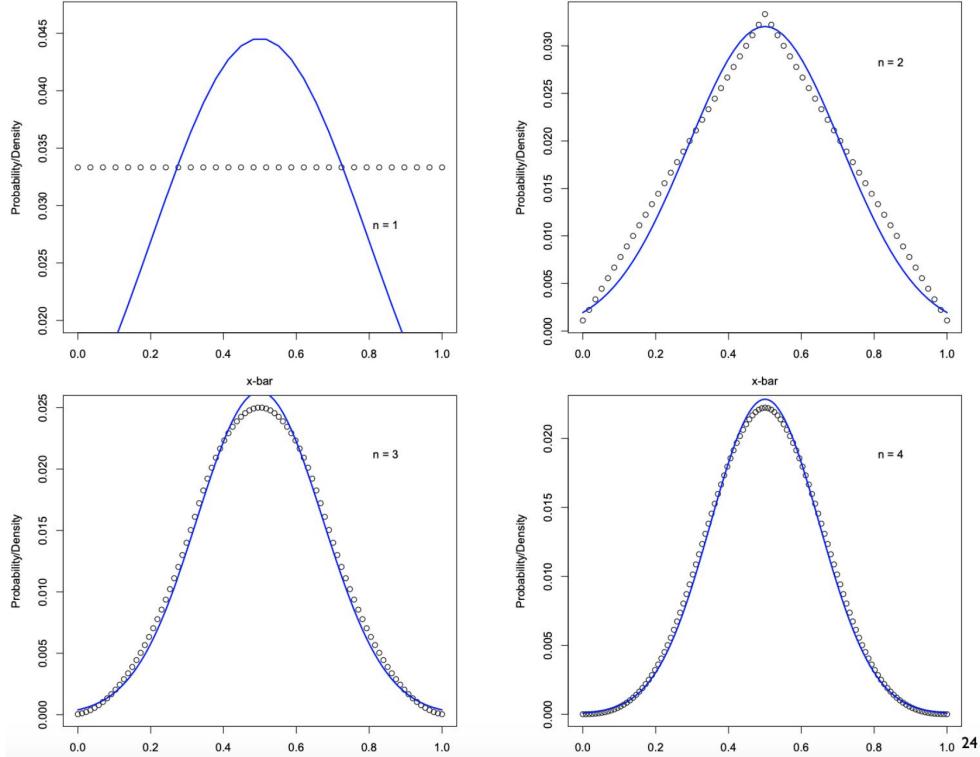
On the next two pages, we'll see some visual "proof" of this surprising result!

Let's see the CLT applied to the (discrete) Uniform distribution.

- The first ( $n = 1$ ) of the four graphs below shows a **discrete**  $\frac{1}{29} \cdot \text{Unif}(0, 29)$  PMF in the dots (and a blue line with the curve of the normal distribution with the same mean and variance). That is,  $\mathbb{P}(X = k) = \frac{1}{30}$  for each value in the range  $\left\{0, \frac{1}{29}, \frac{2}{29}, \dots, \frac{28}{29}, 1\right\}$ .
- The second graph ( $n = 2$ ) has the average of two of these distributions, again with a blue line with the curve of the normal distribution with the same mean and variance. Remember we expected this triangular distribution when summing either discrete or continuous Uniforms. (e.g., when summing two fair 6-sided die rolls, you're most likely to get a 7, and the probability goes down linearly as you approach 2 or 12. See the example in 5.5 if you forgot how we got this!)
- The third ( $n = 3$ ) and fourth ( $n = 4$ ) have the average of 3 and 4 identically distributed random variables respectively, each of the distribution shown in the distribution in the first graph. We can see that as we average more, the sum approaches a normal distribution.

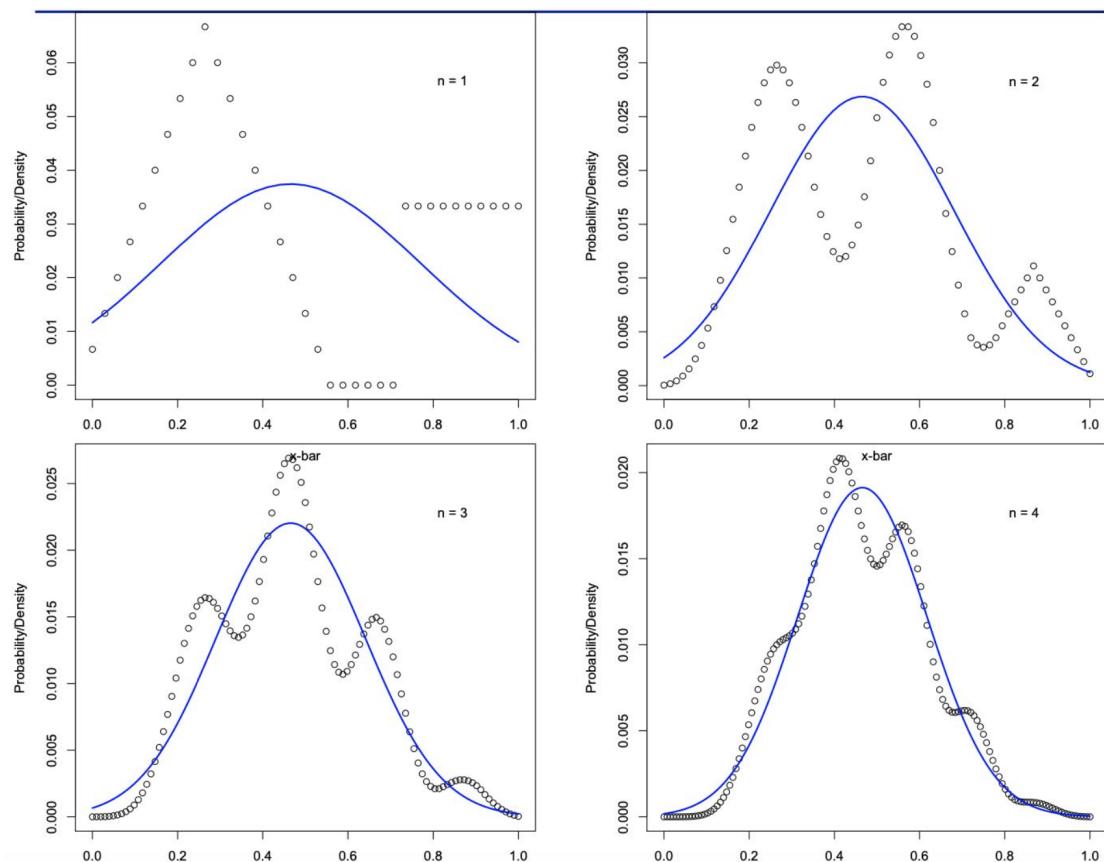
Again, if you don't believe me, you can compute the PMF yourself using convolution: first add two  $\text{Unif}(0, 1)$ , then convolve it with a third, and a fourth!

Despite this being a discrete random variable, when we take an average of many, there become increasingly many values we can get between 0 and 1. The average of these iid discrete rv's approaches a continuous Normal random variable even after just averaging 4 of them!



**Image Credit:** Larry Ruzzo (a previous University of Washington CSE 312 instructor).

You might still be skeptical, because the Uniform distribution is “nice” and already looked pretty “Normal” even with  $n = 2$  samples. We now illustrate the same idea with a strange distribution shown in the first ( $n = 1$ ) of the four graphs below, illustrated with the dots (instead of a “nice” uniform distribution). Even this crazy distribution nearly looks Normal after just averaging 4 of them. This is the power of the CLT!



What we are getting at here is that, regardless of the distribution, as we have more independent and identically distributed random variables, the average follows a Normal distribution (with the same mean and variance as the sample mean).

Now let's see how we can apply the CLT to problems! There were four different equivalent forms (just scaling/shifting) stated, but I find it easier to just look at the problem and decide what's best. Seeing examples is the best way to understand!

### Example(s)

Let's consider the example of flipping a fair coin 40 times independently. What's the probability of getting between 15 to 25 heads? First compute this exactly and then give an approximation using the CLT.

*Solution* Define  $X$  to be the number of heads in the 40 flips. Then we have  $X \sim \text{Bin}(n = 40, p = \frac{1}{2})$ , so we just sum the Binomial PMF:

$$\mathbb{P}(15 \leq X \leq 25) = \sum_{k=15}^{25} \binom{40}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{40-k} \approx 0.9193.$$

Now, let's use the CLT. Since  $X$  can be thought of as the sum of 40 iid  $\text{Ber}(\frac{1}{2})$  RVs, we can apply the CLT. We have  $\mathbb{E}[X] = np = 40(\frac{1}{2}) = 20$  and  $\text{Var}(X) = np(1-p) = 40(\frac{1}{2})(1 - \frac{1}{2}) = 10$ . So we can use the approximation  $X \approx \mathcal{N}(\mu = 20, \sigma^2 = 10)$ .

This gives us the following good but not great approximation:

$$\begin{aligned} \mathbb{P}(15 \leq X \leq 25) &\approx \mathbb{P}(15 \leq \mathcal{N}(20, 10) \leq 25) \\ &= \mathbb{P}\left(\frac{15 - 20}{\sqrt{10}} \leq Z \leq \frac{25 - 20}{\sqrt{10}}\right) && [\text{standardize}] \\ &\approx \mathbb{P}(-1.58 \leq Z \leq 1.58) \\ &= \Phi(1.58) - \Phi(-1.58) \\ &= 0.8862 \end{aligned}$$

□

We'll see how to improve our approximation below!

### 5.7.4 The Continuity Correction

Notice that in the prior example in computing  $\mathbb{P}(15 \leq X \leq 25)$ , we sum over  $25 - 15 + 1 = 11$  terms of the PMF. However, our integral  $\mathbb{P}(15 \leq \mathcal{N}(20, 10) \leq 25)$  has width  $25 - 15 = 10$ . We'll always be off-by-one since the number of integers in  $[a, b]$  is  $(b - a) + 1$  (for integers  $a \leq b$ ) and not  $b - a$  (e.g., the number of integers between  $[12, 15]$  is  $(15 - 12) + 1 = 4 : \{12, 13, 14, 15\}$ ).

The continuity correction says we should add 0.5 in each direction. That is, we should ask for  $\mathbb{P}(a - 0.5 \leq X \leq b + 0.5)$  instead so the width is  $b - a + 1$  instead. Notice that if we do the final calculation, to approximate  $\mathbb{P}(15 \leq X \leq 25)$  using the central limit theorem, now with the continuity correction, we get the following:

### Example(s)

Use the continuity correction to get a better estimate than we did earlier for the coin problem.

*Solution* We'll apply the exact same steps, except changing the bounds from 15 and 25 to 14.5 and 25.5.

$$\begin{aligned}
 \mathbb{P}(15 \leq X \leq 25) &\approx \mathbb{P}(14.5 \leq \mathcal{N}(20, 10) \leq 25.5) && \text{[apply continuity correction]} \\
 &= \mathbb{P}\left(\frac{14.5 - 20}{\sqrt{10}} \leq Z \leq \frac{25.5 - 20}{\sqrt{10}}\right) \\
 &\approx \mathbb{P}(-1.74 \leq Z \leq 1.74) \\
 &= \Phi(1.74) - \Phi(-1.74) \\
 &\approx 0.9182
 \end{aligned}$$

Notice that this is much closer to the exact answer from the first part of the prior example (0.9193) than approximating with the central limit theorem without the continuity correction!  $\square$

#### Definition 5.7.2: The Continuity Correction

When approximating an integer-valued (**discrete**) random variable  $X$  with a continuous one  $Y$  (such as in the CLT), if asked to find a  $\mathbb{P}(a \leq X \leq b)$  for integers  $a \leq b$ , you should compute  $\mathbb{P}(a - 0.5 \leq Y \leq b + 0.5)$  so that the width of the interval being integrated is the same as the number of terms summed over  $(b - a + 1)$ . This is called the **continuity correction**.

**Note:** If you are applying the CLT to sums/averages of *continuous* RVs instead, you should **not** apply the continuity correction.

See the additional exercises below to get more practice with the CLT!

### 5.7.5 Exercises

1. Each day, the number of customers who come to the CSE 312 probability gift shop is approximately  $\text{Poi}(11)$ . Approximate the probability that, after the quarter ends ( $9 \times 7 = 63$  days), that we had over 700 customers.

**Solution:** The total number of customers that come is  $X = X_1 + \dots + X_{63}$ , where each  $X_i \sim \text{Poi}(11)$  has  $\mathbb{E}[X_i] = \text{Var}(X_i) = \lambda = 11$  from the chart. By the CLT,  $X \approx \mathcal{N}(\mu = 63 \cdot 11, \sigma^2 = 63 \cdot 11)$  (sum of the means and sum of the variances). Hence,

$$\begin{aligned}
 \mathbb{P}(X \geq 700) &\approx \mathbb{P}(X \geq 699.5) && \text{[continuity correction]} \\
 &\approx \mathbb{P}(\mathcal{N}(693, 693) \geq 699.5) && \text{[CLT]} \\
 &= \mathbb{P}\left(Z \geq \frac{699.5 - 693}{\sqrt{693}}\right) && \text{[standardize]} \\
 &= 1 - \Phi(0.2469) \\
 &= 1 - 0.598 \\
 &= 0.402
 \end{aligned}$$

Note that you could compute this exactly as well since you know the sum of iid Poissons is Poisson. In fact,  $X \sim \text{Poi}(693)$  (the average rate in 63 days is 693 per 63 days), and you could do a sum which would be very annoying.

2. Suppose I have a flashlight which requires one battery to operate, and I have 18 identical batteries. I want to go camping for a week ( $24 \times 7 = 168$ ) hours. If the lifetime of a single battery is  $\text{Exp}(0.1)$ ,

what's the probability my flashlight can operate for the entirety of my trip?

**Solution:** The total lifetime of the battery is  $X = X_1 + \dots + X_{18}$  where each  $X_i \sim \text{Exp}(0.1)$  has  $\mathbb{E}[X_i] = \frac{1}{0.1} = 10$  and  $\text{Var}(X_i) = \frac{1}{0.1^2} = 100$ . Hence,  $\mathbb{E}[X] = 180$  and  $\text{Var}(X) = 1800$  by linearity of expectation and since variance adds for independent rvs. In fact,  $X \sim \text{Gamma}(r = 18, \lambda = 0.1)$ , but we don't have a closed-form for its CDF. By the CLT,  $X \approx \mathcal{N}(\mu = 180, \sigma^2 = 1800)$ , so

$$\begin{aligned}
 \mathbb{P}(X \geq 168) &\approx \mathbb{P}(\mathcal{N}(180, 1800) \geq 168) && [\text{CLT}] \\
 &= \mathbb{P}\left(Z \geq \frac{168 - 180}{\sqrt{1800}}\right) && [\text{standardize}] \\
 &= 1 - \Phi(-0.28284) \\
 &= \Phi(0.28284) && [\text{symmetry of Normal}] \\
 &= 0.611
 \end{aligned}$$

Note that we don't use the continuity correction here because the RV's we are summing are already continuous RVs.

## Chapter 5: Multiple Random Variables

### 5.8: The Multinomial Distribution

As you've seen, the Binomial distribution is extremely commonly used, and probably the most important discrete distribution. The Normal distribution is certainly the most important continuous distribution. In this section, we'll see how to generalize the Binomial, and in the next, the Normal.

Why do we need to generalize the Binomial distribution? Sometimes, we don't just have two outcomes (success and failure), but we have  $r > 2$  outcomes. In this case, we need to maintain counts of how many times each of the  $r$  outcomes appeared. A single random variable is no longer sufficient; we need a vector of counts!

Actually, the example problems at the end could have been solved in Chapter 1. We will just formalize this situation so that we can use it later!

#### 5.8.1 Random Vectors (RVTRs) and Covariance Matrices

We will first introduce the concept of a random vector, which is just a collection of random variables stacked on top of each other.

##### Definition 5.8.1: Random Vectors

Let  $X_1, \dots, X_n$  be arbitrary random variables, and stack them into a vector like such:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

We call  $\mathbf{X}$  an  $n$ -dimensional **random vector (rvtr)**.

We define the expectation of a random vector just as we would hope, coordinate-wise:

$$\mathbb{E}[\mathbf{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}$$

What about the variance? We cannot just say or compute a single scalar  $\text{Var}(\mathbf{X})$  because what does that mean for a random vector? Actually, we need to define an  $n \times n$  covariance matrix, which stores all pairwise covariances. It is often denoted in one of three ways:  $\Sigma = \text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X})$ .

##### Definition 5.8.2: Covariance Matrices

The **covariance matrix** of a random vector  $\mathbf{X} \in \mathbb{R}^n$  with  $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$  is the matrix denoted  $\Sigma =$

$\text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X})$  whose entries  $\Sigma_{ij} = \text{Cov}(X_i, X_j)$ . The formula for this is:

$$\Sigma = \text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X}) = \mathbb{E}[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T] = \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \boldsymbol{\mu}\boldsymbol{\mu}^T$$

$$\begin{aligned} &= \begin{bmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) & \dots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) & \dots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \dots & \text{Cov}(X_n, X_n) \end{bmatrix} \\ &= \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \dots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \dots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \dots & \text{Var}(X_n) \end{bmatrix} \end{aligned}$$

Notice that the covariance matrix is **symmetric** ( $\Sigma_{ij} = \Sigma_{ji}$ ), and contains variances along the diagonal.

Note: If you know a bit of linear algebra, you might like to know that covariance matrices are always symmetric **positive semi-definite**.

We will not be doing any linear algebra in this class - think of it as just a place to store all the pairwise covariances. Now let us look at an example of a covariance matrix.

### Example(s)

If  $X_1, X_2, \dots, X_n$  are iid with mean  $\mu$  and variance  $\sigma^2$ , then find the mean vector and covariance matrix of the random vector  $\mathbf{X} = (X_1, \dots, X_n)$ .

*Solution* The mean vector is:

$$\mathbb{E}[\mathbf{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} = \mu \mathbf{1}_n$$

where  $\mathbf{1}_n$  denotes the  $n$ -dimensional vector of all 1's. The covariance matrix is (since the diagonal is just the individual variances  $\sigma^2$  and the off-diagonals ( $i \neq j$ ) are all  $\text{Cov}(X_i, X_j) = 0$  due to independence)

$$\Sigma = \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix} = \sigma^2 I_n$$

where  $I_n$  denotes the  $n \times n$  identity matrix. □

Theorem 5.8.25: Properties of Expectation and Variance Hold for RVTRs

An important theorem is that properties of expectation and variance still hold for RVTRs.

Let  $\mathbf{X}$  be an  $n$ -dimensional RVTR,  $A \in \mathbb{R}^{n \times n}$  be a constant matrix,  $\mathbf{b} \in \mathbb{R}^n$  be a constant vector. Then:

$$\mathbb{E}[A\mathbf{X} + \mathbf{b}] = A\mathbb{E}[\mathbf{X}] + \mathbf{b}$$

$$\text{Var}(A\mathbf{X} + \mathbf{b}) = A\text{Var}(\mathbf{X})A^T$$

Since we aren't expecting any linear algebra background, we won't prove this.

## 5.8.2 The Multinomial Distribution

Suppose we have scenario where there are  $r = 3$  outcomes, with probabilities  $p_1, p_2, p_3$  respectively, such that  $p_1 + p_2 + p_3 = 1$ . Suppose we have  $n = 7$  independent trials, and let  $\mathbf{Y} = (Y_1, Y_2, Y_3)$  be the rvtr of counts of each outcome. Suppose we define each  $X_i$  as a one-hot vector (exactly one 1, and the rest 0) as below, so that  $\mathbf{Y} = \sum_{i=1}^n X_i$  (this is exactly like how adding indicators/Bernoulli's gives us a Binomial):

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$
OUTCOME 1	1	0	0	0	0	0	1
OUTCOME 2	0	0	0	1	0	0	0
OUTCOME 3	0	1	1	0	1	1	0

$\Sigma$

1
2
1
4

Now, what is the probability of this outcome (two of outcome 1, one of outcome 2, and four of outcome 3) - that is,  $(Y_1 = 2, Y_2 = 1, Y_3 = 4)$ ? We get the following:

$$p_{Y_1, Y_2, Y_3}(2, 1, 4) = \frac{7!}{2!1!4!} \cdot p_1^2 \cdot p_2^1 \cdot p_3^4 \quad [\text{recall from counting}]$$

$$= \binom{7}{2, 1, 4} \cdot p_1^2 \cdot p_2^1 \cdot p_3^4$$

This describes the joint distribution of the random vector  $\mathbf{Y} = (Y_1, Y_2, Y_3)$ , and its PMF should remind of you of the binomial PMF. We just count the number of ways  $\binom{7}{2, 1, 4}$  to get these counts (multinomial coefficient), and make sure we get each outcome that many times  $p_1^2 p_2^1 p_3^4$ .

Now let us define the Multinomial Distribution more generally.

### Definition 5.8.3: The Multinomial Distribution

Suppose there are  $r$  outcomes, with probabilities  $\mathbf{p} = (p_1, p_2, \dots, p_r)$  respectively, such that  $\sum_{i=1}^r p_i = 1$ . Suppose we have  $n$  independent trials, and let  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_r)$  be the rvtr of counts of each outcome. Then, we say:

$$\mathbf{Y} \sim \text{Mult}_r(n, \mathbf{p})$$

The joint PMF of  $\mathbf{Y}$  is:

$$p_{Y_1, \dots, Y_r}(k_1, \dots, k_r) = \binom{n}{k_1, \dots, k_r} \prod_{i=1}^r p_i^{k_i}, \quad k_1, \dots, k_r \geq 0 \text{ and } \sum_{i=1}^r k_i = n$$

Notice that each  $Y_i$  is marginally  $\text{Bin}(n, p_i)$ . Hence,  $\mathbb{E}[Y_i] = np_i$  and  $\text{Var}(Y_i) = np_i(1 - p_i)$ . Then, we can specify the entire mean vector  $\mathbb{E}[\mathbf{Y}]$  and covariance matrix:

$$\mathbb{E}[\mathbf{Y}] = n\mathbf{p} = \begin{bmatrix} np_1 \\ \vdots \\ np_r \end{bmatrix} \quad \text{Var}(Y_i) = np_i(1 - p_i) \quad \text{Cov}(Y_i, Y_j) = -np_i p_j \quad (\text{for } i \neq j)$$

Notice the covariance is negative, which makes sense because as the number of occurrences of  $Y_i$  increases, the number of occurrences of  $Y_j$  should decrease since they can not occur simultaneously.

*Proof of Multinomial Covariance.* Recall that marginally,  $X_i$  and  $X_j$  are binomial random variables; let's decompose them into their Bernoulli trials. We'll use different dummy indices as we're dealing with covariances.

Let  $X_{ik}$  for  $k = 1, \dots, n$  be indicator/Bernoulli rvs of whether the  $k^{th}$  trial resulted in outcome  $i$ , so that  $X_i = \sum_{k=1}^n X_{ik}$

Similarly, let  $X_{j\ell}$  for  $\ell = 1, \dots, n$  be indicators of whether the  $\ell^{th}$  trial resulted in outcome  $j$ , so that  $X_k = \sum_{\ell=1}^n X_{j\ell}$ .

Before we begin, we should argue that  $\text{Cov}(X_{ik}, X_{j\ell}) = 0$  when  $k \neq \ell$  since  $k$  and  $\ell$  are different trials and are independent.

Furthermore,  $\mathbb{E}[X_{ik} X_{jk}] = 0$  since it's not possible that both outcome  $i$  and  $j$  occur at trial  $k$ .

$$\begin{aligned} \text{Cov}(X_i, X_j) &= \text{Cov}\left(\sum_{k=1}^n X_{ik}, \sum_{\ell=1}^n X_{j\ell}\right) && \text{[indicators]} \\ &= \sum_{k=1}^n \sum_{\ell=1}^n \text{Cov}(X_{ik}, X_{j\ell}) && \text{[covariance works like FOIL]} \\ &= \sum_{k=1}^n \text{Cov}(X_{ik}, X_{jk}) && \text{[independent trials, cross terms are 0]} \\ &= \sum_{k=1}^n \mathbb{E}[X_{ik} X_{jk}] - \mathbb{E}[X_{ik}] \mathbb{E}[X_{jk}] && \text{[def of covariance]} \\ &= \sum_{k=1}^n -p_i p_j && \text{[first expectation is 0]} \\ &= -np_i p_j \end{aligned}$$

Note that in the third line we dropped one of the sums because the indicators across different trials  $k, \ell$  are independent (zero covariance). Hence, we just need to sum when  $k = \ell$ .  $\square$

There is an example of the Multinomial distribution at the end of the section!

### 5.8.3 The Multivariate Hypergeometric (MVHG) Distribution

Suppose there are  $r = 3$  political parties (Green, Democratic, Republican). The senate consists of  $N = 100$  senators:  $K_1 = 45$  Green party members,  $K_2 = 20$  Democrats, and  $K_3 = 35$  Republicans.

We want to choose a committee of  $n = 10$  senators.

Let  $\mathbf{Y} = (Y_1, Y_2, Y_3)$  be the number of each party's members in the committee (G, D, R in that order). What is the probability we get 1 Green party member, 6 Democrats, and 3 Republicans? It turns out is just the following:

$$p_{Y_1, Y_2, Y_3}(1, 6, 3) = \frac{\binom{45}{1} \binom{20}{6} \binom{35}{3}}{\binom{100}{10}}$$

This is very similar to the univariate Hypergeometric distribution! For the denominator, there are  $\binom{100}{10}$  ways to choose 10 senators. For the numerator, we need 1 from the 45 Green party members, 6 from the 20 Democrats, and 3 from the 35 Republicans.

Once again, let us define the MVHG Distribution more generally.

#### Definition 5.8.4: The Multivariate Hypergeometric Distribution

Suppose there are  $r$  different colors of balls in a bag, having  $\mathbf{K} = (K_1, \dots, K_r)$  balls of each color,  $1 \leq i \leq r$ . Let  $N = \sum_{i=1}^r K_i$  be the total number of balls in the bag, and suppose we draw  $n$  without replacement. Let  $\mathbf{Y} = (Y_1, \dots, Y_r)$  be the rvtr such that  $Y_i$  is the number of balls of color  $i$  we drew. We write that:

$$\mathbf{Y} \sim \text{MVHG}_r(N, \mathbf{K}, n)$$

The joint PMF of  $\mathbf{Y}$  is:

$$p_{Y_1, \dots, Y_r}(k_1, \dots, k_r) = \frac{\prod_{i=1}^r \binom{K_i}{k_i}}{\binom{N}{n}}, \quad 0 \leq k_i \leq K_i \text{ for all } 1 \leq i \leq r \text{ and } \sum_{i=1}^r k_i = n$$

Notice that each  $Y_i$  is marginally HypGeo( $N, K_i, n$ ), so  $\mathbb{E}[Y_i] = n \frac{K_i}{N}$  and

$\text{Var}(Y_i) = n \frac{K_i}{N} \cdot \frac{N - K_i}{N} \cdot \frac{N - n}{N - 1}$ . Then, we can specify the entire mean vector  $\mathbb{E}[\mathbf{Y}]$  and covariance matrix:

$$\mathbb{E}[\mathbf{Y}] = n \frac{\mathbf{K}}{N} = \begin{bmatrix} n \frac{K_1}{N} \\ \vdots \\ n \frac{K_r}{N} \end{bmatrix} \quad \text{Var}(Y_i) = n \frac{K_i}{N} \cdot \frac{N - K_i}{N} \cdot \frac{N - n}{N - 1} \quad \text{Cov}(Y_i, Y_j) = -n \frac{K_i}{N} \frac{K_j}{N} \cdot \frac{N - n}{N - 1}$$

*Proof of Hypergeometric Variance.* We'll prove the variance of a univariate Hypergeometric finally (the variance of  $Y_i$ ), but leave the covariance matrix to you (can approach it similarly to the multinomial covariance matrix).

Let  $X \sim \text{HypGeo}(N, K, n)$  (univariate hypergeometric). For  $i = 1, \dots, n$ , let  $X_i$  be the indicator of whether or not we got a success on trial  $i$  (not independent indicators). Then,  $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{K}{N}$  for every trial  $i$ , so  $\mathbb{E}[X] = n \frac{K}{N}$  by linearity of expectation.

First, we have that since  $X_i \sim \text{Ber}\left(\frac{K}{N}\right)$ :

$$\text{Var}(X_i) = p(1-p) = \frac{K}{N} \left(1 - \frac{K}{N}\right)$$

Second, for  $i \neq j$ ,  $\mathbb{E}[X_i X_j] = \mathbb{P}(X_i X_j = 1) = \mathbb{P}(X_i = 1) \mathbb{P}(X_j = 1 \mid X_i = 1) = \frac{K}{N} \cdot \frac{K-1}{N-1}$ , so

$$\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] = \frac{K}{N} \cdot \frac{K-1}{N-1} - \frac{K^2}{N^2}$$

Finally,

$$\begin{aligned} \text{Var}(X) &= \text{Var}\left(\sum_{i=1}^n X_i\right) && [\text{def of } X] \\ &= \text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n X_j\right) && [\text{covariance with self is variance}] \\ &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) && [\text{bilinearity of covariance}] \\ &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) && [\text{split diagonal}] \\ &= n \frac{K}{N} \left(1 - \frac{K}{N}\right) + 2 \binom{n}{2} \left(\frac{K}{N} \cdot \frac{K-1}{N-1} - \frac{K^2}{N^2}\right) && [\text{plug in}] \\ &= n \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1} && [\text{algebra}] \end{aligned}$$

□

#### 5.8.4 Exercises

These won't be very interesting since this could've been done in chapter 1 and 2!

1. Suppose you are fishing in a pond with 3 red fish, 4 green fish, and 5 blue fish.
  - (a) You use a net to scoop up 6 of them. What is the probability you scooped up 2 of each?
  - (b) You "catch and release" until you caught 6 fish (catch 1, throw it back, catch another, throw it back, etc.). What is the probability you caught 2 of each?

**Solution:**

- (a) Let  $(X_1, X_2, X_3)$  be how many red, green, and blue fish I caught respectively. Then,  $X \sim \text{MVHG}_3(N = 12, \mathbf{K} = (3, 4, 5), n = 6)$ , and

$$\mathbb{P}(X_1 = 2, X_2 = 2, X_3 = 2) = \frac{\binom{3}{2} \binom{4}{2} \binom{5}{2}}{\binom{12}{6}}$$

- (b) Let  $(X_1, X_2, X_3)$  be how many red, green, and blue fish I caught respectively. Then,  $X \sim \text{Mult}_3(n = 6, \mathbf{p} = (3/12, 4/12, 5/12))$ , and

$$\mathbb{P}(X_1 = 2, X_2 = 2, X_3 = 2) = \binom{6}{2, 2, 2} \left(\frac{3}{12}\right)^2 \left(\frac{4}{12}\right)^2 \left(\frac{5}{12}\right)^2$$

## Chapter 5: Multiple Random Variables

### 5.9: The Multivariate Normal Distribution

In this section, we will generalize the Normal random variable, the most important continuous distribution! We were able to find the joint PMF for the Multinomial random vector using a counting argument, but how can we find the Multivariate Normal density function? We'll start with the simplest case, and work from there.

#### 5.9.1 The Special Case of Independent Normals

Suppose  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$  and  $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$  are independent normal RVs.

Then by independence, their joint PDF is (recall that  $\exp(z)$  is just another way to write  $e^z$ ):

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_X^2}(x - \mu_X)^2\right) \cdot \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_Y^2}(y - \mu_Y)^2\right), \quad x, y \in \mathbb{R}$$

The mean vector  $\mu$  is given by:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

And the covariance matrix  $\Sigma$  is given by:

$$\Sigma = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$$

Then, we say that  $(X, Y)$  has a bivariate Normal distribution, which we will denote:

$$(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$$

This is nice and all, if we have two independent Normals. But what if they aren't independent?

#### 5.9.2 The Bivariate Normal Distribution

We'll now see how we can construct the joint PDF of two (possibly dependent) Normal RVs, to get the Bivariate Normal PDF.

##### Definition 5.9.1: The Bivariate Normal Distribution

Let  $Z_1, Z_2 \sim \mathcal{N}(0, 1)$  be iid standard Normals, and  $\mu_X, \mu_Y, \sigma_X^2 > 0, \sigma_Y^2 > 0$  and  $-1 \leq \rho \leq 1$  be scalar parameters. We construct from these two RVs a random vector  $(X, Y)$  by the transformations:

1. We construct  $X$  by taking  $Z_1$ , multiplying it by  $\sigma_X$ , and adding  $\mu_X$ :

$$X = \sigma_X Z_1 + \mu_X$$

2. We construct  $Y$  from both  $Z_1$  and  $Z_2$ , as shown below:

$$Y = \sigma_Y(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) + \mu_Y$$

From this transformation, we get that marginally (show this by computing the mean and variance of  $X, Y$  and closure properties of Normal RVs),

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \quad Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$$

Additionally,

$$\rho(X, Y) = \rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \Rightarrow \text{Cov}(X, Y) = \rho \sigma_X \sigma_Y$$

That is, for the the RVTR  $(X, Y)$ ,

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{bmatrix}$$

By using the multivariate change-of-variables formula from 4.4, we can turn the "simple" product of standard normal PDFs into the PDF of the bivariate Normal:

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z}{2(1-\rho^2)}\right), \quad x, y \in \mathbb{R}$$

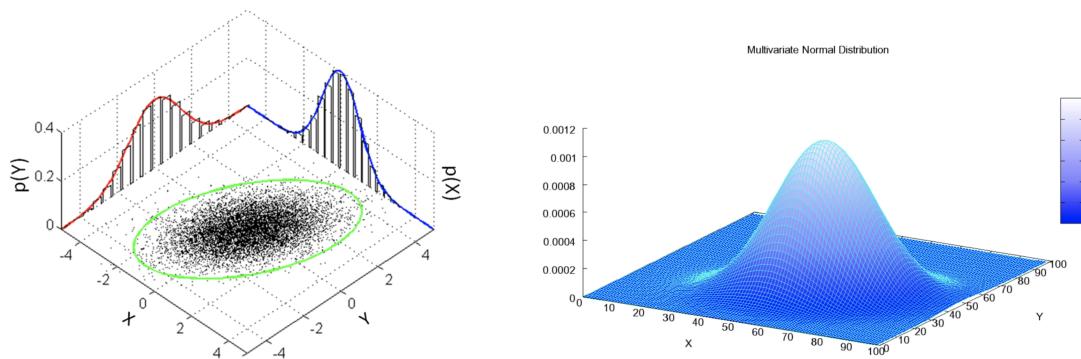
where

$$z = \frac{(x - \mu_X)^2}{\sigma_X^2} - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X \sigma_Y} + \frac{(y - \mu_Y)^2}{\sigma_Y^2}$$

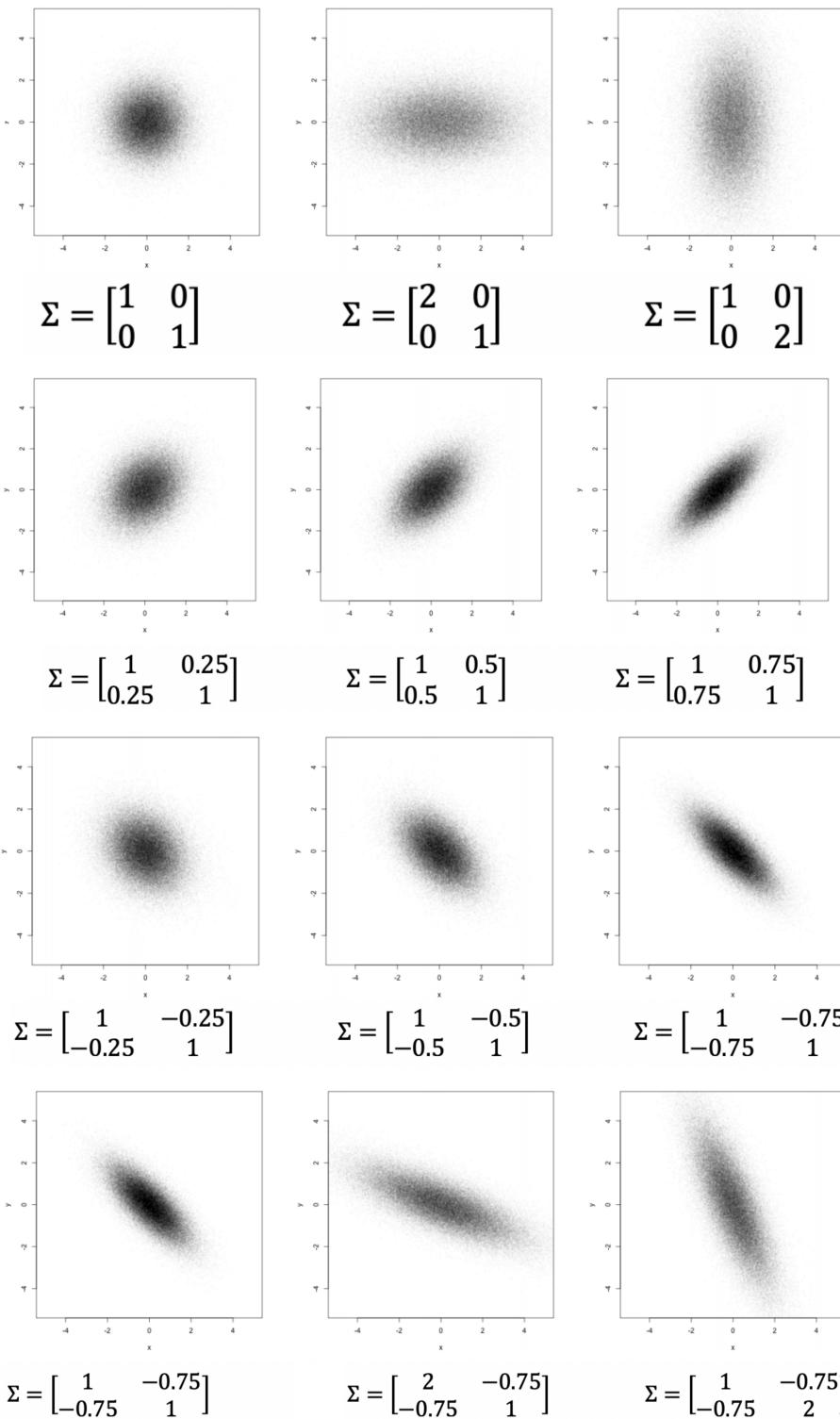
Finally, we write:

$$(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$$

The 2 visualizations below show the density of a bivariate Normal distribution.



Now let's take a look at the effect of different covariance matrices on the distribution (pictures from [Duke](#)):



### 5.9.3 The Multivariate Normal Distribution

#### Definition 5.9.2: The Multivariate Normal Distribution

A random vector  $\mathbf{X} = (X_1, \dots, X_n)$  has a multivariate Normal distribution with mean vector  $\boldsymbol{\mu} \in \mathbb{R}^n$  and (symmetric and positive-definite) covariance matrix  $\Sigma \in \mathbb{R}^{n \times n}$ , written  $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ , if it has the following joint PDF:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right), \quad \mathbf{x} \in \mathbb{R}^n$$

While this PDF may look intimidating, if we recall the PDF of a univariate Normal  $W \sim \mathcal{N}(\mu, \sigma^2)$ :

$$f_W(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(w - \mu)^2\right)$$

We can note that the two formulae are quite similar; we simply extend scalars to vectors and matrices!

Additionally, let us recall that for any RVs  $X$  and  $Y$ :

$$X \perp Y \rightarrow \text{Cov}(X, Y) = 0$$

If  $\mathbf{X} = (X_1, \dots, X_n)$  is Multivariate Normal, the converse also holds:

$$\text{Cov}(X_i, X_j) = 0 \rightarrow X_i \perp X_j$$

Unfortunately, we cannot do example problems as they would require a deeper knowledge of linear algebra, which we do not assume.

## Chapter 5: Multiple Random Variables

### 5.10: Order Statistics

We've talked a lot about the distribution of the sum of random variables, but what about the maximum, minimum, or median? For example, if there are 4 possible buses you could take, and the time until each arrives is independent with an exponential distribution, what is the expected time until the *first* one arrives? Mathematically, this would be  $\mathbb{E}[\min\{X_1, X_2, X_3, X_4\}]$  if the arrival times were  $X_1, X_2, X_3, X_4$ .

In this section, we'll figure out how to find out the density function (and hence expectation/variance) of the minimum, maximum, median, and more!

#### 5.10.1 Order Statistics

We'll first formally define order statistics.

##### Definition 5.10.1: Order Statistics

Suppose  $Y_1, \dots, Y_n$  are iid *continuous* random variables with common PDF  $f_Y$  and common CDF  $F_Y$ . We define  $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$  to be the *sorted* version of this sample. That is,

$$Y_{\min} \equiv Y_{(1)} < Y_{(2)} < \dots < Y_{(n)} \equiv Y_{\max}$$

$Y_{(1)}$  is the smallest value (the minimum), and  $Y_{(n)}$  is the largest value (the maximum), and since they are so commonly used, they have special names  $Y_{\min}$  and  $Y_{\max}$  respectively.

Notice that we can't have equality because with continuous random variables, the probability that any two are equal is 0. So, we don't have to worry about any of these random variables being "less than or equal to" another.

Notice that each  $Y_{(i)}$  is a random variable as well! We call  $Y_{(i)}$  the *i-th order statistic*, i.e. the *i*-th smallest in a sample of size *n*. For example, if we had  $n = 9$  samples,  $Y_{(5)}$  would be the median value. We are interested in finding the distribution of each order statistic, and properties such as expectation and variance as well.

Why are order statistics important? Usually, we take the min, max or median of a set of random variables and do computations with them - so, it would be useful if we had a general formula for the PDF and CDF of the min or max.

We start with an example to find the distribution of  $Y_{(n)} = Y_{\max}$ , the largest order statistic. We'll then extend this to any of the order statistics (not just the max). Again, this means, if we were to repeatedly take the maximum of  $n$  iid RVs, what would the samples look like?

##### Example(s)

Let  $Y_1, Y_2, \dots, Y_n$  be iid continuous random variables with the same CDF  $F_Y$  and PDF  $f_Y$ . What is the distribution of  $Y_{(n)} = Y_{\max} = \max\{Y_1, Y_2, \dots, Y_n\}$  the largest order statistic?

*Solution*

We'll employ our typical strategy and work with probabilities instead of densities, so we'll start with the CDF:

$$\begin{aligned}
 F_{Y_{\max}}(y) &= \mathbb{P}(Y_{\max} \leq y) && [\text{def of CDF}] \\
 &= \mathbb{P}\left(\bigcap_{i=1}^n \{Y_i \leq y\}\right) && [\text{max is } \leq y \text{ if and only if all are}] \\
 &= \prod_{i=1}^n \mathbb{P}(Y_i \leq y) && [\text{independence}] \\
 &= \prod_{i=1}^n F_Y(y) && [\text{def of CDF}] \\
 &= F_Y^n(y) && [\text{identically distributed, all have same CDF}]
 \end{aligned}$$

We can differentiate the CDF to find the PDF:

$$\begin{aligned}
 f_{Y_{\max}}(y) &= F'_{Y_{\max}}(y) \\
 &= \frac{d}{dy}(F_Y^n(y)) \\
 &= nF_Y^{n-1}(y)f_Y(y) && [\text{chain rule of calculus and } \frac{d}{dx}x^n = nx^{n-1}]
 \end{aligned}$$

Let's take a step back and see what we just did here. We just computed the density function of the maximum of  $n$  iid random variables, denoted  $Y_{\max} = Y_{(n)}$ . We now need to find the density of any arbitrary ranked  $Y_{(i)}$ .  $\square$

Theorem 5.10.26: Order Statistics

Suppose  $Y_1, \dots, Y_n$  are iid *continuous* random variables with common PDF  $f_Y$  and common CDF  $F_Y$ . We define  $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$  to be the *sorted* version of this sample. That is,

$$Y_{\min} \equiv Y_{(1)} < Y_{(2)} < \dots < Y_{(n)} \equiv Y_{\max}$$

. The density function of  $Y_{(i)}$  is

$$f_{Y_{(i)}}(y) = \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot f_Y(y), y \in \Omega_Y$$

Now, using the same intuition as before, we'll use an informal argument to find the density of a general  $Y_{(i)}$ ,  $f_{Y_{(i)}}(y)$ . For example, this might help find the distribution of the minimum  $f_{Y_{(1)}}$  or the median.

*Proof of Density of Order Statistics.* The formula above may remind you of a multinomial distribution, and you would be correct! Let's consider what it means for  $Y_{(i)} = y$  (the  $i$ -th smallest value in the sample of  $n$  to equal a particular value  $y$ ).

- One of the values needs to be exactly  $y$

- $i - 1$  of the values need to be smaller than  $y$  (this happens for each with probability  $F_Y(y)$ )
- the other  $n - i$  values need to be greater than  $y$  (this happens for each with probability  $1 - F_Y(y)$ )

Now, we have 3 distinct types of objects: 1 that is exactly  $y$ ,  $i - 1$  which are less than  $y$  and  $n - i$  which are greater. Using multinomial coefficients and the above, we see that

$$f_{Y_{(i)}}(y) = \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot f_Y(y)$$

Note that this isn't a probability; it is a density, so there is something flawed with how we approached this problem. For a more rigorous approach, we just have to make a slight modification, but use the same idea.

*Re-Proof (Rigorous)* This time, we'll find  $\mathbb{P}(y - \frac{\varepsilon}{2} \leq Y_{(i)} \leq y + \frac{\varepsilon}{2})$  and use the fact that this is approximately equal to  $\varepsilon f_{Y_{(i)}}(y)$  for small  $\varepsilon > 0$  (Riemann integral (rectangle) approximation from 4.1).

We have very similar cases:

- One of the values needs to be between  $y - \frac{\varepsilon}{2}$  and  $y + \frac{\varepsilon}{2}$  (this happens with probability approximately  $\varepsilon f_Y(y)$ , again by Riemann approximation).
- $i - 1$  of the values need to be smaller than  $y - \frac{\varepsilon}{2}$  (this happens for each with probability  $F_Y(y - \frac{\varepsilon}{2})$ )
- the other  $n - i$  values need to be greater than  $y + \frac{\varepsilon}{2}$  (this happens for each with probability  $1 - F_Y(y + \frac{\varepsilon}{2})$ )

Now these are actually probabilities (not densities), so we get

$$\mathbb{P}\left(y - \frac{\varepsilon}{2} \leq Y_{(i)} \leq y + \frac{\varepsilon}{2}\right) \approx \varepsilon f_{Y_{(i)}}(y) = \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot (\varepsilon f_Y(y))$$

Dividing both sides by  $\varepsilon > 0$  gives the same result as earlier!

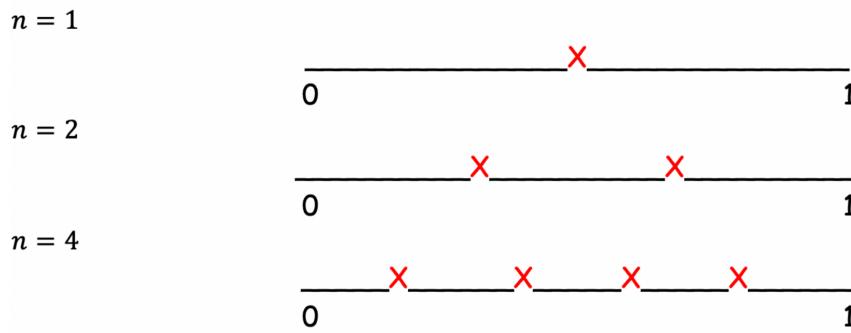
□

Let's verify this formula with our maximum that we derived earlier by plugging in  $n$  for  $i$ :

$$f_{Y_{\max}}(y) = f_{Y_{(n)}}(y) = \binom{n}{n-1, 1, 0} \cdot [F_Y(y)]^{n-1} \cdot [1 - F_Y(y)]^0 \cdot f_Y(y) = nF_Y^{n-1}(y)f_Y(y)$$

### Example(s)

If  $Y_1, \dots, Y_n$  are iid  $\text{Unif}(0, 1)$ , where do we "expect" the points to end up? That is, find  $\mathbb{E}[Y_{(i)}]$  for any  $i$ . You may find this picture with different values of  $n$  useful for intuition.



*Solution*

Intuitively, from the picture, if  $n = 1$ , we expect the single point to end up at  $\frac{1}{2}$ . If  $n = 2$ , we expect the two points to end up at  $\frac{1}{3}$  and  $\frac{2}{3}$ . If  $n = 4$ , we expect the four points to end up at  $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$  and  $\frac{4}{5}$ .

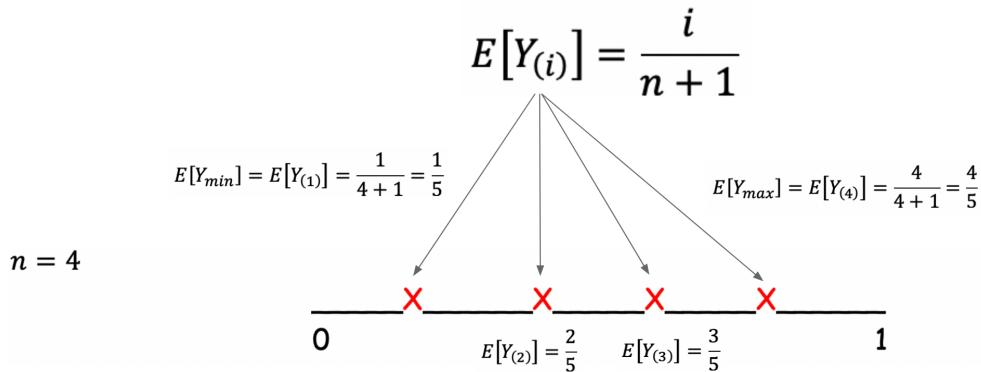
Let's prove this formally. Recall, if  $Y \sim \text{Unif}(0, 1)$  (continuous), then  $f_Y(y) = 1$  for  $y \in [0, 1]$  and  $F_Y(y) = y$  for  $y \in [0, 1]$ . By the order statistics formula,

$$\begin{aligned} f_{Y_{(i)}}(y) &= \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot f_Y(y) \\ &= \binom{n}{i-1, 1, n-i} \cdot [y]^{i-1} \cdot [1 - y]^{n-i} \cdot 1 \end{aligned}$$

Using the PDF, we find the expectation:

$$\mathbb{E}[Y_{(i)}] = \int_0^1 y \binom{n}{i-1, 1, n-i} \cdot [y]^{i-1} \cdot [1 - y]^{n-i} dy = \frac{i}{n+1}$$

Here is a picture which may help you figure out what the formulae you just computed mean!



Now let's do our bus example from earlier.

□

## Example(s)

At 5pm each day, four buses make their way to the HUB bus stop. Each bus would be acceptable to take you home. The time in hours (after 5pm) that each arrives at the stop is independent with  $Y_1, Y_2, Y_3, Y_4 \sim \text{Exp}(\lambda = 6)$  (on average, it takes 1/6 of an hour (10 minutes) for each bus to arrive).

1. On Mondays, you want to get home ASAP, so you arrive at the bus stop at 5pm sharp. What is the expected time until the *first* one arrives?
2. On Tuesdays, you have a lab meeting that runs until 5:15 and are worried you may not catch any bus. What is the probability you miss all the buses?

*Solution* The first question asks about the smallest order statistic  $Y_{(1)} = Y_{\min}$  since we care about the first bus. The second question asks about the largest order statistic  $Y_{(4)}$  since we care about the last bus. Let's compute the general formula for order statistics first so we can apply it to both parts of the problem.

Recall, if  $Y \sim \text{Exp}(\lambda = 6)$  (continuous), then  $f_Y(y) = 6e^{-6y}$  for  $y \in [0, \infty)$  and  $F_Y(y) = 1 - e^{-6y}$  for  $y \in [0, \infty)$ . By the order statistics formula,

$$\begin{aligned} f_{Y_{(i)}}(y) &= \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot f_Y(y) \\ &= \binom{n}{i-1, 1, n-i} \cdot [1 - e^{-6y}]^{i-1} \cdot [e^{-6y}]^{n-i} \cdot 6e^{-6y} \end{aligned}$$

1. For the first part, we want  $\mathbb{E}[Y_{(1)}]$ , so we plug in  $i = 1$  (and  $n = 4$ ) to the above formula to get:

$$f_{Y_{(1)}}(y) = \binom{4}{1-1, 1, 4-1} \cdot [1 - e^{-6y}]^{1-1} \cdot [e^{-6y}]^{4-1} \cdot 6e^{-6y} = 4[e^{-18y}]6e^{-6y} = 24e^{-24y}$$

Now we can use the PDF to find the expectation normally. However, notice that the PDF is that of an  $\text{Exp}(\lambda = 24)$  distribution, so it has expectation  $1/24$ . That is, the expected time until the first bus arrives is  $1/24$  an hour, or 2.5 minutes.

Let's talk about something amazing here. We found that  $\min\{Y_1, Y_2, Y_3, Y_4\} \sim \text{Exp}(\lambda = 4 \cdot 6)$ ; that the minimum of exponentials is distributed as an exponential with the sum of the rates! Why might this be true? If we have  $Y_1, Y_2, Y_3, Y_4 \sim \text{Exp}(6)$ , that means on average, 6 buses of each type arrive each hour, for a total of 24. That just means we can model our waiting time in this regime with an average of 24 buses per hour, to get that the time until the first bus has an  $\text{Exp}(6 + 6 + 6 + 6)$  distribution!

2. For finding the maximum, we just plug in  $i = n = 4$  (and  $n = 4$ ), to get

$$f_{Y_{(4)}}(y) = \binom{4}{4-1, 1, 4-4} \cdot [1 - e^{-6y}]^{4-1} \cdot [e^{-6y}]^{4-4} \cdot 6e^{-6y} = [1 - e^{-6y}]^3 6e^{-6y}$$

Unfortunately, this is as simplified as it gets, and we don't get the nice result that the maximum of exponentials is exponential. To find the desired quantity, we just need to compute the probability the last bus comes before 5:15 (which is 0.25 hours - be careful of units!):

$$\mathbb{P}(Y_{\max} \leq 0.25) = \int_0^{0.25} f_{Y_{\max}}(y) dy = \int_0^{0.25} [1 - e^{-6y}]^3 6e^{-6y} dy$$

□

## Chapter 5: Multiple Random Variables

### 5.11: Proof of the CLT

In this optional section, we'll prove the Central Limit Theorem, one of the most fundamental and amazing results in all of statistics, using MGFs!

#### 5.11.1 Properties of Moment Generating Functions

Let's first recall the properties of MGFs (this is just copied from 5.6):

##### Theorem 5.11.27: Properties and Uniqueness of Moment Generating Functions

For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we will denote  $f^{(n)}(x)$  to be the  $n^{\text{th}}$  derivative of  $f(x)$ . Let  $X, Y$  be independent random variables, and  $a, b \in \mathbb{R}$  be scalars. Then MGFs satisfy the following properties:

1.  $M'_X(0) = \mathbb{E}[X]$ ,  $M''_X(0) = \mathbb{E}[X^2]$ , and in general  $M_X^{(n)} = \mathbb{E}[X^n]$ . This is why we call  $M_X$  a *moment generating* function, as we can use it to generate the moments of  $X$ .
2.  $M_{aX+b}(t) = e^{tb} M_X(at)$ .
3. If  $X \perp Y$ , then  $M_{X+Y}(t) = M_X(t)M_Y(t)$ .
4. **(Uniqueness)** The following are equivalent:
  - (a)  $X$  and  $Y$  have the same distribution.
  - (b)  $f_X(z) = f_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (c)  $F_X(z) = F_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (d) There is an  $\varepsilon > 0$  such that  $M_X(t) = M_Y(t)$  for all  $t \in (-\varepsilon, \varepsilon)$  (they match on a small interval around  $t = 0$ ).

That is  $M_X$  uniquely identifies a distribution, just like PDFs or CDFs do.

#### 5.11.2 Proof of the Central Limit Theorem (CLT)

Here is a restatement of the CLT from 5.7 that we will prove:

##### Theorem 5.11.28: The Central Limit Theorem (CLT)

Let  $X_1, \dots, X_n$  be a sequence of independent and identically distributed random variables with mean  $\mu$  and (finite) variance  $\sigma^2$ . Then, the standardized sample mean approaches the standard Normal distribution:

$$\text{As } n \rightarrow \infty, \quad \bar{Z}_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \rightarrow \mathcal{N}(0, 1)$$

*Proof of The Central Limit Theorem.* Our strategy will be to compute the MGF of  $\bar{Z}_n$  and exploit properties of the MGF (especially uniqueness) to show that it must have a standard Normal distribution!

Suppose  $\mu = 0$  (without loss of generality), so:

$$\mathbb{E}[X_i^2] = \text{Var}(X_i) + \mathbb{E}[X_i]^2 = \sigma^2$$

Now, let:

$$\bar{Z}_n = \frac{\bar{X}_n}{\sigma/\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n X_i$$

Note there is no typo above: the  $\frac{1}{n}$  from  $\bar{X}_n$  changes the division by  $\sqrt{n}$  to a multiplication.

We will show  $M_{\bar{Z}_n}(t) \rightarrow e^{t^2/2}$  (the standard normal MGF) and hence,  $\bar{Z}_n \rightarrow \mathcal{N}(0, 1)$  by uniqueness of the MGF.

1. First, for an **arbitrary** random variable  $Y$ , since the MGF exists in  $(-\varepsilon, \varepsilon)$  under “most” conditions, we can use the 2<sup>nd</sup> order Taylor series expansion around 0 (quadratic approximation to a function):

$$\begin{aligned} M_Y(s) &\approx M_Y(0) \cdot \frac{s^0}{0!} + M'_Y(0) \cdot \frac{s^1}{1!} + M''_Y(0) \cdot \frac{s^2}{2!} \\ &= \mathbb{E}[Y^0] + \mathbb{E}[Y]s + \mathbb{E}[Y^2] \frac{s^2}{2} && [\text{Since } M_Y^{(n)}(0) = \mathbb{E}[Y^n]] \\ &= 1 + \mathbb{E}[Y]s + \mathbb{E}[Y^2] \frac{s^2}{2} && [\text{Since } Y^0 = 1] \end{aligned}$$

2. Now, let  $M_X$  denote the common MGF of all the  $X_i$ ’s (since they are iid).

$$\begin{aligned} M_{\bar{Z}_n}(t) &= M_{\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n X_i}(t) && [\text{Definition of } \bar{Z}_n] \\ &= M_{\sum_{i=1}^n X_i} \left( \frac{t}{\sigma\sqrt{n}} \right) && [\text{By Property 2 of MGFs above, where } a = \frac{1}{\sigma\sqrt{n}}, b = 0] \\ &= \left[ M_X \left( \frac{t}{\sigma\sqrt{n}} \right) \right]^n && [\text{By Property 3 of MGFs above}] \end{aligned}$$

3. Recall Step 1, and now let  $Y = X$  and  $s = \frac{t}{\sigma\sqrt{n}}$  so we get a Taylor approximation of  $M_X$ . Then:

$$\begin{aligned} M_X \left( \frac{t}{\sigma\sqrt{n}} \right) &\approx 1 + \mathbb{E}[X] \frac{t}{\sigma\sqrt{n}} + \mathbb{E}[X^2] \frac{\left( \frac{t}{\sigma\sqrt{n}} \right)^2}{2} && [\text{Step 1}] \\ &= 1 + 0 + \sigma^2 \frac{t^2}{2\sigma^2 n} && [\text{Since } \mathbb{E}[X] = 0 \text{ and } \mathbb{E}[X^2] = \sigma^2] \\ &= 1 + \frac{t^2/2}{n} \end{aligned}$$

4. Now we combine Steps 2 and 3:

$$\begin{aligned}
 M_{\bar{Z}_n}(t) &= \left[ M_X \left( \frac{t}{\sigma\sqrt{n}} \right) \right]^n && [\text{step 2}] \\
 &\approx \left( 1 + \frac{t^2/2}{n} \right)^n && [\text{step 3}] \\
 &\rightarrow e^{t^2/2} && \left[ \text{Since } \left( 1 + \frac{x}{n} \right)^n \rightarrow e^x \right]
 \end{aligned}$$

Hence,  $\bar{Z}_n$  has the same MGF as that of a standard normal, so must follow that distribution!  $\square$

## Chapter 6: Concentration Inequalities

### 6.1: Markov and Chebyshev Inequalities

When reasoning about some random variable  $X$ , it's not always easy or possible to calculate/know its exact PMF/PDF. We might not know much about  $X$  (maybe just its mean and variance), but we can still provide **concentration inequalities** to get a bound of how likely it is for  $X$  to be far from its mean  $\mu$  (of the form  $\mathbb{P}(|X - \mu| > \alpha)$ ), or how likely for this random variable to be very large (of the form  $\mathbb{P}(X \geq k)$ ).

You might ask when we would only know the mean/variance but not the PMF/PDF? Some of our distributions that we use (like Exponential for bus waiting time), are just modelling assumptions and are probably incorrect. If we measured how long it took for the bus to arrive over many days, we could *estimate* its mean and variance! That is, we have no idea the true distribution of daily bus waiting times but can get good estimates for the mean and variance. We can use these concentration inequalities to bound the probability that we wait too long for a bus knowing just those two quantities and nothing else!

#### 6.1.1 Markov's Inequality

We'll start with our weakest inequality, Markov's inequality. This one only requires us to know the mean, and nothing else! Again, if we didn't know the PMF/PDF of what we cared about, we could use the sample mean as a good estimate for the true mean (by the Law of Large Numbers from 5.7), and our inequality/bound would be pretty accurate still!

This first example will help build intuition for why Markov's inequality is true.

##### Example(s)

The score distribution of an exam is modelled by a random variable  $X$  with range  $\Omega_X = [0, 110]$  (with 10 points for extra credit). Give an upper bound on the proportion of students who score at least 100 when the average is 50? When the average is 25?

*Solution* What would you guess? If the average is  $\mathbb{E}[X] = 50$ , an upper bound on the proportion of students who score at least 100 should be 50% right? If more than 50% of students scored a 100 (or higher), the average would already be 50% since all scores must be nonnegative ( $\geq 0$ ). Mathematically, we just argued that:

$$\mathbb{P}(X \geq 100) \leq \frac{\mathbb{E}[X]}{100} = \frac{50}{100} = \frac{1}{2}$$

This sounds reasonable - if say 70% of the class were to get 100 or higher, the average would already be at least 70%, even if everyone else got a zero. The best bound we can get is 50% - and that requires everyone else to get a zero.

If the average is  $\mathbb{E}[X] = 25$ , an upper bound on the proportion of students who score at least 100 is:

$$\mathbb{P}(X \geq 100) \leq \frac{\mathbb{E}[X]}{100} = \frac{25}{100} = \frac{1}{4}$$

Similarly, if we had more than 30% students get 100 or higher, the average would already be at least 30%, even if everyone else got a zero.  $\square$

That's literally the entirety of the idea for Markov's inequality.

Theorem 6.1.29: Markov's Inequality

Let  $X \geq 0$  be a **non-negative** random variable (discrete or continuous), and let  $k > 0$ . Then:

$$\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}[X]}{k}$$

Equivalently (plugging in  $k\mathbb{E}[X]$  for  $k$  above):

$$\mathbb{P}(X \geq k\mathbb{E}[X]) \leq \frac{1}{k}$$

*Proof of Markov's Inequality.* Below is the proof when  $X$  is continuous. The proof for discrete RVs is similar (just change all the integrals into summations).

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty x f_X(x) dx && [\text{because } X \geq 0] \\ &= \int_0^k x f_X(x) dx + \int_k^\infty x f_X(x) dx && [\text{split integral at some } 0 \leq k \leq \infty] \\ &\geq \int_k^\infty x f_X(x) dx && \left[ \int_0^k x f_X(x) dx \geq 0 \text{ because } k \geq 0, x \geq 0 \text{ and } f_X(x) \geq 0 \right] \\ &\geq \int_k^\infty k f_X(x) dx && [\text{because } x \geq k \text{ in the integral}] \\ &= k \int_k^\infty f_X(x) dx \\ &= k \mathbb{P}(X \geq k) \end{aligned}$$

□

So just knowing that the random variable is non-negative and what its expectation is, we can bound the probability that it is “very large”. We know nothing else about the exam distribution! Note there is no bound we can derive if  $X$  could be negative. Always check that  $X$  is indeed nonnegative before applying this bound!

The following example demonstrates how to use Markov's inequality, and how loose it can be in some cases.

Example(s)

A coin is weighted so that its probability of landing on heads is 20%, independently of other flips. Suppose the coin is flipped 20 times. Use Markov's inequality to bound the probability it lands on heads at least 16 times.

*Solution* We actually do know this distribution; the number of heads is  $X \sim \text{Bin}(n = 20, p = 0.2)$ . Thus,  $\mathbb{E}[X] = np = 20 \cdot 0.2 = 4$ . By Markov's inequality:

$$\mathbb{P}(X \geq 16) \leq \frac{\mathbb{E}[X]}{16} = \frac{4}{16} = \frac{1}{4}$$

Let's compare this to the actual probability that this happens:

$$\mathbb{P}(X \geq 16) = \sum_{k=16}^{20} \binom{20}{k} 0.2^k \cdot 0.8^{20-k} \approx 1.38 \cdot 10^{-8}$$

This is not a good bound, since we only assume to know the expected value. Again, we knew the exact distribution, but chose not to use any of that information (the variance, the PMF, etc.).  $\square$

### Example(s)

Suppose the expected runtime of QuickSort is  $2n \log(n)$  operations/comparisons (we can show this using linearity of expectation with dependent indicator variables). Use Markov's inequality to bound the probability that QuickSort runs for longer than  $20n \log(n)$  time.

*Solution* Let  $X$  be the runtime of QuickSort, with  $\mathbb{E}[X] = 2n \log(n)$ . Then, since  $X$  is non-negative, we can use Markov's inequality:

$$\begin{aligned} \mathbb{P}(X \geq 20n \log(n)) &\leq \frac{\mathbb{E}[X]}{20n \log(n)} && \text{[Markov's inequality]} \\ &= \frac{2n \log(n)}{20n \log(n)} \\ &= \frac{1}{10} \end{aligned}$$

So we know there's at most 10% probability that QuickSort takes this long to run. Again, we can get this bound despite not knowing anything except its expectation!  $\square$

### 6.1.2 Chebyshev's Inequality

Chebyshev's inequality unlike Markov's inequality does not require that the random variable is non-negative. However, it also requires that we know the variance in addition to the mean. The goal of Chebyshev's inequality is to bound the probability that the RV is far from its mean (in either direction). This generally gives a stronger bound than Markov's inequality; if we know the variance of a random variable, we should be able to control how much it deviates from its mean better!

We'll actually prove the Weak Law of Large Numbers as well!

#### Theorem 6.1.30: Chebyshev's Inequality

Let  $X$  be any random variable with expected value  $\mu = \mathbb{E}[X]$  and finite variance  $\text{Var}(X)$ . Then, for any real number  $\alpha > 0$ :

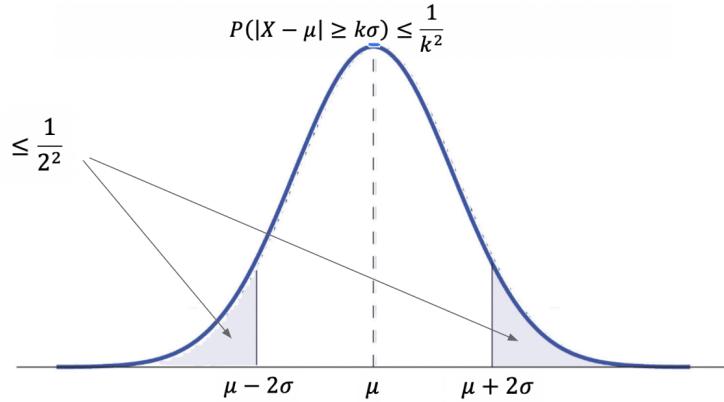
$$\mathbb{P}(|X - \mu| \geq \alpha) \leq \frac{\text{Var}(X)}{\alpha^2}$$

Equivalently (plugging in  $k\sigma$  for  $\alpha$  above, where  $\sigma = \sqrt{\text{Var}(X)}$ ):

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

This is used to bound the probability of being in the *tails*. Here is a picture of Chebyshev's inequality

bounding the probability that a Gaussian  $X \sim \mathcal{N}(\mu, \sigma^2)$  is more than  $k = 2$  standard deviations from its mean:



*Proof of Chebyshev's Inequality.*  $X$  is a random variable, so  $(X - \mathbb{E}[X])^2$  is a **non-negative** random variable. Hence, we can apply Markov's inequality.

$$\begin{aligned}
 \mathbb{P}(|X - \mathbb{E}[X]| \geq \alpha) &= \mathbb{P}((X - \mathbb{E}[X])^2 \geq \alpha^2) && \text{[square both sides]} \\
 &\leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{\alpha^2} && \text{[Markov's inequality]} \\
 &= \frac{\text{Var}(X)}{\alpha^2} && \text{[def of variance]}
 \end{aligned}$$

□

While in principle Chebyshev's inequality asks about distance from the mean in either direction, it can still be used to give a bound on how often a random variable can take large values, and will usually give much better bounds than Markov's inequality. This is expected, since we also assume to know the variance - and if the variance is small, we know the RV can't deviate too far from its mean.

### Example(s)

Let's revisit the example in Markov's inequality section earlier in which we toss a weighted coin independently with probability of landing heads  $p = 0.2$ . Upper bound the probability it lands on heads at least 16 times out of 20 flips using Chebyshev's inequality.

*Solution* Because  $X \sim \text{Bin}(n = 20, p = 0.2)$ :

$$\mathbb{E}[X] = np = 20 \cdot 0.2 = 4$$

and:

$$\text{Var}(X) = np(1 - p) = 20 \cdot 0.2 \cdot (1 - 0.2) = 3.2$$

Note that since Chebyshev's asks about the difference in either direction of the RV from its mean, we must weaken our statement first to include the probability  $X \leq -8$ . The reason we chose  $-8$  is because

Chebyshev's inequality is symmetric about the mean (difference of 12;  $4 \pm 12$  gives the interval  $[-8, 16]$ ):

$$\begin{aligned}
 \mathbb{P}(X \geq 16) &\leq \mathbb{P}(X \geq 16 \cup X \leq -8) && \text{[adding another event can only increase probability]} \\
 &= \mathbb{P}(|X - 4| \geq 12) && \text{[def of abs value]} \\
 &= \mathbb{P}(|X - \mathbb{E}[X]| \geq 12) && \text{[}\mathbb{E}[X] = 4\text{]} \\
 &\leq \frac{\text{Var}(X)}{12^2} && \text{[Chebyshev's inequality]} \\
 &= \frac{3.2}{12^2} = \frac{1}{45}
 \end{aligned}$$

This is a much better bound than given by Markov's inequality, but still far from the actual probability. This is because Chebyshev's inequality only takes the mean and variance into account. There is so much more information about a RV than just these two quantities!  $\square$

We can actually use Chebyshev's inequality to prove an important result from 5.7: The Weak Law of Large Numbers. The proof is so short!

### 6.1.3 Proof of the Weak Law of Large Numbers

Theorem 6.1.31: Weak Law of Large Numbers

Let  $X_1, X_2, \dots, X_n$  be a sequence of iid random variables with mean  $\mu$ . Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  be the sample mean. Then,  $\bar{X}_n$  converges in probability to  $\mu$ . That is, for any  $\epsilon > 0$ :

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\bar{X}_n - \mu| > \epsilon) = 0$$

*Proof.* By the property of the expectation and variance of sample mean consisting of  $n$  iid variables:  $\mathbb{E}[\bar{X}_n] = \mu$  and  $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$  (from 5.7). By Chebyshev's inequality:

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\bar{X}_n - \mu| > \epsilon) \leq \lim_{n \rightarrow \infty} \frac{\text{Var}(\bar{X}_n)}{\epsilon^2} = \lim_{n \rightarrow \infty} \frac{\sigma^2}{n\epsilon^2} = 0$$

$\square$

## Chapter 6: Concentration Inequalities

### 6.2: The Chernoff Bound

The more we know about a distribution, the stronger concentration inequality we can derive. We know that Markov's inequality is weak, since we only use the expectation of a random variable to get the probability bound. Chebyshev's inequality is a bit stronger, because we incorporate the variance into the probability bound. However, as we showed in the example in 6.1, these bounds are still pretty "loose". (They are tight in some cases though).

What if we know even more? In particular, its PMF/PDF and hence MGF? That will allow us to have an even stronger bound. The Chernoff bound is derived using a combination of Markov's inequality and moment generating functions.

#### 6.2.1 The Chernoff Bound for the Binomial Distribution

Here is the idea for the Chernoff bound. We will only derive it for the Binomial distribution, but the same idea can be applied to any distribution.

Let  $X$  be any random variable.  $e^{tX}$  is always a non-negative random variable. Thus, for any  $t > 0$ , using Markov's inequality and the definition of MGF:

$$\begin{aligned}\mathbb{P}(X \geq k) &= \mathbb{P}(e^{tX} \geq e^{tk}) && [\text{since } t > 0. \text{ if } t < 0, \text{ flip the inequality.}] \\ &\leq \frac{\mathbb{E}[e^{tX}]}{e^{tk}} && [\text{Markov's inequality}] \\ &= \frac{M_X(t)}{e^{tk}} && [\text{def of MGF}]\end{aligned}$$

(Note that the first line requires  $t > 0$ , otherwise it would change to  $\mathbb{P}(e^{tX} \leq e^{tk})$ . This is because  $e^t > 1$  for  $t > 0$  so we get something like  $2^X$  which is monotone increasing. If  $t < 0$ , then  $e^t < 1$  so we get something like  $0.3^X$  which is monotone decreasing.)

Now the right hand side holds for (uncountably) infinitely many  $t$ . For example, if we plugged in  $t = 0.5$  we might get  $\frac{M_X(t)}{e^{tk}} = 0.53$  and if we plugged in  $t = 3.26$  we might get 0.21. Since  $\mathbb{P}(X \geq k)$  has to be less than all the possible values when plugging in different  $t > 0$ , it in particular must be less than the **minimum** of all the values.

$$\mathbb{P}(X \geq k) \leq \min_{t>0} \frac{M_X(t)}{e^{tk}}$$

This is good - if we can minimize the right hand side, we can get a very tight/strong bound.

We'll now focus our attention to deriving the Chernoff bound when  $X$  has a Binomial distribution. Everything above applies generally though.

## Theorem 6.2.32: Chernoff Bound for Binomial Distribution

Let  $X \sim \text{Bin}(n, p)$  and let  $\mu = \mathbb{E}[X]$ . For any  $0 < \delta < 1$ :

Upper tail bound:

$$\mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right)$$

Lower tail bound:

$$\mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$$

where  $\exp(x) = e^x$ .

The Chernoff bound will allow us to bound the probability that  $X$  is larger than some multiple of its mean, or less than or equal to it. These are the *tails* of a distribution as you go farther in either direction from the mean. For example, we might want to bound the probability that  $X \geq 1.5\mu$  or  $X \leq 0.1\mu$ .

I think it's completely acceptable if you'd like to not read the proof, as it is very involved algebraically. You can still use the result regardless!

*Proof of Chernoff Bound for Binomial.*

If  $X = \sum_{i=1}^n X_i$  where  $X_1, X_2, \dots, X_n$  are iid variables, then since the MGF of the (independent) sum equals the product of the MGFs. Taking our general result from above and using this fact, we get:

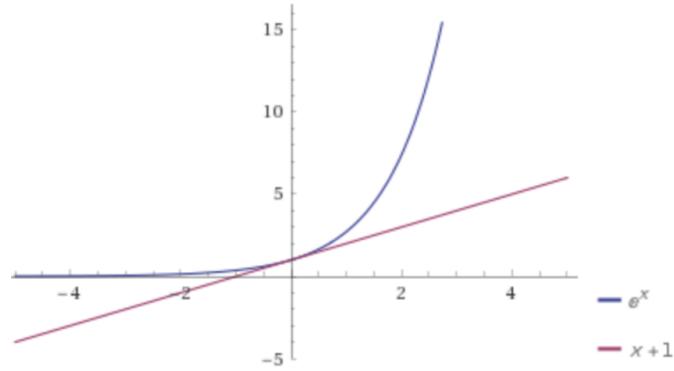
$$\mathbb{P}(X \geq k) \leq \min_{t>0} \frac{M_X(t)}{e^{tk}} = \min_{t>0} \frac{\prod_{i=1}^n M_{X_i}(t)}{e^{tk}}$$

Let's derive a Chernoff bound for  $X \sim \text{Bin}(n, p)$ , which has the form  $\mathbb{P}(X \geq (1 + \delta)\mu)$  for  $\delta > 0$ . For example with  $\delta = 4$ , you may want to bound  $\mathbb{P}(X \geq 5\mathbb{E}[X])$ .

Recall  $X = \sum_{i=1}^n X_i$  where  $X_i \sim \text{Ber}(p)$  are iid, with  $\mu = \mathbb{E}[X] = np$ .

$$\begin{aligned} M_{X_i}(t) &= \mathbb{E}[e^{tX_i}] && [\text{def of MGF}] \\ &= e^{t \cdot 1} p_{X_i}(1) + e^{t \cdot 0} p_{X_i}(0) && [\text{LOTUS}] \\ &= pe^t + 1(1 - p) && [X_i \sim \text{Ber}(p)] \\ &= 1 + p(e^t - 1) \\ &\leq e^{p(e^t - 1)} && [1 + x \leq e^x \text{ with } x = p(e^t - 1)] \end{aligned}$$

See here for a pictorial proof that  $1 + x \leq e^x$  for any real number  $x$  (just plot the two functions). Alternatively, use the Taylor series for  $e^x$  to argue this. We use this bound for algebra convenience coming up soon.



Now using the result from earlier and plugging in the MGF for the  $\text{Ber}(p)$  distribution, we get:

$$\begin{aligned}
 \mathbb{P}(X \geq k) &\leq \min_{t>0} \frac{\prod_{i=1}^n M_{X_i}(t)}{e^{tk}} && \text{[from earlier]} \\
 &\leq \min_{t>0} \frac{\left(e^{p(e^t-1)}\right)^n}{e^{tk}} && \text{[MGF of } \text{Ber}(p), n \text{ times]} \\
 &= \min_{t>0} \frac{e^{np(e^t-1)}}{e^{tk}} && \text{[algebra]} \\
 &= \min_{t>0} \frac{e^{\mu(e^t-1)}}{e^{tk}} && \text{[} \mu = np \text{]}
 \end{aligned}$$

For our bound, we want something like  $\mathbb{P}(X \geq (1 + \delta)\mu)$ , so our  $k = (1 + \delta)\mu$ . To minimize the RHS and get the tightest bound, the best bound we get is by choosing  $t = \ln(1 + \delta)$  after some terrible algebra (take the derivative and set to 0). We simply plug in  $k$  and our optimal value of  $t$  to the above equation:

$$\mathbb{P}(X \geq (1 + \delta)\mu) \leq \frac{e^{\mu(e^{\ln(1+\delta)}-1)}}{e^{(1+\delta)\mu \ln(1+\delta)}} = \frac{e^{\mu((1+\delta)-1)}}{\left(e^{\ln(1+\delta)}\right)^{(1+\delta)\mu}} = \frac{e^{\delta\mu}}{(1+\delta)^{(1+\delta)\mu}} = \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$$

Again, we wanted to choose  $t$  that minimizes our upper bound for the tail probability. Taking the derivative with respect to  $t$  tells us we should plug in  $t = \ln(1 + \delta)$  to minimize that quantity. This would actually be pretty annoying to plug into a calculator.

We actually can show that the final RHS is  $\leq \exp\left(\frac{-\delta^2\mu}{2 + \delta}\right)$  with some more messy algebra. Additionally, if we restrict  $0 < \delta < 1$ , we can simplify this even more to the bound provided earlier:

$$\mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(\frac{-\delta^2\mu}{3}\right)$$

The proof of the lower tail is entirely analogous, except optimizing over  $t < 0$  when the inequality flips. It proceeds by taking  $t = \ln(1 - \delta)$ .

We also get a lower tail bound:

$$\mathbb{P}(X \leq (1 - \delta)\mu) \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}}\right)^\mu \leq \left(\frac{e^{-\delta}}{e^{-\delta + \frac{\delta^2}{2}}}\right)^\mu = \exp\left(\frac{-\delta^2\mu}{2}\right)$$

□

You may wonder, why are we bounding  $\mathbb{P}(X \geq (1 + \delta)\mu)$ , when we can just sum the PMF of a binomial to get an exact answer? The reason is, it is very computationally expensive to compute the binomial PMF! For example, if  $X \sim \text{Bin}(n = 20000, p = 0.1)$ , then by plugging in the PMF, we get

$$\mathbb{P}(X = 13333) = \binom{20000}{13333} 0.1^{13333} 0.9^{20000-13333} = \frac{20000!}{13333!(20000-13333)!} 0.1^{13333} 0.9^{20000-13333}$$

(Actually,  $n = 20000$  isn't even that large.) You have to multiply 20,000 numbers on the second two terms, and it multiplies to a number that is infinitesimally small. For the first term (binomial coefficient), computing  $20000!$  is impossible - in fact, it is so large you can't even imagine. You would have to cleverly interleave multiplying a factorial vs the probability, to keep the value in an acceptable range for the computer. Then, sum up a bunch of these....

This is why we have/need the Poisson approximation, the Normal approximation (CLT), and the Chernoff bound for the Binomial!

### Example(s)

Suppose  $X \sim \text{Bin}(500, 0.2)$ . Use Markov's inequality and the Chernoff bound to bound  $P(X \geq 150)$ , and compare the results.

*Solution* We have:

$$\mathbb{E}[X] = np = 500 \cdot 0.2 = 100$$

$$\text{Var}(X) = np(1-p) = 500 \cdot 0.2 \cdot 0.8 = 80$$

Using Markov's Inequality:

$$P(X \geq 150) \leq \frac{\mathbb{E}[X]}{150} = \frac{100}{150} \approx 0.6667$$

Using the Chernoff Bound (with  $\delta = 0.5$ ):

$$P(X \geq 150) = P(X \geq (1 + 0.5) \cdot 100) \leq e^{-\frac{0.5^2 \cdot 100}{3}} \approx 0.00024$$

The Chernoff bound is much stronger! It isn't a fair comparison necessarily, because the Chernoff bound required knowing the MGF (and hence the distribution), whereas Markov only required knowing the mean (and that it was non-negative).  $\square$

These examples give you an overall comparison of all three inequalities we learned so far!

### Example(s)

Suppose the number of red lights Alex encounters each day to work is on average 4.8 (according to historical trips to work). Alex really will be late if he encounters 8 or more red lights. Let  $X$  be the number of lights he gets on a given day.

1. Give a bound for  $\mathbb{P}(X \geq 8)$  using Markov's inequality.
2. Give a bound for  $\mathbb{P}(X \geq 8)$  using Chebyshev's inequality, if we also assume  $\text{Var}(X) = 2.88$ .
3. Give a bound for  $\mathbb{P}(X \geq 8)$  using the Chernoff bound. Assume that  $X \sim \text{Bin}(12, 0.4)$  - that there are 12 traffic lights, and each is independently red with probability 0.4.
4. Compute  $\mathbb{P}(X \geq 8)$  exactly using the assumption from the previous part.

5. Compare the three bounds and their assumptions.

1. Since  $X$  is nonnegative and we know its expectation, we can apply Markov's inequality:

$$\mathbb{P}(X \geq 8) \leq \frac{\mathbb{E}[X]}{8} = \frac{4.8}{8} = 0.6$$

2. Since we know  $X$ 's variance, we can apply Chebyshev's inequality after some manipulation. We have to do this to match the form required:

$$\mathbb{P}(X \geq 8) \leq \mathbb{P}(X \geq 8) + \mathbb{P}(X \leq 1.6) = \mathbb{P}(|X - 4.8| \geq 3.2)$$

The reason we chose  $\leq 1.6$  is so it looks like  $\mathbb{P}(|X - \mu| \geq \alpha)$ . Now, applying Chebyshev's gives:

$$\leq \frac{\text{Var}(X)}{3.2^2} = \frac{2.88}{3.2^2} = 0.28125$$

3. Actually,  $X \sim \text{Bin}(12, 0.4)$  also has  $\mathbb{E}[X] = np = 4.8$  and  $\text{Var}(X) = np(1-p) = 2.88$  (what a coincidence). The Chernoff bound requires something of the form  $\mathbb{P}(X \geq (1+\delta)\mu)$ , so we first need to solve for  $\delta$ :  $(1+\delta)4.8 = 8$  so that  $\delta = 2/3$ . Now,

$$\mathbb{P}(X \geq 8) = \mathbb{P}(X \geq (1+2/3) \cdot 4.8) \leq \exp\left(\frac{-(2/3)^2 4.8}{3}\right) \approx 0.4911$$

4. The exact probability can be found summing the Binomial PMF:

$$\mathbb{P}(X \geq 8) = \sum_{k=8}^{12} \binom{12}{k} 0.4^k 0.6^{12-k} \approx 0.0573$$

5. Actually it's usually the case that the bounds are tighter/better as we move down the list Markov, Chebyshev, Chernoff. But in this case Chebyshev's gave us the tightest bound, even after being weakened by including some additional  $\mathbb{P}(X \leq 1.6)$ . Chernoff bounds will typically be better for farther tails - 8 isn't considered too far from the mean 4.8.

It's also important to note that we found out more information progressively - we can't blindly apply all these inequalities every time. We need to make sure the conditions for the bound being valid are satisfied.

Even our best bound of 0.28125 was 5-6x larger than the true probability of 0.0573.

## Chapter 6: Concentration Inequalities

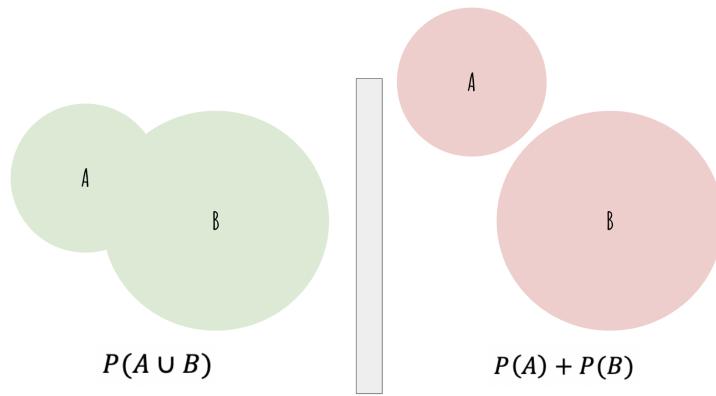
### 6.3: Even More Inequalities

In this section, we will talk about a potpourri of remaining concentration bounds. More specifically, the union bound, Jensen's inequality for convex functions, and Hoeffding's inequality.

#### 6.3.1 The Union Bound

Suppose there are many bad events  $B_1, \dots, B_n$ , and we don't want any of them to happen. They may or may not be independent. Can we bound the probability that any (at least one) bad event occurs?

The intuition for the union bound is fairly simple. Suppose we have two events  $A$  and  $B$ . Then  $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$  since the event space of  $A$  and  $B$  may overlap:



We will now define the Union Bound more formally.

##### Theorem 6.3.33: The Union Bound

Let  $E_1, E_2, \dots, E_n$  be a collection of events. Then:

$$\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mathbb{P}(E_i)$$

Additionally, if  $E_1, E_2, \dots$  is a (countably) infinite collection of events, then:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

We can prove the union bound using induction.

*Proof of Union Bound by Induction.*

**Base Case:** For  $n = 2$  events, by inclusion-exclusion, we know

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &\leq \mathbb{P}(A) + \mathbb{P}(B) \quad [\text{since } \mathbb{P}(A \cap B) \geq 0]\end{aligned}$$

**Inductive Hypothesis:** Suppose it's true for  $n$  events,  $\mathbb{P}(E_1 \cup \dots \cup E_n) \leq \sum_{i=1}^n \mathbb{P}(E_i)$ .

**Inductive Step:** We will show it for  $n + 1$ .

$$\begin{aligned}\mathbb{P}(E_1 \cup \dots \cup E_n \cup E_{n+1}) &= \mathbb{P}((E_1 \cup \dots \cup E_n) \cup E_{n+1}) && [\text{associativity of } \cup] \\ &= \mathbb{P}(E_1 \cup \dots \cup E_n) + \mathbb{P}(E_{n+1}) && [\text{base case}] \\ &\leq \sum_{i=1}^n \mathbb{P}(E_i) + \mathbb{P}(E_{n+1}) && [\text{inductive hypothesis}] \\ &= \sum_{i=1}^{n+1} \mathbb{P}(E_i)\end{aligned}$$

□

The union bound, though seemingly trivial, can actually be quite useful.

### Example(s)

This will relate to the earlier question of bounding the probability of at least one bad event happening.

Suppose the probability Alex is late to teaching class on a given day is at most 0.01. Bound the probability that Alex is late at least once over a 30-class quarter. Do **not** make any independence assumptions.

### *Solution*

Let  $A_i$  be the event Alex is late to class on day  $i$  for  $i = 1, \dots, 30$ . Then, by the union bound,

$$\begin{aligned}\mathbb{P}(\text{late at least once}) &= \mathbb{P}\left(\bigcup_{i=1}^{30} A_i\right) \\ &\leq \sum_{i=1}^{30} \mathbb{P}(A_i) && [\text{union bound}] \\ &\leq \sum_{i=1}^{30} 0.01 && [\mathbb{P}(A_i) \leq 0.01] \\ &= 0.30\end{aligned}$$

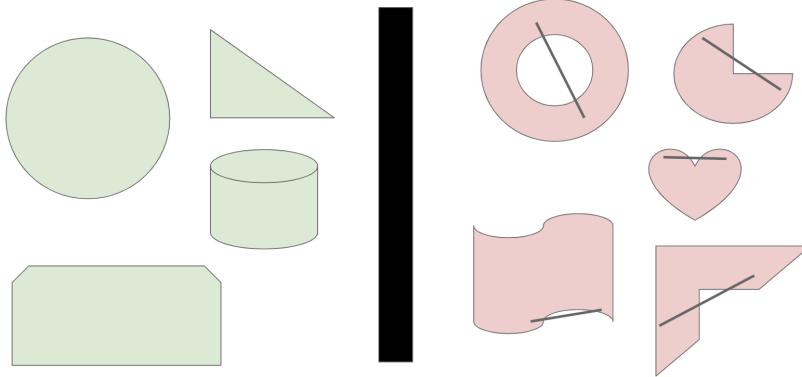
Sometimes it may be useless though; imagine I asked instead about over a 200-day period. Then the union bound would've given me a bound of 2.0 which is not helpful since probabilities have to be at most 1 already...

□

### 6.3.2 Convex Sets and Functions

Our next inequality is called Jensen's inequality, and deals with convex functions. So first, we need to define what that means. Before convex functions though, we need to discuss convex sets.

Let's look at some examples of convex (left) and non-convex (right) sets:



The sets on the left hand side are said to be **convex** because if you take any two points in the set and draw the line segment between them, it is always contained in the set. The sets on the right hand side are non-convex because I found two endpoints in the set, but the line segment connecting them is not completely contained in the set.

How can we describe this mathematically? Well for *any* two points  $x, y \in S$ , the set of points between them must be entirely contained in  $S$ . The set of points making up the line segment between two points  $x, y$  can be described as a weighted average  $(1 - p)x + py$  for  $p \in [0, 1]$ . If  $p = 0$ , we just get  $x$ ; if  $p = 1$ , we just get  $y$ , and if  $p = 1/2$ , we get the midpoint  $(x + y)/2$ . So  $p$  controls the fraction of the way we are from  $x$  to  $y$ .

#### Definition 6.3.1: Convex Sets

A set  $S \subseteq \mathbb{R}^n$  is a **convex set** if for any  $x, y \in S$ , the entire line segment between them is contained in  $S$ . That is, for any two points  $x, y \in S$ ,

$$\overline{xy} = \{(1 - p)x + py : p \in [0, 1]\} \subseteq S$$

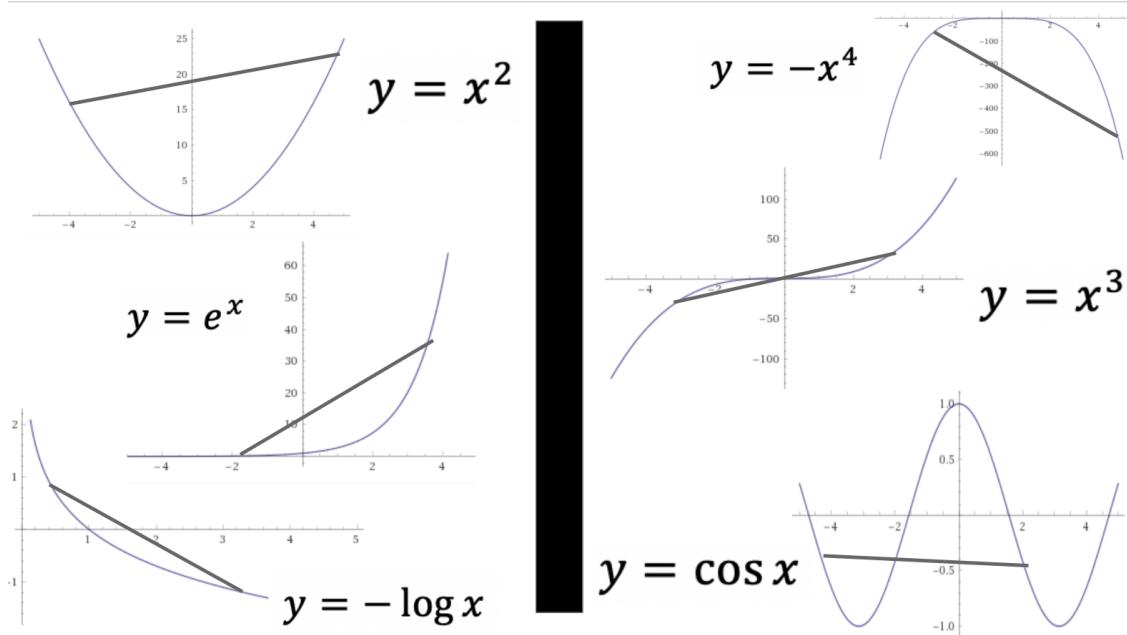
Equivalently, for any points  $x_1, \dots, x_m \in S$ , convex polyhedron formed by the “corners” is contained in  $S$ . (This sounds complicated, but if  $m = 3$ , it just says the triangle formed by the 3 corners completely lies in the set  $S$ . If  $m = 4$ , the quadrilateral formed by the 4 corners completely lies in the set  $S$ .) The points in the convex polyhedron are described by taking weighted average of the points, where the weights are non-negative and sum to 1. (This should remind you of a probability distribution!)

$$\left\{ \sum_{i=1}^m p_i x_i : p_1, \dots, p_m \geq 0 \text{ and } \sum_{i=1}^m p_i = 1 \right\} \subseteq S$$

Here are some examples of convex sets:

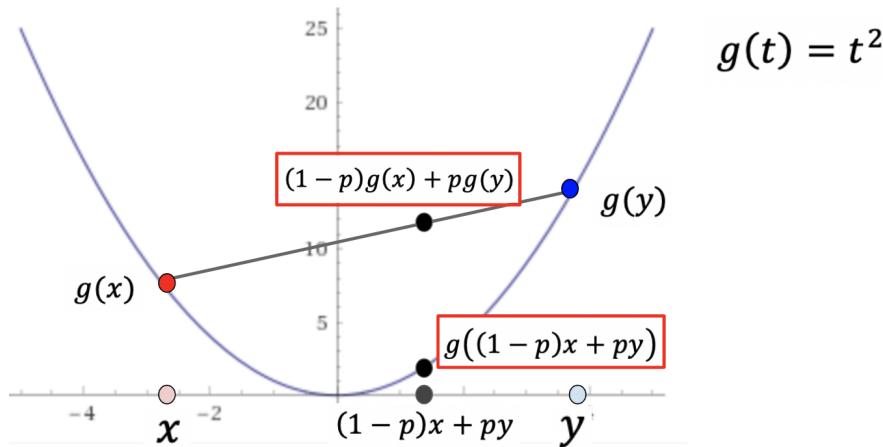
1. Any interval  $([a, b], (a, b)$ , etc.) in  $\mathbb{R}$  is a convex set (and the only convex sets in  $\mathbb{R}$  are intervals).
2. The circle  $C = \{(x, y) : x^2 + y^2 \leq 1\}$  in  $\mathbb{R}^2$  is a convex set.
3. Any  $n$ -dimensional box  $B = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$  is a convex set.

Now, onto convex *functions*. Let's take a look at some convex (left) and non-convex (right) functions:



The functions on the left (convex) have the property that, for **any** two points on the function curve, the line segment connecting them lies **above** the function always. The functions on the right don't have this property: you can see that some or all of the line segment is below the function.

Let's try to formalize what this means. For the convex function  $g(t) = t^2$  below, we can see that any line drawn connecting 2 points of the function clearly lies above the function itself and so it is convex. Look at any two points on the curve  $g(x)$  and  $g(y)$ . Pick a point on the  $x$ -axis between  $x$  and  $y$ , call it  $(1-p)x + py$  where  $p \in [0, 1]$ . The function value at this point is  $g((1-p)x + py)$ . The corresponding point above it on the line segment connecting  $g(x)$  and  $g(y)$  is actually the weighted average  $(1-p)g(x) + pg(y)$ . Hence, a function  $g$  is convex if it satisfies the following for any  $x, y$  and  $p \in [0, 1]$ :  $g((1-p)x + py) \leq (1-p)g(x) + pg(y)$



### Definition 6.3.2: Convex Functions

Let  $S \subseteq \mathbb{R}^n$  be a *convex set* (a convex function must have the domain being a convex set). A function  $g : S \rightarrow \mathbb{R}$  is a **convex function** if for any line segment connecting  $g(x)$  and  $g(y)$ , the function  $g$  lies entirely below the line. Mathematically, for any  $p \in [0, 1]$  and  $x, y \in \mathbb{R}$ ,

$$g((1-p)x + py) \leq (1-p)g(x) + pg(y)$$

Equivalently, for any  $m$  points  $x_1, \dots, x_m \in S$ , and  $p_1, \dots, p_m \geq 0$  such that  $\sum_{i=1}^m p_i = 1$ ,

$$g\left(\sum_{i=1}^m p_i x_i\right) \leq \sum_{i=1}^m p_i g(x_i)$$

Here are some examples of convex functions:

1.  $g(x) = x^2$
2.  $g(x) = x$
3.  $g(x) = -\log(x)$
4.  $g(x) = e^x$

### 6.3.3 Jensen's Inequality

Now after learning about convex sets and functions, we can learn Jensen's inequality, which relates  $\mathbb{E}[g(X)]$  and  $g(\mathbb{E}[X])$  for convex functions. Remember we said many times that these two quantities were never equal (use LOTUS to compute  $\mathbb{E}[g(X)]$ )!

#### Theorem 6.3.34: Jensen's Inequality

Let  $X$  be any random variable, and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. Then,

$$g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)]$$

*Proof of Jensen's Inequality.* We will only prove it in the case  $X$  is a discrete random variable (not a random vector), and with finite range (not countably infinite). However, this inequality does hold for any random variable.

The proof follows immediately from the definition of a convex function. Since  $X$  has finite range, let  $\Omega_X = \{x_1, \dots, x_n\}$  and  $p_X(x_i) = p_i$ . By definition of a convex function (see above),

$$\begin{aligned}
 g(\mathbb{E}[X]) &= g\left(\sum_{i=1}^n p_i x_i\right) && [\text{def of expectation}] \\
 &\leq \sum_{i=1}^n p_i g(x_i) && [\text{def of convex function}] \\
 &= \mathbb{E}[g(X)] && [\text{LOTUS}]
 \end{aligned}$$

□

### Example(s)

Show that variance of any random variable  $X$  is always non-negative using Jensen's inequality.

*Solution* We already know that  $\text{Var}(X) = \mathbb{E}[(X - \mu)^2] \geq 0$  since  $(X - \mu)^2$  is a non-negative RV, but let's prove it a different way.

We know  $g(t) = t^2$  is a convex function, so by Jensen's inequality,

$$\mathbb{E}[X]^2 = g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)] = \mathbb{E}[X^2]$$

Hence  $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \geq 0$ . □

### 6.3.4 Hoeffding's Inequality

One final inequality that is commonly used is called Hoeffding's inequality. We'll state it without proof since it is quite complicated. The proof uses Jensen's inequality and ideas from the proof of the Chernoff bound (MGFs)!

#### Definition 6.3.3: Hoeffding's Inequality

Let  $X_1, \dots, X_n$  be independent random variables, where each  $X_i$  is bounded:  $a_i \leq X_i \leq b_i$  and let  $\bar{X}_n$  be their sample mean. Then,

$$\mathbb{P}(|\bar{X}_n - \mathbb{E}[\bar{X}_n]| \geq t) \leq 2 \exp\left(\frac{-2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

where  $\exp(x) = e^x$ .

In the case  $X_1, \dots, X_n$  are iid (so  $a \leq X_i \leq b$  for all  $i$ ) with mean  $\mu$ , then

$$\mathbb{P}(|\bar{X}_n - \mu| \geq t) \leq 2 \exp\left(\frac{-2n^2t^2}{n(b-a)^2}\right) = 2 \exp\left(\frac{-2nt^2}{(b-a)^2}\right)$$

## Example(s)

Suppose an email company ColdMail is responsible for delivering 100 emails per day. ColdMail has a bad day if it takes longer than 190 seconds to deliver all 100 emails, and a bad week if there is even one bad day in the week.

The time it takes to send an email on average is 1 second, with a worst-case time of 5 seconds; independently of other emails. (Note we don't know anything else like its PDF).

1. Give an upper bound for the probability that ColdMail has a bad day.
2. Give an upper bound for the probability that ColdMail has a bad week.

*Solution*

1. In this scenario, we may use Hoeffding's inequality since we have  $X_1, \dots, X_{100}$  the (independent) times to send each email bounded in the interval  $[0, 5]$  seconds, with  $\mathbb{E}[\bar{X}_{100}] = 1$ . Asking that the total time to be at least 190 seconds is the same as asking the mean time to be at least 1.9 seconds.

Like we did for Chebyshev, we have to massage (and weaken) a little bit to get in the same form as required for Hoeffding's:

$$\mathbb{P}(\bar{X}_{100} \geq 1.9) \leq \mathbb{P}(\bar{X}_{100} \geq 1.9 \cup \bar{X}_{100} \leq 0.1) = \mathbb{P}(|\bar{X}_{100} - 1| \geq 0.9)$$

Applying Hoeffding's (since  $\mathbb{E}[\bar{X}_n] = 1$ ):

$$\mathbb{P}(\bar{X}_{100} \geq 1.9) \leq \mathbb{P}(|\bar{X}_{100} - 1| \geq 0.9) \leq 2 \exp\left(\frac{-2 \cdot 100 \cdot 0.9^2}{(5 - 0)^2}\right) \approx 0.0031$$

2. For  $i = 1, \dots, 7$ , let  $B_i$  be the event we had a bad day on day  $i$ . Then,

$$\begin{aligned} \mathbb{P}(\text{bad week}) &= \mathbb{P}\left(\bigcup_{i=1}^7 B_i\right) \\ &\leq \sum_{i=1}^7 \mathbb{P}(B_i) && [\text{union bound}] \\ &\leq \sum_{i=1}^7 0.0031 && [\text{Hoeffding in previous part}] \\ &\approx 0.0215 \end{aligned}$$

You might be tempted to use the CLT (and you should when you can), as it would probably give a better bound than Hoeffding's. But we didn't know the variances, so we wouldn't know which Normal to use. Hoeffding's gives us a way!  $\square$

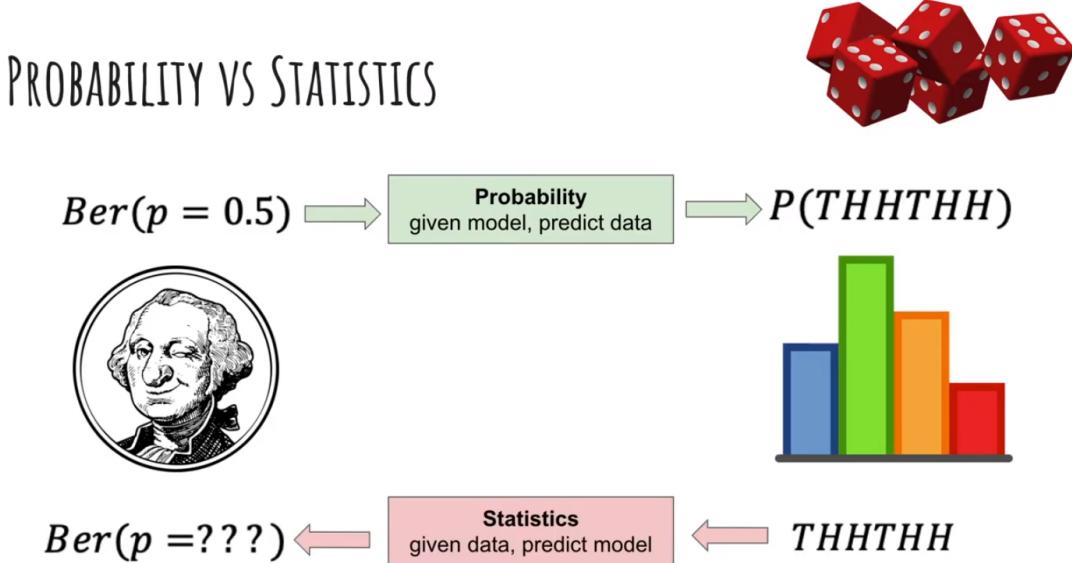
## Chapter 7: Statistical Estimation

### 7.1: Maximum Likelihood Estimation

#### 7.1.1 Probability vs Statistics

Before we start, we need to make an important distinction: what is the difference between probability and statistics? What we've been doing up until this point is probability. We're given a model, in this picture  $Ber(p = 0.5)$  (our assumption), and we're trying to find the probability of some data. So, given this model, what is the probability of THHTHH, or  $P(THHTHH)$ ? That's something you know how to do now!

What we're going to focus now is going the opposite way. Given a coin with unknown probability of heads is, I flip it a few times and I get THHTHH. How can I use this data to predict/estimate this value of  $p$ ?



#### 7.1.2 Likelihood

Let's say I give you and your classmates each 5 minutes with a coin with unknown probability of heads  $p$ . Whoever has the closest estimate will get an A+ in the class. What do you do in your precious 5 minutes, and what do you give as your estimate?

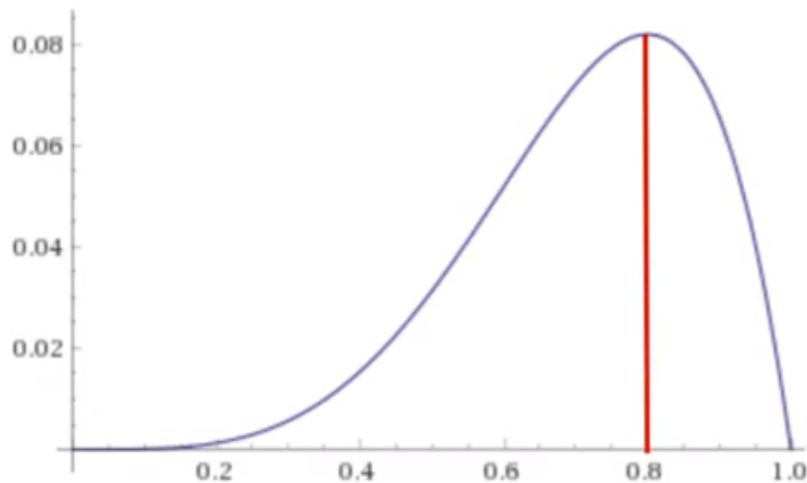
I don't know about you, but I would flip the coin as many times as I can, and return the total number of heads over the total number of flips, or

$$\frac{\text{Heads}}{\text{Heads} + \text{Tails}}$$

which actually turns out to be a really good estimate.

To make things concrete, let's say you saw 4 heads and 1 tail. You tell me that  $\hat{p} = \frac{4}{5}$  (the hat above the  $p$  just means it is an estimate). How can you argue, objectively, that this is the "best" estimate?

Is there some objective function that it maximizes? It turns out yes,  $\frac{4}{5}$  maximizes this blue curve, which is called the likelihood of the data. The  $x$ -axis has the different possible values of  $p$ , and the  $y$ -axis has the probability of seeing the data if the coin had probability of heads  $p$ .



You assume a model (Bernoulli in our case) with unknown parameter  $\theta$  (the probability of heads), and receive iid samples  $\mathbf{x} = (x_1, \dots, x_n) \sim \text{Ber}(\theta)$  (in this example, each  $x_i$  is either 1 or 0). The likelihood of the data given a parameter  $\theta$  is defined as the probability of seeing the data, given  $\theta$ , or:

$$\begin{aligned} L(\mathbf{x} \mid \theta) &= \mathbb{P}(\text{seeing data} \mid \theta) && [\text{def of likelihood}] \\ &= \mathbb{P}(x_1, \dots, x_n \mid \theta) && [\text{plug in data}] \\ &= \prod_{i=1}^n p_X(x_i \mid \theta) && [\text{independence}] \end{aligned}$$

(**Note:** When estimating unknown parameters, we typically use  $\theta$  instead of  $p$ ,  $\lambda$ ,  $\mu$ , etc.)

#### Definition 7.1.1: Realization / Sample

A realization/sample  $x$  of a random variable  $X$  is the value that is actually observed (will always be in  $\Omega_X$ ).

For example, for Bernoulli, a realization is either 0 or 1, and for Geometric, some positive integer  $\geq 1$ .

#### Definition 7.1.2: Likelihood

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid samples from probability mass function  $p_X(t \mid \theta)$  (if  $X$  is discrete), or from density  $f_X(t \mid \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). We define the likelihood of  $\mathbf{x}$  given  $\theta$  to be the "probability" of observing  $\mathbf{x}$  if the true parameter is  $\theta$ .

If  $X$  is discrete,

$$L(\mathbf{x} \mid \theta) = \prod_{i=1}^n p_X(x_i \mid \theta)$$

If  $X$  is continuous,

$$L(\mathbf{x} \mid \theta) = \prod_{i=1}^n f_X(x_i \mid \theta)$$

In the continuous case, we have to multiply densities, because the probability of seeing a particular value with a continuous random variable is always 0. We can do this because the density preserves relative probabilities; i.e.,  $\frac{\mathbb{P}(X \approx u)}{\mathbb{P}(X \approx v)} \approx \frac{f_X(u)}{f_X(v)}$ . For example, if  $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 5)$ , the realization  $x = -503.22$  has much lower density/likelihood than  $x = 3.12$ .

### Example(s)

Give the likelihoods for each of the samples, and take a guess at which value of  $\theta$  maximizes the likelihood!

1. Suppose  $\mathbf{x} = (x_1, x_2, x_3) = (1, 0, 1)$  are iid samples from  $\text{Ber}(\theta)$  (recall  $\theta$  is the probability of a success).
2. Suppose  $\mathbf{x} = (x_1, x_2, x_3, x_4) = (3, 0, 2, 7)$  are iid samples from  $\text{Poi}(\theta)$  (recall  $\theta$  is the historical average number of events in a unit of time).
3. Suppose  $\mathbf{x} = (x_1, x_2, x_3) = (3.22, 1.81, 2.47)$  are iid samples from  $\text{Exp}(\theta)$  (recall  $\theta$  is the historical average number of events in a unit of time).

### Solution

1. The samples mean we got a success, then a failure, then a success. The likelihood is the “probability” of observing the data.

$$L(\mathbf{x} \mid \theta) = \prod_{i=1}^3 p_X(x_i \mid \theta) = p_X(1 \mid \theta) \cdot p_X(0 \mid \theta) \cdot p_X(1 \mid \theta) = \theta(1 - \theta)\theta = \theta^2(1 - \theta)$$

Since we observed two successes out of three trials, my guess for the maximum likelihood estimate would be  $\hat{\theta} = \frac{2}{3}$ .

2. The samples mean we observed 3 events in the first unit of time, then 0 in the second, then 2 in the third, then 7 in the fourth. The likelihood is the “probability” of observing the data (just multiplying Poisson PMFs  $p_X(k \mid \lambda) = e^{-\lambda} \frac{\lambda^k}{k!}$ ).

$$\begin{aligned} L(\mathbf{x} \mid \theta) &= \prod_{i=1}^4 p_X(x_i \mid \theta) = p_X(3 \mid \theta) \cdot p_X(0 \mid \theta) \cdot p_X(2 \mid \theta) \cdot p_X(7 \mid \theta) \\ &= \left( e^{-\theta} \frac{\theta^3}{3!} \right) \left( e^{-\theta} \frac{\theta^0}{0!} \right) \left( e^{-\theta} \frac{\theta^2}{2!} \right) \left( e^{-\theta} \frac{\theta^7}{7!} \right) \end{aligned}$$

Since there were a total of  $3 + 0 + 2 + 7 = 12$  events over 4 units of time (samples), my guess for the maximum likelihood estimate would be  $\hat{\theta} = \frac{12}{4} = 3$  events per unit time.

3. The samples mean we waited until three events happened  $(x_1, x_2, x_3)$ , and it took 3.22 units of time until the first event, 1.81 until the second, and 2.47 until the third. The likelihood is the “probability” of observing the data. The likelihood is the “probability” of observing the data (just multiplying Exponential PDFs  $f_X(y | \lambda) = \lambda e^{-\lambda y}$ ).

$$L(\mathbf{x} | \theta) = \prod_{i=1}^3 f_X(x_i | \theta) = f_X(x_1 | \theta) \cdot f_X(x_2 | \theta) \cdot f_X(x_3 | \theta) = (\theta e^{-3.22\theta}) (\theta e^{-1.81\theta}) (\theta e^{2.47\theta})$$

Since it took an average of  $\frac{3.22 + 1.81 + 2.47}{3} = 2.5$  units of time to observe each events, my guess for the maximum likelihood estimate would be  $\hat{\theta} = \frac{3}{7.5} = 0.4$  events happen per unit of time.

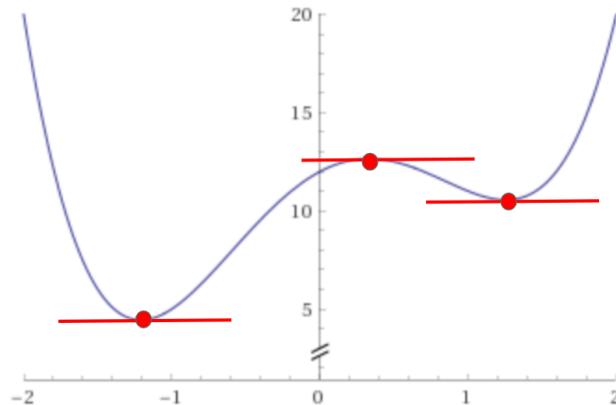
□

### 7.1.3 Maximum Likelihood Estimation

Now, we'll formally define what the maximum likelihood estimator of an unknown parameter is. Intuitively, it is just the value of  $\theta$  which maximizes the “probability” of seeing the data  $L(\mathbf{x} | \theta)$ .

In the previous three scenarios, we set up the likelihood of the data. Now, the only thing left to do is find out which value of  $\theta$  maximizes the likelihood. Everything else in this section is just explaining how to use calculus to optimize this likelihood! There is no more “probability” or “statistics” involved in the remaining pages.

Before we move on, we have to go back and review calculus really quickly. How do we optimize a function? Each of these three points is a local optima; what do they have in common? Their derivative is 0. We're going to try and set the derivative of our likelihood to 0, so we can solve for the optimum value.



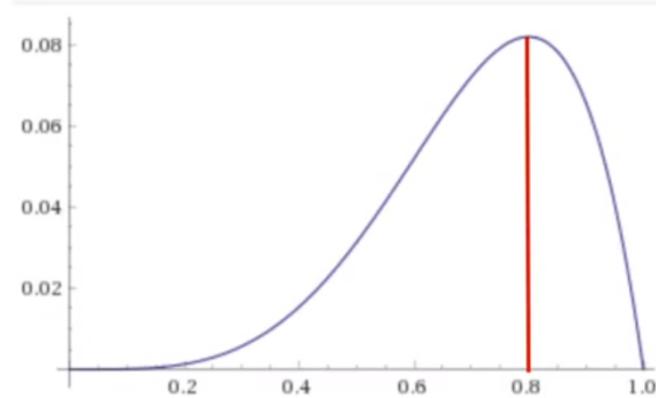
#### Example(s)

Suppose  $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 1, 0)$  are iid samples from the  $\text{Ber}(\theta)$  distribution with unknown parameter  $\theta$ . Find the maximum likelihood estimator  $\hat{\theta}$  of  $\theta$ .

*Solution* The data  $(1, 1, 1, 1, 0)$  can be thought of the sequence  $HHHHT$ , which has likelihood (assuming the probability of heads is  $\theta$ )

$$\begin{aligned} L(HHHHT \mid \theta) &= \theta^4(1 - \theta) \\ &= \theta^4 - \theta^5 \end{aligned}$$

The plot of the likelihood with  $\theta$  on the  $x$ -axis and  $L(HHHHT \mid \theta)$  on the  $y$ -axis is:



and we can actually see the  $\theta$  which maximizes the likelihood is  $\hat{\theta} = 4/5$ . But sometimes we can't plot the likelihood, so we will solve for this analytically now.

We want to find the  $\theta$  which maximizes this likelihood, so we take the derivative with respect to  $\theta$  and set it to 0:

$$\begin{aligned} \frac{\partial}{\partial \theta} L(\mathbf{x} \mid \theta) &= 4\theta^3 - 5\theta^4 \\ &= \theta^3(4 - 5\theta) \end{aligned}$$

Now, when we set the derivative to 0 (remember the optimum points occur when the derivative is 0), we replace  $\theta$  with  $\hat{\theta}$  because we are now estimating  $\theta$ . After solving for  $\theta$ , we end up with

$$\hat{\theta}^3(4 - 5\hat{\theta}) = 0 \rightarrow \hat{\theta} = \frac{4}{5} \text{ or } 0$$

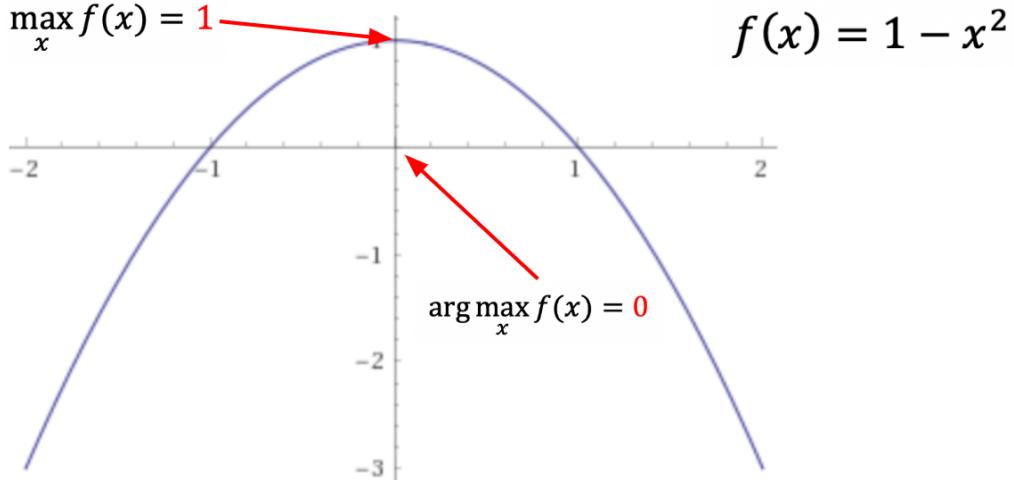
We switch  $\theta$  to  $\hat{\theta}$  when we set the derivative to 0, as that is when we start estimating. To see which is the maximizer, you can just plug in the candidates (0 and  $4/5$ ) and the endpoints (0 and 1: the min and max possible values of  $\theta$ )! That is, compute the likelihood at 0,  $4/5$ , 1 and see which is largest.  $\square$

To summarize, we defined  $\hat{\theta}_{MLE} = \arg \max_{\theta} L(\mathbf{x} \mid \theta)$ , the **argument** (input)  $\theta$  that **maximizes** the likelihood function. The difference between max and argmax is as follows. Here is a function,

$$f(x) = 1 - x^2$$

where the maximum value is 1, it's the highest value this function could ever achieve. The argmax, on the other hand, is 0, because argmax just means the *argument* (input) that maximizes the function. So, which  $x$

actually achieved  $f(x) = 1$ ? Well that was  $x = 0$ . And so, in MLE, we're trying to find the  $\theta$  that maximizes the likelihood, and we don't care what the maximum value of the likelihood is. We didn't even compute it! We just care that the argmax is  $\frac{4}{5}$ .



#### Definition 7.1.3: Maximum Likelihood Estimation (MLE)

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t | \theta)$  (if  $X$  is discrete), or from density  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). We define the **maximum likelihood estimator**  $\hat{\theta}_{MLE}$  of  $\theta$  to be the parameter which maximizes the likelihood (or equivalently, the log-likelihood) of the data.

$$\hat{\theta}_{MLE} = \arg \max_{\theta} L(\mathbf{x} | \theta) = \arg \max_{\theta} \ln L(\mathbf{x} | \theta)$$

The (usual) recipe to find the MLE goes as follows:

1. Compute the likelihood and log-likelihood of data.
2. Take the partial derivative(s) with respect to  $\theta$  and set to 0. Solve the equation(s).
3. Optionally, verify  $\hat{\theta}_{MLE}$  is indeed a (local) maximizer by checking that the second derivative at  $\hat{\theta}_{MLE}$  is negative (if  $\theta$  is a single parameter), or the Hessian (matrix of second partial derivatives) is negative semi-definite (if  $\theta$  is a vector of parameters).

##### 7.1.3.1 Optimizing Function vs Log(Function)

You may have noticed we also included this “log-likelihood” that we hadn’t talked about earlier. In the next section, we’ll do several more examples of maximum likelihood estimation, and you’ll see that taking the log makes our derivatives easier. Recall that the likelihood is the product of PDFs or PMFs, and taking the derivative of a product is quite annoying, especially with more than 2 terms:

$$\frac{d}{dx} (f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$$

whereas the derivative of the sum is just the sum of the derivatives:

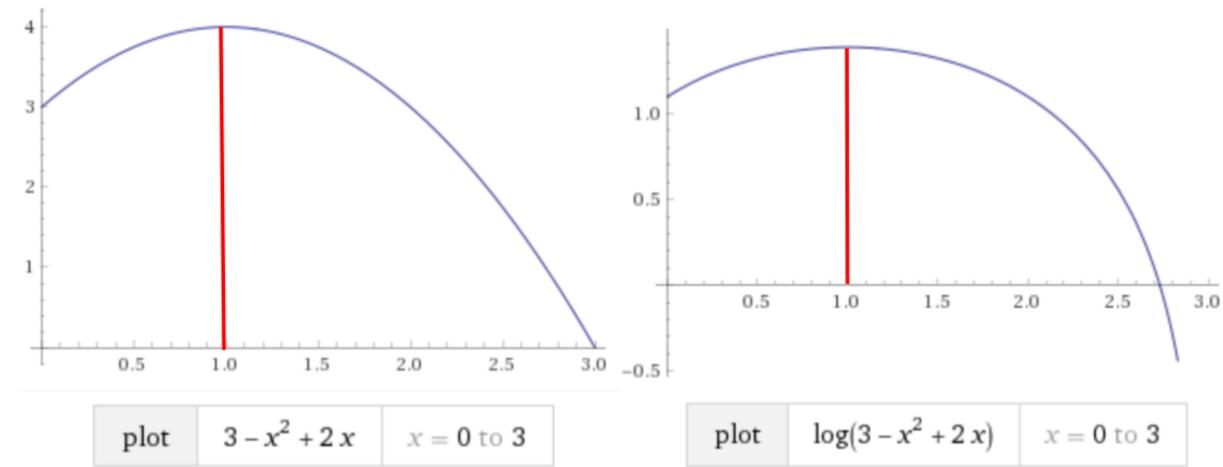
$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Taking the log of a product (such as the likelihood) results in the sum of logs because of log properties:

$$\log(a \cdot b \cdot c) = \log(a) + \log(b) + \log(c)$$

We see now why we might **want** to take the log of the likelihood before differentiating it, but why **can** we?

Below there are two images: the left image is a function, and the right image is the log of that function.



The values are different (see the  $y$ -axis), but if you look at the  $x$ -axis, it happens that both functions are maximized at 1 (the argmax's are the same). Log is a monotone increasing function, so it preserves order, so whatever was the maximizer (argmax) in the original function, will also be maximizer in the log function.

See below to see what happens when you apply the natural log ( $\ln$ ) to a product in our likelihood scenario! And see the next section 7.2 for examples of maximum likelihood estimation in action.

#### Definition 7.1.4: Log-Likelihood

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t | \theta)$  (if  $X$  is discrete), or from density  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). We define the likelihood of  $\mathbf{x}$  given  $\theta$  to be the probability of observing  $\mathbf{x}$  if the true parameter is  $\theta$ .

If  $X$  is discrete,

$$\ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \ln p_X(x_i | \theta)$$

If  $X$  is continuous,

$$\ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \ln f_X(x_i | \theta)$$

## Chapter 7: Statistical Estimation

### 7.2: Maximum Likelihood Examples

We spend an entire section just doing examples because maximum likelihood is such a fundamental concept used everywhere (especially machine learning). I promise that the idea is simple: find  $\theta$  that maximizes the likelihood of the data. The computation and notation can be confusing at first though.

#### 7.2.1 MLE Example (Poisson)

##### Example(s)

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from  $\text{Poi}(\theta)$ . (These values might look like  $x_1 = 13, x_2 = 5, x_3 = 6$ , etc...) What is the MLE of  $\theta$ ?

*Solution* Remember that we discussed that the sample mean might be a good estimate of  $\theta$ . If we observed 20 events over 5 units of time, a good estimate for  $\lambda$ , the average number of events per unit of time, would be  $\frac{20}{5} = 4$ . This turns out to be the maximum likelihood estimate!

Let's follow the recipe provided in 7.1.

1. **Compute the likelihood and log-likelihood of data.** To do this, we take the following product of the Poisson PMFs at each sample  $x_i$ , over all the data points:

$$L(\mathbf{x} \mid \theta) = \prod_{i=1}^n p_X(x_i \mid \theta) = \prod_{i=1}^n e^{-\theta} \frac{\theta^{x_i}}{x_i!}$$

Again, this is the probability of seeing  $x_1$ , then  $x_2$ , and so on. This function is pretty hard to differentiate, so to make it easier, let's compute the log-likelihood instead, using the following identities:

$$\log(ab) = \log(a) + \log(b) \quad \log(a/b) = \log(a) - \log(b) \quad \log(a^b) = b \log(a)$$

In most cases, we'll want to optimize the log-likelihood instead of the likelihood (since we don't want to use the product rule of calculus)!

$$\begin{aligned} \ln L(\mathbf{x} \mid \theta) &= \ln \left( \prod_{i=1}^n e^{-\theta} \frac{\theta^{x_i}}{x_i!} \right) && [\text{def of likelihood}] \\ &= \sum_{i=1}^n \ln \left[ e^{-\theta} \frac{\theta^{x_i}}{x_i!} \right] && [\text{log of product is sum of logs}] \\ &= \sum_{i=1}^n [\ln(e^{-\theta}) + \ln(\theta^{x_i}) - \ln x_i!] && [\text{log of product is sum of logs}] \\ &= \sum_{i=1}^n [-\theta + x_i \ln \theta - \ln x_i!] && [\text{other log properties}] \end{aligned}$$

2. Take the partial derivative(s) with respect to  $\theta$  and set to 0. Solve the equation(s).

Now we want to take the derivative of the log likelihood with respect to  $\theta$ , so the derivative of  $-\theta$  is just  $-1$ , and the derivative of  $x_i \ln \theta$  is just  $\frac{x_i}{\theta}$ , because remember  $x_i$  is a constant with respect to  $\theta$ .

$$\frac{\partial}{\partial \theta} \ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \left[ -1 + \frac{x_i}{\theta} \right]$$

And now we want to set the derivative equal to 0, and solve for  $\theta$ , and  $\hat{\theta}$  is actually the estimate that we solve for. We do some algebra, and get  $\frac{1}{n} \sum_{i=1}^n x_i$ , which is actually just the sample mean!

$$\sum_{i=1}^n \left[ -1 + \frac{x_i}{\theta} \right] = 0 \rightarrow -n + \frac{1}{\hat{\theta}} \sum_{i=1}^n x_i = 0 \rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$$

3. Optionally, verify  $\hat{\theta}_{MLE}$  is indeed a (local) maximizer by checking that the second derivative at  $\hat{\theta}_{MLE}$  is negative (if  $\theta$  is a single parameter), or the Hessian (matrix of second partial derivatives) is negative semi-definite (if  $\theta$  is a vector of parameters).

We want to take the second derivative also, because otherwise we don't know if this is a maximum or a minimum. We differentiate the first derivative  $\sum_{i=1}^n [-1 + \frac{x_i}{\theta}]$  again with respect to  $\theta$ , and we notice that because  $\theta^2$  is always positive, the negative of that is always negative, so the second derivative is always less than 0, so that means that it's concave down everywhere. This means that anywhere the derivative is zero is a global maximum, so we've successfully found the global maximum of our likelihood equation.

$$\frac{\partial^2}{\partial \theta^2} \ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \left[ -\frac{x_i}{\theta^2} \right] < 0 \rightarrow \text{concave down everywhere}$$

□

### 7.2.2 MLE Example (Exponential)

#### Example(s)

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from  $\text{Exp}(\theta)$ . (These values might look like  $x_1 = 1.354, x_2 = 3.198, x_3 = 4.312$ , etc...) What is the MLE of  $\theta$ ?

*Solution* Now that we've seen one example, we'll just follow the procedure given in the previous section.

1. Compute the likelihood and log-likelihood of data.

Since we have a continuous distribution, our likelihood is the product of the PDFs:

$$L(\mathbf{x} | \theta) = \prod_{i=1}^n f_X(x_i | \theta) = \prod_{i=1}^n \theta e^{-\theta x_i}$$

The log-likelihood is

$$\ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \ln (\theta e^{-\theta x_i}) = \sum_{i=1}^n [\ln(\theta) - \theta x_i]$$

2. Take the partial derivative(s) with respect to  $\theta$  and set to 0. Solve the equation(s).

$$\frac{\partial}{\partial \theta} \ln L(\mathbf{x} \mid \theta) = \sum_{i=1}^n \left[ \frac{1}{\theta} - x_i \right]$$

Now, we set the derivative to 0 and solve (here we replace  $\theta$  with  $\hat{\theta}$ ):

$$\sum_{i=1}^n \left[ \frac{1}{\hat{\theta}} - x_i \right] = 0 \rightarrow \frac{n}{\hat{\theta}} - \sum_{i=1}^n x_i = 0 \rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^n x_i}$$

This is just the inverse of the sample mean! This makes sense because if the average waiting time was  $1/2$  hours, then the average rate per unit of time  $\lambda$  should be  $\frac{1}{1/2} = 2$  per hour!

3. Optionally, verify  $\hat{\theta}_{MLE}$  is indeed a (local) maximizer by checking that the second derivative at  $\hat{\theta}_{MLE}$  is negative (if  $\theta$  is a single parameter), or the Hessian (matrix of second partial derivatives) is negative semi-definite (if  $\theta$  is a vector of parameters). The second derivative of the log-likelihood just requires us to take one more derivative:

$$\frac{\partial^2}{\partial \theta^2} \ln L(\mathbf{x} \mid \theta) = \sum_{i=1}^n \left[ \frac{-1}{\theta^2} \right] < 0$$

Since the second derivative is negative everywhere, the function is concave down, and any critical point is a global maximum!

□

### 7.2.3 MLE Example (Uniform)

#### Example(s)

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from (continuous)  $\text{Unif}(0, \theta)$ . (These values might look like  $x_1 = 2.325, x_2 = 1.1242, x_3 = 9.262$ , etc...) What is the MLE of  $\theta$ ?

*Solution* It turns out our usual procedure won't work on this example, unfortunately. We'll explain why once we run into the problem!

To compute the likelihood, we first need the individual density functions. Recall

$$f_X(x \mid \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Let's actually define an indicator function for whether or not some boolean condition  $A$  is true or false:

$$I_A = \begin{cases} 1 & A \text{ is true} \\ 0 & A \text{ is false} \end{cases}$$

This way, we can rewrite the uniform density in one line as ( $1/\theta$  for  $0 \leq x \leq \theta$  and 0 otherwise):

$$f_X(x \mid \theta) = \frac{1}{\theta} I_{\{0 \leq x \leq \theta\}}$$

First, we take the product over all data points of the density at that data point, and plug in the density of the uniform distribution. How do we simplify this? First of all, we notice that in every term in the product, there is still a  $\frac{1}{\theta}$ , so multiply it by itself  $n$  times and get  $\frac{1}{\theta^n}$ . How do we multiply indicators? If we want the product of 1's and 0's to be 1, they ALL have to be 1. So,

$$I_{\{0 \leq x_1 \leq \theta\}} \cdot I_{\{0 \leq x_2 \leq \theta\}} \cdots I_{\{0 \leq x_n \leq \theta\}} = I_{\{0 \leq x_1, \dots, x_n \leq \theta\}}$$

and our likelihood is

$$L(\mathbf{x} \mid \theta) = \prod_{i=1}^n f_X(x_i \mid \theta) = \prod_{i=1}^n \frac{1}{\theta} I_{\{0 \leq x_i \leq \theta\}} = \frac{1}{\theta^n} I_{\{0 \leq x_1, \dots, x_n \leq \theta\}}$$

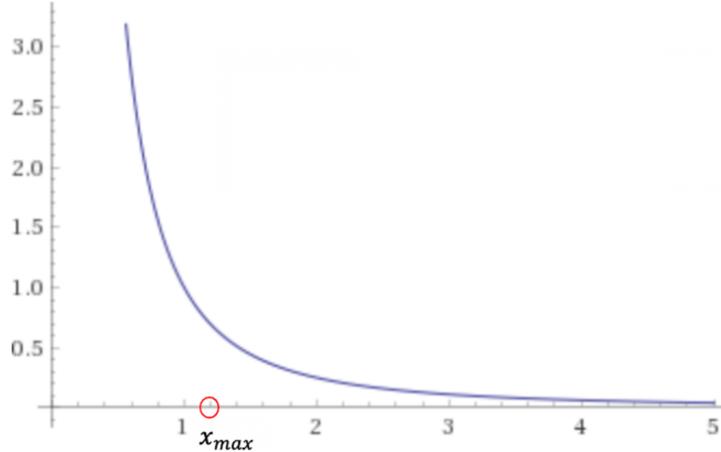
We could take the log-likelihood before differentiating, but this function isn't too bad-looking, so let's take the derivative of this. The  $I_{\{0 \leq x_1, \dots, x_n \leq \theta\}}$  just says the function is  $\frac{1}{\theta^n}$  when it the condition is true and 0 otherwise. So our derivative will just be the derivative of  $\frac{1}{\theta^n}$  when that condition is true and 0 otherwise.

$$\frac{d}{d\theta} L(\mathbf{x} \mid \theta) = -\frac{n}{\theta^{n+1}} I_{\{0 \leq x_1, \dots, x_n \leq \theta\}}$$

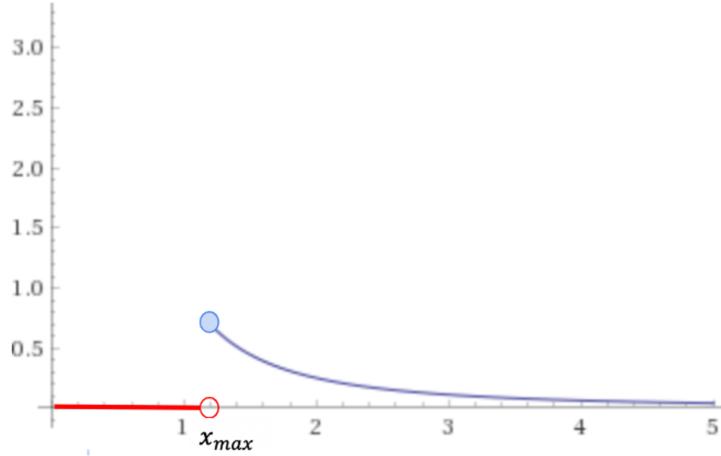
Now, let's set the derivative equal to 0 and solve for  $\theta$ .

$$-\frac{n}{\theta^{n+1}} = 0 \rightarrow \theta = ???$$

There seems to be no value of  $\theta$  that solves this, what's going on? Let's plot the likelihood. First, we plot just  $(\frac{1}{\theta})^n$  (not quite the likelihood) where  $\theta$  is on the  $x$ -axis:



Above is a graph of  $\frac{1}{\theta^n}$ , and so if we wanted to maximize this function, we should choose  $\theta = 0$ . But remember that the likelihood, was  $\frac{1}{\theta^n} I_{\{0 \leq x_1, \dots, x_n \leq \theta\}}$ , which can also be written as  $\frac{1}{\theta^n} I_{\{x_{\max} \leq \theta\}}$ , because all the samples are  $\leq \theta$  if and only if the maximum is. Below is the graph of the actual likelihood:



Notice that multiplying by the indicator function just kept the function as is when the condition was true,  $x_{\max} \leq \theta$ , but zeroed it out otherwise. So now we can see that our maximum likelihood estimator should be  $\hat{\theta}_{MLE} = x_{\max} = \max\{x_1, x_2, \dots, x_n\}$ , since it achieves the highest value.

Why? Remember  $x_1, \dots, x_n \sim \text{Unif}(0, \theta)$ , so  $\theta$  has to be at least as large as the biggest  $x_i$ , because if it's not as large as the biggest  $x_i$ , then it would have been impossible for that uniform to produce that largest  $x_i$ . For example, if our samples were  $x_1 = 2.53, x_2 = 8.55, x_3 = 4.12$ , our  $\theta$  had to be at least 8.55 (the maximum sample), because if it were 7 for example, then  $\text{Unif}(0, 7)$  could not possibly generate the sample 8.55.

So our likelihood remember  $\frac{1}{\theta^n}$  would have preferred as small a  $\theta$  as possible to maximize it, but subject to  $\theta \geq x_{\max}$ . Therefore the “compromise” was reached by making them equal!

I'd like to point out this is a special case because the range of the uniform distribution depends on its parameter(s)  $a, b$  (the range of  $\text{Unif}(a, b)$  is  $[a, b]$ ). On the other hand, most of our distributions like Poisson or Exponential have the same range no matter what value the value of their parameters. For example, the range of  $\text{Poi}(\lambda)$  is always  $\{0, 1, 2, \dots\}$  and the range of  $\text{Exp}(\lambda)$  is always  $[0, \infty)$ , independent of  $\lambda$ .

Therefore, most MLE problems will be similar to the first two examples rather than this complicated one!

□

## Chapter 7: Statistical Estimation

### 7.3: Method of Moments Estimation

#### 7.3.1 Sample Moments

Maximum likelihood estimation (MLE) as you saw had a nice intuition but mathematically is a bit tedious to solve. We'll learn a different technique for estimating parameters called the Method of Moments (MoM). The early definitions and strategy may be confusing at first, but we provide several examples which hopefully makes things clearer!

Recall the definition of a moment from 5.6:

##### Definition 7.3.1: Moments (Review)

Let  $X$  be a random variable and  $c \in \mathbb{R}$  a scalar. Then: The  $k^{\text{th}}$  moment of  $X$  is:

$$\mathbb{E}[X^k]$$

and the  $k^{\text{th}}$  moment of  $X$  (about  $c$ ) is:

$$\mathbb{E}[(X - c)^k]$$

Usually, we are interested in the first moment of  $X$ :  $\mu = \mathbb{E}[X]$ , and the second moment of  $X$  about  $\mu$ :  $\text{Var}(X) = \mathbb{E}[(X - \mu)^2]$ .

Now since we are in the statistics portion of the class, we will define a sample moment.

##### Definition 7.3.2: Sample Moments

Let  $X$  be a random variable, and  $c \in \mathbb{R}$  a scalar. Let  $x_1, \dots, x_n$  be iid realizations (samples) from  $X$ . The  $k^{\text{th}}$  **sample moment of  $X$**  is

$$\frac{1}{n} \sum_{i=1}^n x_i^k$$

The  $k^{\text{th}}$  **sample moment of  $X$  (about  $c$ )** is

$$\frac{1}{n} \sum_{i=1}^n (x_i - c)^k$$

For example, the first sample moment is just the **sample mean**, and the second sample moment about the sample mean is the **sample variance**.

### 7.3.2 Method of Moments (MoM)

Recall that the first four moments tell us a lot about the distribution (see 5.6). The first moment is the expectation or mean, and the second moment tells us the variance.

Suppose we only need to estimate one parameter  $\theta$  (you might have to estimate two for example  $\theta = (\mu, \sigma^2)$  for the  $\mathcal{N}(\mu, \sigma^2)$  distribution). The idea behind **Method of Moments (MoM)** estimation is that: to find a good estimator, we should have the true and sample moments match as best we can. That is, I should choose the parameter  $\theta$  such that the first true moment  $\mathbb{E}[X]$  is equal to the first sample moment  $\bar{x}$ . Examples always make things clearer!

#### Example(s)

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from  $X \sim \text{Unif}(0, \theta)$  (continuous). (These values might look like  $x_1 = 3.21, x_2 = 5.11, x_3 = 4.33$ , etc.) What is the MoM estimator of  $\theta$ ?

*Solution* We then set the first true moment to the first sample moment as follows (recall that  $\mathbb{E}[\text{Unif}(a, b)] = \frac{a+b}{2}$ ):

$$\mathbb{E}[X] = \frac{\theta}{2} = \frac{1}{n} \sum_{i=1}^n x_i$$

Solving for  $\theta$  we get:

$$\hat{\theta}_{MoM} = \frac{2}{n} \sum_{i=1}^n x_i$$

That's all there is to it! Much simpler than MLE right?

This estimator makes sense intuitively once you think about it for a bit: if we take the sample mean of a bunch of  $\text{Unif}(0, \theta)$  rvs, we expect to get close to the true mean:  $\frac{1}{n} \sum_{i=1}^n x_i \rightarrow \theta/2$  (by the Law of Large Numbers). Hence, a good estimator for  $\theta$  would just be twice the sample mean!

Notice that in this case, the MoM estimator **disagrees with the MLE** we derived in 7.2!

$$\frac{2}{n} \sum_{i=1}^n x_i = \hat{\theta}_{MoM} \neq \hat{\theta}_{MLE} = x_{\max}$$

□

What if you had two parameters instead of just one? Well, then you would set the first true moment equal to the first sample moment (as we just did), but also the second true moment equal to the second sample moment! We'll see an example of this below. But basically, if we have  $k$  parameters to estimate, we need  $k$  equations to solve for these  $k$  unknowns!

**Definition 7.3.3: Method of Moments Estimation**

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations (samples) from probability mass function  $p_X(t; \theta)$  (if  $X$  is discrete), or from density  $f_X(t; \theta)$  (if  $X$  continuous), where  $\theta$  is a parameter (or vector of parameters).

We then define the **method of moments (MoM)** estimator  $\hat{\theta}_{MoM}$  of  $\theta = (\theta_1, \dots, \theta_k)$  to be a solution (if it exists) to the  $k$  simultaneous equations where, for  $j = 1, \dots, k$ , we set the  $j^{th}$  (true) moment equal to the  $j^{th}$  sample moment:

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{n} \sum_{i=1}^n x_i \\ &\dots \\ \mathbb{E}[X^k] &= \frac{1}{n} \sum_{i=1}^n x_i^k\end{aligned}$$

**Example(s)**

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from  $X \sim \text{Exp}(\theta)$ . (These values might look like  $x_1 = 3.21, x_2 = 5.11, x_3 = 4.33$ , etc.) What is the MoM estimator of  $\theta$ ?

*Solution* We have  $k = 1$  (since only one parameter). **We then set the first true moment to the first sample moment** as follows (recall that  $\mathbb{E}[\text{Exp}(\lambda)] = \frac{1}{\lambda}$ ):

$$\mathbb{E}[X] = \frac{1}{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$$

Solving for  $\theta$  (just taking inverse), we get:

$$\hat{\theta}_{MoM} = \frac{1}{\frac{1}{n} \sum_{i=1}^n x_i}$$

Notice that in this case, the MoM estimator **agrees with the MLE** (Maximum Likelihood Estimator), hooray!

$$\hat{\theta}_{MoM} = \hat{\theta}_{MLE} = \frac{1}{\frac{1}{n} \sum_{i=1}^n x_i}$$

Isn't this way better/easier than MLE? □

**Example(s)**

Let's say  $x_1, x_2, \dots, x_n$  are iid samples from  $X \sim \text{Poi}(\theta)$ . (These values might look like  $x_1 = 13, x_2 = 5, x_3 = 4$ , etc.) What is the MoM estimator of  $\theta$ ?

*Solution* We have  $k = 1$  (since only one parameter). **We then set the first true moment to the first**

**sample moment** as follows (recall that  $\mathbb{E}[\text{Poi}(\lambda)] = \lambda$ ):

$$\mathbb{E}[X] = \theta = \frac{1}{n} \sum_{i=1}^n x_i$$

“Solving” for  $\theta$ , we get:

$$\hat{\theta}_{MoM} = \frac{1}{n} \sum_{i=1}^n x_i$$

In this case, again, the MoM estimator **agrees with the MLE!** Again, much easier than MLE :).  $\square$

Now, we’ll do an example where there is more than one parameter.

### Example(s)

Let’s say  $x_1, x_2, \dots, x_n$  are iid samples from  $X \sim \mathcal{N}(\theta_1, \theta_2)$ . (These values might look like  $x_1 = -2.321, x_2 = 1.112, x_3 = -5.221$ , etc.) What is the MoM estimator of the vector  $\theta = (\theta_1, \theta_2)$  ( $\theta_1$  is the mean, and  $\theta_2$  is the variance)?

*Solution* We have  $k = 2$  (since now we have two parameters  $\theta_1 = \mu$  and  $\theta_2 = \sigma^2$ ). Notice  $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ , so rearranging we get  $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2$ . Let’s solve for  $\theta_1$  first:

Again, we set the first true moment to the first sample moment:

$$\mathbb{E}[X] = \theta_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

“Solving” for  $\theta_1$ , we get:

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n x_i$$

Now let’s use our result for  $\hat{\theta}_1$  to solve for  $\hat{\theta}_2$  (recall that  $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = \theta_2 + \theta_1^2$ )

$$\mathbb{E}[X^2] = \theta_2 + \theta_1^2 = \frac{1}{n} \sum_{i=1}^n x_i^2$$

Solving for  $\theta_2$ , and plugging in our result for  $\hat{\theta}_1$ , we get:

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \left( \frac{1}{n} \sum_{i=1}^n x_i \right)^2$$

If you were to use maximum likelihood to estimate the mean and variance of a Normal distribution, you would get the same result!  $\square$

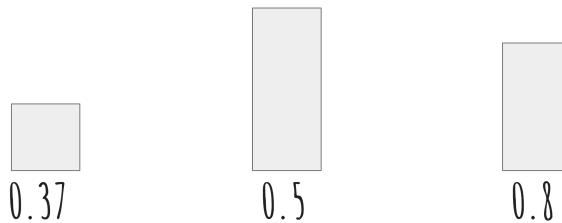
## Chapter 7: Statistical Estimation

### 7.4: The Beta and Dirichlet Distributions

We'll take a quick break after learning two ways (MLE and MoM) to estimate unknown parameters! In the next section, we'll learn yet another approach. But that approach requires us to learn at least one other distribution, the Beta distribution, which will be the focus of this section.

#### 7.4.1 The Beta Random Variable

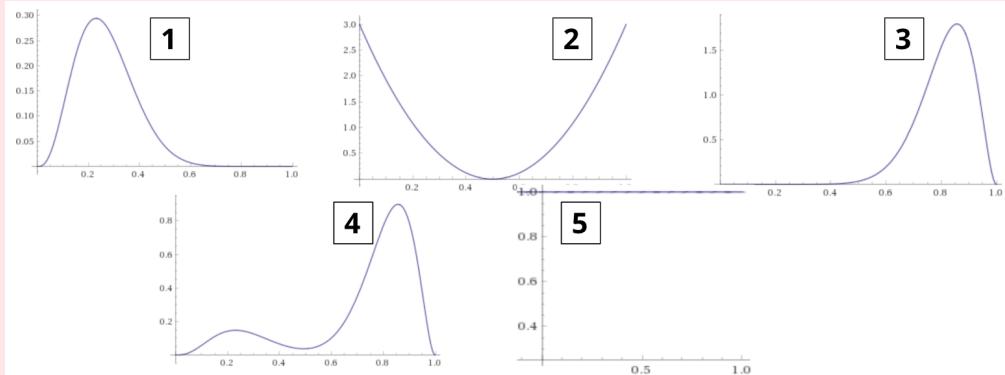
Suppose you want to model your belief on the unknown probability  $X$  of heads. You could assign, for example, a probability distribution as follows:



This figure below shows that you **believe** that  $X = \mathbb{P}(\text{head})$  is most likely to be 0.5, somewhat likely to be 0.8, and least likely to be 0.37. That is,  $X$  is a *discrete* random variable with range  $\Omega_X = \{0.37, 0.5, 0.8\}$  and  $p_X(0.37) + p_X(0.5) + p_X(0.8) = 1$ . This is a probability distribution on a probability of heads!

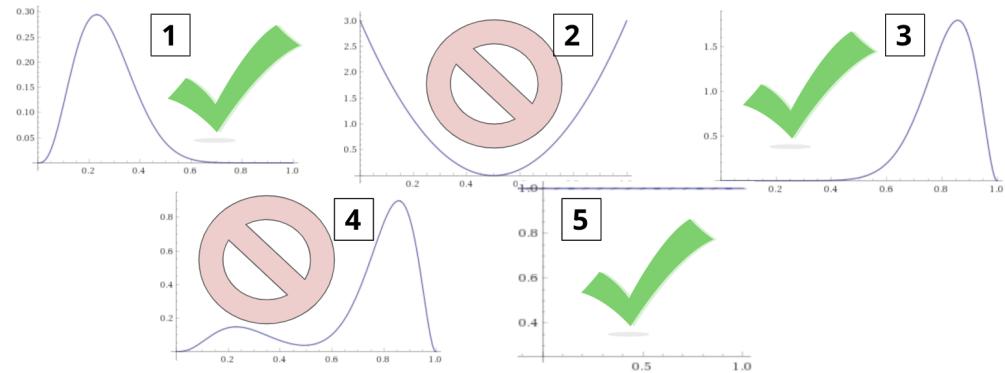
Now what if we want  $\mathbb{P}(\text{head})$  to be open to any value in  $[0, 1]$  (which we should want; having it be just one of three values is arbitrary and unrealistic)? The answer is that we need a **continuous** random variable (with range  $[0, 1]$  because probabilities can be any number within this range)! Let's try to see how we might define a new distribution which might do a good job modelling this belief! Let's see which of the following shapes might be appropriate (or not).

## Example(s)



Suppose you flipped the coin  $n$  times and observed  $k$  heads. Which of the above density functions have a “shape” which would be *reasonable* to model your belief?

*Solution* Here is the answer:



It's important to note that Distributions 2 and 4 are **invalid**, because there is no possible sequence of flips that could result in the belief that is "bi-modal" (have two peaks in the graph of the distribution). Your belief should have a single peak at your highest belief, and go down on both sides from there.

For instance, if you believe that the probability of (getting heads) is most likely around 0.25, we have Distribution 1 in the figure above. Similarly, if you think that it's most likely around 0.85, we have Distribution 3. Or, more interestingly, if you have NO idea what the probability might be and you want to make every probability equally likely, you could use a **Uniform distribution** like in Distribution 5.

□

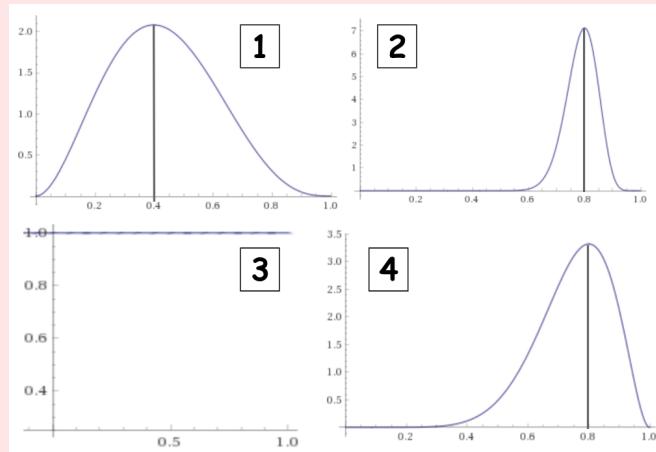
Let's have some practice with concrete numbers now.

## Example(s)

If you flip a coin with unknown probability of heads  $X$ , what does your belief distribution look like if:

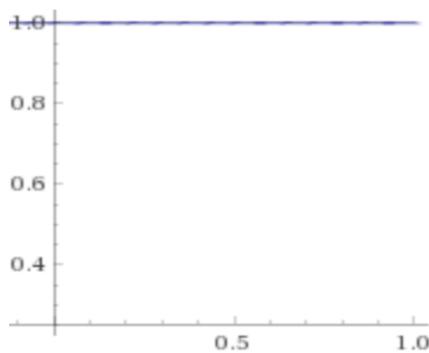
- You didn't observe anything?
- You observed 8 heads and 2 tails?
- You observed 80 heads and 20 tails?
- You observed 2 heads and 3 tails?

Match the four distributions below to the four scenarios above. Note the vertical bars in each distribution represents where the **mode** (the point with highest density) is, as that's probably what we want to estimate as our probability of heads!



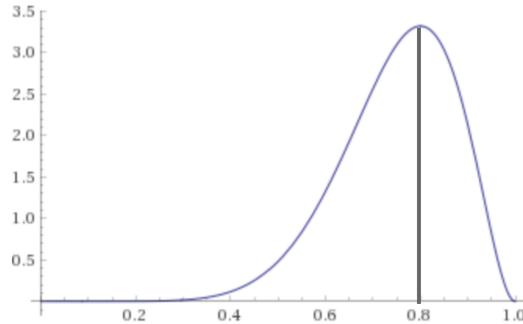
## Solution

- You didn't observe anything? **Answer:** Distribution 3.



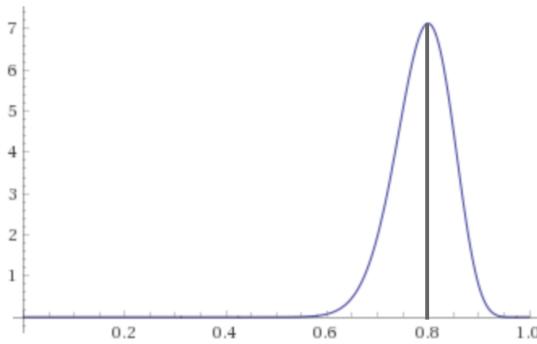
**Explanation:** Since we haven't observed anything yet, we shouldn't have preference over any particular value. This is encoded as a continuous  $\text{Unif}(0, 1)$  distribution.

- You observed 8 heads and 2 tails? **Answer:** Distribution 4.



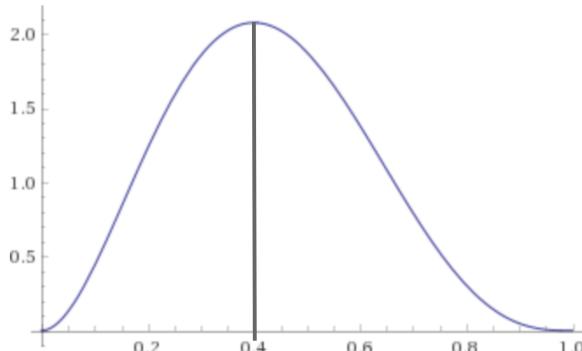
**Explanation:** We expect  $\mathbb{P}(\text{head})$  to be around  $\frac{8}{8+2} = 0.8$ , so either distribution 2 or 4 is reasonable. BUT we have much **uncertainty** (since we only flipped it 10 times) so we have a wider distribution. Note that 0.8 is the MODE, not the mean.

- You observed 80 heads and 20 tails? **Answer:** Distribution 2.



**Explanation:** We expect  $\mathbb{P}(\text{head})$  to be around  $\frac{80}{80+20} = 0.8$  again, but now since we have way more flips, we can be more certain that the probability is more likely to be 0.8 (thus the "spread" is smaller than the previous).

- You observed 2 heads and 3 tails? **Answer:** Distribution 1.



**Explanation:** We expect  $\mathbb{P}(\text{head})$  to be around  $\frac{2}{2+3} = 0.4$ , but since 5 flips are rather limited, we have much uncertainty in the actual distribution, therefore the "spread" is quite large!

□

There is a continuous distribution/rv with range  $[0, 1]$  that parametrizes probability distributions over a probability just like this, based on two parameters  $\alpha$  and  $\beta$ , which allow you to account for how many heads and tails you've seen!

#### Definition 7.4.1: Beta RV

$X \sim \text{Beta}(\alpha, \beta)$ , if and only if  $X$  has the following density function (and range  $\Omega_X = [0, 1]$ ):

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$X$  is typically the belief distribution about some unknown probability of success, where we pretend we've seen  $\alpha-1$  successes and  $\beta-1$  failures. Hence the mode (most likely value of the probability/point with highest density)  $\arg \max_{x \in [0,1]} f_X(x)$ , is

$$\text{mode}[X] = \frac{\alpha - 1}{(\alpha - 1) + (\beta - 1)}$$

Also note the following:

- The first term in the pdf,  $\frac{1}{B(\alpha, \beta)}$  is just a normalizing constant (ensures the pdf to integrates to 1). It is called the Beta function, and so our random variable is called a Beta random variable.
- There is an annoying “off-by-1” issue:  $(\alpha - 1)$  heads and  $\beta - 1$  tails), so when choosing these parameters, be careful (examples below)!
- $x$  is the probability of success, and  $(1 - x)$  is the probability of failure.

## 7.4.2 Beta Random Variable Examples

### Example(s)

If you flip a coin with unknown probability of heads  $X$ , identify the parameters of the most appropriate Beta distribution to model your belief:

- You didn't observe anything?
- You observed 8 heads and 2 tails?
- You observed 80 heads and 20 tails?
- You observed 2 heads and 3 tails?

### Solution

- You didn't observe anything?  $\text{Beta}(0 + 1, 0 + 1) \equiv \text{Beta}(1, 1) \equiv \text{Unif}(0, 1) \rightarrow \text{NO mode}$  (because it follows the Uniform distribution; every point has same density).

- You observed 8 heads and 2 tails?  $\text{Beta}(8+1, 2+1) \equiv \text{Beta}(9, 3) \rightarrow \text{mode} = \frac{(9-1)}{(9-1)+(3-1)} = \frac{8}{10}$
- You observed 80 heads and 20 tails?  $\text{Beta}(80+1, 20+1) \equiv \text{Beta}(81, 21) \rightarrow \text{mode} = \frac{(81-1)}{(81-1)+(21-1)} = \frac{80}{100}$
- You observed 2 heads and 3 tails?  $\text{Beta}(2+1, 3+1) \equiv \text{Beta}(3, 4) \rightarrow \text{mode} = \frac{(3-1)}{(3-1)+(4-1)} = \frac{2}{5}$

Note all the off-by-1's in the parameters! □

### 7.4.3 The Dirichlet Random Vector

The Dirichlet random vector generalizes the Beta random variable to having a belief distribution over  $p_1, p_2, \dots, p_r$  (like in the multinomial distribution so  $\sum p_i = 1$ ), and has  $r$  parameters  $\alpha_1, \alpha_2, \dots, \alpha_r$ . It has the similar interpretation of pretending you've seen  $\alpha_i - 1$  outcomes of type  $i$ .

**Definition 7.4.2: Dirichlet RV**

$X \sim \text{Dir}(\alpha_1, \alpha_2, \dots, \alpha_r)$ , if and only if  $X$  has the following density function:

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{1}{B(\alpha)} \prod_{i=1}^r x_i^{\alpha_i-1}, & x_i \in (0, 1) \text{ and } \sum_{i=1}^r x_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

This is a generalization of the Beta random variable from 2 outcomes to  $r$ . The random vector  $X$  is typically the belief distribution about some unknown probabilities of the different outcomes, where we pretend we saw  $\alpha_1 - 1$  outcomes of type 1,  $\alpha_2 - 1$  outcomes of type 2,  $\dots$ , and  $\alpha_r - 1$  outcomes of type  $r$ . Hence, the mode of the distribution is the vector,  $\arg \max_{x \in [0,1]^d \text{ and } \sum x_i = 1} f_{\mathbf{X}}(\mathbf{x})$ , is

$$\text{mode}[\mathbf{X}] = \left( \frac{\alpha_1 - 1}{\sum_{i=1}^r (\alpha_i - 1)}, \frac{\alpha_2 - 1}{\sum_{i=1}^r (\alpha_i - 1)}, \dots, \frac{\alpha_r - 1}{\sum_{i=1}^r (\alpha_i - 1)} \right)$$

Also note the following:

- Similar to the Beta RV, the first term in the pdf,  $\frac{1}{B(\alpha)}$  is just a normalizing constant (ensures the pdf integrates to 1), where  $\alpha = (\alpha_1, \dots, \alpha_r)$ .
- Notice that this is the probability distribution over the random vector  $x_i$ 's, which is the vector of probabilities, so they must sum to 1 ( $\sum_{i=1}^r x_i = 1$ ).

## Chapter 7: Statistical Estimation

### 7.5: Maximum A Posteriori Estimation

We've seen two ways now to estimate unknown parameters of a distribution. Maximum likelihood estimation (MLE) says that we should find the parameter  $\theta$  that maximizes the likelihood ("probability") of seeing the data, whereas the method of moments (MoM) says that we should match as many moments as possible (mean, variance, etc.). Now, we learn yet another (and final) technique for estimation that will cover (there are many more...).

#### 7.5.1 Maximum A Posteriori (MAP) Estimation

Maximum a Posteriori (MAP) estimation is quite different from the estimation techniques we learned so far (MLE/MoM), because it allows us to **incorporate prior knowledge** into our estimate. Suppose you wanted to estimate the unknown probability of heads on a coin  $\theta$ : using MLE, you may flip the head 20 times and observe 13 heads, giving an estimate of 13/20. But what if your friend had flipped the coin before and observed 10 heads and 2 tails: how can you (formally) incorporate her information into your estimate? Or what if you just believed in general that coins were more likely to be fair  $\theta = 0.5$  than unfair? We'll see how to do this below!

##### 7.5.1.1 Intuition

In Maximum Likelihood Estimation (MLE), we used iid samples  $\mathbf{x} = (x_1, \dots, x_n)$  from some distribution with unknown parameter(s)  $\theta$ , in order to estimate  $\theta$ .

$$\hat{\theta}_{MLE} = \arg \max_{\theta} L(\mathbf{x} \mid \theta) = \arg \max_{\theta} \prod_{i=1}^n f_X(x_i \mid \theta)$$

Note: Recall that, using the English description, how we found  $\hat{\theta}_{MLE}$  is: we computed this likelihood, which is the probability of seeing the data given the parameter  $\theta$ , and we chose the "best"  $\theta$  that maximized this likelihood.

You might have been thinking: shouldn't we be trying to maximize " $\mathbb{P}(\theta \mid x)$ " instead? Well, this doesn't make sense **unless  $\Theta$  is a R.V.**! And this is where Maximum A Posteriori (MAP) Estimation comes in.

So far, for MLE and MoM estimation, we assumed  $\theta$  was fixed but unknown. This is called the **Frequentist framework** where we only estimate our parameter based on **data alone**, and  $\theta$  is not a random variable. Now, we are in the **Bayesian framework**, meaning that our unknown parameter is a random variable  $\Theta$ . This means, we will have some belief distribution  $\pi_\Theta(\theta)$  (think of this as a density function over all possible values of the parameter), and after observing data  $\mathbf{x}$ , we will have a new/updated belief distribution  $\pi_\Theta(\theta \mid \mathbf{x})$ . Let's see a picture of what MAP is going to do first, before getting more into the math and formalism.

## Example(s)

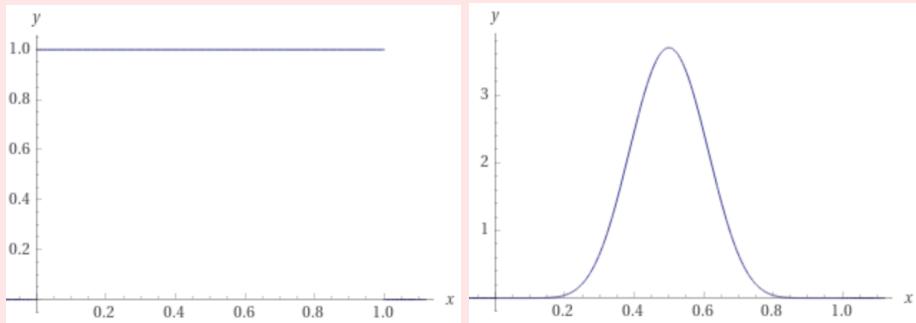
We'll see the idea of MAP being applied to our typical coin example. Suppose we are trying to estimate the unknown parameter for the probability of heads on a coin: that is,  $\theta$  in  $\text{Ber}(\theta)$ . We are going to treat the parameter as a *random variable* (before in MLE/MoM we treated it as a *fixed* unknown quantity), so we'll call it  $\Theta$  (capitalized  $\theta$ ).

1. **We must have a prior belief distribution  $\pi_\Theta(\theta)$  over possible values that  $\Theta$  could take on.**

The range of  $\Theta$  in our case is  $\Omega_\Theta = [0, 1]$ , because the probability of heads must be in this interval. Hence, when we plot the density function of  $\Theta$ , the  $x$ -axis will range from 0 to 1.

On a piece of paper, please sketch a density function that you might have for this probability of heads without yet seeing any data (coin flips). There are two reasonable shapes for this PDF:

- The  $\text{Unif}(0, 1) = \text{Beta}(1, 1)$  distribution (left picture below).
- Some Beta distribution where  $\alpha = \beta$ , since most coins in this world are fair. Let's say  $\text{Beta}(11, 11)$ ; meaning we pretend we've seen 10 heads and 10 tails (right picture below).



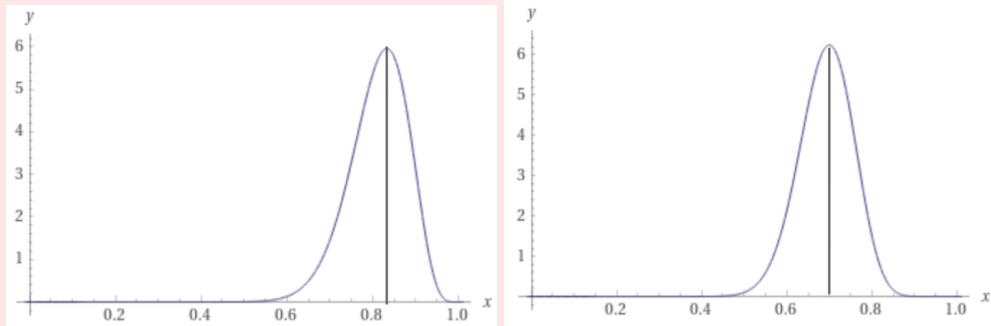
2. **We will observe some iid samples  $\mathbf{x} = (x_1, \dots, x_n)$ .**

Again, for the Bernoulli distribution, these will be a sequence of  $n$  1's and 0's representing heads or tails. Suppose we observed  $n = 30$  samples, in which  $\sum_{i=1}^n x_i = 25$  were heads and  $n - \sum_{i=1}^n x_i = 5$  were tails.

3. **We will combine our prior knowledge and the data to create a posterior belief distribution  $\pi_\Theta(\theta | \mathbf{x})$ .**

Sketch two density functions for this posterior: one using the  $\text{Beta}(1, 1)$  prior above, and one using the  $\text{Beta}(11, 11)$  prior above. We'll compare these.

- If our prior distribution was  $\Theta \sim \text{Beta}(1, 1)$  (meaning we pretend we didn't see anything yet), then our posterior distribution should be  $\Theta | \mathbf{x} \sim \text{Beta}(26, 6)$  (meaning we saw 25 heads and 5 tails total).
- If our prior distribution was  $\Theta \sim \text{Beta}(11, 11)$  (meaning we saw 10 heads and 10 tails beforehand), then our posterior distribution should be  $\Theta | \mathbf{x} \sim \text{Beta}(36, 16)$  (meaning we saw 35 heads and 15 tails total).



4. We'll give our **MAP** estimate as the **mode** of this posterior distribution. Hence, the name "Maximum a Posteriori".

- If we used the  $\Theta \sim \text{Beta}(1, 1)$  prior, we ended up with the  $\Theta | \mathbf{x} \sim \text{Beta}(26, 6)$  posterior, and our MAP estimate is defined to be the **mode** of the distribution, which occurs at  $\hat{\theta}_{MAP} = \frac{25}{30} \approx 0.833$  (left picture above). You may notice that this would give the same as the MLE: we'll examine this more later!
- If we used the  $\Theta \sim \text{Beta}(11, 11)$  prior, we ended up with the  $\Theta | \mathbf{x} \sim \text{Beta}(36, 16)$  posterior, our MAP estimate is defined to be the **mode** of the distribution, which occurs at  $\hat{\theta}_{MAP} = \frac{35}{50} = 0.70$  (right picture above).

Hopefully you now see the process and idea behind MAP: We have a prior belief on our unknown parameter, and after observing data, we update our belief distribution and take the mode (most likely value)! Our estimate definitely depends on the prior distribution we choose (which is often arbitrary).

### 7.5.1.2 Derivation

We chose a Beta prior, and ended up with a Beta posterior, which made sense intuitively given our definition of the Beta distribution. But how do we prove this? We'll see the math behind MAP now (quite short), and see the same example again but mathematically rigorous now.

**MAP Idea:** Actually, unknown parameter(s) is a random variable  $\Theta$ . We have a **prior** distribution (prior belief on  $\Theta$  before seeing data)  $\pi_\Theta(\theta)$  and **posterior** distribution (given data; updated belief on  $\Theta$  after observing some data)  $\pi_\Theta(\theta | \mathbf{x})$ .

By Bayes' Theorem,

$$\pi_\Theta(\theta | \mathbf{x}) = \frac{L(\mathbf{x} | \theta) \pi_\Theta(\theta)}{\mathbb{P}(\mathbf{x})} \propto L(\mathbf{x} | \theta) \pi_\Theta(\theta)$$

Recall that  $\pi_\Theta$  is just a PDF or PMF over possible values of  $\Theta$ . In other words, now we are maximizing the **posterior distribution**  $\pi_\Theta(\theta | x)$ , where  $\Theta$  has a PMF/PDF. That is, we are finding the **mode** of the density/mass function. Note that since the denominator  $\mathbb{P}(x)$  in the expression above **does not** depend on  $\theta$ , we can just maximize the numerator  $L(\mathbf{x} | \theta) \pi_\Theta(\theta)$ ! Therefore:

$$\hat{\theta}_{MAP} = \arg \max_{\theta} \pi_{\Theta}(\theta \mid \mathbf{x}) = \arg \max_{\theta} L(\mathbf{x} \mid \theta) \pi_{\Theta}(\theta)$$

#### Definition 7.5.1: Maximum A Posteriori (MAP) Estimation

Let  $x = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t; \Theta = \theta)$  (if  $X$  discrete), or from density  $f_X(t; \Theta = \theta)$  (if  $X$  continuous), where  $\Theta$  is the random variable representing the parameter (or vector of parameters). We define the Maximum A Posteriori (MAP) estimator  $\hat{\theta}_{MAP}$  of  $\Theta$  to be the parameter which maximizes the **posterior** distribution of  $\Theta$  given the data.

$$\hat{\theta}_{MAP} = \arg \max_{\theta} \pi_{\Theta}(\theta \mid \mathbf{x}) = \arg \max_{\theta} L(\mathbf{x} \mid \theta) \pi_{\Theta}(\theta)$$

That is, it's exactly the same as maximum likelihood, except instead of just maximizing the likelihood, we are maximizing the likelihood multiplied by the prior!

Now we'll see a similar coin-flipping example, but deriving the MAP estimate mathematically and building even more intuition. I encourage you to try each part out before reading the answers!

#### 7.5.1.3 Example

##### Example(s)

- (a) Suppose our samples are  $\mathbf{x} = (0, 0, 1, 1, 0)$ , from  $\text{Ber}(\theta)$ , where  $\theta$  is unknown. Assume  $\theta$  is unrestricted; that is,  $\theta \in (0, 1)$ . What is the MLE for  $\theta$ ?
- (b) Suppose we impose the restriction that  $\theta \in \{0.2, 0.5, 0.7\}$ . What is the MLE for  $\theta$ ?
- (c) Assume  $\Theta$  is restricted as in part (b) (but now a random variable for MAP). Suppose we have a (discrete) prior  $\pi_{\Theta}(0.2) = 0.1, \pi_{\Theta}(0.5) = 0.01$ , and  $\pi_{\Theta}(0.7) = 0.89$ . What is the MAP for  $\theta$ ?
- (d) Show that we can make the MAP whatever we like, by finding a prior over  $\{0.2, 0.5, 0.7\}$  so that the MAP is 0.2, another so that it is 0.5, and another so that it is 0.7.
- (e) Typically, for the Bernoulli/Binomial distribution, if we use MAP, we want to be able to get any value  $\in (0, 1)$ , not just ones in a finite set such as  $\{0.2, 0.5, 0.7\}$ . So we need a (continuous) prior distribution with range  $(0, 1)$  instead of our discrete one. We assign  $\Theta \sim \text{Beta}(\alpha, \beta)$  with parameters  $\alpha, \beta > 0$  and density  $\pi_{\Theta}(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$  for  $\theta \in (0, 1)$ . Recall the mode of a  $W \sim \text{Beta}(\alpha, \beta)$  random variable is  $\frac{\alpha-1}{(\alpha-1)+(\beta-1)}$  (the mode is the value with highest density  $\arg \max_w f_W(w)$ ).

Suppose  $x_1, \dots, x_n$  are iid from a Bernoulli distribution with unknown parameter. Recall the MLE is  $\frac{k}{n}$ , where  $k = \sum x_i$  (the total number of successes). Show that the posterior  $\pi_{\Theta}(\theta \mid x)$  has a  $\text{Beta}(k + \alpha, n - k + \beta)$  distribution, and find the MAP estimator.

- (f) Recall that  $\text{Beta}(1, 1) \equiv \text{Unif}(0, 1)$  (pretend we saw 1 – 1 heads and 1 – 1 tails ahead of time). If we used this as the prior, how would the MLE and MAP compare?

- (g) Since the posterior is also a Beta Distribution, we call Beta the **conjugate prior** to the Bernoulli/Binomial distribution's parameter  $p$ . Interpret  $\alpha, \beta$  as to how they affect our estimate. This is a really special property: if the prior distribution multiplied by the likelihood results in a posterior distribution in the same family (with different parameters), then we say that distribution is the conjugate prior to the distribution we are estimating.
- (h) As the number of samples goes to infinity, what is the relationship between the MLE and MAP? What does this say about our **prior** when  $n$  is small, or  $n$  is large?
- (i) Which do you think is "better", MLE or MAP?

*Solution*

- (a) Suppose our samples are  $\mathbf{x} = (0, 0, 1, 1, 0)$ , from  $\text{Ber}(\theta)$ , where  $\theta$  is unknown. Assume  $\theta$  is unrestricted; that is,  $\theta \in (0, 1)$ . What is the MLE for  $\theta$ ?

- Answer:  $\frac{2}{5}$ . We just find the likelihood of the data, which is the probability of observing 2 heads and 3 tails, and find the  $\theta$  that maximizes it.

$$L(\mathbf{x} \mid \theta) = \theta^2(1-\theta)^3$$

$$\hat{\theta}_{MLE} = \arg \max_{\theta \in [0,1]} \theta^2(1-\theta)^3 = \frac{2}{5}$$

- (b) Suppose we impose the restriction that  $\theta \in \{0.2, 0.5, 0.7\}$ . What is the MLE for  $\theta$ ?

- Answer: 0.5. We need to find which of the three acceptable  $\theta$  values maximizes the likelihood, and since there are only finitely many, we can just plug them all in and compare!

$$L(\mathbf{x} \mid 0.2) = (0.2^2 0.8^3) = 0.02048$$

$$L(\mathbf{x} \mid 0.5) = (0.5^2 0.5^3) = 0.03125$$

$$L(\mathbf{x} \mid 0.7) = (0.7^2 0.3^3) = 0.01323$$

$$\hat{\theta}_{MLE} = \arg \max_{\theta \in \{0.2, 0.5, 0.7\}} L(\mathbf{x} \mid \theta) = 0.5$$

- (c) Assume  $\Theta$  is restricted as in part (b) (but now a random variable for MAP). Suppose we have a (discrete) prior  $\pi_\Theta(0.2) = 0.1, \pi_\Theta(0.5) = 0.01$ , and  $\pi_\Theta(0.7) = 0.89$ . What is the MAP for  $\theta$ ?

- Answer: 0.7. Instead of maximizing just the likelihood, we need to maximize the likelihood times the prior. Again, since there are only finitely many values, we just plug them in!

$$\pi_\Theta(0.2 \mid x) = L(\mathbf{x} \mid 0.2)\pi_\Theta(0.2) = (0.2^2 0.8^3)(0.1) = 0.0020480$$

$$\pi_\Theta(0.5 \mid x) = L(\mathbf{x} \mid 0.5)\pi_\Theta(0.5) = (0.5^2 0.5^3)(0.01) = 0.0003125$$

$$\pi_\Theta(0.7 \mid x) = L(\mathbf{x} \mid 0.7)\pi_\Theta(0.7) = (0.7^2 0.3^3)(0.89) = 0.0117747$$

Note the effect of this prior - by setting  $\pi_\Theta(0.7)$  so high and the other two values, we actually get a different maximizer. This is the effect of the prior on the MAP estimate (which was completely arbitrary)!

- (d) Show that we can make the MAP whatever we like, by finding a prior over  $\{0.2, 0.5, 0.7\}$  so that the MAP is 0.2, another so that it is 0.5, and another so that it is 0.7.

- Answer: Choose  $\pi_\Theta(\theta) = 1$  for the  $\theta$  you want! This shows that the prior really does make a difference, and that MAP and MLE are indeed different techniques.

- (e) Typically, for the Bernoulli/Binomial distribution, if we use MAP, we want to be able to get any value  $\in (0, 1)$ , not just ones in a finite set such as  $\{0.2, 0.5, 0.7\}$ . So we need a (continuous) prior distribution with range  $(0, 1)$  instead of our discrete one. We assign  $\Theta \sim \text{Beta}(\alpha, \beta)$  with parameters  $\alpha, \beta > 0$  and density  $\pi_\Theta(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$  for  $\theta \in (0, 1)$ . Recall the mode of a  $W \sim \text{Beta}(\alpha, \beta)$  random variable is  $\frac{\alpha-1}{(\alpha-1)+(\beta-1)}$  (the mode is the value with highest density  $\arg \max_w f_W(w)$ ).

Suppose  $x_1, \dots, x_n$  are iid from a Bernoulli distribution with unknown parameter. Recall the MLE is  $\frac{k}{n}$ , where  $k = \sum x_i$  (the total number of successes). Show that the posterior  $\pi_\Theta(\theta | x)$  has a  $\text{Beta}(k + \alpha, n - k + \beta)$  distribution, and find the MAP estimator.

- Answer:  $\hat{\theta}_{MAP} = \frac{k + (\alpha - 1)}{n + (\alpha - 1) + (\beta - 1)}$ . We first have to write out what the posterior distribution is, which is proportional to just the prior times the likelihood:

$$\pi_\Theta(\theta | x) \propto L(\mathbf{x} | \theta) \cdot \pi_\Theta(\theta)$$

$$= \binom{n}{k} \theta^k (1-\theta)^{n-k} \cdot \left( \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \right)$$

$$\propto \theta^{(k+\alpha)-1} (1-\theta)^{(n-k+\beta)-1}$$

The first to second line comes from noticing  $L(\mathbf{x} | \theta)$  is just the probability of seeing exactly  $k$  successes out of  $n$  (binomial PMF), and plugging in our equation for  $\pi_\Theta$  (beta density). The second to third line comes from dropping the normalizing constants (that don't depend on  $\theta$ ), which we can do because we only care to maximize this over  $\theta$ . If you stare closely at that last equation, it actually proportional to the PDF of a Beta distribution with different parameters! Our posterior is hence  $\text{Beta}(k + \alpha, n - k + \beta)$  since PDFs uniquely define a distribution (there is only one normalizing constant that would make it integrate to 1). The MAP estimator is the mode of this posterior Beta distribution, which is given by the formula:

$$\hat{\theta}_{MAP} = \frac{k + \alpha - 1}{(k + \alpha - 1) + (n - k + \beta - 1)} = \frac{k + (\alpha - 1)}{n + (\alpha - 1) + (\beta - 1)}$$

Try staring at this to see why this might make sense. We'll explain it more in part (g)!

- (f) Recall that  $\text{Beta}(1, 1) \equiv \text{Unif}(0, 1)$  (pretend we saw 1 - 1 heads and 1 - 1 tails ahead of time). If we used this as the prior, how would the MLE and MAP compare?

- Answer: They would be the same! From our previous question, if  $\alpha = \beta = 1$ , then

$$\hat{\theta}_{MAP} = \frac{k + (\alpha - 1)}{n + (\alpha - 1) + (\beta - 1)} = \frac{k}{n} = \hat{\theta}_{MLE}$$

This is because we don't have any prior information essentially, by saying each value is equally likely!

(g) Since the posterior is also a Beta Distribution, we call Beta the **conjugate prior** to the Bernoulli/Binomial distribution's parameter  $p$ . Interpret  $\alpha, \beta$  as to how they affect our estimate. This is a really special property: if the prior distribution multiplied by the likelihood results in a posterior distribution in the same family (with different parameters), then we say that distribution is the conjugate prior to the distribution we are estimating.

- Answer: The interpretation is: pretend we saw  $\alpha - 1$  heads ahead of time, and  $\beta - 1$  tails ahead of time. Then our **total** number of heads is  $k + (\alpha - 1)$  (real + fake) and our **total** number of trials is  $n + (\alpha + \beta - 2)$  (real + fake), so that's our estimate! That's how prior information was factored in to our estimator, rather than just using what we actually saw in the data.

(h) As the number of samples goes to infinity, what is the relationship between the MLE and MAP? What does this say about our **prior** when  $n$  is small, or  $n$  is large?

- Answer: They become equal! The prior is important if we don't have much data, but as we get more, the evidence overwhelms the prior. You can imagine that if we only flipped the coin 5 times, the prior would play a huge role in our estimate. But if we flipped the coin 10,000 times, any (small) prior wouldn't really change our estimate.

(i) Which do you think is "better", MLE or MAP?

- Answer: There is no right answer. There are two main schools in statistics: Bayesians and Frequentists.
- Frequentists prefer MLE since they don't believe you should be putting a prior belief on anything, and you should only make judgment based on what you've seen. They believe the parameter being estimated is a **fixed quantity**.
- On the other hand, Bayesians prefer MAP, since they can incorporate their prior knowledge into the estimation. Hence the parameter being estimated is a **random variable**, and we seek the mode - the value with the highest probability or density. An example would be estimating the probability of heads of a coin - is it reasonable to assume it is more likely fair than not? If so, what distribution should we put on the parameter space?
- Anyway, in the long run, the prior "washes out", and the only thing that matters is the likelihood; the observed data. For small sample sizes like this, the prior significantly influences the MAP estimate. However, as the number of samples goes to infinity, the MAP and MLE are equal.

□

### 7.5.2 Exercises

1. Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid samples from  $\text{Exp}(\Theta)$  where  $\Theta$  is a random variable (not fixed). Note that the range of  $\Theta$  should be  $\Omega_\Theta = [0, \infty)$  (the average rate of events per unit time), so any prior we choose should have this range.
  - (a) Using the prior  $\Theta \sim \text{Gamma}(r, \lambda)$  (for some arbitrary but known parameters  $r, \lambda > 0$ ), show that the posterior distribution  $\Theta | \mathbf{x}$  also follows a Gamma distribution and identify its parameters (by computing  $\pi_\Theta(\theta | \mathbf{x})$ ). Then, explain this sentence: "The Gamma distribution is the conjugate prior for the rate parameter of the Exponential distribution". Hint: This can be done in just a few lines!

- (b) Now derive the MAP estimate for  $\Theta$ . The mode of a  $\text{Gamma}(s, \nu)$  distribution is  $\frac{s-1}{\nu}$ . Hint: This should be just one line using your answer to part (a).
- (c) Explain how this MAP estimate differs from the MLE estimate (recall for the Exponential distribution it was just the inverse sample mean  $\frac{n}{\sum_{i=1}^n x_i}$ ), and provide an interpretation of  $r$  and  $\lambda$  as to how they affect the estimate.

**Solution:**

- (a) Remember that the posterior is proportional to likelihood times prior, and the density of  $Y \sim \text{Exp}(\theta)$  is  $f_Y(y | \theta) = \theta e^{-\theta y}$ :

$$\begin{aligned} \pi_\Theta(\theta | \mathbf{x}) &\propto L(\mathbf{x} | \theta) \pi_\Theta(\theta) && [\text{def of posterior}] \\ &= \left( \prod_{i=1}^n \theta e^{-\theta x_i} \right) \cdot \frac{\lambda^r}{(r-1)!} \theta^{r-1} e^{-\lambda \theta} && [\text{def of } \text{Exp}(\theta) \text{ likelihood + } \text{Gamma}(r, \lambda) \text{ pdf}] \\ &\propto \theta^n e^{-\theta \sum x_i} \theta^{r-1} e^{-\lambda \theta} && [\text{algebra, drop constants}] \\ &= \theta^{(n+r)-1} e^{-(\lambda + \sum x_i) \theta} \end{aligned}$$

Therefore  $\Theta | \mathbf{x} \sim \text{Gamma}(n+r, \lambda + \sum x_i)$ , since the final line above is proportional to the pdf for the gamma distribution (minus normalizing constant).

It is the conjugate prior because, assuming a Gamma prior for the Exponential likelihood, we end up with a Gamma posterior. That is, the prior and posterior are in the same family of distributions (Gamma) with different parameters.

- (b) Just citing the mode of a Gamma given, we get

$$\hat{\theta}_{MAP} = \frac{n+r-1}{\lambda + \sum x_i}$$

- (c) We see how the estimate changes from the MLE of  $\hat{\theta}_{MLE} = \frac{n}{\sum x_i}$ : pretend we saw  $r-1$  extra events over  $\lambda$  units of time. (Instead of waiting for  $n$  events, we waited for  $n+r-1$ , and instead of  $\sum x_i$  as our total time, we now have  $\lambda + \sum x_i$  units of time).

## Chapter 7: Statistical Estimation

### 7.6: Properties of Estimators I

Now that we have all these techniques to compute estimators, you might be wondering which one is the “best”. Actually, a better question would be: how can we determine which *estimator* is “better” (rather than the technique)? There are even more different ways to estimate besides MLE/MoM/MAP, and in different scenarios, different techniques may work better. In these notes, we will consider some properties of estimators that allow us to compare their “goodness”.

#### 7.6.1 Bias

The first estimator property we’ll cover is Bias. The bias of an estimator measures whether or not in expectation, the estimator will be equal to the true parameter.

##### Definition 7.6.1: Bias

Let  $\hat{\theta}$  be an estimator for  $\theta$ . The **bias** of  $\hat{\theta}$  as an estimator for  $\theta$  is

$$\text{Bias}(\hat{\theta}, \theta) = \mathbb{E}[\hat{\theta}] - \theta$$

If

- $\text{Bias}(\hat{\theta}, \theta) = 0$ , or equivalently  $\mathbb{E}[\hat{\theta}] = \theta$ , then we say  $\hat{\theta}$  is an **unbiased** estimator of  $\theta$ .
- $\text{Bias}(\hat{\theta}, \theta) > 0$ , then  $\hat{\theta}$  typically overestimates  $\theta$ .
- $\text{Bias}(\hat{\theta}, \theta) < 0$ , then  $\hat{\theta}$  typically underestimates  $\theta$ .

Let’s go through some examples!

##### Example(s)

First, recall that, if  $x_1, \dots, x_n$  are iid realizations from  $\text{Poi}(\theta)$ , then the MLE and MoM were both the sample mean.

$$\hat{\theta} = \hat{\theta}_{MLE} = \hat{\theta}_{MoM} = \frac{1}{n} \sum_{i=1}^n x_i$$

Show that  $\hat{\theta}$  is an unbiased estimator of  $\theta$ .

*Solution*

$$\begin{aligned}
 \mathbb{E}[\hat{\theta}] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n x_i\right] \\
 &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[x_i] && [\text{LoE}] \\
 &= \frac{1}{n} \sum_{i=1}^n \theta && [\mathbb{E}[\text{Poi}(\theta)] = \theta] \\
 &= \frac{1}{n} n\theta \\
 &= \theta
 \end{aligned}$$

This makes sense: the average of your samples should be “on-target” for the true average!  $\square$

Example(s)

First, recall that, if  $x_1, \dots, x_n$  are iid realizations from (continuous)  $\text{Unif}(0, \theta)$ , then

$$\hat{\theta}_{MLE} = x_{\max} \quad \hat{\theta}_{MoM} = 2 \cdot \frac{1}{n} \sum_{i=1}^n x_i$$

Sure,  $\hat{\theta}_{MLE}$  maximizes the likelihood, so in a way  $\hat{\theta}_{MLE}$  is better than  $\hat{\theta}_{MoM}$ . But, what are the biases of these estimators? Before doing any computation: do you think  $\hat{\theta}_{MLE}$  and  $\hat{\theta}_{MoM}$  are overestimates, underestimates, or unbiased?

*Solution* I actually think  $\hat{\theta}_{MoM}$  is spot-on since the average of the samples should be close to  $\theta/2$ , and multiplying by 2 would seem to give the true  $\theta$ . On the other hand,  $\hat{\theta}_{MLE}$  might be a bit of an underestimate, since we probably wouldn’t have  $\theta$  be exactly the largest (maybe a little larger).

• **Bias of the maximum likelihood estimator.**

Recall from 5.10 that the density of the largest order statistic (i.e. the maximum of the sample) is

$$f_{X_{\max}}(y) = n F_X^{n-1}(y) f_X(y) = n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}$$

You could also instead first find the CDF of  $X_{\max}$  as

$$F_{X_{\max}}(y) = \mathbb{P}(X_{\max} \leq y) = \mathbb{P}(X_i \leq y)^n = F_X(y)^n = \left(\frac{y}{\theta}\right)^n$$

since the max is less than or equal to a value if and only if each of them is, then take the derivative. Using this density function we can compute the expected value of the  $\hat{\theta}_{MLE}$  as follows:

$$\mathbb{E}[\hat{\theta}_{MLE}] = \mathbb{E}[X_{\max}] = \int_0^\theta y \left(n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\right) dy = \frac{n}{\theta^n} \int_0^\theta y^n dy = \frac{n}{\theta^n} \left[\frac{1}{n+1} y^{n+1}\right]_0^\theta = \frac{n}{n+1} \theta$$

This makes sense because if I had 3 samples from  $\text{Unif}(0, 1)$  for example, I would expect them at  $1/4, 2/4, 3/4$ , and so it would be  $\frac{n}{n+1}$  as my expected max. Similarly, if I had 4 samples, then I would expect them at  $1/5, 2/5, 3/5, 4/5$ , and so it would again be  $\frac{n}{n+1}$  as my expected max.

Finally,

$$\text{Bias}(\hat{\theta}_{MLE}, \theta) = \mathbb{E}[\hat{\theta}_{MLE}] - \theta = \frac{n}{n+1}\theta - \theta = -\frac{1}{n+1}\theta$$

- **Bias of the method of moments estimator.**

$$\mathbb{E}[\hat{\theta}_{MOM}] = \mathbb{E}\left[2 \cdot \frac{1}{n} \sum_{i=1}^n x_i\right] = \frac{2}{n} \sum_{i=1}^n \mathbb{E}[x_i] = \frac{2}{n} n \frac{\theta}{2} = \theta$$

$$\text{Bias}(\hat{\theta}_{MOM}, \theta) = \mathbb{E}[\hat{\theta}_{MOM}] - \theta = \theta - \theta = 0$$

- **Analysis of Results**

This means that  $\hat{\theta}_{MLE}$  typically underestimates  $\theta$  and  $\hat{\theta}_{MOM}$  is an unbiased estimator of  $\theta$ . But something isn't quite right...

Suppose the samples are  $x_1 = 1, x_2 = 9, x_3 = 2$ . Then, we would have

$$\hat{\theta}_{MLE} = \max\{1, 9, 2\} = 9 \quad \hat{\theta}_{MOM} = \frac{2}{3}(1 + 9 + 2) = 8$$

However, based on our sample, the MoM estimator is impossible. If the actual parameter were 8, then that means that the distribution we pulled the sample from is  $\text{Unif}(0, 8)$ , in which case the likelihood that we get a 9 is 0. But we did see a 9 in our sample. So, even though  $\hat{\theta}_{MOM}$  is unbiased, it still yields an impossible estimate. This just goes to show that finding the right estimator is actually quite tricky.

A good solution would be to “de-bias” the MLE by scaling it appropriately. If you decided to have a new estimator based on the MLE:

$$\hat{\theta} = \frac{n+1}{n} \hat{\theta}_{MLE}$$

you would now get an unbiased estimator that can't be wrong! But now it does not maximize the likelihood anymore...

Actually, the MLE is what we say to be “**asymptotically unbiased**”, meaning unbiased in the limit. This is because

$$\text{Bias}(\hat{\theta}_{MLE}, \theta) = -\frac{1}{n+1}\theta \rightarrow 0$$

as  $n \rightarrow \infty$ . So usually we might just leave it because we can't seem to win...

□

### Example(s)

Recall that if  $x_1, \dots, x_n \sim \text{Exp}(\theta)$  are iid, our MLE and MoM estimates were both the inverse sample mean:

$$\hat{\theta} = \hat{\theta}_{MLE} = \hat{\theta}_{MOM} = \frac{1}{\bar{x}} = \frac{n}{\sum_{i=1}^n x_i}$$

What can you say about the bias of this estimator?

*Solution*

$$\begin{aligned}
\mathbb{E}[\hat{\theta}] &= \mathbb{E}\left[\frac{n}{\sum_{i=1}^n x_i}\right] \\
&\geq \frac{n}{\sum_{i=1}^n \mathbb{E}[x_i]} && [\text{Jensen's inequality}] \\
&= \frac{n}{\sum_{i=1}^n \frac{1}{\theta}} && \left[\mathbb{E}[\text{Exp}(\theta)] = \frac{1}{\theta}\right] \\
&= \frac{n}{n \frac{1}{\theta}} \\
&= \theta
\end{aligned}$$

The inequality comes from Jensen's (section 6.3): since  $g(x_1, \dots, x_n) = \frac{1}{\sum_{i=1}^n x_i}$  is convex (at least in the positive octant when all  $x_i \geq 0$ ), we have that  $\mathbb{E}[g(x_1, \dots, x_n)] \geq g(\mathbb{E}[x_1], \mathbb{E}[x_2], \dots, \mathbb{E}[x_n])$ . It is convex for a reason similar to that  $\frac{1}{x}$  is a convex function. So  $\mathbb{E}[\hat{\theta}] \geq \theta$  systematically, and we typically have an overestimate.  $\square$

## 7.6.2 Variance and Mean Squared Error

We are often also interested in how much a estimator varies (we would like it to be unbiased and have small variance to that it is more accurate). One metric that captures this property of estimators is an estimators variance.

The variance of an estimator  $\hat{\theta}$  is

$$\text{Var}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

This is just the definition of variance applied to the random variable  $\hat{\theta}$  and isn't actually a new definition.

But maybe instead of just computing the variance, we want a slightly different metric which instead measures the squared difference from the *actual* estimator and not just its expectation:

$$\mathbb{E}[(\hat{\theta} - \theta)^2]$$

We call this property the mean squared error (MSE), and it is related to both bias and variance! Look closely at the difference: if  $\hat{\theta}$  is unbiased, then  $\mathbb{E}[\hat{\theta}] = \theta$  and the MSE and variance are actually equal!

### Definition 7.6.2: Mean Squared Error

The mean squared error of an estimator  $\hat{\theta}$  of  $\theta$  is

$$\text{MSE}(\hat{\theta}, \theta) = \mathbb{E}[(\hat{\theta} - \theta)^2]$$

If  $\hat{\theta}$  is an unbiased estimator of  $\theta$  (i.e.  $\mathbb{E}[\hat{\theta}] = \theta$ ), then you can see that  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta})$ . In fact, in general  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2$ .

This leads to what is known as the “Bias-Variance Tradeoff” in machine learning and statistics. Usually, we want to minimize MSE, and these two quantities are often inversely related. That is, decreasing one leads to an increase in the other, and finding the balance will minimize the MSE. It’s hard to see why that might be the case since we aren’t working with as complex of estimators (we’re just learning the basics!).

*Proof of Alternate MSE Formula.* We will prove that  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2$ .

$$\begin{aligned}
 \text{MSE}(\hat{\theta}, \theta) &= \mathbb{E}[(\hat{\theta} - \theta)^2] && [\text{def of MSE}] \\
 &= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}] + \mathbb{E}[\hat{\theta}] - \theta\right)^2\right] && [\text{add and subtract } \mathbb{E}[\hat{\theta}]] \\
 &= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}]\right)^2\right] + 2\mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}]\right)\left(\mathbb{E}[\hat{\theta}] - \theta\right)\right] + \mathbb{E}\left[\left(\mathbb{E}[\hat{\theta}] - \theta\right)^2\right] && [(a+b)^2 = a^2 + 2ab + b^2] \\
 &= \text{Var}(\hat{\theta}) + 0 + \mathbb{E}[\text{Bias}(\hat{\theta}, \theta)^2] && [\text{def of var, bias, } \mathbb{E}[\hat{\theta} - \mathbb{E}[\hat{\theta}]] = 0] \\
 &= \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2
 \end{aligned}$$

□

It is highly desirable that the MSE of an estimator is low! We want a small difference between  $\hat{\theta}$  and  $\theta$ . Use the formula above to compute MSE:  $\text{Var}(\hat{\theta})$  is something we learned how to compute a long time ago, and there are several examples of bias computations above.

### Example(s)

First, recall that, if  $x_1, \dots, x_n$  are iid realizations from  $\text{Poi}(\theta)$ , then the MLE and MoM were both the sample mean.

$$\hat{\theta} = \hat{\theta}_{MLE} = \hat{\theta}_{MoM} = \frac{1}{n} \sum_{i=1}^n x_i$$

Compute the MSE of  $\hat{\theta}$  as an estimator of  $\theta$ .

*Solution* To compute the MSE, let’s compute the bias and variance separately. Earlier, we showed that

$$\text{Bias}(\hat{\theta}, \theta) = \mathbb{E}[\hat{\theta}] - \theta = \theta - \theta = 0$$

Now for the variance:

$$\begin{aligned}
 \text{Var}(\hat{\theta}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
 &= \frac{1}{n} \sum_{i=1}^n \text{Var}(x_i) && [\text{variance adds if independent}] \\
 &= \frac{1}{n^2} \sum_{i=1}^n \theta && [\text{Var}(\text{Poi}(\theta)) = \theta] \\
 &= \frac{1}{n^2} n\theta \\
 &= \frac{\theta}{n}
 \end{aligned}$$

Finally, using both of those results:

$$\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2 = \frac{\theta}{n} + 0^2 = \frac{\theta}{n}$$

□

## Chapter 7: Statistical Estimation

### 7.7: Properties of Estimators II

We'll discuss even more desirable properties of estimators. Last time we talked about bias, variance, and MSE. Bias measured whether or not, in expectation, our estimator was equal to the true value of  $\theta$ . MSE measured the expected squared difference between our estimator and the true value of  $\theta$ . If our estimator was unbiased, then the MSE of our estimator was precisely the variance.

#### 7.7.1 Consistency

##### Definition 7.7.1: Consistency

An estimator  $\hat{\theta}_n$  (depending on  $n$  iid samples) of  $\theta$  is said to be **consistent** if it converges (in probability) to  $\theta$ . That is, for any  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = 0$$

Basically, as  $n \rightarrow \infty$ ,  $\hat{\theta}_n$  in the limit will be extremely close to  $\theta$ .

As usual, we'll do some examples to see how to show this.

##### Example(s)

Recall that, if  $x_1, \dots, x_n$  are iid realizations from (continuous)  $\text{Unif}(0, \theta)$ , then

$$\hat{\theta}_n = \hat{\theta}_{n, MoM} = 2 \cdot \frac{1}{n} \sum_{i=1}^n x_i$$

Let  $\varepsilon > 0$ . Show that  $\hat{\theta}_n$  is a consistent estimator of  $\theta$ .

##### *Solution*

Since  $\hat{\theta}_n$  is unbiased, we have that

$$\mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = \mathbb{P}(|\hat{\theta}_n - \mathbb{E}[\hat{\theta}_n]| > \varepsilon)$$

because we can replace  $\theta$  with the expected value of the estimator. Now, we can apply Chebyshev's inequality (6.1) to see that

$$\mathbb{P}(|\hat{\theta}_n - \mathbb{E}[\hat{\theta}_n]| > \varepsilon) \leq \frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2}$$

Now, we can take out the  $2^2$  from the estimator's expression and are left only with the variance of the sample

mean, which is always just  $\frac{\sigma^2}{n} = \frac{\text{Var}(x_i)}{n}$ .

$$\mathbb{P}(|\hat{\theta}_n - \mathbb{E}[\hat{\theta}_n]| > \varepsilon) \leq \frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2} = \frac{2^2 \text{Var}(\frac{1}{n} \sum_{i=1}^n x_i)}{\varepsilon^2} = \frac{4 \cdot \text{Var}(x_i)/n}{\varepsilon^2}$$

So now we take the limit with this expression.

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) \leq \lim_{n \rightarrow \infty} \frac{4 \cdot \text{Var}(x_i)/n}{\varepsilon^2} = 0$$

So,  $\hat{\theta}_{n,MoM}$  is a consistent estimator of  $\theta$ . □

We're also going to show that the MLE estimator is consistent!

### Example(s)

Recall that, if  $x_1, \dots, x_n$  are iid realizations from (continuous)  $\text{Unif}(0, \theta)$ , then

$$\hat{\theta}_n = \hat{\theta}_{n,MLE} = \max\{x_1, \dots, x_n\}$$

Let  $\varepsilon > 0$ . Show that  $\hat{\theta}_n$  is a consistent estimator of  $\theta$ .

### Solution

In this case, we cannot use Chebyshev's inequality unfortunately, because the maximum likelihood estimator is not unbiased. The CDF for  $\hat{\theta}_n$  is

$$F_{\hat{\theta}_n}(t) = \mathbb{P}(\hat{\theta}_n \leq t)$$

which is the probability that each individual sample is less than  $t$  because only in that case will the max be less than  $t$ , and we have independence so we can say

$$\mathbb{P}(\hat{\theta}_n \leq t) = \mathbb{P}(X_1 \leq t) \mathbb{P}(X_2 \leq t) \dots \mathbb{P}(X_n \leq t)$$

This is just the CDF of  $X_i$  to the  $n$ -th power, where the CDF of  $\text{Unif}(0, \theta)$  is just  $\frac{t}{\theta}$  (see the distribution sheet):

$$F_{\hat{\theta}_n}(t) = F_X^n(t) = \begin{cases} 0, & t < 0 \\ (\frac{t}{\theta})^n, & 0 \leq t \leq \theta \\ 1, & t > \theta \end{cases}$$

There are two ways we can have the absolute value from before be greater than epsilon

$$\mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = \mathbb{P}(\hat{\theta}_n > \theta + \varepsilon) + \mathbb{P}(\hat{\theta}_n < \theta - \varepsilon)$$

The first term is 0, because there's no way our estimator is greater than  $\theta + \varepsilon$ , as it's never going to be greater than  $\theta$  by definition (the samples are between 0 and  $\theta$  so there's no way the max of the samples is greater than  $\theta$ ). So, now we can just use the CDF on the right term, and just plug in for  $t$ :

$$\mathbb{P}(\hat{\theta}_n > \theta + \varepsilon) + \mathbb{P}(\hat{\theta}_n < \theta - \varepsilon) = \mathbb{P}(\hat{\theta}_n < \theta - \varepsilon) = \begin{cases} (\frac{\theta - \varepsilon}{\theta})^n, & \varepsilon < \theta \\ 0, & \varepsilon \geq \theta \end{cases}$$

We can assume that  $\varepsilon$  is less than  $\theta$  because we really only care when  $\varepsilon$  is very very small, so we have that

$$\mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n$$

Thus, when we take the limit as  $n$  approaches infinity, we see that in the parenthesis, we have a number less than 1, and we raise it to the  $n$ -th power, so it goes to 0

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = 0$$

So,  $\hat{\theta}_{n,MLE}$  is also a consistent estimator of  $\theta$ .

□

Now we've seen that, even though the MLE and MoM estimators of  $\theta$  given iid samples from  $\text{Unif}(0, \theta)$  are different, they are both consistent! That means, as  $n \rightarrow \infty$ , they will both converge to the true parameter  $\theta$ . This is clearly a good property of an estimator.

### 7.7.2 Consistency vs Unbiasedness

You may be wondering, what's the difference between consistency and unbiasedness? I, for one, was very confused about the difference for a while as well. There is, in fact, a subtle difference, which we'll see by comparing estimators for  $\theta$  in the continuous  $\text{Unif}(0, \theta)$  distribution.

Unbiased?	Consistent?	Example
Yes	Yes	$\hat{\theta}_{MoM}$
Yes	No	$2X_1$
No	Yes	$\hat{\theta}_{MLE}$
No	No	$1/X_1^2$

1. For instance, an unbiased and consistent estimator was the MoM for the uniform distribution:  $\hat{\theta}_{n,MoM} = 2\bar{x}$ . We proved it was unbiased in 7.6, meaning it is correct in expectation. It converges to the true parameter (consistent) since the variance goes to 0.
2. However, if you ignore all the samples and just take the first one and multiply it by 2,  $\hat{\theta} = 2X_1$ , it is unbiased (as it is  $2 \cdot \frac{\theta}{2}$ ), but it's not consistent; our estimator doesn't get better and better with more  $n$  because we're not using all  $n$  samples. Consistency requires that as we get more samples, we approach the true parameter.
3. Biased but consistent, on the other hand, was the MLE estimator. We showed its expectation was  $\frac{n}{n+1}\theta$ , which is actually "asymptotically unbiased" since  $\mathbb{E}[\hat{\theta}_{n,MLE}] = \frac{n}{n+1}\theta \rightarrow \theta$  as  $n \rightarrow \infty$ . It does get better and better as  $n \rightarrow \infty$ .
4. Neither unbiased nor consistent would just be some random expression, such as  $\hat{\theta} = \frac{1}{X_1^2}$ .

### 7.7.3 Efficiency

To take about our last topic, efficiency, we first have to define Fisher Information. Efficiency says that our estimator has as low variance as possible. This property combined with consistency and unbiasedness mean that our estimator is on target (unbiased), converges to the true parameter (consistent), and does so as fast as possible (efficient).

#### 7.7.3.1 Fisher Information

##### Definition 7.7.2: Fisher Information

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t | \theta)$  (if  $X$  is discrete), or from density function  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). The **Fisher Information** of the parameter  $\theta$  is defined to be:

$$I(\theta) = n \cdot \mathbb{E} \left[ \left( \frac{\partial \ln L(\mathbf{x} | \theta)}{\partial \theta} \right)^2 \right] = -\mathbb{E} \left[ \frac{\partial^2 \ln L(\mathbf{x} | \theta)}{\partial \theta^2} \right]$$

where  $L(\mathbf{x} | \theta)$  denotes the likelihood of the data given parameter  $\theta$  (defined in 7.1). From Wikipedia, it “is a way of measuring the amount of information that an observable random variable  $X$  carries about an unknown parameter  $\theta$  upon which the probability  $X$  depends”.

That written definition is definitely a mouthful, but if you stop and parse it, you’ll see it’s not too bad to compute. We always take the second derivative of the log-likelihood to confirm that our MLE was a maximizer; now all you have to do is take the expectation to get the Fisher Information. There’s no way though that I can interpret the negative expected value of the second derivative of the log-likelihood, it’s just too gross and messy.

#### 7.7.3.2 The Cramer-Rao Lower Bound (CRLB) and Efficiency

Why did we define that nasty Fisher information? (Actually, it’s much worse when  $\theta$  is a vector instead of a single number, as the second derivative becomes a matrix of second partial derivatives). It would be great if the mean squared error of an estimator  $\hat{\theta}$  was as low as possible. The Cramer-Rao Lower Bound actually gives a lower bound on the variance on any unbiased estimator  $\hat{\theta}$  for  $\theta$ . That is, if  $\hat{\theta}$  is any unbiased estimator for  $\theta$ , there is a minimum possible variance (variance = MSE for unbiased estimators). And if your estimator achieves this lowest possible variance, it is said to be **efficient**. This is also a highly desirable property of estimators. The bound is called the Cramer-Rao Lower Bound.

##### Definition 7.7.3: Cramer-Rao Lower Bound (CRLB)

Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t | \theta)$  (if  $X$  is discrete), or from density function  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). If  $\hat{\theta}$  is an *unbiased* estimator for  $\theta$ , then

$$\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

where  $I(\theta)$  is the Fisher information defined earlier. What this is saying is, for any unbiased estimator  $\hat{\theta}$  for  $\theta$ , the variance (=MSE) is at least  $\frac{1}{I(\theta)}$ . If we achieve this lower bound, meaning our variance is exactly equal to  $\frac{1}{I(\theta)}$ , then we have the best variance possible for our estimate. That is, we have the **minimum variance unbiased estimator (MVUE)** for  $\theta$ .

Since we want to find the lowest variance possible, we can look at this through the frame of finding the estimator's efficiency.

#### Definition 7.7.4: Efficiency

Let  $\hat{\theta}$  be an unbiased estimator of  $\theta$ . The **efficiency** of  $\hat{\theta}$  is

$$e(\hat{\theta}, \theta) = \frac{I(\theta)^{-1}}{\text{Var}(\hat{\theta})} \leq 1$$

This will always be between 0 and 1 because if your variance is equal to the CRLB, then it equals 1, and anything greater will result in a smaller value. A larger variance will result in a smaller efficiency, and we want our efficiency to be as high as possible (1).

An *unbiased* estimator is said to be **efficient** if it achieves the CRLB - meaning  $e(\hat{\theta}, \theta) = 1$ . That is, it could not possibly have a lower variance. Again, the CRLB is not guaranteed for biased estimators.

That was super complicated - let's see how to verify the MLE of  $\text{Poi}(\theta)$  is efficient. It looks scary - but it's just messy algebra!

#### Example(s)

Recall that, if  $x_1, \dots, x_n$  are iid realizations from  $X \sim \text{Poi}(\theta)$  (recall  $\mathbb{E}[X] = \text{Var}(X) = \theta$ ), then

$$\hat{\theta} = \hat{\theta}_{\text{MLE}} = \hat{\theta}_{\text{MoM}} = \frac{1}{n} \sum_{i=1}^n x_i$$

Is  $\hat{\theta}$  efficient?

#### Solution

First, you have to check that it's unbiased, as the CRLB only holds for unbiased estimators...

$$\mathbb{E}[\hat{\theta}] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n x_i\right] = \mathbb{E}[x_i] = \theta$$

...which it is! Otherwise, we wouldn't be able to use this bound. We also need to compute the variance. The variance of the sample mean (the estimator) is just  $\frac{\sigma^2}{n}$ , and the variance of a Poisson is just  $\theta$ .

$$\text{Var}(\hat{\theta}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{\text{Var}(x_i)}{n} = \frac{\theta}{n}$$

Then, we're going to compute that weird Fisher Information, which gives us the CRLB, and see if our variance matches. Remember, we take the second derivative of the log-likelihood, which we did earlier in 7.2

so we're just going to copy over the answer.

$$\frac{\partial^2}{\partial \theta^2} \ln L(\mathbf{x} \mid \theta) = \sum_{i=1}^n -\frac{x_i}{\theta^2}$$

Then, we need to take the expected value of this. It turns out, with some algebra, you get  $-\frac{n}{\theta}$ .

$$\mathbb{E} \left[ \frac{\partial^2 \ln L(\mathbf{x} \mid \theta)}{\partial \theta^2} \right] = \mathbb{E} \left[ \sum_{i=1}^n -\frac{x_i}{\theta^2} \right] = -\frac{1}{\theta^2} \sum_{i=1}^n \mathbb{E}[x_i] = -\frac{1}{\theta^2} n\theta = -\frac{n}{\theta}$$

Our Fisher Information was the **negative** expected value of the second derivative of the log-likelihood, so we just flip the sign to get  $\frac{n}{\theta}$ .

$$I(\theta) = -\mathbb{E} \left[ \frac{\partial^2 \ln L(\mathbf{x} \mid \theta)}{\partial \theta^2} \right] = \frac{n}{\theta}$$

Finally, our efficiency is the inverse of the Fisher Information over the variance:

$$e(\hat{\theta}, \theta) = \frac{I(\theta)^{-1}}{\text{Var}(\hat{\theta})} = \frac{\left(\frac{n}{\theta}\right)^{-1}}{\frac{\theta}{n}} = 1$$

Thus, we've shown that, since our efficiency is 1, our estimator is efficient. That is, it has the best possible variance among all unbiased estimators of  $\theta$ . This, again, is a really good property that we want to have.

To reiterate, this means we cannot possibly do better in terms of mean squared error. Our bias is 0, and our variance is as low as it can possibly go. The sample mean is the unequivocally best estimator for a Poisson distribution, in terms of efficiency, in terms of bias, and MSE (it also happens to be consistent, so there are a lot of good things).

As you can see, showing efficiency is just a bunch of tedious calculations!

□

## Chapter 7: Statistical Estimation

### 7.8: Properties of Estimators III

The final property of estimators we will discuss is called sufficiency. Just like we want our estimators to be consistent and efficient, we also want them to be sufficient.

#### 7.8.1 Sufficiency

We first must define what a statistic is.

##### Definition 7.8.1: Statistic

A **statistic** is any function  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  of samples  $\mathbf{x} = (x_1, \dots, x_n)$ . Examples include:

- $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  (the sum)
- $T(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i$  (the mean)
- $T(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$  (the max/largest value)
- $T(x_1, \dots, x_n) = x_1$  (just take the first sample)
- $T(x_1, \dots, x_n) = 7$  (ignore all samples)

All estimators are statistics because they take in our  $n$  data points and produce a single number. We'll see an example which intuitively explains what it means for a statistic to be sufficient.

Suppose we have iid samples  $\mathbf{x} = (x_1, \dots, x_n)$  from a known distribution with unknown parameter  $\theta$ . Imagine we have two people:

- **Statistician A:** Knows the entire sample, gets  $n$  quantities:  $\mathbf{x} = (x_1, \dots, x_n)$ .
- **Statistician B:** Knows  $T(x_1, \dots, x_n) = t$ , a single number which is a function of the samples. For example, the sum or the maximum of the samples.

Heuristically,  $T(x_1, \dots, x_n)$  is a sufficient statistic if Statistician B can do just as good a job as Statistician A, given “less information”. For example, if the samples are from the Bernoulli distribution, knowing  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  (the number of heads) is just as good as knowing all the individual outcomes, since a good estimate would be the number of heads over the number of total trials! Hence, we don’t actually care the ORDER of the outcomes, just how many heads occurred! The word “sufficient” in English roughly means “enough”, and so this terminology was well-chosen.

Now for the formal definition:

### Definition 7.8.2: Sufficient Statistic

A statistic  $T = T(X_1, \dots, X_n)$  is a **sufficient statistic** if the conditional distribution of  $X_1, \dots, X_n$  given  $T = t$  and  $\theta$  does not depend on  $\theta$ .

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n \mid T = t, \theta) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n \mid T = t)$$

(if  $X_1, \dots, X_n$  are continuous rather than discrete, replace the probability with a density).

To motivate the definition, we'll go back to the previous example. Again, statistician A has all the samples  $x_1, \dots, x_n$  but statistician B only has the single number  $t = T(x_1, \dots, x_n)$ . The idea is, Statistician B only knows  $T = t$ , but since  $T$  is sufficient, doesn't need  $\theta$  to generate new samples  $X'_1, \dots, X'_n$  from the distribution. This is because  $\mathbb{P}(X_1 = x_1, \dots, X_n = x_n \mid T = t, \theta) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n \mid T = t)$  and since she knows  $T = t$ , she knows the conditional distribution (can generate samples)! Now Statistician B has  $n$  iid samples from the distribution, just like Statistician A. So using these samples  $X'_1, \dots, X'_n$ , statistician B can do just a good a job as statistician A with samples  $X_1, \dots, X_n$  (on average). So no one is at any disadvantage. :)

This definition is hard to check, but it turns out that there is a criterion that helps us determine whether a statistic is sufficient:

#### Theorem 7.8.35: Neyman-Fisher Factorization Criterion

Let  $x_1, \dots, x_n$  be iid random samples with likelihood  $L(x_1, \dots, x_n \mid \theta)$ . A statistic  $T = T(x_1, \dots, x_n)$  is sufficient if and only if there exist non-negative functions  $g$  and  $h$  such that:

$$L(x_1, \dots, x_n \mid \theta) = g(x_1, \dots, x_n) \cdot h(T(x_1, \dots, x_n), \theta)$$

That is, the likelihood of the data can be split into a product of two terms: the first term  $g$  can depend on the entire data, but not  $\theta$ , and the second term  $h$  can depend on  $\theta$ , but **only on the data through the sufficient statistic  $T$** . (In other words,  $T$  is the only thing that allows the data  $x_1, \dots, x_n$  and  $\theta$  to interact!) That is, we don't have access to the  $n$  individual quantities  $x_1, \dots, x_n$ ; just the single number ( $T$ , the sufficient statistic).

If you are reading this for the first time, you might not think this is any better... You may be very confused right now, but let's see some examples to clear things up!

But basically, you want to split the likelihood into a product of two terms/functions:

1. For the first term  $g$ , you are allowed to know each individual sample if you want, but NOT  $\theta$ .
2. For the second term  $h$ , you can only know the sufficient statistic (single number)  $T(x_1, \dots, x_n)$  and  $\theta$ . You may not know each individual  $x_i$ .

#### Example(s)

Let  $x_1, \dots, x_n$  be iid random samples from  $\text{Unif}(0, \theta)$  (continuous). Show that the MLE  $\hat{\theta} = T(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$  is a sufficient statistic. (The reason this is true is because we don't need to know each individual sample to have a good estimate for  $\theta$ ; we just need to know the largest!)

*Solution* We saw the likelihood of this continuous uniform in 7.2, which we'll just rewrite:

$$L(x_1, \dots, x_n | \theta) = \prod_{i=1}^n \frac{1}{\theta} I_{\{x_i \leq \theta\}} = \frac{1}{\theta^n} I_{\{x_1, \dots, x_n \leq \theta\}} = \frac{1}{\theta^n} I_{\{\max\{x_1, \dots, x_n\} \leq \theta\}} = \frac{1}{\theta^n} I_{\{T(x_1, \dots, x_n) \leq \theta\}}$$

Choose

$$g(x_1, \dots, x_n) = 1$$

and

$$h(T(x_1, \dots, x_n), \theta) = \frac{1}{\theta^n} I_{\{T(x_1, \dots, x_n) \leq \theta\}}$$

By the Neyman-Fisher Factorization Criterion,  $\hat{\theta}_{MLE} = T = \max\{x_1, \dots, x_n\}$  is sufficient. This is a good property of an estimator!

Notice there is no need for a  $g$  term (that's why it is = 1), because there is no term in the likelihood which just has the data (without  $\theta$ ).

For the  $h$  term, notice that we just need to know the max of the samples  $T(x_1, \dots, x_n)$  to compute  $h$ : we don't actually need to know each individual  $x_i$ .

Notice that here the only interaction between the data and parameter  $\theta$  happens through the sufficient statistic (the max of all the values).  $\square$

### Example(s)

Let  $x_1, \dots, x_n$  be iid random samples from  $\text{Poi}(\theta)$ . Show that  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  is a sufficient statistic, and hence the MLE  $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$  is sufficient as well. (The reason this is true is because we don't need to know each individual sample to have a good estimate for  $\theta$ ; we just need to know how many events happened total!)

*Solution* We take our Poisson likelihood and split it into smaller terms:

$$\begin{aligned} L(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n e^{-\theta} \frac{\theta^{x_i}}{x_i!} = \left( \prod_{i=1}^n e^{-\theta} \right) \left( \prod_{i=1}^n \theta^{x_i} \right) \left( \prod_{i=1}^n \frac{1}{x_i!} \right) = \frac{e^{-n\theta} \theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \\ &= \frac{1}{\prod_{i=1}^n x_i!} \cdot e^{-n\theta} \theta^{T(x_1, \dots, x_n)} \end{aligned}$$

Choose

$$g(x_1, \dots, x_n) = \frac{1}{\prod_{i=1}^n x_i!}$$

and

$$h(T(x_1, \dots, x_n), \theta) = e^{-n\theta} \theta^{T(x_1, \dots, x_n)}$$

By the Neyman-Fisher Factorization Criterion,  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  is sufficient. The mean  $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n x_i}{n} = \frac{T(x_1, \dots, x_n)}{n}$  is as well, since knowing the total number of events and the average number of events is equivalent (since we know  $n$ !).

Notice here we had the  $g$  term handle some function of only  $x_1, \dots, x_n$  but not  $\theta$ .

For the  $h$  term though, we do have  $\theta$  but don't need the individual samples  $x_1, \dots, x_n$  to compute  $h$ . Imagine being just given  $T(x_1, \dots, x_n)$ : now you have enough information to compute  $h$ !

Notice that here the only interaction between the data and parameter  $\theta$  happens through the sufficient statistic (the sum/mean of all the values). We don't actually need to know each individual  $x_i$ .  $\square$

### Example(s)

Let  $x_1, \dots, x_n$  be iid random samples from  $\text{Ber}(\theta)$ . Show that  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  is a sufficient statistic, and hence the MLE  $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$  is sufficient as well. (The reason this is true is because we don't need to know each individual sample to have a good estimate for  $\theta$ ; we just need to know how many heads happened total!)

*Solution* The Bernoulli likelihood comes by using the PMF  $p_X(k) = \theta^k (1 - \theta)^{1-k}$  for  $k \in \{0, 1\}$ . We get this by observing that  $\text{Ber}(\theta) = \text{Bin}(1, \theta)$ .

$$\begin{aligned} L(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n \theta^{x_i} (1 - \theta)^{1-x_i} = \left( \prod_{i=1}^n \theta^{x_i} \right) \left( \prod_{i=1}^n (1 - \theta)^{1-x_i} \right) \\ &= \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i} = \theta^{T(x_1, \dots, x_n)} (1 - \theta)^{n - T(x_1, \dots, x_n)} \end{aligned}$$

Choose

$$g(x_1, \dots, x_n) = 1$$

and

$$h(T(x_1, \dots, x_n), \theta) = \theta^{T(x_1, \dots, x_n)} (1 - \theta)^{n - T(x_1, \dots, x_n)}$$

By the Neyman-Fisher Factorization Criterion,  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  is sufficient. The mean  $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n x_i}{n} = \frac{T(x_1, \dots, x_n)}{n}$  is as well, since knowing the total number of heads and the sample proportion of heads is equivalent (since we know  $n$ !).

Notice that here the only interaction between the data and parameter  $\theta$  happens through the sufficient statistic (the sum/mean of all the values). We don't actually need to know each individual  $x_i$ .  $\square$

## 7.8.2 Properties of Estimators Summary

Here are all the properties of estimators we've talked about from 7.6 to 7.8 (now), in one place!

### Definition 7.8.3: Bias

Let  $\hat{\theta}$  be an estimator for  $\theta$ . The **bias** of  $\hat{\theta}$  as an estimator for  $\theta$  is

$$\text{Bias}(\hat{\theta}, \theta) = \mathbb{E}[\hat{\theta}] - \theta$$

As estimator is **unbiased** if  $\text{Bias}(\hat{\theta}, \theta) = 0$  or equivalently,  $\mathbb{E}[\hat{\theta}] = \theta$ .

#### Definition 7.8.4: Mean Squared Error (MSE)

The **mean squared error** of an estimator  $\hat{\theta}$  of  $\theta$  measures the expected squared error from the true value  $\theta$ , and decomposes into a bias term and variance term. This term results in the phrase "Bias-Variance Tradeoff" - sometimes these are opposing forces and minimizing MSE is a result of choosing the right balance.

$$\text{MSE}(\hat{\theta}, \theta) = \mathbb{E}[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + \text{Bias}^2(\hat{\theta}, \theta)$$

If  $\hat{\theta}$  is an unbiased estimator of  $\theta$ , then the MSE reduces to just:  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta})$ .

#### Definition 7.8.5: Consistency

An estimator  $\hat{\theta}_n$  (depending on  $n$  iid samples) of  $\theta$  is **consistent** if it converges (in probability) to  $\theta$ . That is, for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon) = 0$$

#### Definition 7.8.6: Efficiency

An *unbiased* estimator  $\hat{\theta}$  is **efficient** if it achieves the **Cramer-Rao Lower Bound**, meaning it has the lowest variance possible.

$$e(\hat{\theta}, \theta) = \frac{I(\theta)^{-1}}{\text{Var}(\hat{\theta})} = 1 \iff \text{Var}(\hat{\theta}) = \frac{1}{I(\theta)} = \frac{1}{-\mathbb{E}\left[\frac{\partial^2 \ln L(\mathbf{x}|\theta)}{\partial \theta^2}\right]}$$

#### Definition 7.8.7: Sufficiency

An estimator  $\hat{\theta} = T(x_1, \dots, x_n)$  is **sufficient** if it satisfies the **Neyman-Fisher Factorization Criterion**. That is, there exist non-negative functions  $g$  and  $h$  such that:

$$L(x_1, \dots, x_n | \theta) = g(x_1, \dots, x_n) \cdot h(\hat{\theta}, \theta)$$

## Chapter 8: Statistical Inference

### 8.1: Confidence Intervals

We've talked about several ways to estimate unknown parameters, and desirable properties. But there is just one problem now: even if our estimator had all the good properties, the probability that our estimator for  $\theta$  is *exactly* correct is 0, since  $\theta$  is continuous (a decimal number)! We'll see how we can construct confidence intervals around our estimator, so that we can argue that  $\hat{\theta}$  is *close to*  $\theta$  with high probability.

#### 8.1.1 Confidence Intervals Motivation

Confidence intervals are used in the Frequentist setting, which means the population parameters are assumed to be unknown but will always be **fixed**, not random variables. Credible intervals, on the other hand, are a Bayesian version of a Frequentist's confidence interval which is discussed in the next section 8.2.

When doing point estimation (such as MLE, MoM), the probability that our answer is correct (over the randomness in our iid samples) is 0:

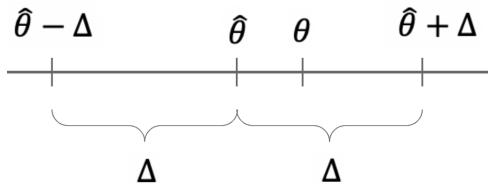
$$\mathbb{P}(\hat{\theta} = \theta) = 0$$

because  $\theta$  is a real number and can take uncountably many values. Hence the probability we are *exactly* correct is zero, though we may be very close.

Instead, we can give an interval (often but not always centered at our point estimate  $\hat{\theta}$ ), such that  $\theta$  falls into it with high probability, like 95%.

$$\mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) = 0.95$$

The confidence interval for  $\theta$  can be illustrated in the below picture. We will explain how to interpret a confidence interval at a specific confidence level soon.



Note that we can write this in any of the following three equivalent ways, as they all represent the probability that  $\hat{\theta}$  and  $\theta$  differ by no more than some amount  $\Delta$ :

$$\mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) = \mathbb{P}(|\hat{\theta} - \theta| \leq \Delta) = \mathbb{P}(\hat{\theta} \in [\theta - \Delta, \theta + \Delta]) = 0.95$$

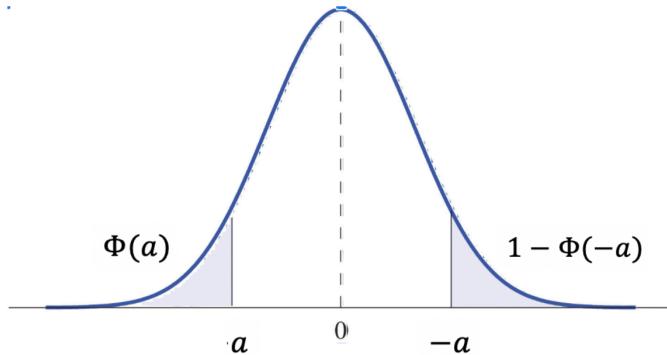
Note the first and third equivalent statements especially (swapping  $\hat{\theta}$  and  $\theta$ ).

### 8.1.2 Review: The Standard Normal CDF

Before we construct confidence intervals, we need to review the standard normal CDF. It turns out, the Normal distribution frequently appears since our estimators are usually the sample mean (at least for our common distributions), and the Central Limit Theorem applies!

We have learned about the CDF of normal distribution. If  $Z \sim \mathcal{N}(0, 1)$ , we denote the CDF  $\Phi(a) = F_Z(a) = P(Z \leq a)$ , since it's so commonly used. There is no closed-form formula, so one way to find a z-score associated with a percentage is to look up in a z-table.

Note:  $\Phi(a) = 1 - \Phi(-a)$  by symmetry.



Suppose we want a (centered) interval, where the probability of being in that interval is 95%.

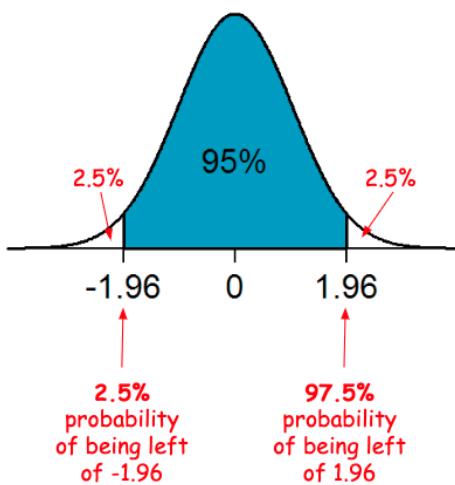
**Left bound:** the probability of being less than the left bound is 2.5%.

**Right bound:** the probability of being greater than the right bound is 2.5%. Thus, the probability of being less than the right bound should be 97.5%.

Note the following two equivalent statements that say that  $P(Z \leq 1.96) = 0.975$  (where  $\Phi^{-1}$  is the inverse CDF of the standard normal):

$$\Phi(1.96) = 0.975$$

$$\Phi^{-1}(0.975) = 1.96$$



### 8.1.3 Confidence Intervals

Let's start by doing an example.

#### Example(s)

Suppose  $x_1, \dots, x_n$  are iid samples from  $\text{Poi}(\theta)$  where  $\theta$  is unknown. Our MLE and MoM estimates agreed at the sample mean:  $\hat{\theta} = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ . Create an interval centered at  $\hat{\theta}$  which contains  $\theta$  with probability 95%.

*Solution* Recall that if  $W \sim \text{Poi}(\theta)$ , then  $\mathbb{E}[W] = \text{Var}(W) = \theta$ , and so our estimator (the sample mean)  $\hat{\theta} = \bar{x}$  has  $\mathbb{E}[\hat{\theta}] = \theta$  and  $\text{Var}(\hat{\theta}) = \frac{\text{Var}(x_i)}{n} = \frac{\theta}{n}$ . Thus, by the Central Limit Theorem,  $\hat{\theta}$  is approximately Normally distributed:

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i \approx \mathcal{N}\left(\theta, \frac{\theta}{n}\right)$$

If we standardize, we get that

$$\frac{\hat{\theta} - \theta}{\sqrt{\theta/n}} \approx \mathcal{N}(0, 1)$$

To construct our 95% confidence interval, we want  $\mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) = 0.95$

$$\begin{aligned} \mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) &= \mathbb{P}(\theta - \Delta \leq \hat{\theta} \leq \theta + \Delta) && \text{[one of 3 equivalent statements]} \\ &= \mathbb{P}(-\Delta \leq \hat{\theta} - \theta \leq \Delta) \\ &= \mathbb{P}\left(-\frac{\Delta}{\sqrt{\theta/n}} \leq \frac{\hat{\theta} - \theta}{\sqrt{\theta/n}} \leq \frac{\Delta}{\sqrt{\theta/n}}\right) \\ &= \mathbb{P}\left(-\frac{\Delta}{\sqrt{\theta/n}} \leq Z \leq \frac{\Delta}{\sqrt{\theta/n}}\right) && \text{[CLT]} \\ &= 0.95 \end{aligned}$$

Because  $\frac{\Delta}{\sqrt{\theta/n}}$  represents the right bound, and the probability of being less than the right bound is 97.5% for a 95% interval (see the above picture again). Thus:

$$\frac{\Delta}{\sqrt{\theta/n}} = \Phi^{-1}(0.975) = 1.96 \implies \Delta = 1.96 \sqrt{\frac{\theta}{n}}$$

Since we don't know  $\theta$ , we plug in our estimator  $\hat{\theta}$ , and get

$$[\hat{\theta} - \Delta, \hat{\theta} + \Delta] = \left[ \hat{\theta} - 1.96 \sqrt{\frac{\hat{\theta}}{n}}, \hat{\theta} + 1.96 \sqrt{\frac{\hat{\theta}}{n}} \right]$$

That is, since  $\hat{\theta}$  is normally distributed with mean  $\theta$ , we just need to find the  $\Delta$  so that  $\hat{\theta} \pm \Delta$  contains 95% of the area in a Normal distribution. The way to do so is to find  $\Phi^{-1}(0.975) = 1.96$ , and go  $\pm 1.96$  standard deviations of  $\hat{\theta}$  in each direction!  $\square$

### Definition 8.1.1: Confidence Interval

Suppose you have iid samples  $x_1, \dots, x_n$  from some distribution with unknown parameter  $\theta$ , and you have some estimator  $\hat{\theta}$  for  $\theta$ .

A  $100(1 - \alpha)\%$  **confidence interval** for  $\theta$  is an interval (typically but not always) centered at  $\hat{\theta}$ ,  $[\hat{\theta} - \Delta, \hat{\theta} + \Delta]$ , such that the probability (over the randomness in the samples  $x_1, \dots, x_n$ )  $\theta$  lies in the interval is  $1 - \alpha$ :

$$\mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) = 1 - \alpha$$

If  $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$  is the sample mean, then  $\hat{\theta}$  is approximately normal by the CLT, and a  $100(1 - \alpha)\%$  confidence interval is given by the formula:

$$\left[ \hat{\theta} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \hat{\theta} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right]$$

where  $z_{1-\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2})$  and  $\sigma$  is the true standard deviation of a single sample (which may need to be estimated).

It is important to note that this last formula ONLY works when  $\hat{\theta}$  is the sample mean (otherwise we can't use the CLT); you'll need to find some other strategy if it isn't.

If we wanted a 95% interval, then that corresponds to  $\alpha = 0.05$ , since  $100(1 - \alpha) = 95$ . We were then looking up the inverse Phi table at  $(1 - \alpha/2) = (1 - 0.05/2) = 0.975$  to get our desired number of standard deviations in each direction of 1.96.

If we wanted a 98% interval, then that corresponds to  $\alpha = 0.02$  since  $100(1 - \alpha) = 98$ . We then would look up  $\Phi^{-1}(0.99)$  since  $1 - \alpha/2 = 0.99$ , because if there is to be 98% of the area in the middle, there is 1% to the left and right!

### Example(s)

Construct a 99% confidence interval for  $\theta$  (the unknown probability of success) in  $\text{Ber}(\theta)$  given  $n = 400$  iid samples  $x_1, \dots, x_{400}$  where  $\sum_{i=1}^n x_i = 136$  (observed 136 successes out of 400).

#### *Solution*

Recall for Bernoulli distribution  $\text{Ber}(\theta)$ , our MLE/MoM estimator was the sample mean:

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{136}{400} = 0.34$$

Because we want to construct a  $99\% = 100(1 - \alpha)\%$  confidence interval:

$$\alpha = 1 - \frac{99}{100} = 0.01$$

A 99% confidence interval would use the z-score:

$$z_{1-\alpha/2} = z_{1-0.01/2} = z_{0.995} = \Phi^{-1}(0.995) \approx 2.576$$

The population standard deviation  $\sigma$  is unknown, but we'll approximate it using the standard deviation of  $\text{Ber}(\theta)$  as follows (since  $\text{Var}(\text{Ber}(\theta)) = \theta(1 - \theta)$ ):

$$\sigma = \sqrt{\theta(1 - \theta)} \approx \sqrt{\hat{\theta}(1 - \hat{\theta})} = \sqrt{0.34(1 - 0.34)} = 0.474$$

Thus, our 99% confidence interval for  $\theta$  is:

$$\left[ 0.34 - 2.576 \frac{0.474}{\sqrt{400}}, 0.34 + 2.576 \frac{0.474}{\sqrt{400}} \right] = [0.279, 0.401]$$

□

### 8.1.4 Interpreting Confidence Intervals

How can we interpret our 99% confidence interval  $[0.279, 0.401]$  from the above example?

**Incorrect:** There is a 99% probability that  $\theta$  falls in the confidence interval  $[\hat{\theta} - \Delta, \hat{\theta} + \Delta] = [0.279, 0.401]$

This is incorrect because there is no randomness here:  $\theta$  is a fixed parameter.  $\theta$  is either in the interval or out of it; there's nothing probabilistic about it.

**Correct:** If we repeat this process several times (getting  $n$  samples each time and constructing different confidence intervals), about 99% of the confidence intervals we construct will contain  $\theta$ .

Notice the subtle difference! Alternatively, *before* you receive samples, you can say that there is a 99% probability (over the randomness in the samples) that  $\theta$  will fall into our to-be-constructed confidence interval  $[\hat{\theta} - \Delta, \hat{\theta} + \Delta]$ . Once you plug in the numbers though, you cannot say that anymore.

## Chapter 8: Statistical Inference

### 8.2: Credible Intervals

#### 8.2.1 Credible Intervals

Now we will assume we are in the **Bayesian setting**, which means our unknown parameter  $\Theta$  will always be some random variable, not a fixed quantity. If we give a single point estimate like we do in MAP, we will never be exactly correct. Therefore, just like we did in the Frequentist setting with confidence intervals, we might want to give an interval instead of a single number. These are called credible intervals instead, and serve the same purpose!

Actually, since  $\Theta$  is a random variable, finding an interval where the probability is at least 90% for example involves just looking at the PDF/CDF! An example best illustrates this.

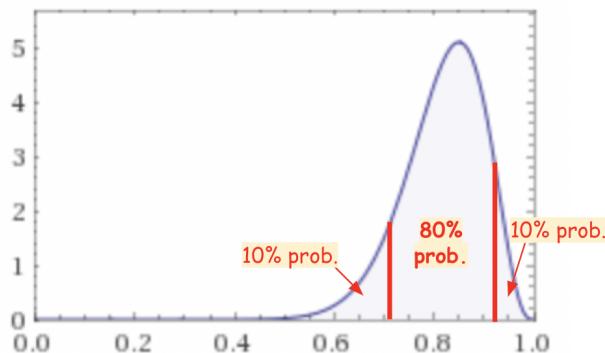
#### Example(s)

Construct a 80% credible interval for  $\Theta$  (the unknown) probability of success in  $\text{Ber}(\Theta)$  given iid  $n = 12$  samples  $\mathbf{x} = (x_1, x_2, \dots, x_{12})$  where  $\sum_{i=1}^n x_i = 11$  (observed 11 successes out of 12). Suppose our prior is  $\Theta \sim \text{Beta}(\alpha = 7, \beta = 3)$  (i.e., pretend we saw 6 successes and 2 failures ahead of time).

*Solution* From lecture 7.5 (MAP), we showed that choosing a Beta prior for  $\Theta$  leads to a Beta posterior of  $\Theta \mid \mathbf{x} \sim \text{Beta}(11 + 7, 1 + 3) = \text{Beta}(18, 4)$  and our MAP was then  $\frac{18-1}{(18-1)+(4-1)} = \frac{17}{20}$  (since we saw 17 total successes, and 3 total failures.)

We want an interval  $[a, b]$  such that  $\mathbb{P}(a \leq \Theta \leq b) = 0.8$

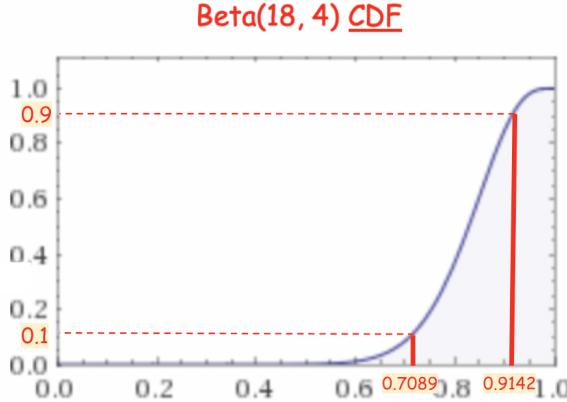
Beta(18, 4) PDF



If we look at the Beta PDF, we are looking for such an interval that the probability that we fall in this area is 80%. If the area is centered, then the area to the left of that should have probability of 10%, and the area to the right of that should also have probability 10%.

This is equivalent of looking for  $P(\Theta \leq a) = 0.1$  and  $P(\Theta \leq b) = 0.9$ . These information are given in the

CDF of the Beta distribution. Note that on the  $x$ -axis we have the range of the Beta distribution  $[0, 1]$ , and on the  $y$ -axis, we have the cumulative probability of being to the left by integrating the PDF from above.



Let  $F_{\text{Beta}}$  denote the CDF of this Beta(18,4) distribution. Then, choose  $a = F_{\text{Beta}}^{-1}(0.1) \approx 0.7089$  and  $b = F_{\text{Beta}}^{-1}(0.9) \approx 0.9142$ , so our credible interval is  $[0.7089, 0.9142]$ .

Note that the MAP was  $\frac{17}{20} = 0.85$ , which is not at the center! We could have chosen any  $a, b$  where the area between them is 80%, but we set the areas to the left and right to be equal.

In order to compute the inverse CDF, we can use the `scipy.stats` library as follows:

---

```

1 >>> from scipy.stats import beta
2 >>> beta.ppf(0.1, 18, 4) # inverse cdf
3 0.70898031757137514

```

---

□

That's all there is to it! Just find the PDF/CDF of your posterior distribution (hopefully you chose a conjugate prior), and look up the inverse CDF at points  $a$  and  $b$  such that  $b - a$  is your desired confidence level of your credible interval.

#### Definition 8.2.1: Credible Intervals

Suppose you have iid samples  $\mathbf{x} = (x_1, \dots, x_n)$  from some distribution with unknown parameter  $\Theta$ . You are in the **Bayesian setting**, so you have chosen a prior distribution for the RV  $\Theta$ .

A  $100(1 - \alpha)\%$  **credible interval** for  $\Theta$  is an interval  $[a, b]$  such that the probability (over the randomness in  $\Theta$ ) that  $\Theta$  lies in the interval is  $1 - \alpha$ :

$$P(\Theta \in [a, b]) = 1 - \alpha$$

If we've chosen the appropriate conjugate prior for the sampling distribution (like Beta for Bernoulli), the posterior is easy to compute. Say the CDF of the posterior is  $F_Y$ . Then, a  $100(1 - \alpha)\%$  credible interval is given by

$$\left[ F_Y^{-1} \left( \frac{\alpha}{2} \right), F_Y^{-1} \left( 1 - \frac{\alpha}{2} \right) \right]$$

Again, this is one which has equal area to the left and right of the interval, but there are infinitely many possible credible intervals you can create.

### 8.2.2 Interpreting Credible Intervals

How can we interpret a 80% credible interval  $[0.7089, 0.9142]$  for parameter  $\Theta$ ?

**Correct:** There is an 80% probability that  $\Theta$  falls in the credible interval  $[0.7089, 0.9142]$ . Written out,

$$P(\Theta \in [0.7089, 0.9142]) = 0.8$$

This is correct because not  $\Theta$  is a random variable, and it makes sense to say!

Contrast this with the interpretation of a confidence interval, where  $\theta$  is a fixed number.

## Chapter 8: Statistical Inference

### 8.3: Introduction to Hypothesis Testing

Hypothesis testing allows us to “statistically prove” claims. For example, if a drug company wants to claim that their new drug reduces the risk of cancer, they might perform a hypothesis test. Or if a company wanted to argue that their academic prep program leads to a higher SAT score. A lot of business decisions are reliant on this statistical method of hypothesis testing, and we’ll see how to conduct them properly below.

#### 8.3.1 Hypothesis Testing (Idea)

Suppose we have this Magician Mark, who says

**Magician Mark:** I have here a fair coin.

And then an audience member, a skeptical statistician named Stacy, engages him in a conversation:

**Skeptical Statistician Stacy:** I don’t believe you. Can we examine it?

**Magician Mark:** Be my guest.

**Skeptical Statistician Stacy:** I’m going to flip your coin 100 times and see how many heads I get.  
[Stacy flips the coin 100 times and sees 99 heads.]

You cannot be telling the truth, there’s no way this coin is fair!

**Magician Mark:** Wait I was just unlucky, I swear I’m not lying!

So let’s give Mark the **benefit of the doubt**. We’ll compute the probability that we observed an outcome *as least as extreme* as this, **given that Mark isn’t lying**.

If Mark isn’t lying, then the coin is fair, so the number of heads observed should be  $X \sim \text{Bin}(100, 0.5)$ , because there are 100 independent trials and a 50% of heads since it’s fair. So, the probability that we observe at least 99 heads (because we’re looking for something *as least as extreme*), is the sum of the probability of 99 heads and the probability of 100 heads. You just sum the Binomial PMF and you get:

$$\mathbb{P}(X \geq 99) = \binom{100}{99} (0.5)^{99} (1 - 0.5)^1 + \binom{100}{100} (0.5)^{100} = \frac{101}{2^{100}} \approx 7.96 \times 10^{-29} \approx 0$$

Basically, if the coin were fair, the probability of what we just observed (99 heads or more) is basically 0. This is strong statistical evidence that the coin is NOT fair. Our assumption was that the coin is fair, but if this were the case, observing such an extreme outcome would be extremely unlikely. Hence, our assumption is probably wrong.

So, this is like a ”Probabilistic Proof by Contradiction”!

### 8.3.2 Hypothesis Testing (Example)

There is a formal procedure for a hypothesis test, which we will illustrate by example. There are many types of hypothesis tests, each with different uses, but we'll get into that later! You'll see the CLT often appear in the most fundamental/commonly conducted hypothesis tests.

1. **Make a claim (like "Airplane food is good", "Pineapples belong on pizza", etc...)**
  - Our example will be that SuperSAT Prep claims that their program helps students perform better on the SAT. (The average SAT score as of June 2020 was: 1059 out of 1600, and the standard deviation of SAT scores was 210).
2. **Set up a null hypothesis  $H_0$  and alternative hypothesis  $H_A$ .**
  - (a) Alternative hypothesis can be one-sided or two-sided.
    - Let  $\mu$  be the true mean of the SAT scores of students of SuperSAT Prep.
    - Our **null hypothesis** is that  $H_0 : \mu = 1059$ , which is our "baseline", "no effect", "benefit of the doubt". We're going to assume that the true mean of our scores is the same as the nationwide scores (for the sake of contradiction).
    - Our **alternative hypothesis** is what we want to show, which is  $H_A : \mu > 1059$ , or that SuperSAT Prep is good and that their test takers are (strictly) better off. So, our alternative will assert that  $\mu > 1059$ .
    - This is called a **one-sided hypothesis**. The other one-sided hypothesis would be  $\mu < 1059$  (if we wanted to argue that SuperSAT Prep makes students worse off).
    - A **two-sided hypothesis** would be that  $\mu \neq 1059$ , because it's two sides (less than or greater than). This is if we wanted to argue that SuperSAT Prep makes some difference for better or worse.
3. **Choose a significance level  $\alpha$  (usually  $\alpha = 0.05$  or  $0.01$ ).**
  - Let's choose  $\alpha = 0.05$  and explain this more later!
4. **Collect data.**
  - We observe 100 students from SuperSAT Prep,  $x_1, \dots, x_{100}$ . It turns out, the sample mean of the scores,  $\bar{x}$ , is  $\bar{x} = 1113$ .
5. **Compute a p-value,  $p = \mathbb{P}(\text{observing data at least as extreme as ours} \mid H_0 \text{ is true})$ .**
  - Again, since we're assuming  $H_0$  is true (that SuperSAT has no effect), our true mean  $\mu$  is 1059 (again we do this in hopes of reaching a "probabilistic contradiction"). By the CLT, since  $n = 100$  is large, the distribution of the sample mean of 100 samples is approximately normal with mean 1059, and variance  $\frac{210^2}{100}$  (because the variance of a single test taker was given to be  $\sigma^2 = 210^2$ , and so the variance of the sample mean is  $\frac{\sigma^2}{n}$ ):
$$\bar{X} \approx \mathcal{N}(\mu = 1059, \sigma^2 = \frac{210^2}{100})$$

So, then, the p-value is the probability that if we took an arbitrary sample mean, that it would be at least as extreme as the one we computed, which was 1113. So, we can just standardize, look up a  $\Phi$  table like always, which is a procedure you know how to do:

$$p = \mathbb{P}(\bar{X} \geq \bar{x}) = \mathbb{P}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right) = \mathbb{P}\left(Z \geq \frac{1113 - 1059}{210/\sqrt{100}}\right) = \mathbb{P}(Z \geq 2.14) \approx 0.0162$$

We end up getting that our p-value is 0.0162

**6. State your conclusion. Include an interpretation in the context of the problem.**

- (a) If  $p < \alpha$ , "reject" the null hypothesis  $H_0$  in favor of the alternative  $H_A$ . (Because, given the null hypothesis is true, the probability of what we saw happening (or something more extreme) is  $p$  which is less than some small number  $\alpha$ .)
- (b) Otherwise, "fail to reject" the null hypothesis  $H_0$ .

- Since  $p = 0.0162 < 0.05 = \alpha$ , we'll reject the null hypothesis  $H_0$  at the  $\alpha = 0.05$  significance level. We can say that there is strong statistical evidence to suggest that SuperSAT Prep actually helps students perform better on the SAT.

Notice that if we had chosen  $\alpha = 0.01$  earlier instead of 0.05, we would have a different conclusion: Since  $p = 0.0162 > 0.01 = \alpha$ , we fail to reject the null hypothesis at the  $\alpha = 0.01$  significance level. There is insufficient evidence to prove that SuperSAT Prep actually helps students perform better.

Note that **we'll NEVER say we "accept" the null hypothesis.** If you recall the coin example, if we had observed 55 heads instead of 99, that wouldn't have been improbable. We wouldn't have called the magician a liar, but it does NOT imply that  $p = 0.5$ . It could have been 0.54 or 0.58, for example.

### 8.3.3 Hypothesis Testing Procedure

The formal hypothesis testing procedure is summarized as follows:

1. Make a claim (like "Airplane food is good", "Pineapples belong on pizza", etc...)
2. Set up a null hypothesis  $H_0$  and alternative hypothesis  $H_A$ .
  - (a) Alternative hypothesis can be one-sided or two-sided.
  - (b) The null hypothesis is usually a "baseline", "no effect", or "benefit of the doubt".
  - (c) The alternative is what you want to "prove", and is opposite the null.
3. Choose a significance level  $\alpha$  (usually  $\alpha = 0.05$  or  $0.01$ ).
4. Collect data.
5. Compute a p-value,  $p = \mathbb{P}(\text{observing data at least as extreme as ours} \mid H_0 \text{ is true})$ .
6. State your conclusion. Include an interpretation in the context of the problem.
  - (a) If  $p < \alpha$ , "reject" the null hypothesis  $H_0$  in favor of the alternative  $H_A$ .
  - (b) Otherwise, "fail to reject" the null hypothesis  $H_0$ .

### 8.3.4 Exercises

1. You want to determine whether or not more than 3/4 of Americans would vote for George Washington for President in 2020 (if he were still alive). In a random poll sampling  $n = 137$  Americans, we collected responses  $x_1, \dots, x_n$  (each is 1 or 0, if they would vote for him or not). We observe 131 “yes” responses:  $\sum_{i=1}^n x_i = 131$ . Perform a hypothesis test and state your conclusion.

**Solution:** We have our claim that “Over 3/4 of Americans would vote for George Washington for President in 2020 (if he were still alive).

Let  $p$  denote the true proportion of Americans that would vote for Washington. Then our null and alternative hypotheses are:

$$H_0 : p = 0.75$$

$$H_A : p > 0.75$$

Let's test these hypotheses at the  $\alpha = 0.01$  significance level.

We know by the CLT that the sample mean is approximately  $\bar{X} \sim \mathcal{N}\left(\mu = 0.75, \sigma^2 = \frac{0.75(1-0.75)}{137}\right) = \mathcal{N}(0.75, \sigma^2 = 0.037^2)$  (since  $X_i \sim \text{Ber}(p)$ :  $\mathbb{E}[X_i] = p = 0.75$  under the null hypothesis, and  $\text{Var}(X_i) = p(1-p) = 0.75(1-0.75)$  and we know  $\mathbb{E}[\bar{X}] = \mathbb{E}[X_i] = p$  and  $\text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{0.75(1-0.75)}{137}$ ).

Hence our p-value (observing data at least as extreme), is

$$\mathbb{P}(\bar{X} \geq \bar{x}) = \mathbb{P}\left(\mathcal{N}(0.75, \sigma^2 = 0.037^2) \geq \frac{131}{137}\right) = \mathbb{P}\left(Z \geq \frac{131/137 - 0.75}{0.037}\right) = \mathbb{P}(Z \geq 5.42643) \approx 0$$

With a p-value so close to 0 (and certainly  $< \alpha = 0.01$ ), we reject the null hypothesis that (only) 75% of Americans would vote for Washington. There is strong evidence that this proportion is actually larger.

**Note:** Again, what we did was: assume  $p = 0.75$  (null hypothesis), then note that the probability of observing data so extreme (in fact very close to 100% of people), was nearly 0. Hence, we reject this null hypothesis because what we observed would've been so unlikely if it were true.

## Chapter 9: Applications to Computer Science

### 9.1: Python Programming

#### 9.1.1 Python

In Progress.

## Chapter 9: Applications to Computer Science

### 9.2: Probability via Simulation

#### 9.2.1 Python

In Progress.

## Chapter 9: Applications to Computer Science

### 9.3: The Naive Bayes Classifier

#### 9.3.1 Naive Bayes

In Progress.

## Chapter 9: Applications to Computer Science

### 9.4: Bloom Filters

#### 9.4.1 Bloom Filters

In Progress.

## Chapter 9: Applications to Computer Science

### 9.5: Distinct Elements

#### 9.5.1 Distinct Elements

In Progress.

## Chapter 9: Applications to Computer Science

### 9.6: Markov Chain Monte Carlo (MCMC)

#### 9.6.1 MCMC

In Progress.

## Chapter 9: Applications to Computer Science

### 9.7: Bootstrapping

#### 9.7.1 Bootstrapping

In Progress.

## Chapter 9: Applications to Computer Science

### 9.8: Multi-Armed Bandits

#### 9.8.1 Bandits

In Progress.

# Probability & Statistics with Applications to Computing

## Key Definitions and Theorems

### 1 Combinatorial Theory

#### 1.1 So You Think You Can Count?

**The Sum Rule:** If an experiment can either end up being one of  $N$  outcomes, or one of  $M$  outcomes (where there is no overlap), then the total number of possible outcomes is:  $N + M$ .

**The Product Rule:** If an experiment has  $N_1$  outcomes for the first stage,  $N_2$  outcomes for the second stage, ..., and  $N_m$  outcomes for the  $m^{\text{th}}$  stage, then the total number of outcomes of the experiment is  $N_1 \times N_2 \times \dots \times N_m = \prod_{i=1}^m N_i$ .

**Permutation:** The number of orderings of  $N$  **distinct** objects is  $N! = N \cdot (N - 1) \cdot (N - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$ .

**Complementary Counting:** Let  $\mathcal{U}$  be a (finite) universal set, and  $S$  a subset of interest. Then,  $|S| = |\mathcal{U}| - |\mathcal{U} \setminus S|$ .

#### 1.2 More Counting

**$k$ -Permutations:** If we want to *pick* (**order matters**) only  $k$  out of  $n$  distinct objects, the number of ways to do so is:

$$P(n, k) = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

**$k$ -Combinations/Binomial Coefficients:** If we want to *choose* (**order doesn't matter**) only  $k$  out of  $n$  distinct objects, the number of ways to do so is:

$$C(n, k) = \binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n!}{k!(n - k)!}$$

**Multinomial Coefficients:** If we have  $k$  distinct types of objects ( $n$  total), with  $n_1$  of the first type,  $n_2$  of the second, ..., and  $n_k$  of the  $k$ -th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!n_2!\dots n_k!}$$

**Stars and Bars/Divider Method:** The number of ways to distribute  $n$  indistinguishable balls into  $k$  distinguishable bins is

$$\binom{n + (k - 1)}{k - 1} = \binom{n + (k - 1)}{n}$$

#### 1.3 No More Counting Please

**Binomial Theorem:** Let  $x, y \in \mathbb{R}$  and  $n \in \mathbb{N}$  a positive integer. Then:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ .

**Principle of Inclusion-Exclusion (PIE):**

2 events:  $|A \cup B| = |A| + |B| - |A \cap B|$

3 events:  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$k$  events: singles - doubles + triples - quads + ...

**Pigeonhole Principle:** If there are  $n$  pigeons we want to put into  $k$  holes (where  $n > k$ ), then at least one pigeonhole must contain at least 2 (or to be precise,  $\lceil n/k \rceil$ ) pigeons.

**Combinatorial Proofs:** To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

## 2 Discrete Probability

### 2.1 Discrete Probability

**Key Probability Definitions:** The **sample space** is the set  $\Omega$  of all possible outcomes of an experiment. An **event** is any subset  $E \subseteq \Omega$ . Events  $E$  and  $F$  are **mutually exclusive** if  $E \cap F = \emptyset$ .

#### Axioms of Probability & Consequences:

1. **(Axiom: Nonnegativity)** For any event  $E$ ,  $\mathbb{P}(E) \geq 0$ .
  2. **(Axiom: Normalization)**  $\mathbb{P}(\Omega) = 1$ .
  3. **(Axiom: Countable Additivity)** If  $E$  and  $F$  are mutually exclusive, then  $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ .
1. **(Corollary: Complementation)**  $\mathbb{P}(E^C) = 1 - \mathbb{P}(E)$
  2. **(Corollary: Monotonicity)** If  $E \subseteq F$ , then  $\mathbb{P}(E) \leq \mathbb{P}(F)$
  3. **(Corollary: Inclusion-Exclusion)**  $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

**Equally Likely Outcomes:** If  $\Omega$  is a sample space such that each of the unique outcome elements in  $\Omega$  are equally likely, then for any event  $E \subseteq \Omega$ :  $\mathbb{P}(E) = |E|/|\Omega|$ .

### 2.2 Conditional Probability

**Conditional Probability:**  $\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

**Bayes Theorem:**  $\mathbb{P}(A | B) = \frac{\mathbb{P}(B | A) \mathbb{P}(A)}{\mathbb{P}(B)}$

**Partition:** Non-empty events  $E_1, \dots, E_n$  **partition** the sample space  $\Omega$  if they are both:

- **(Exhaustive)**  $E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^n E_i = \Omega$  (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all  $i \neq j$ ,  $E_i \cap E_j = \emptyset$  (none of them overlap)

Note that for any event  $E$ ,  $E$  and  $E^C$  always form a partition of  $\Omega$ .

**Law of Total Probability (LTP):** If events  $E_1, \dots, E_n$  partition  $\Omega$ , then for any event  $F$ :

$$\mathbb{P}(F) = \sum_{i=1}^n \mathbb{P}(F \cap E_i) = \sum_{i=1}^n \mathbb{P}(F | E_i) \mathbb{P}(E_i)$$

**Bayes Theorem with LTP:** Let events  $E_1, \dots, E_n$  partition the sample space  $\Omega$ , and let  $F$  be another event. Then:

$$\mathbb{P}(E_1 | F) = \frac{\mathbb{P}(F | E_1) \mathbb{P}(E_1)}{\sum_{i=1}^n \mathbb{P}(F | E_i) \mathbb{P}(E_i)}$$

## 2.3 Independence

**Chain Rule:** Let  $A_1, \dots, A_n$  be events with nonzero probabilities. Then:

$$\mathbb{P}(A_1, \dots, A_n) = \mathbb{P}(A_1) \mathbb{P}(A_2 | A_1) \mathbb{P}(A_3 | A_1 A_2) \cdots \mathbb{P}(A_n | A_1, \dots, A_{n-1})$$

**Independence:**  $A$  and  $B$  are **independent** if any of the following equivalent statements hold:

1.  $\mathbb{P}(A | B) = \mathbb{P}(A)$
2.  $\mathbb{P}(B | A) = \mathbb{P}(B)$
3.  $\mathbb{P}(A, B) = \mathbb{P}(A) \mathbb{P}(B)$

**Mutual Independence:** We say  $n$  events  $A_1, A_2, \dots, A_n$  are **(mutually) independent** if, for *any* subset  $I \subseteq [n] = \{1, 2, \dots, n\}$ , we have

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

This equation is actually representing  $2^n$  equations since there are  $2^n$  subsets of  $[n]$ .

**Conditional Independence:**  $A$  and  $B$  are **conditionally independent given an event  $C$**  if any of the following equivalent statements hold:

1.  $\mathbb{P}(A | B, C) = \mathbb{P}(A | C)$
2.  $\mathbb{P}(B | A, C) = \mathbb{P}(B | C)$
3.  $\mathbb{P}(A, B | C) = \mathbb{P}(A | C) \mathbb{P}(B | C)$

## 3 Discrete Random Variables

### 3.1 Discrete Random Variables Basics

**Random Variable (RV):** A random variable (RV)  $X$  is a numeric function of the outcome  $X : \Omega \rightarrow \mathbb{R}$ . The set of possible values  $X$  can take on is its **range/support**, denoted  $\Omega_X$ .

If  $\Omega_X$  is finite or countable infinite (typically integers or a subset),  $X$  is a **discrete RV**. Else if  $\Omega_X$  is uncountably large (the size of real numbers),  $X$  is a **continuous RV**.

**Probability Mass Function (PMF):** For a discrete RV  $X$ , assigns probabilities to values in its range. That is  $p_X : \Omega_X \rightarrow [0, 1]$  where:  $p_X(k) = \mathbb{P}(X = k)$ .

**Expectation:** The **expectation** of a discrete RV  $X$  is:  $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$ .

## 3.2 More on Expectation

**Linearity of Expectation (LoE):** For any random variables  $X, Y$  (possibly dependent):

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

**Law of the Unconscious Statistician (LOTUS):** For a discrete RV  $X$  and function  $g$ ,  $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$ .

## 3.3 Variance

**Linearity of Expectation with Indicators:** If asked only about the expectation of a RV  $X$  which is some sort of “count” (and not its PMF), then you may be able to write  $X$  as the sum of possibly dependent **indicator** RVs  $X_1, \dots, X_n$ , and apply LoE, where for an indicator RV  $X_i$ ,  $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1)$ .

**Variance:**  $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ .

**Standard Deviation (SD):**  $\sigma_X = \sqrt{\text{Var}(X)}$ .

**Property of Variance:**  $\text{Var}(aX + b) = a^2\text{Var}(X)$ .

## 3.4 Zoo of Discrete Random Variables Part I

**Independence:** Random variables  $X$  and  $Y$  are **independent**, denoted  $X \perp Y$ , if for all  $x \in \Omega_X$  and all  $y \in \Omega_Y$ :  $\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$ .

**Independent and Identically Distributed (iid):** We say  $X_1, \dots, X_n$  are said to be **independent and identically distributed (iid)** if all the  $X_i$ ’s are independent of each other, and have the same distribution (PMF for discrete RVs, or CDF for continuous RVs).

**Variance Adds for Independent RVs:** If  $X \perp Y$ , then  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ .

**Bernoulli Process:** A **Bernoulli process** with parameter  $p$  is a sequence of independent coin flips  $X_1, X_2, X_3, \dots$  where  $\mathbb{P}(\text{head}) = p$ . If flip  $i$  is heads, then we encode  $X_i = 1$ ; otherwise,  $X_i = 0$ .

**Bernoulli/Indicator Random Variable:**  $X \sim \text{Bernoulli}(p)$  (Ber( $p$ ) for short) iff  $X$  has PMF:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$  and  $\text{Var}(X) = p(1 - p)$ . An example of a Bernoulli/indicator RV is one flip of a coin with  $\mathbb{P}(\text{head}) = p$ . By a clever trick, we can write

$$p_X(k) = p^k (1 - p)^{1-k}, \quad k = 0, 1$$

**Binomial Random Variable:**  $X \sim \text{Binomial}(n, p)$  (Bin( $n, p$ ) for short) iff  $X$  has PMF

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k \in \Omega_X = \{0, 1, \dots, n\}$$

$\mathbb{E}[X] = np$  and  $\text{Var}(X) = np(1 - p)$ .  $X$  is the sum of  $n$  iid Ber( $p$ ) random variables. An example of a Binomial RV is the number of heads in  $n$  independent flips of a coin with  $\mathbb{P}(\text{head}) = p$ . Note that  $\text{Bin}(1, p) \equiv \text{Ber}(p)$ . As  $n \rightarrow \infty$  and  $p \rightarrow$

0, with  $np = \lambda$ , then  $\text{Bin}(n, p) \rightarrow \text{Poi}(\lambda)$ . If  $X_1, \dots, X_n$  are independent Binomial RV's, where  $X_i \sim \text{Bin}(N_i, p)$ , then  $X = X_1 + \dots + X_n \sim \text{Bin}(N_1 + \dots + N_n, p)$ .

### 3.5 Zoo of Discrete Random Variables Part II

Uniform Random Variable (Discrete):  $X \sim \text{Uniform}(a, b)$  (Unif( $a, b$ ) for short), for integers  $a \leq b$ , iff  $X$  has PMF:

$$p_X(k) = \frac{1}{b-a+1}, \quad k \in \Omega_X = \{a, a+1, \dots, b\}$$

$\mathbb{E}[X] = \frac{a+b}{2}$  and  $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$ . This represents each *integer* in  $[a, b]$  to be equally likely. For example, a single roll of a fair die is Unif(1, 6).

Geometric Random Variable:  $X \sim \text{Geometric}(p)$  (Geo( $p$ ) for short) iff  $X$  has PMF:

$$p_X(k) = (1-p)^{k-1} p, \quad k \in \Omega_X = \{1, 2, 3, \dots\}$$

$\mathbb{E}[X] = \frac{1}{p}$  and  $\text{Var}(X) = \frac{1-p}{p^2}$ . An example of a Geometric RV is the number of independent coin flips up to and including the first head, where  $\mathbb{P}(\text{head}) = p$ .

Negative Binomial Random Variable:  $X \sim \text{NegativeBinomial}(r, p)$  (NegBin( $r, p$ ) for short) iff  $X$  has PMF:

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k \in \Omega_X = \{r, r+1, r+2, \dots\}$$

$\mathbb{E}[X] = \frac{r}{p}$  and  $\text{Var}(X) = \frac{r(1-p)}{p^2}$ .  $X$  is the sum of  $r$  iid Geo( $p$ ) random variables. An example of a Negative Binomial RV is the number of independent coin flips up to and including the  $r$ -th head, where  $\mathbb{P}(\text{head}) = p$ . If  $X_1, \dots, X_n$  are independent Negative Binomial RV's, where  $X_i \sim \text{NegBin}(r_i, p)$ , then  $X = X_1 + \dots + X_n \sim \text{NegBin}(r_1 + \dots + r_n, p)$ .

### 3.6 Zoo of Discrete Random Variables Part III

Poisson Random Variable:  $X \sim \text{Poisson}(\lambda)$  (Poi( $\lambda$ ) for short) iff  $X$  has PMF:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \Omega_X = \{0, 1, 2, \dots\}$$

$\mathbb{E}[X] = \lambda$  and  $\text{Var}(X) = \lambda$ . An example of a Poisson RV is the number of people born during a particular minute, where  $\lambda$  is the average birth rate per minute. If  $X_1, \dots, X_n$  are independent Poisson RV's, where  $X_i \sim \text{Poi}(\lambda_i)$ , then  $X = X_1 + \dots + X_n \sim \text{Poi}(\lambda_1 + \dots + \lambda_n)$ .

Hypergeometric Random Variable:  $X \sim \text{HyperGeometric}(N, K, n)$  (HypGeo( $N, K, n$ ) for short) iff  $X$  has PMF:

$$p_X(k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}, \quad k \in \Omega_X = \{\max\{0, n+K-N\}, \dots, \min\{K, n\}\}$$

$\mathbb{E}[X] = n \frac{K}{N}$  and  $\text{Var}(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)}$ . This represents the number of successes drawn, when  $n$  items are drawn from a bag with  $N$  items ( $K$  of which are successes, and  $N - K$  failures) *without* replacement. If we did this with replacement, then this scenario would be represented as  $\text{Bin}\left(n, \frac{K}{N}\right)$ .

## 4 Continuous Random Variables

### 4.1 Continuous Random Variables Basics

**Probability Density Function (PDF):** The **probability density function (PDF)** of a continuous RV  $X$  is the function  $f_X : \mathbb{R} \rightarrow \mathbb{R}$ , such that the following properties hold:

- $f_X(z) \geq 0$  for all  $z \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(t) dt = 1$
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(w) dw$

**Cumulative Distribution Function (CDF):** The **cumulative distribution function (CDF)** of ANY random variable (discrete or continuous) is defined to be the function  $F_X : \mathbb{R} \rightarrow \mathbb{R}$  with  $F_X(t) = \mathbb{P}(X \leq t)$ . If  $X$  is a *continuous* RV, we have:

- $F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^t f_X(w) dw$  for all  $t \in \mathbb{R}$
- $\frac{d}{du} F_X(u) = f_X(u)$

#### Univariate: Discrete to Continuous:

	Discrete	Continuous
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t < x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
Expectation/LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$

### 4.2 Zoo of Continuous RVs

**Uniform Random Variable (Continuous):**  $X \sim \text{Uniform}(a, b)$  (Unif( $a, b$ ) for short) iff  $X$  has PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in \Omega_X = [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{a+b}{2}$  and  $\text{Var}(X) = \frac{(b-a)^2}{12}$ . This represents each real number from  $[a, b]$  to be equally likely. Do NOT confuse this with its discrete counterpart!

**Exponential Random Variable:**  $X \sim \text{Exponential}(\lambda)$  (Exp( $\lambda$ ) for short) iff  $X$  has PDF:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \in \Omega_X = [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$  and  $\text{Var}(X) = \frac{1}{\lambda^2}$ .  $F_X(x) = 1 - e^{-\lambda x}$  for  $x \geq 0$ . The exponential RV is the continuous analog of the geometric RV: it represents the waiting time to the next event, where  $\lambda > 0$  is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential RV is also memoryless:

$$\text{for any } s, t \geq 0, \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

**Gamma Random Variable:**  $X \sim \text{Gamma}(r, \lambda)$  (Gam( $r, \lambda$ ) for short) iff  $X$  has PDF:

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x \in \Omega_X = [0, \infty)$$

$\mathbb{E}[X] = \frac{r}{\lambda}$  and  $\text{Var}(X) = \frac{r}{\lambda^2}$ .  $X$  is the sum of  $r$  iid  $\text{Exp}(\lambda)$  random variables. In the above PDF, for positive integers  $r$ ,  $\Gamma(r) = (r-1)!$  (a normalizing constant). An example of a Gamma RV is the waiting time until the  $r$ -th event in the Poisson process. If  $X_1, \dots, X_n$  are independent Gamma RV's, where  $X_i \sim \text{Gam}(r_i, \lambda)$ , then  $X = X_1 + \dots + X_n \sim \text{Gam}(r_1 + \dots + r_n, \lambda)$ . It also serves as a conjugate prior for  $\lambda$  in the Poisson and Exponential distributions.

### 4.3 The Normal/Gaussian Random Variable

**Normal (Gaussian, “bell curve”) Random Variable:**  $X \sim \mathcal{N}(\mu, \sigma^2)$  iff  $X$  has PDF:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \Omega_X = \mathbb{R}$$

$\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ . The “standard normal” random variable is typically denoted  $Z$  and has mean 0 and variance 1: if  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$ . The CDF has no closed form, but we denote the CDF of the standard normal as  $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leq z)$ . Note from symmetry of the probability density function about  $z = 0$  that:  $\Phi(-z) = 1 - \Phi(z)$ .

**Closure of the Normal Under Scale and Shift:** If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ . In particular, we can always scale/shift to get the standard Normal:  $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$ .

**Closure of the Normal Under Addition:** If  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$  and  $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$  are *independent*, then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

### 4.4 Transforming Continuous RVs

**Steps to compute PDF of  $Y = g(X)$  from  $X$  (via CDF):** Suppose  $X$  is a *continuous* RV.

1. Write down the range  $\Omega_X$ , PDF  $f_X$ , and CDF  $F_X$ .
2. Compute the range  $\Omega_Y = \{g(x) : x \in \Omega_X\}$ .
3. Start computing the CDF of  $Y$  on  $\Omega_Y$ ,  $F_Y(y) = \mathbb{P}(g(X) \leq y)$ , in terms of  $F_X$ .
4. Differentiate the CDF  $F_Y(y)$  to get the PDF  $f_Y(y)$  on  $\Omega_Y$ .  $f_Y$  is 0 outside  $\Omega_Y$ .

**Explicit Formula to compute PDF of  $Y = g(X)$  from  $X$  (Univariate Case):** Suppose  $X$  is a *continuous* RV. If  $Y = g(X)$  and  $g : \Omega_X \rightarrow \Omega_Y$  is *strictly monotone* and *invertible* with inverse  $X = g^{-1}(Y) = h(Y)$ , then

$$f_Y(y) = \begin{cases} f_X(h(y)) \cdot |h'(y)| & \text{if } y \in \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

**Explicit Formula to compute PDF of  $Y = g(X)$  from  $X$  (Multivariate Case):** Let  $\mathbf{X} = (X_1, \dots, X_n)$ ,  $\mathbf{Y} = (Y_1, \dots, Y_n)$  be continuous random vectors (each component is a continuous rv) with the same dimension  $n$  (so  $\Omega_{\mathbf{X}}, \Omega_{\mathbf{Y}} \subseteq \mathbb{R}^n$ ), and  $\mathbf{Y} = g(\mathbf{X})$  where  $g : \Omega_{\mathbf{X}} \rightarrow \Omega_{\mathbf{Y}}$  is invertible and differentiable, with differentiable inverse  $\mathbf{X} = g^{-1}(\mathbf{y}) = h(\mathbf{y})$ . Then,

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h(\mathbf{y})) \left| \det \left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right) \right|$$

where  $\left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right) \in \mathbb{R}^{n \times n}$  is the Jacobian matrix of partial derivatives of  $h$ , with

$$\left( \frac{\partial h(\mathbf{y})}{\partial \mathbf{y}} \right)_{ij} = \frac{\partial (h(\mathbf{y}))_i}{\partial y_j}$$

## 5 Multiple Random Variables

### 5.1 Joint Discrete Distributions

**Cartesian Product of Sets:** The **Cartesian product** of sets  $A$  and  $B$  is denoted:  $A \times B = \{(a, b) : a \in A, b \in B\}$ .

**Joint PMFs:** Let  $X, Y$  be discrete random variables. The joint PMF of  $X$  and  $Y$  is:

$$p_{X,Y}(a, b) = \mathbb{P}(X = a, Y = b)$$

The joint range is the set of pairs  $(c, d)$  that have nonzero probability:

$$\Omega_{X,Y} = \{(c, d) : p_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that the probabilities in the table must sum to 1:

$$\sum_{(s,t) \in \Omega_{X,Y}} p_{X,Y}(s, t) = 1$$

Further, note that if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function, then LOTUS extends to the multidimensional case:

$$\mathbb{E}[g(X, Y)] = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} g(x, y) p_{X,Y}(x, y)$$

**Marginal PMFs:** Let  $X, Y$  be discrete random variables. The marginal PMF of  $X$  is:  $p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a, b)$ .

**Independence (DRVs):** Discrete RVs  $X, Y$  are **independent**, written  $X \perp Y$ , if for all  $x \in \Omega_X$  and  $y \in \Omega_Y$ :  $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ .

**Variance Adds for Independent RVs:** If  $X \perp Y$ , then:  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ .

### 5.2 Joint Continuous Distributions

**Joint PDFs:** Let  $X, Y$  be continuous random variables. The joint PDF of  $X$  and  $Y$  is:

$$f_{X,Y}(a, b) \geq 0$$

The joint range is the set of pairs  $(c, d)$  that have nonzero density:

$$\Omega_{X,Y} = \{(c, d) : f_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that the double integral over all values must be 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u, v) du dv = 1$$

Further, note that if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function, then LOTUS extends to the multidimensional case:

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s, t) f_{X,Y}(s, t) ds dt$$

The joint PDF must satisfy the following (similar to univariate PDFs):

$$\mathbb{P}(a \leq X < b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dy dx$$

**Marginal PDFs:** Let  $X, Y$  be continuous random variables. The marginal PDF of  $X$  is:  $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$ .

**Independence of Continuous Random Variables:** Continuous RVs  $X, Y$  are independent, written  $X \perp Y$ , if for all  $x \in \Omega_X$  and  $y \in \Omega_Y$ ,  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ .

### 5.3 Conditional Distributions

**Conditional PMFs and PDFs:** If  $X, Y$  are discrete, the conditional PMF of  $X$  given  $Y$  is:

$$p_{X|Y}(a | b) = \mathbb{P}(X = a | Y = b) = \frac{p_{X,Y}(a, b)}{p_Y(b)} = \frac{p_{Y|X}(b | a)p_X(a)}{p_Y(b)}$$

Similarly for continuous RVs, but with  $f$ 's instead of  $p$ 's (PDFs instead of PMFs).

**Conditional Expectation:** If  $X$  is discrete (and  $Y$  is either discrete or continuous), then we define the conditional expectation of  $g(X)$  given (the event that)  $Y = y$  as:

$$\mathbb{E}[g(X) | Y = y] = \sum_{x \in \Omega_X} g(x)p_{X|Y}(x | y)$$

If  $X$  is continuous (and  $Y$  is either discrete or continuous), then

$$\mathbb{E}[g(X) | Y = y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x | y)dx$$

Notice that these sums and integrals are **over  $x$**  (not  $y$ ), since  $\mathbb{E}[g(X) | Y = y]$  is a function of  $y$ .

**Law of Total Expectation (LTE):** Let  $X, Y$  be jointly distributed random variables.

If  $Y$  is discrete (and  $X$  is either discrete or continuous), then:

$$\mathbb{E}[g(X)] = \sum_{y \in \Omega_Y} \mathbb{E}[g(X) | Y = y] p_Y(y)$$

If  $Y$  is continuous (and  $X$  is either discrete or continuous), then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} \mathbb{E}[g(X) | Y = y] f_Y(y) dy$$

Basically, for  $\mathbb{E}[g(X)]$ , we take a weighted average of  $\mathbb{E}[g(X) | Y = y]$  over all possible values of  $y$ .

### Multivariate: Discrete to Continuous:

	Discrete	Continuous
<b>Joint Dist</b>	$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$
<b>Joint CDF</b>	$F_{X,Y}(x, y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t, s)$	$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(t, s) ds dt$
<b>Normalization</b>	$\sum_{x,y} p_{X,Y}(x, y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$
<b>Marginal Dist</b>	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$
<b>Expectation</b>	$\mathbb{E}[g(X, Y)] = \sum_{x,y} g(x, y)p_{X,Y}(x, y)$	$\mathbb{E}[g(X, Y)] = \int \int g(x, y)f_{X,Y}(x, y) dx dy$
<b>Conditional Dist</b>	$p_{X Y}(x y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X Y}(x y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
<b>Conditional Exp</b>	$\mathbb{E}[X Y = y] = \sum_x x p_{X Y}(x y)$	$\mathbb{E}[X Y = y] = \int_{-\infty}^{\infty} x f_{X Y}(x y) dx$
<b>Independence</b>	$\forall x, y, p_{X,Y}(x, y) = p_X(x)p_Y(y)$	$\forall x, y, f_{X,Y}(x, y) = f_X(x)f_Y(y)$

## 5.4 Covariance and Correlation

**Covariance:** The covariance of  $X$  and  $Y$  is:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Covariance satisfies the following properties:

1. If  $X \perp Y$ , then  $\text{Cov}(X, Y) = 0$  (but not necessarily vice versa).
2.  $\text{Cov}(X, X) = \text{Var}(X)$ . (Just plug in  $Y = X$ ).
3.  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ . (Multiplication is commutative).
4.  $\text{Cov}(X + c, Y) = \text{Cov}(X, Y)$ . (Shifting doesn't and shouldn't affect the covariance).
5.  $\text{Cov}(aX + bY, Z) = a \cdot \text{Cov}(X, Z) + b \cdot \text{Cov}(Y, Z)$ . This can be easily remembered like the distributive property of scalars  $(aX + bY)Z = a(XZ) + b(YZ)$ .
6.  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$ , and hence if  $X \perp Y$ , then  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ .
7.  $\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j)$ . That is covariance works like FOIL (first, outer, inner, last) for multiplication of sums  $((a + b + c)(d + e)) = ad + ae + bd + be + cd + ce$ .

**(Pearson) Correlation:** The **(Pearson) correlation** of  $X$  and  $Y$  is:  $\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$ .

It is always true that  $-1 \leq \rho(X, Y) \leq 1$ . That is, correlation is just a normalized version of covariance. Most notably,  $\rho(X, Y) = \pm 1$  if and only if  $Y = aX + b$  for some constants  $a, b \in \mathbb{R}$ , and then the sign of  $\rho$  is the same as that of  $a$ .

**Variance of Sums of RVs:** Let  $X_1, \dots, X_n$  be any RVs (independent or not). Then,

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)$$

## 5.5 Convolution

### Law of Total Probability for Random Variables:

Discrete version: If  $X, Y$  are discrete:

$$p_X(x) = \sum_y p_{X,Y}(x, y) = \sum_y p_{X|Y}(x | y)p_Y(y)$$

Continuous version: If  $X, Y$  are continuous:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy = \int_{-\infty}^{\infty} f_{X|Y}(x | y)f_Y(y)dy$$

**Convolution:** Let  $X, Y$  be *independent* RVs, and  $Z = X + Y$ .

Discrete version: If  $X, Y$  are discrete:

$$p_Z(z) = \sum_{x \in \Omega_X} p_X(x)p_Y(z - x)$$

Continuous version: If  $X, Y$  are continuous:

$$f_Z(z) = \int_{x \in \Omega_X} f_X(x)f_Y(z - x)dx$$

## 5.6 Moment Generating Functions

**Moments:** Let  $X$  be a random variable and  $c \in \mathbb{R}$  a scalar. Then: The  $k$ -th **moment** of  $X$  is  $\mathbb{E}[X^k]$  and the  $k$ -th moment of  $X$  (about  $c$ ) is:  $\mathbb{E}[(X - c)^k]$ .

**Moment Generating Functions (MGFs):** The **moment generating function (MGF)** of  $X$  is a function of a dummy variable  $t$  (use LOTUS to compute this):  $M_X(t) = \mathbb{E}[e^{tX}]$ .

**Properties and Uniqueness of Moment Generating Functions:** For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we will denote  $f^{(n)}(x)$  to be the  $n$ -th derivative of  $f(x)$ . Let  $X, Y$  be *independent* random variables, and  $a, b \in \mathbb{R}$  be scalars. Then MGFs satisfy the following properties:

1.  $M'_X(0) = \mathbb{E}[X]$ ,  $M''_X(0) = \mathbb{E}[X^2]$ , and in general  $M_X^{(n)}(0) = \mathbb{E}[X^n]$ . This is why we call  $M_X$  a *moment generating* function, as we can use it to generate the moments of  $X$ .
2.  $M_{aX+b}(t) = e^{tb}M_X(at)$ .
3. If  $X \perp Y$ , then  $M_{X+Y}(t) = M_X(t)M_Y(t)$ .
4. (**Uniqueness**) The following are equivalent:
  - (a)  $X$  and  $Y$  have the same distribution.
  - (b)  $f_X(z) = f_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (c)  $F_X(z) = F_Y(z)$  for all  $z \in \mathbb{R}$ .
  - (d) There is an  $\varepsilon > 0$  such that  $M_X(t) = M_Y(t)$  for all  $t \in (-\varepsilon, \varepsilon)$ .

That is  $M_X$  uniquely identifies a distribution, just like PDFs/PMFs or CDFs do.

## 5.7 Limit Theorems

**The Sample Mean + Properties:** Let  $X_1, X_2, \dots, X_n$  be a sequence of iid RVs with mean  $\mu$  and variance  $\sigma^2$ . The **sample mean** is:  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . Further,  $\mathbb{E}[\bar{X}_n] = \mu$  and  $\text{Var}(\bar{X}_n) = \sigma^2/n$

**The Law of Large Numbers (LLN):** Let  $X_1, \dots, X_n$  be iid RVs with the same mean  $\mu$ . As  $n \rightarrow \infty$ , the sample mean  $\bar{X}_n$  converges to the true mean  $\mu$ .

**The Central Limit Theorem (CLT):** Let  $X_1, \dots, X_n$  be a sequence of iid RVs with mean  $\mu$  and (finite) variance  $\sigma^2$ . Then as  $n \rightarrow \infty$ ,

$$\bar{X}_n \rightarrow \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

The mean or variance are not a surprise; the importance of the CLT is, regardless of the distribution of  $X_i$ 's, the sample mean approaches a Normal distribution as  $n \rightarrow \infty$ .

**The Continuity Correction:** When approximating an integer-valued (*discrete*) random variable  $X$  with a *continuous* one  $Y$  (such as in the CLT), if asked to find a  $\mathbb{P}(a \leq X \leq b)$  for integers  $a \leq b$ , you should use  $\mathbb{P}(a - 0.5 \leq Y \leq b + 0.5)$  so that the width of the interval being integrated is the same as the number of terms summed over ( $b - a + 1$ ).

## 5.8 The Multinomial Distribution

**Random Vectors (RVTRs):** Let  $X_1, \dots, X_n$  be random variables. We say  $\mathbf{X} = (X_1, \dots, X_n)^T$  is a **random vector**. Expectation is defined pointwise:  $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_n])^T$ .

**Covariance Matrices:** The **covariance matrix** of a random vector  $\mathbf{X} \in \mathbb{R}^n$  with  $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$  is the matrix  $\Sigma = \text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X})$  whose entries  $\Sigma_{ij} = \text{Cov}(X_i, X_j)$ . The formula for this is:

$$\begin{aligned}\Sigma &= \text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X}) = \mathbb{E}[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T] = \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \boldsymbol{\mu}\boldsymbol{\mu}^T \\ &= \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \dots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \dots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \dots & \text{Var}(X_n) \end{bmatrix}\end{aligned}$$

Notice that the covariance matrix is **symmetric** ( $\Sigma_{ij} = \Sigma_{ji}$ ), and has variances on the diagonal.

**The Multinomial Distribution:** Suppose there are  $r$  outcomes, with probabilities  $\mathbf{p} = (p_1, p_2, \dots, p_r)$  respectively, such that  $\sum_{i=1}^r p_i = 1$ . Suppose we have  $n$  independent trials, and let  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_r)$  be the rvtr of counts of each outcome. Then, we say  $\mathbf{Y} \sim \text{Mult}_r(n, \mathbf{p})$ :

The joint PMF of  $\mathbf{Y}$  is:

$$p_{Y_1, \dots, Y_r}(k_1, \dots, k_r) = \binom{n}{k_1, \dots, k_r} \prod_{i=1}^r p_i^{k_i}, \quad k_1, \dots, k_r \geq 0 \text{ and } \sum_{i=1}^r k_i = n$$

Notice that each  $Y_i$  is marginally  $\text{Bin}(n, p_i)$ . Hence,  $\mathbb{E}[Y_i] = np_i$  and  $\text{Var}(Y_i) = np_i(1 - p_i)$ . Then, we can specify the entire mean vector  $\mathbb{E}[\mathbf{Y}]$  and covariance matrix:

$$\mathbb{E}[\mathbf{Y}] = n\mathbf{p} = \begin{bmatrix} np_1 \\ \vdots \\ np_r \end{bmatrix} \quad \text{Var}(Y_i) = np_i(1 - p_i) \quad \text{Cov}(Y_i, Y_j) = -np_i p_j$$

**The Multivariate Hypergeometric (MVHG) Distribution:** Suppose there are  $r$  different colors of balls in a bag, having  $\mathbf{K} = (K_1, \dots, K_r)$  balls of each color,  $1 \leq i \leq r$ . Let  $N = \sum_{i=1}^r K_i$  be the total number of balls in the bag, and suppose we draw  $n$  without replacement. Let  $\mathbf{Y} = (Y_1, \dots, Y_r)$  be the rvtr such that  $Y_i$  is the number of balls of color  $i$  we drew. We write that  $\mathbf{Y} \sim \text{MVHG}_r(N, \mathbf{K}, n)$ . The joint PMF of  $\mathbf{Y}$  is:

$$p_{Y_1, \dots, Y_r}(k_1, \dots, k_r) = \frac{\prod_{i=1}^r \binom{K_i}{k_i}}{\binom{N}{n}}, \quad 0 \leq k_i \leq K_i \text{ for all } 1 \leq i \leq r \text{ and } \sum_{i=1}^r k_i = n$$

Notice that each  $Y_i$  is marginally  $\text{HypGeo}(N, K_i, n)$ , so  $\mathbb{E}[Y_i] = n \frac{K_i}{N}$  and  $\text{Var}(Y_i) = n \frac{K_i}{N} \cdot \frac{N - K_i}{N} \cdot \frac{N - n}{N - 1}$ . The mean vector  $\mathbb{E}[\mathbf{Y}]$  and covariance matrix are:

$$\mathbb{E}[\mathbf{Y}] = n \frac{\mathbf{K}}{N} = \begin{bmatrix} n \frac{K_1}{N} \\ \vdots \\ n \frac{K_r}{N} \end{bmatrix} \quad \text{Var}(Y_i) = n \frac{K_i}{N} \cdot \frac{N - K_i}{N} \cdot \frac{N - n}{N - 1} \quad \text{Cov}(Y_i, Y_j) = -n \frac{K_i}{N} \frac{K_j}{N} \cdot \frac{N - n}{N - 1}$$

## 5.9 The Multivariate Normal Distribution

**Properties of Expectation and Variance Hold for RVTRs:** Let  $\mathbf{X}$  be an  $n$ -dimensional RVTR,  $A \in \mathbb{R}^{n \times n}$  be a constant matrix,  $\mathbf{b} \in \mathbb{R}^n$  be a constant vector. Then:  $\mathbb{E}[A\mathbf{X} + \mathbf{b}] = A\mathbb{E}[\mathbf{X}] + \mathbf{b}$  and  $\text{Var}(A\mathbf{X} + \mathbf{b}) = A\text{Var}(\mathbf{X})A^T$ .

**The Multivariate Normal Distribution:** A random vector  $\mathbf{X} = (X_1, \dots, X_n)$  has a multivariate Normal distribution with mean vector  $\boldsymbol{\mu} \in \mathbb{R}^n$  and (symmetric and positive-definite) covariance matrix  $\Sigma \in \mathbb{R}^{n \times n}$ , written  $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ , if it has the following joint PDF:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right), \quad \mathbf{x} \in \mathbb{R}^n$$

Additionally, let us recall that for any RVs  $X$  and  $Y$ :  $X \perp Y \rightarrow \text{Cov}(X, Y) = 0$ . If  $\mathbf{X} = (X_1, \dots, X_n)$  is Multivariate Normal, the converse also holds:  $\text{Cov}(X_i, X_j) = 0 \rightarrow X_i \perp X_j$ .

## 5.10 Order Statistics

**Order Statistics:** Suppose  $Y_1, \dots, Y_n$  are iid *continuous* random variables with common PDF  $f_Y$  and common CDF  $F_Y$ . We sort the  $Y_i$ 's such that  $Y_{\min} \equiv Y_{(1)} < Y_{(2)} < \dots < Y_{(n)} \equiv Y_{\max}$ .

Notice that we can't have equality because with continuous random variables, the probability that any two are equal is 0. Notice that each  $Y_{(i)}$  is a random variable as well! We call  $Y_{(i)}$  the **ith order statistic**, i.e. the  $i$ th smallest in a sample of size  $n$ . The density function of each  $Y_{(i)}$  is

$$f_{Y_{(i)}}(y) = \binom{n}{i-1, 1, n-i} \cdot [F_Y(y)]^{i-1} \cdot [1 - F_Y(y)]^{n-i} \cdot f_Y(y), y \in \Omega_Y$$

# 6 Concentration Inequalities

## 6.1 Markov and Chebyshev Inequalities

**Markov's Inequality:** Let  $X \geq 0$  be a **non-negative** RV, and let  $k > 0$ . Then:  $\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}[X]}{k}$ .

**Chebyshev's Inequality:** Let  $X$  be any RV with expected value  $\mu = \mathbb{E}[X]$  and finite variance  $\text{Var}(X)$ . Then, for any real number  $\alpha > 0$ . Then,  $\mathbb{P}(|X - \mu| \geq \alpha) \leq \frac{\text{Var}(X)}{\alpha^2}$ .

## 6.2 The Chernoff Bound

**Chernoff Bound for Binomial:** Let  $X \sim \text{Bin}(n, p)$  and let  $\mu = \mathbb{E}[X]$ . For any  $0 < \delta < 1$ :

$$\mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right) \quad \text{and} \quad \mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$$

## 6.3 Even More Inequalities

**The Union Bound:** Let  $E_1, E_2, \dots, E_n$  be a collection of events. Then:  $\mathbb{P}(\bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n \mathbb{P}(E_i)$ . A similar statement also holds if the number of events is *countably* infinite.

**Convex Sets:** A set  $S \subseteq \mathbb{R}^n$  is a **convex set** if for any  $x_1, \dots, x_m \in S$

$$\left\{ \sum_{i=1}^m p_i x_i : p_1, \dots, p_m \geq 0 \text{ and } \sum_{i=1}^m p_i = 1 \right\} \subseteq S$$

**Convex Functions:** Let  $S \subseteq \mathbb{R}^n$  be a convex set. A function  $g : S \rightarrow \mathbb{R}$  is a **convex function** if for any  $x_1, \dots, x_m \in S$ , and  $p_1, \dots, p_m \geq 0$  such that  $\sum_{i=1}^m p_i = 1$ ,

$$g\left(\sum_{i=1}^m p_i x_i\right) \leq \sum_{i=1}^m p_i g(x_i)$$

**Jensen's Inequality:** Let  $X$  be any RV, and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be convex. Then,  $g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)]$ .

**Hoeffding's Inequality:** Let  $X_1, \dots, X_n$  be independent random variables, where each  $X_i$  is bounded:  $a_i \leq X_i \leq b_i$  and let  $\bar{X}_n$  be their sample mean. Then,

$$\mathbb{P}(|\bar{X}_n - \mathbb{E}[\bar{X}_n]| \geq t) \leq 2 \exp\left(\frac{2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

In the case  $X_1, \dots, X_n$  are iid (so  $a \leq X_i \leq b$  for all  $i$ ) with mean  $\mu$ , then

$$\mathbb{P}(|\bar{X}_n - \mu| \geq t) \leq 2 \exp\left(\frac{2n^2t^2}{n(b-a)^2}\right) = 2 \exp\left(\frac{2nt^2}{(b-a)^2}\right)$$

## 7 Statistical Estimation

### 7.1 Maximum Likelihood Estimation

**Realization / Sample:** A **realization/sample**  $x$  of a random variable  $X$  is the value that is actually observed (will always be in  $\Omega_X$ ).

**Likelihood:** Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from PMF  $p_X(t | \theta)$  (if  $X$  is discrete), or from density  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). We define the **likelihood** of  $\mathbf{x}$  given  $\theta$  to be the “probability” of observing  $\mathbf{x}$  if the true parameter is  $\theta$ . The **log-likelihood** is just the log of the likelihood, which is typically easier to optimize.

If  $X$  is discrete,

$$L(\mathbf{x} | \theta) = \prod_{i=1}^n p_X(x_i | \theta) \quad \ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \ln p_X(x_i | \theta)$$

If  $X$  is continuous,

$$L(\mathbf{x} | \theta) = \prod_{i=1}^n f_X(x_i | \theta) \quad \ln L(\mathbf{x} | \theta) = \sum_{i=1}^n \ln f_X(x_i | \theta)$$

**Maximum Likelihood Estimator (MLE):** Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from probability mass function  $p_X(t | \theta)$  (if  $X$  is discrete), or from density  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). We define the **maximum likelihood estimator (MLE)**  $\hat{\theta}_{MLE}$  of  $\theta$  to be the parameter which maximizes the likelihood/log-likelihood:

$$\hat{\theta}_{MLE} = \arg \max_{\theta} L(\mathbf{x} | \theta) = \arg \max_{\theta} \ln L(\mathbf{x} | \theta)$$

### 7.2 MLE Examples

### 7.3 Method of Moments Estimation

**Sample Moments:** Let  $X$  be a random variable, and  $c \in \mathbb{R}$  a scalar. Let  $x_1, \dots, x_n$  be iid realizations (samples) from  $X$ .

The  $k^{th}$  **sample moment** of  $X$  is:  $\frac{1}{n} \sum_{i=1}^n x_i^k$ .

The  $k^{th}$  **sample moment** of  $X$  (about  $c$ ) is:  $\frac{1}{n} \sum_{i=1}^n (x_i - c)^k$ .

**Method of Moments Estimation:** Let  $x = (x_1, \dots, x_n)$  be iid realizations (samples) from PMF  $p_X(t; \theta)$  (if  $X$  is discrete), or from density  $f_X(t; \theta)$  (if  $X$  continuous), where  $\theta$  is a parameter (or vector of parameters).

We then define the **Method of Moments (MoM)** estimator  $\hat{\theta}_{MoM}$  of  $\theta = (\theta_1, \dots, \theta_k)$  to be a solution (if it exists) to the  $k$  simultaneous equations where, for  $j = 1, \dots, k$ , we set the  $j^{th}$  true and sample moments equal:

$$\mathbb{E}[X] = \frac{1}{n} \sum_{i=1}^n x_i \quad \dots \quad \mathbb{E}[X^k] = \frac{1}{n} \sum_{i=1}^n x_i^k$$

## 7.4 The Beta and Dirichlet Distributions

**Beta Random Variable:**  $X \sim \text{Beta}(\alpha, \beta)$ , if and only if  $X$  has the following PDF:

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, & x \in \Omega_X = [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

$X$  is typically the belief distribution about some unknown probability of success, where we pretend we've seen  $\alpha - 1$  successes and  $\beta - 1$  failures. Hence the mode (most likely value of the probability/point with highest density)  $\arg \max_{x \in [0, 1]} f_X(x)$ , is

$$\text{mode}[X] = \frac{\alpha-1}{(\alpha-1) + (\beta-1)}$$

Also note that there is an annoying “off-by-1” issue: ( $\alpha - 1$  heads and  $\beta - 1$  tails), so when choosing these parameters, be careful! It also serves as a conjugate prior for  $p$  in the Bernoulli and Geometric distributions.

**Dirichlet RV:**  $X \sim \text{Dir}(\alpha_1, \alpha_2, \dots, \alpha_r)$ , if and only if  $X$  has the following density function:

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{1}{B(\alpha)} \prod_{i=1}^r x_i^{\alpha_i-1}, & x_i \in (0, 1) \text{ and } \sum_{i=1}^r x_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

This is a generalization of the Beta random variable from 2 outcomes to  $r$ . The random vector  $X$  is typically the belief distribution about some unknown probabilities of the different outcomes, where we pretend we saw  $a_1 - 1$  outcomes of type 1,  $a_2 - 1$  outcomes of type 2, ..., and  $a_r - 1$  outcomes of type  $r$ . Hence, the mode of the distribution is the vector,  $\arg \max_{\mathbf{x} \in [0, 1]^d \text{ and } \sum x_i = 1} f_{\mathbf{X}}(\mathbf{x})$ , is

$$\text{mode}[\mathbf{X}] = \left( \frac{\alpha_1-1}{\sum_{i=1}^r (\alpha_i-1)}, \frac{\alpha_2-1}{\sum_{i=1}^r (\alpha_i-1)}, \dots, \frac{\alpha_r-1}{\sum_{i=1}^r (\alpha_i-1)} \right)$$

## 7.5 Maximum A Posteriori Estimation

**Maximum A Posteriori (MAP) Estimation:** Let  $x = (x_1, \dots, x_n)$  be iid realizations from PMF  $p_X(t; \Theta = \theta)$  (if  $X$  discrete), or from density  $f_X(t; \Theta = \theta)$  (if  $X$  continuous), where  $\Theta$  is the random variable representing the parameter (or vector of parameters). We define the **Maximum A Posteriori (MAP)** estimator  $\hat{\theta}_{MAP}$  of  $\Theta$  to be the parameter which maximizes the **posterior** distribution of  $\Theta$  given the data (the mode).

$$\hat{\theta}_{MAP} = \operatorname{argmax}_{\theta} \pi_{\Theta}(\theta | \mathbf{x}) = \operatorname{argmax}_{\theta} L(\mathbf{x} | \theta) \pi_{\Theta}(\theta)$$

## 7.6 Properties of Estimators I

**Bias:** Let  $\hat{\theta}$  be an estimator for  $\theta$ . The **bias** of  $\hat{\theta}$  as an estimator for  $\theta$  is  $\text{Bias}(\hat{\theta}, \theta) = \mathbb{E}[\hat{\theta}] - \theta$ . If  $\text{Bias}(\hat{\theta}, \theta) = 0$ , or equivalently  $\mathbb{E}[\hat{\theta}] = \theta$ , then we say  $\hat{\theta}$  is an **unbiased** estimator of  $\hat{\theta}$ .

**Mean Squared Error (MSE):** The **mean squared error (MSE)** of an estimator  $\hat{\theta}$  of  $\theta$  is  $\text{MSE}(\hat{\theta}, \theta) = \mathbb{E}[(\hat{\theta} - \theta)^2]$ .

If  $\hat{\theta}$  is an unbiased estimator of  $\theta$  (i.e.  $\mathbb{E}[\hat{\theta}] = \theta$ ), then you can see that  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta})$ . In fact, in general  $\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2$ .

## 7.7 Properties of Estimators II

**Consistency:** An estimator  $\hat{\theta}_n$  (depending on  $n$  iid samples) of  $\theta$  is said to be **consistent** if it converges (in probability) to  $\theta$ . That is, for any  $\varepsilon > 0$ ,  $\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \varepsilon) = 0$ .

**Fisher Information:** Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from PMF  $p_X(t | \theta)$  (if  $X$  is discrete), or from density function  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). The **Fisher Information** of a parameter  $\theta$  is defined to be

$$I(\theta) = n \cdot \mathbb{E} \left[ \left( \frac{\partial \ln L(\mathbf{x} | \theta)}{\partial \theta} \right)^2 \right] = -\mathbb{E} \left[ \frac{\partial^2 \ln L(\mathbf{x} | \theta)}{\partial \theta^2} \right]$$

**Cramer-Rao Lower Bound (CRLB):** Let  $\mathbf{x} = (x_1, \dots, x_n)$  be iid realizations from PMF  $p_X(t | \theta)$  (if  $X$  is discrete), or from density function  $f_X(t | \theta)$  (if  $X$  is continuous), where  $\theta$  is a parameter (or vector of parameters). If  $\hat{\theta}$  is an *unbiased* estimator for  $\theta$ , then

$$\text{MSE}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

That is, for any unbiased estimator  $\hat{\theta}$  for  $\theta$ , the variance (=MSE) is at least  $\frac{1}{I(\theta)}$ . If we achieve this lower bound, meaning our variance is exactly equal to  $\frac{1}{I(\theta)}$ , then we have the best variance possible for our estimate. Hence, it is the **minimum variance unbiased estimator (MVUE)** for  $\theta$ .

**Efficiency:** Let  $\hat{\theta}$  be an unbiased estimator of  $\theta$ . The efficiency of  $\hat{\theta}$  is  $e(\hat{\theta}, \theta) = \frac{I(\theta)^{-1}}{\text{Var}(\hat{\theta})} \leq 1$ .

An estimator is said to be **efficient** if it achieves the CRLB - meaning  $e(\hat{\theta}, \theta) = 1$ .

## 7.8 Properties of Estimators III

**Statistic:** A **statistic** is any function  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  of samples  $\mathbf{x} = (x_1, \dots, x_n)$ . For example,  $T(x_1, \dots, x_n) = \sum_{i=1}^n x_i$  (the sum),  $T(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$  (the max/largest value),  $T(x_1, \dots, x_n) = x_1$  (just take the first sample)

**Sufficiency:** A statistic  $T = T(X_1, \dots, X_n)$  is a **sufficient statistic** if the conditional distribution of  $X_1, \dots, X_n$  given  $T = t$  and  $\theta$  does not depend on  $\theta$ .

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n | T = t, \theta) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n | T = t)$$

**Neyman-Fisher Factorization Criterion (NFFC):** Let  $x_1, \dots, x_n$  be iid random samples with likelihood  $L(x_1, \dots, x_n | \theta)$ . A statistic  $T = T(x_1, \dots, x_n)$  is sufficient if and only if there exist non-negative functions  $g$  and  $h$  such that:

$$L(x_1, \dots, x_n | \theta) = g(x_1, \dots, x_n) \cdot h(T(x_1, \dots, x_n), \theta)$$

## 8 Statistical Inference

### 8.1 Confidence Intervals

**Confidence Interval:** Suppose you have iid samples  $x_1, \dots, x_n$  from some distribution with unknown parameter  $\theta$ , and you have some estimator  $\hat{\theta}$  for  $\theta$ .

A  $100(1 - \alpha)\%$  **confidence interval** for  $\theta$  is an interval (typically but not always) centered at  $\hat{\theta}$ ,  $[\hat{\theta} - \Delta, \hat{\theta} + \Delta]$ , such that the probability (over the randomness in the samples  $x_1, \dots, x_n$ )  $\theta$  lies in the interval is  $1 - \alpha$ :

$$\mathbb{P}(\theta \in [\hat{\theta} - \Delta, \hat{\theta} + \Delta]) = 1 - \alpha$$

If  $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$  is the sample mean, then  $\hat{\theta}$  is approximately normal by the CLT, and a  $100(1 - \alpha)\%$  confidence interval is given by the formula:

$$\left[ \hat{\theta} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}, \hat{\theta} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \right]$$

where  $z_{1-\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2})$  and  $\sigma$  is the true standard deviation of a single sample (which may need to be estimated).

### 8.2 Credible Intervals

**Credible Intervals:** Suppose you have iid samples  $\mathbf{x} = (x_1, \dots, x_n)$  from some distribution with unknown parameter  $\Theta$ . You are in the **Bayesian setting**, so you have chosen a prior distribution for the RV  $\Theta$ .

A  $100(1 - \alpha)\%$  **credible interval** for  $\Theta$  is an interval  $[a, b]$  such that the probability (over the randomness in  $\Theta$ ) that  $\Theta$  lies in the interval is  $1 - \alpha$ :

$$P(\Theta \in [a, b]) = 1 - \alpha$$

If we've chosen the appropriate conjugate prior for the sampling distribution (like Beta for Bernoulli), the posterior is easy to compute. Say the CDF of the posterior is  $F_Y$ . Then, a  $100(1 - \alpha)\%$  credible interval is given by

$$\left[ F_Y^{-1}\left(\frac{\alpha}{2}\right), F_Y^{-1}\left(1 - \frac{\alpha}{2}\right) \right]$$

### 8.3 Introduction to Hypothesis Testing

#### Hypothesis Testing Procedure:

1. Make a claim (like "Airplane food is good", "Pineapples belong on pizza", etc...)
2. Set up a null hypothesis  $H_0$  and alternative hypothesis  $H_A$ .
  - (a) Alternative hypothesis can be one-sided or two-sided.
  - (b) The null hypothesis is usually a "baseline", "no effect", or "benefit of the doubt".
  - (c) The alternative is what you want to "prove", and is opposite the null.
3. Choose a significance level  $\alpha$  (usually  $\alpha = 0.05$  or  $0.01$ ).
4. Collect data.
5. Compute a p-value,  $p = \mathbb{P}(\text{observing data at least as extreme as ours} \mid H_0 \text{ is true})$ .
6. State your conclusion. Include an interpretation in the context of the problem.
  - (a) If  $p < \alpha$ , "reject" the null hypothesis  $H_0$  in favor of the alternative  $H_A$ .
  - (b) Otherwise, "fail to reject" the null hypothesis  $H_0$ .

Discrete Distributions					
Distribution	Parameters	Possible Description	Range $\Omega_X$	$\mathbb{E}[\mathbf{X}]$	$\text{Var}(\mathbf{X})$
Uniform (disc)	$X \sim \text{Unif}(a, b)$ for $a, b \in \mathbb{Z}$ and $a \leq b$	Equally likely to be any <i>integer</i> in $[a, b]$	$\{a, \dots, b\}$	$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$
Bernoulli	$X \sim \text{Ber}(p)$ for $p \in [0, 1]$	Takes value 1 with prob $p$ and 0 with prob $1-p$	$\{0, 1\}$	$p$	$p(1-p)$
Binomial	$X \sim \text{Bin}(n, p)$ for $n \in \mathbb{N}$ , and $p \in [0, 1]$	Sum of $n$ iid $\text{Ber}(p)$ rvs. # of heads in $n$ independent coin flips with $P(\text{head}) = p$ .	$\{0, 1, \dots, n\}$	$np$	$\binom{n}{k} p^k (1-p)^{n-k}$
Poisson	$X \sim \text{Poi}(\lambda)$ for $\lambda > 0$	# of events that occur in <b>one</b> unit of time independently with rate $\lambda$ per unit time	$\{0, 1, \dots\}$	$\lambda$	$e^{-\lambda} \frac{\lambda^k}{k!}$
Geometric	$X \sim \text{Geo}(p)$ for $p \in [0, 1]$	# of independent Bernoulli trials with parameter $p$ <i>up to</i> (w/o replacement) from $N$ items that contain $K$ successes in total	$\{1, 2, \dots\}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$X \sim \text{HypGeo}(N, K, n)$ for $n, K \leq N$ and $n, K, N \in \mathbb{N}$	Sum of $r$ iid $\text{Geo}(p)$ rws. # of independent flips until $r^{\text{th}}$ head with $P(\text{head}) = p$	$\{\max(0, n+K-N), \dots, \min(n, K)\}$	$\frac{n}{N} \frac{K}{N}$	$n \frac{K(N-K)(N-n)}{N^2(N-1)} \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$
Negative Binomial	$X \sim \text{NegBin}(r, p)$ for $r \in \mathbb{N}, p \in [0, 1]$	$\{r, r+1, \dots\}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\binom{k-1}{r-1} p^r (1-p)^{k-r}$
Multinomial	$\mathbf{X} \sim \text{Mult}_r(n, \mathbf{p})$ for $r, n \in \mathbb{N}$ and $\mathbf{p} = (p_1, p_2, \dots, p_r)$ , $\sum_{i=1}^r p_i = 1$	Generalization of the Binomial distribution, $n$ trials with $r$ categories each with probability $p_i$	$\begin{bmatrix} k_1 \in \{0, \dots, n\}, \\ i \in \{1, \dots, r\} \\ \text{and } \sum k_i = n \end{bmatrix}$	$\mathbb{E}[\mathbf{X}] = n\mathbf{p} = \begin{bmatrix} np_1 \\ \vdots \\ np_r \end{bmatrix}$	$\text{Var}(X_i) = np_i(1-p_i)$ $\text{Cov}(X_i, X_j) = -np_i p_j, i \neq j$ $\binom{n}{k_1, \dots, k_r} \prod_{i=1}^r p_i^{k_i}$
Multivariate Hypergeometric	$\mathbf{X} \sim \text{MVHG}_r(N, \mathbf{K}, n)$ for $r, n \in \mathbb{N}$ , $\mathbf{K} \in \mathbb{N}^r$ and $N = \sum_{i=1}^r K_i$	Generalization of the Hypergeometric distribution, $n$ draws from $r$ categories each with $K_i$ successes (w/out replacement)	$k_i \in \{0, \dots, K_i\},$ $i \in \{1, \dots, r\}$ and $\sum k_i = n$	$\mathbb{E}[\mathbf{X}] = n \frac{\mathbf{K}}{N} = \begin{bmatrix} n \frac{K_1}{N} \\ \vdots \\ n \frac{K_r}{N} \end{bmatrix}$	$\text{Var}(X_i) = \frac{n \frac{K_i}{N} \cdot \frac{N-K_i}{N} \cdot \frac{N-n}{N-1}}{N-1}$ $\text{Cov}(X_i, X_j) = -n \frac{K_i K_j}{N^2} \cdot \frac{N-n}{N-1}, i \neq j$ $\frac{\prod_{i=1}^r \binom{K_i}{k_i}}{\binom{N}{n}}$

Continuous Distributions							
Distribution	Parameters	Possible Description	Range $\Omega_X$	$\mathbb{E}[\mathbf{X}]$	Var( $\mathbf{X}$ )	PDF $f_X(x)$ for $x \in \Omega_X$	CDF $(\mathbf{F}_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}))$
Uniform	$X \sim \text{Unif}(a, b)$ for $a < b$	Equally likely to be any real number in $[a, b]$	$[a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{b-a}$	$\begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b \end{cases}$
Exponential	$X \sim \text{Exp}(\lambda)$ for $\lambda > 0$	Time until first event in Poisson process	$[0, \infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\lambda e^{-\lambda x}$	$\begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$
Normal	$X \sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$ , and $\sigma^2 > 0$	Standard bell curve	$(-\infty, \infty)$	$\mu$	$\sigma^2$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$
Gamma	$X \sim \text{Gam}(r, \lambda)$ for $r, \lambda > 0$	Sum of $r$ iid $\text{Exp}(\lambda)$ rvs. Time to $r^{\text{th}}$ event in Poisson process. Conjugate prior for $\text{Exp}$ , Poi parameter $\lambda$	$[0, \infty)$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$	$\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$	Note: $\Gamma(r) = (r-1)!$ for integers $r$ .
Beta	$X \sim \text{Beta}(\alpha, \beta)$ for $\alpha, \beta > 0$	Conjugate prior for Ber, Bin, Geo, NegBin parameter $p$	$(0, 1)$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	
Dirichlet	$\mathbf{X} \sim \text{Dir}(\alpha_1, \alpha_2, \dots, \alpha_r)$ for $\alpha_i, r > 0$ and $r \in \mathbb{N}, \alpha_i \in \mathbb{R}$	Generalization of Beta distribution. Conjugate prior for Multinomial parameter $\mathbf{P}$	$x_i \in (0, 1); \sum_{i=1}^r x_i = 1$	$\mathbb{E}[X_i] = \frac{\alpha_i}{\sum_{j=1}^r \alpha_j}$	$\frac{1}{B(\alpha)} \prod_{i=1}^r x_i^{\alpha_i-1},$ $x_i \in (0, 1), \sum_{i=1}^r x_i = 1$		
Multivariate Normal	$\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$	Generalization of Normal distribution	$\mathbb{R}^n$	$\boldsymbol{\mu}$	$\boldsymbol{\Sigma}$	$\frac{1}{(2\pi)^{n/2}  \boldsymbol{\Sigma} ^{1/2}} \cdot \exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}))$	