Primal and Dual Generalized Eigenvalue Problems for Power Systems Small-Signal Stability Analysis

Federico Milano, *Fellow, IEEE*, and Ioannis Dassios

Abstract—The paper presents a comprehensive study of small-signal stability analysis of power systems based on matrix pencils and the generalized eigenvalue problem. Both primal and dual formulations of the generalized eigenvalue problem are considered and solved through a variety of state-of-the-art solvers. The paper also discusses the impact on the performance of the solvers of two formulations of the equations modelling the power systems, namely, the explicit and semi-implicit form of differential-algebraic equations. The case study illustrates the theoretical aspects and numerical features of these formulations and solvers through two real-world systems, namely, a 1,479-bus model of the all-island Irish system, and a 21,177-bus model of the ENTSO-E network.

Index Terms—Small-signal stability analysis, generalized eigenvalue problem, duality, differential algebraic equations, matrix pencil.

I. INTRODUCTION

A. Motivations

IN HIS seminal monograph, Wilkinson wrote: *The eigenvalue problem has a deceptively simple formulation and the background theory has been known for many years; yet the determination of accurate solutions presents a wide variety of challenging problems* [1]. After half a century, this statement is still actual. The ability to solve large eigenvalue problems is crucial in several fields of applied mathematics, physics and engineering, e.g., [2]–[5]. Eigenvalue analysis is also a fundamental tool of the small-signal stability analysis of power systems.

There are several variants of the eigenvalue problem. The simplest form, which is the one commonly solved for power system small-signal stability analysis, is formulated as the problem of finding a complex scalar s and a non-zero vector v that solve:

$$Av = sv \tag{1}$$

where A is a square real matrix of order $n \times n$. For every solution of (1), s is an *eigenvalue*, and v is the associated *eigenvector*. Equation (1) is known as Linear Eigenvalue Problem (LEP).

This paper provides a thorough discussion of an alternative formulation, namely, the Generalized Eigenvalue Problem (GEP):

$$Av = sBv \tag{2}$$

where B is a real $n \times n$ matrix, and its dual version:

$$B\hat{v} = \hat{s}A\hat{v} \tag{3}$$

where $\hat{s}$ and $\hat{v}$ are the eigenvalue and corresponding eigenvector, respectively, of the dual GEP. Both theoretical and computational aspects of (2) and (3) and their impact on power system small-signal stability analysis are discussed in the paper.

B. Literature and Software Review

In several applied research fields, the generalized formulation (2) has been successfully utilized to solve large eigenvalue problems (see, for example, [6]–[15]). In power system analysis, however, the eigenvalue problem has been mostly formulated as in (1) [16]–[23]. A recent paper that discusses the primal GEP for power system analysis is [24]. The importance of preserving sparsity of the state matrix has been first identified in [17] and then assumed in following works. Since power systems are modelled as a set of Differential Algebraic Equations (DAEs), preserving sparsity implies considering an *augmented matrix* of the linearized system that has the size of the combined vector of state and algebraic variables. In [17]–[19], available algorithms to solve (1) are adapted in order to allow adopting the augmented matrix. This basically consists in solving the subsystem of algebraic variables and substituting them into the linearized differential equations. In this paper, we show that the GEP formulation (2) is a more natural and general way to preserve sparsity, because, in (2), there is no need to distinguish between state and algebraic variables.

The dual problem (3) has been considered in recent years in applied mathematics [25] and in signal processing applications [26], [27]. These references focus on the Hermitian GEP, for which A and/or B are symmetric or Hermitian. However, the structure of power systems always leads to a non-Hermitian A and, hence, the dual eigenvalue problem (3) is never Hermitian.

The most common formulation of power system DAEs is explicit. Explicit DAEs (Ex-DAEs) show a diagonal B, where diagonal elements are 1 or 0. This leads to an eigenvalue problem where A is non-Hermitian but B is Hermitian and positive semi-definite. On the other hand, if a Semi-Implicit DAE (SI-DAE) formulation is used as in [28], neither A nor B are Hermitian. Hence, solvers that admit a fully non-Hermitian GEP

are needed. LAPACK [29] can handle the most generic form of GEPs, but works only with dense matrices and is thus suitable for small size problems. The GPU-based library MAGMA [30], which is a port of LAPACK routines, currently only support LEPs as in (1).

ARPACK [31] provides a state-of-art implementation of the Arnoldi iteration, including the implicit shift-&-invert and Caley transformation, but requires that B is positive semi-definite and can thus solve only (1) and (2) for the Ex-DAE formulation.

To the best of our knowledge, there are currently only three libraries that allows solving fully non-Hermitian GEPs and, thus, can handle equally well problems (1), (2) and (3). These are SLEPc [32], Z-PARES [33] and FEAST [34]. SLEPc includes Jacobi-Davidson, generalized Davidson and Krylov-Schur methods that are basically variants of the Arnoldi iteration. Z-PARES and FEAST implement a numerical quadrature computation and solves several linear systems along a complex contour and computes eigenvalues inside such a contour. The advantage of this procedure is that the size of each quadrature problem is smaller than that of the original GEP. Moreover, the quadrature problems can be solved in parallel as they are fully decoupled from each other.

Commercial software tools, such as PSS/E and DSATools, implement explicit DAEs. For small networks, e.g., for less than 800 state variables, these software packages use QR-based eigenvalue analysis as in LAPACK. For larger networks, the NEVA (Netomac Eigenvalue Analysis) module of PSS/E provides an implicit inverse iteration and a transfer function-based dominant pole method, which, in turn, is solved using subspace techniques, such as the Krylov subspace method. Apart from the QR algorithm, DSATools includes implicitly restarted Arnoldi and an enhanced version of the AESOPS algorithm [35], which is an iterative method to find local modes–typically those of a given machine–and is not comparable to the techniques considered here. Thus, except for the enhanced AESOPS algorithm, this paper considers standard industry-grade techniques through available open-source libraries.

C. Contributions

The paper presents a comprehensive study of the solution of the GEP for the small-signal stability analysis of power systems. Several aspects are covered leading to the following specific contributions.

1) The formulation of the small-signal stability analysis in terms of a GEP considering both Ex-DAE and SI-DAE formulations of power system models.
2) A rigorous proof of the equivalence of the primal and dual GEPs.
3) A fair and complete comparison of state-of-art solvers of non-Hermitian GEP problems.

D. Contents

The remainder of the paper is organized as follows. Section II defines the primal and dual GEPs and provides a theoretical proofs of their equivalence. Section III presents power system models based on DAEs. These are the conventional explicit model and a semi-implicit formulation, as recently proposed in [28]. Section IV presents a variety of state-of-art solvers for large non-Hermitian GEPs. These are LAPACK, MAGMA, ARPACK, SLEPc, FEAST, and Z-PARES. Section V presents numerical results based on two real-world systems, namely a 1,479-bus model of the all-island Irish system, and a 21,177-bus model of the ENTSO-E network. Finally, Section VI duly draws conclusions.

II. PRIMAL AND DUAL GENERALIZED EIGENVALUE PROBLEMS

This section provides relevant definitions on matrix pencils and GEPs and presents a theorem that shows the equivalency between the primal and dual GEPs. The main objectives of this section is to state the mathematical equivalency of the primal and dual GEPs and of the properties of the eigenvectors associated with null, finite and infinite eigenvalues.

It is important to note that, while we can prove that the primal and dual GEPs are equivalent from the mathematical point of view, the two problems behave differently when solved numerically and have certainly different computational burdens. For example, A is generally less sparse than B. Numerical techniques are thus expected to performs differently and, possibly, to provide different results, depending on the GEP formulation, as discussed in Section V.

Let us consider the following singular system of differential equations of the form

$$Bz'(t) = Az(t) \tag{4}$$

where $B, A \in \mathbb{R}^{n \times n}$, $z : [0, +\infty) \to \mathbb{R}^n$ and the matrix B is singular. Note that B is always singular for DAEs and hence (4) is of interest for power systems modelled as it is discussed in the next section. The importance of systems of type (4) has been further emphasized by their role in defining notions of duality. The notion of *dual configuration* and of the *dual problem* originates from projective geometry. The essence of the *dual system* defined next, is of similar nature to that of projective geometry may be stated for autonomous differential systems. Using this principle, if a proposition is true on one system, the dual proposition is true for the dual system. We define the system

$$A\hat{z}'(t) = B\hat{z}(t) \tag{5}$$

Definition 1: The system (4) will be referred as the *primal system* and (5) will be defined as the proper *dual system*, or simply dual.

Definition 2: Given $B, A \in \mathbb{R}^{n \times m}$ and an arbitrary $s \in \mathbb{C}$, a *matrix pencil* is a family of matrices $sB - A$, parametrized by a complex number s. The pencil is then called:

1) Regular when $n = m$ and $\det(sB - A) = \mathrm{p}(s) \not\equiv 0$. Where $\mathrm{p}(s)$ is a polynomial of order equal to the finite eigenvalues of the pencil;
2) Singular when $n \neq m$ or $n = m$ and $\det(sB - A) \equiv 0$.

In this article, since B, A are defined based on a physical dynamical system (see Section III), the resulting pencil $sB - A$ is always *regular*. Hence, hereinafter, $n = m$ will be always assumed. A regular pencil has p finite eigenvalues (FEs) and q infinite eigenvalues (IEs), with $p + q = n$. The FEs are the zeros of the polynomial $\det(sB - A) = \mathrm{p}(s)$ which is of order

p. The existence of IEs can be seen by means of the generalized eigenvalue problem $s\boldsymbol{B}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}$ in the reciprocal form $\boldsymbol{B}\boldsymbol{u} = s^{-1}\boldsymbol{A}\boldsymbol{u}$. Since $\boldsymbol{B}$ is singular there exists a null vector $\boldsymbol{u}$ such that $\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0}_{n,1}$ and consequently $s^{-1}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}_{n,1}$, so that $\boldsymbol{u}$ is an eigenvector of the reciprocal problem corresponding to eigenvalue $s^{-1} = 0$, i.e., $s \to \infty$. Since $\boldsymbol{B}$ is singular, the pencil $\boldsymbol{B} - \hat{s}\boldsymbol{A}$ has always the zero eigenvalue with algebraic multiplicity $q \geq 1$ and consequently q is the algebraic multiplicity of the infinite eigenvalue of $s\boldsymbol{B} - \boldsymbol{A}$.

The corresponding pencil of (4) is $s\boldsymbol{B} - \boldsymbol{A}$ and of (5) $\boldsymbol{B} - \hat{s}\boldsymbol{A}$. It is clear that the essence of the above type of duality depends on the relationships between the associated pencils. The study of duality between (4) and (5) is reduced to an investigation of the links between their pencils. The main result is that, if the solution of one of those systems is known, the solution of the other two systems can be represented, without further computation.

The complete set of invariants of $s\boldsymbol{B} - \boldsymbol{A}$ is of the following type:

1) Zero eigenvalues of algebraic multiplicity p_0;
2) Non-zero finite eigenvalues a_i of algebraic multiplicity p_i;
3) Infinite eigenvalues of algebraic multiplicity q.

Where $\sum_{i=0}^{\nu} p_i = p$ (ν is the number of the distinct non-zero eigenvalues).

The duality between (4) and (5) or between their pencils $s\boldsymbol{B} - \boldsymbol{A}$, $\boldsymbol{B} - \hat{s}\boldsymbol{A}$, can be seen as a consequence of the special bilinear or projective transformation on a straight line:

$$s \longrightarrow \frac{1}{\hat{s}},$$

which clearly transforms the points 0, $s_i \neq 0$, ∞ of the compacted complex plane to the points ∞, $\frac{1}{s_i}$, 0, correspondingly. The notions of duality may be qualified algebraically in terms of relationships between the strict equivalence invariants of the associated pencils. These relationships are summarized below.

1) A zero eigenvalue of $s\boldsymbol{B} - \boldsymbol{A}$ is an infinite eigenvalue of $\boldsymbol{B} - \hat{s}\boldsymbol{A}$ and vice versa;
2) A non-zero finite eigenvalue a_i of $s\boldsymbol{B} - \boldsymbol{A}$ defines a nonzero finite eigenvalue $\frac{1}{a_i}$ of $\boldsymbol{B} - \hat{s}\boldsymbol{A}$ and vice versa;
3) An infinite eigenvalue of $s\boldsymbol{B} - \boldsymbol{A}$ is a zero eigenvalue of $\boldsymbol{B} - \hat{s}\boldsymbol{A}$ and vice versa.

The Appendix provides a rigorous discussion on the equivalence of the primal and dual GEPs and presents a theorem that formalizes such equivalence.

III. POWER SYSTEM MODELS FOR SMALL-SIGNAL STABILITY ANALYSIS

A. Explicit Differential-Algebraic Equations

Let us recall first conventional DAE models, described by the following equations:

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$$
$$\boldsymbol{0}_{q,1} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \tag{6}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^p$) are the differential equations; $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^q$) are the algebraic equations; $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^p$) are

the state variables; and $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathbb{R}^q$) are the algebraic variables. We also assume that (6) is autonomous, i.e., does not depend explicitly on time t. Finally, $\boldsymbol{0}_{i,j}$ is the zero matrix of i rows and j columns.

Assume that a stationary solution of (6) is known and has the form:

$$\boldsymbol{0}_{p,1} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0)$$
$$\boldsymbol{0}_{q,1} = \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0) \tag{7}$$

Then, differentiating (6) at the stationary solution yields:

$$\Delta \boldsymbol{x}' = \boldsymbol{f}_{\boldsymbol{x}}\Delta \boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{y}}\Delta \boldsymbol{y} \tag{8}$$
$$\boldsymbol{0}_{q,1} = \boldsymbol{g}_{\boldsymbol{x}}\Delta \boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{y}}\Delta \boldsymbol{y} \tag{9}$$

The state matrix of the system, namely $\boldsymbol{A}_s$, is defined as

$$\boldsymbol{A}_s = \boldsymbol{f}_{\boldsymbol{x}} - \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}} \tag{10}$$

which can be defined only if $\boldsymbol{g}_{\boldsymbol{y}}$ is non-singular. The characteristic equation to be solved based on the state matrix $\boldsymbol{A}_s$ is (1) with matrix pencil $s\boldsymbol{I}_p - \boldsymbol{A}_s$ and provides p finite eigenvalues and p associated eigenvectors.

In [17] and most of the following papers on power system small-signal stability analysis, the state matrix $\boldsymbol{A}_s$ is not utilized because it tends to be dense even if matrices $\boldsymbol{f}_{\boldsymbol{x}}$, $\boldsymbol{f}_{\boldsymbol{y}}$, $\boldsymbol{g}_{\boldsymbol{y}}$ and $\boldsymbol{g}_{\boldsymbol{x}}$ are sparse. Hence, the approach adopted in [17] and references thereafter has been to use the system *augmented matrix*, as follows:

$$\boldsymbol{A}_c = \begin{bmatrix} \boldsymbol{f}_{\boldsymbol{x}} & \boldsymbol{f}_{\boldsymbol{y}} \\ \boldsymbol{g}_{\boldsymbol{x}} & \boldsymbol{g}_{\boldsymbol{y}} \end{bmatrix} \tag{11}$$

which, however, complicates the computation of the eigenvalues, as these are defined only for a subset of $\boldsymbol{A}_c$. The characteristic equation defined in [17] is:

$$\begin{bmatrix} \boldsymbol{f}_{\boldsymbol{x}} - s\boldsymbol{I}_p & \boldsymbol{f}_{\boldsymbol{y}} \\ \boldsymbol{g}_{\boldsymbol{x}} & \boldsymbol{g}_{\boldsymbol{y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{0}_{q,1} \end{bmatrix} \tag{12}$$

where $\boldsymbol{v}$ is the eigenvector associated with the eigenvalue s and $\boldsymbol{y}$ is required to solve the augmented problem but is immaterial for the computation of the eigenvalues s. Two remarks are relevant: (i) the assumption that $\boldsymbol{y}$ is decoupled from $\boldsymbol{x}$ is correct but was not proven in [17]–such a proof is a contribution of this paper and is given in the Appendix; and (ii) the solution of (12) requires computing the term:

$$\boldsymbol{g}_{\boldsymbol{y}} - \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{f}_{\boldsymbol{x}} - s\boldsymbol{I}_p)^{-1}\boldsymbol{f}_{\boldsymbol{y}} \tag{13}$$

as many times as the number of iterations of the algorithm that solves the eigenvalue problem.

In this paper we propose a more general formulation for the characteristic equation of (6), which consists in considering a generalized eigenvalue problem as in (2), with matrix pencil $s\boldsymbol{B}_c - \boldsymbol{A}_c$, where

$$\boldsymbol{B}_c = \begin{bmatrix} \boldsymbol{I}_p & \boldsymbol{0}_{p,q} \\ \boldsymbol{0}_{q,p} & \boldsymbol{0}_{q,q} \end{bmatrix} \tag{14}$$

and we assume an augmented state vector $\boldsymbol{z} = [\boldsymbol{x}^T, \boldsymbol{y}^T]^T$. The matrix $\boldsymbol{B}_c$ is positive semi-definite, which allows using most

available algorithms, e.g., ARPACK, to solve the primal GEP based on the matrix pencil $s\boldsymbol{B}_c - \boldsymbol{A}_c$. The dual GEP based on the matrix pencil $\boldsymbol{B}_c - \hat{s}\boldsymbol{A}_c$, however, cannot be solved with commonly-used libraries as $\boldsymbol{A}_c$ is never Hermitian and cannot be guaranteed to be positive semidefinite.

It is also important to note that the GEP formulation is more general than the conventional (12) as the identity matrix $\boldsymbol{I}_p$ can be substituted, without any change in the resulting problem to be solved, with a diagonal matrix $\boldsymbol{\Gamma}_p$, whose i-th diagonal element γ_i is 1 if the dynamic of the state variable x_i is enabled, 0 otherwise. Hence, one can still use (2), with matrix pencil $s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$, where:

$$\boldsymbol{B}_\gamma = \begin{bmatrix} \boldsymbol{\Gamma}_p & \boldsymbol{0}_{p,q} \\ \boldsymbol{0}_{q,p} & \boldsymbol{0}_{q,q} \end{bmatrix} \qquad (15)$$

On the other hand, switching a state variable to an algebraic one is involved if considering the augmented formulation (12).

B. Semi-Implicit Differential-Algebraic Equations

In [28], a semi-implicit formulation of the DAEs describing power system models has been proposed, as follows:

$$\boldsymbol{T}\boldsymbol{x}' = \tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{y})$$
$$\boldsymbol{R}\boldsymbol{x}' = \tilde{\boldsymbol{g}}(\boldsymbol{x}, \boldsymbol{y}) \qquad (16)$$

where $\tilde{\boldsymbol{f}}$ ($\tilde{\boldsymbol{f}} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^p$) are the differential equations; $\tilde{\boldsymbol{g}}$ ($\boldsymbol{g} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^q$) are the algebraic equations; $\boldsymbol{T}$ is a $p \times p$ matrix and $\boldsymbol{R}$ is a $q \times p$ matrix.

Linearizing (16) at an equilibrium point, leads to:

$$\boldsymbol{T}\Delta\boldsymbol{x}' = \tilde{\boldsymbol{f}}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \tilde{\boldsymbol{f}}_{\boldsymbol{y}}\Delta\boldsymbol{y} \qquad (17)$$

$$\boldsymbol{R}\Delta\boldsymbol{x}'p = \tilde{\boldsymbol{g}}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \tilde{\boldsymbol{g}}_{\boldsymbol{y}}\Delta\boldsymbol{y} \qquad (18)$$

Reference [28] shows that the Jacobian matrices that appear in the right-hand-side of (17)-(18) are sparser than those that appear in (8)-(9) and that the semi-implicit formulation (16) has several advantages with respect to (6) for the transient stability analysis of power systems. The interested reader can find in [28] a thorough discussion on the semi-implicit formulation.

The resulting primal and dual matrix pencils based on (17)-(18) are $s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$ and $\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$, respectively, where

$$\tilde{\boldsymbol{A}}_c = \begin{bmatrix} \tilde{\boldsymbol{f}}_{\boldsymbol{x}} & \tilde{\boldsymbol{f}}_{\boldsymbol{y}} \\ \tilde{\boldsymbol{g}}_{\boldsymbol{x}} & \tilde{\boldsymbol{g}}_{\boldsymbol{y}} \end{bmatrix}, \qquad \tilde{\boldsymbol{B}}_c = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{0}_{p,q} \\ \boldsymbol{R} & \boldsymbol{0}_{q,q} \end{bmatrix} \qquad (19)$$

Since matrix $\boldsymbol{T}$ is, in general, not diagonal and not positive semi-definite, and $\boldsymbol{R} \neq \boldsymbol{0}_{q,p}$, both primal and dual GEPs are non-Hermitian. However, since $\tilde{\boldsymbol{A}}_c$ is sparser than $\boldsymbol{A}_c$,[1] solving the GEP for the semi-implicit formulation (16) can be more efficient than solving the GEP for the standard explicit DAE formulation given in (6). This point is further discussed in Section V.

IV. SOLVERS FOR THE GENERALIZED EIGENVALUE PROBLEM

A. Solvers Based on QR Factorization

For dense matrices, the standard *de facto* for the solution of both LEP and GEP is LAPACK [29]. This public-domain library implements a QR factorization that has the ability to find all finite eigenvalues and eigenvectors. In recent year, it has been developed a GPU-based implementation of LAPACK, namely, MAGMA, that provides an efficient GPU-based implementation of LAPACK functions [30]. The QR factorization is known to have computational complexity $O(n^3)$ and is thus impractical for large systems. Moreover, LAPACK and MAGMA work only with dense matrices.[2] Note also that MAGMA can currently solve only LEPs as in (1). LAPACK and MAGMA are considered in this paper only for comparison with the other libraries discussed in the remainder of this section.

B. Arnoldi Iteration and its Variants

The Arnoldi iteration is widely considered the most efficient method to compute a reduced number of *dominant* eigenvalues of a non-Hermitian pencil $s\boldsymbol{I}_n - \boldsymbol{A}$. The most used implementation of such a method, ARPACK [31], also allows to solve GEPs, i.e., $s\boldsymbol{B} - \boldsymbol{A}$, but requires that $\boldsymbol{B}$ is positive semi-definite.

Since the dominant eigenvalues of a matrix are those with largest modulus, which are not of interest for small-signal stability analysis, the Arnoldi iteration cannot be used as is. In this paper, two methods are considered, namely, the shift-&-invert and the Caley transformations. The shift-&-invert transformation utilizes a shift point σ and finds the eigenvalues of the shifted-inverse matrix $(\boldsymbol{A} - \sigma\boldsymbol{B})^{-1}$, where $\sigma \in \mathbb{C}$ and dominant eigenvalues are found in the neighborhood of σ. The Caley transformation is similar to the shift-&-invert approach and consists in using two shift points, σ and κ, and solves the matrix $(\boldsymbol{A} - \sigma\boldsymbol{B})^{-1}(\boldsymbol{A} - \kappa\boldsymbol{B})$. This method has the advantage to map the symmetry axis with abscissa $(\sigma + \kappa)/2$ to a unit circle where dominant eigenvalues are those with modulus greater than 1. The interested reader can find more details on the mathematical aspects of the shift-&-invert and Caley transformations in the ARPACK documentation [31], and practical implications for the determination of the eigenvalues of a physical system in [19].

In the case study discussed in Section V, some variants or alternative approaches of the Arnoldi iteration, provided by the software tool SLEPc, are considered, as follows:

1) *Krylov-Schur*, a variation of Arnoldi with a very effective restarting technique. In the case of symmetric problems, this is equivalent to the thick-restart Lanczos method.
2) *Generalized Davidson*, a simple iteration based on the subspace expansion by the preconditioned residual.
3) *Jacobi-Davidson*, a preconditioned eigensolver with an effective correction equation.

Similarly to the Arnoldi iteration, the methods above also allow using the shift-&-invert and Caley transformations. The SLEPc solvers indicated above allow solving fully non-Hermitian

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[1]In particular, [28] shows that $\tilde{\boldsymbol{A}}_c$ has about 15% less non-zero values than $\boldsymbol{A}_c$ for large real-world power systems.

[2]Recently, a support for sparse matrices has bee added to MAGMA but this support is limited to the Hermitian eigenvalue problem, which is not of interest in this paper.
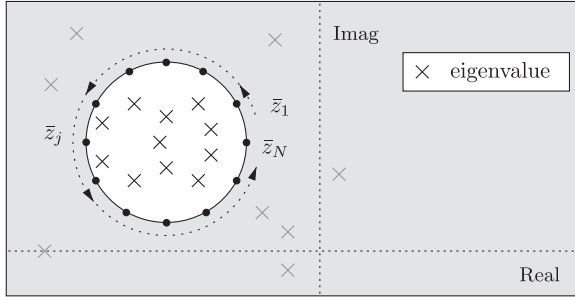
Fig. 1. FEAST and Z-PARES compute eigenvalues within the a contour path on the complex plane defined by a set of quadrature points $\bar{z}_1, \ldots, \bar{z}_N$.

GEPs, but requires a careful tuning of the shifting quantities σ and/or κ to avoid singularities and/or poor conditioning of the matrices that have to be inverted during the iterative solution process.

C. FEAST and Z-PARES

Both libraries FEAST [34] and Z-PARES [33] solves the GEP (2) for eigenvalues located inside a search contour, which can be an interval, an ellipse or a custom user-defined path in the complex plane. The first step is to define a set of N quadrature points on the contour, say $\{\bar{z}_1, \ldots, \bar{z}_j, \ldots \bar{z}_N\}$ (see Fig. 1), then the basis of the subspace that is utilized to compute the eigenpairs are computed by solving the following linear system with multiple right hand sides:

$$(\bar{z}_j\boldsymbol{B} - \boldsymbol{A})\boldsymbol{Z}_j = \boldsymbol{B}\boldsymbol{V}, \qquad j = 1, 2, \ldots, N \qquad (20)$$

for $\boldsymbol{Z}_j$, where $\boldsymbol{V}$ and $\boldsymbol{Z}_j$ are $n \times L$ matrices. $\boldsymbol{V}$ is called *source matrix*. The N problems (20) are independent and thus can be straightforwardly parallelized.

Note that ARPACK, FEAST and Z-PARES delegate the solution of linear systems and matrix vector multiplications to external libraries through the Reverse Communication Interface (RCI). In this paper, ARPACK, FEAST and Z-PARES are coupled with the libraries KLU,[3] PARDISO [36], and MUMPS [37], respectively, for sparse matrix operations and factorization.

D. Remarks on Time and Space Complexity of the Solvers

All solvers considered in this paper heavily rely on matrix factorization, which is also the most consuming part both in terms of time and space complexity. Then, the differences in the performance of the solvers are due the number of times such a factorization has to be performed and the size and representation of the matrices to be factorized.

The most demanding solver is LAPACK as it utilizes a dense matrix representation and the QR factorization algorithm. The space complexity of the QR factorization is $O(N^2)$ while the time complexity is $O(N^3)$, which explains why LAPACK is expected to be not suitable for very large systems. MAGMA has same space complexity as LAPACK, but time complexity is scaled down thanks to parallelization. The concrete amount of the scaling depends on the number of cores of the GPU.

[3] Available at faculty.cse.tamu.edu/davis/suitesparse.html

TABLE I
LIBRARIES TO SOLVE THE EIGENVALUE PROBLEM

Library	Version	Dependency	Version
LAPACK	3.6.1	ATLAS	3.10.2
MAGMA	2.0.1	CUDA	7.5
ARPACK	3.3.0	KLU	1.3.8
SLEPc	3.7.1	PETSc	3.7.1
FEAST	3.0	INTEL-MKL	2016
Z-PARES	0.9.6a	MUMPS	5.0.1
		OPENMPI	1.10.2

All other algorithms considered in this paper utilize a sparse matrix factorization (mostly based on the LU algorithm) which drastically reduces the space complexity as matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ obtained for power systems are very sparse (see Tables II and IV in Section V). Then, the time complexity of the factorization of a sparse matrix can be estimated as $O(N^{1.5})$. Finally, Z-PARES and FEAST are expected to be more efficient than other algorithms as they factorize matrices of reduced size with respect to the full system matrices. This is confirmed by the simulation results presented in the following section.

V. CASE STUDIES

In this section, two real-world systems are considered, namely the all-island Irish system and the ENTSO-E network. These systems show different features which are complementary for the analysis carried out in this case study. The all-island Irish system includes both conventional and non-conventional generation, which consists mainly of wind power plants and hence is useful to test the ability of eigenvalue problem solvers for a spectrum which is not limited to the classical electromechanical modes of synchronous machines. The ENTSO-E network, on the other hand, is a large system and allows properly discussing the performance of the numerical libraries considered in this paper.

All simulations are obtained using Dome, a Python-based power system software tool [38]. The Dome version utilized in this case study is based on Python 3.4.3; ATLAS 3.10.2 for dense vector and matrix operations; CVXOPT 1.1.8 for sparse matrix operations; and SUITESPARSE 4.5.3 for sparse matrix factorization. The libraries used for solving the eigenvalue problems and their main dependencies for matrix operations and factorization are summarized in Table I.

The libraries considered in this case study have been carefully tested on several small test systems–e.g., the New England 39-bus 10-machine system and the IEEE 145-bus 50-machine system–and proved to provide nearly identical results for the LEP and primal and dual GEPs and for both the Ex-DAE and SI-DAE formulations. We can thus assume that is fair to say that the results presented in the following subsections are accurate and properly reflect the performance of each library.

Simulations were executed on a server mounting two quad-core Intel Xeon 3.50 GHz CPUs, 1 GB NVidia Quadro 2000 GPU, 12 GB of RAM, and running a 64-bit Linux OS.

TABLE II
STATISTICS FOR THE ALL-ISLAND IRISH SYSTEM

Matrix	Size	NNZ	NNZ [%]
$\boldsymbol{A}_s$	5,669,161	894,675	15.781%
$\boldsymbol{A}_c$	97,199,881	41,197	0.0423%
$\boldsymbol{B}_\gamma$	97,199,881	2,381	0.0024%
$\tilde{\boldsymbol{A}}_c$	97,199,881	40,373	0.0415%
$\tilde{\boldsymbol{B}}_c$	97,199,881	2,625	0.0027%

TABLE III
SMALL-SIGNAL STABILITY ANALYSIS OF THE ALL-ISLAND IRISH SYSTEM

Library	Pencil	Which	Time [s]	Notes
LAPACK	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	All	26.03	OK
	$s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$	All	8,039.94	24 eigs. > 0
	$s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$	All	6,568.81	2 eigs. > 0
	$\boldsymbol{B}_\gamma - \hat{s}\boldsymbol{A}_c$	All	6,060.84	OK
	$\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$	All	6,876.06	OK
MAGMA	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	All	17.27	OK
ARPACK	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	50 SM	99.89	2 eigs. > 0
	$s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$	50 SM	5,746.77	OK
SLEPc	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	50 SM	122.65	no eig., diverge
	$s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$	50 SM	1.31	no eig., diverge
	$s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$	50 SM	1.38	no eig., diverge
	$\boldsymbol{B}_\gamma - \hat{s}\boldsymbol{A}_c$	50 LM	180.32	no eig., diverge
	$\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$	50 LM	179.49	no eig., diverge
FEAST	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	$\bar{c}, r$	362.95	no eig., diverge
	$s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$	$\bar{c}, r$	5,327.41	39 eigs., max. # iters.
	$s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$	$\bar{c}, r$	232.06	40 eigs., max. # iters.
	$\boldsymbol{B}_\gamma - \hat{s}\boldsymbol{A}_c$	$\bar{c}, r$	5,025.13	55 eigs., 1 eig. > 0
	$\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$	$\bar{c}, r$	240.04	54 eigs., 1 eig. > 0
Z-PARES	$s\boldsymbol{I}_p - \boldsymbol{A}_s$	$\bar{c}, r$	56.87	250 eigs., 12 eigs. > 0
	$s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$	$\bar{c}, r$	59.46	250 eigs., 4 eigs. > 0
	$s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$	$\bar{c}, r$	14.13	253 eigs., 4 eigs. > 0
	$\boldsymbol{B}_\gamma - \hat{s}\boldsymbol{A}_c$	$\bar{c}, r$	57.36	6 eigs., 2 eigs. > 0
	$\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$	$\bar{c}, r$	12.89	146 eigs., 3 eigs. > 0

SM: smallest magnitude; LM: largest magnitude; $\bar{c} = (-0.1, 5)$; $r = 5.25$.

A. All-Island Irish System

In this subsection, the all-island Irish transmission system is utilized to compare the solvers for the generalized eigenvalue problem that are discussed in Section IV. The topology and the steady-state data of the system are based on the actual real-world system provided by the Irish TSO, EirGrid. However, all dynamic data are estimated based on the knowledge of the various power plant technologies used. The dynamic model of the Irish system includes both conventional and wind power generation. The system consists of 1,479 buses, 1,851 transmission lines and transformers, 245 loads, 22 conventional synchronous power plants modeled with 6th order synchronous machine models with AVRs and turbine governors, 6 PSSs and 176 wind power plants, of which 142 are equipped with doubly-fed induction generators and 34 with constant-speed wind turbines. Statistics for the all-island Irish system are shown in Table II.

Table III shows the performance of the solvers considered in this paper for different pencils. The size of the system is not big enough to prevent using LAPACK and MAGMA libraries, which work on dense matrices. For ARPACK and SLEPc libraries, $\sigma =$
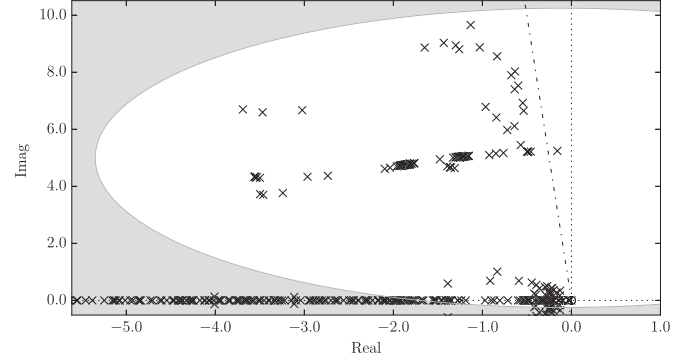

Fig. 2. Rightmost eigenvalues for the all-island Irish system. Solution obtained using LAPACK and the standard eigenvalue problem (1) with the pencil $s\boldsymbol{I}_p - \boldsymbol{A}_s$. The dash-dot line indicate the locus of eigenvalues with 5% damping.

-0.01 and $\kappa = -0.01$ has been used. Note, however, that, in our tests, the Caley transformation performed similarly to the shift-&-invert transformation and provided basically same results. For the SLEPc library, the Krylov-Schur, Jacobi-Davidson and generalized Jacobi proved to work similarly and results for the Krylov-Schur solver only are shown. Finally, for the libraries FEAST and Z-PARES, the search contour is a circle with center $\bar{c} = (-0.1, 5)$ and radius $r = 5.25$. The rightmost eigenvalues obtained with LAPACK solving (1) with the pencil $s\boldsymbol{I}_p - \boldsymbol{A}_s$ is shown in Fig. 2. The figure also shows the search contour used with FEAST and Z-PARES solved. It can be noted that the solution shows a poorly damped eigenvalue (i.e., an eigenvalue whose damping is lower than 5%) and no positive eigenvalue.

Fig. 3 shows the solution obtained with Z-PARES using pencils $s\boldsymbol{I}_p - \boldsymbol{A}_s$ and $s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$. As it can be observed, the solution of pencil $s\boldsymbol{I}_p - \boldsymbol{A}_s$ retains the poorly damped eigenvalue, but also shows several spurious positive eigenvalues. On the other hand, the solution obtained using pencil $s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$ does not show the poorly damped eigenvalue and some of the positive eigenvalues.

As expected, LAPACK does not perform well when dealing with GEPs. This is a consequence of the high computational complexity of the QR factorization. The solution of the LEP, however, only requires 26.03 s, which is very competitive,[4] considering the fact that LAPACK returns the full spectrum. In this case, also MAGMA performs well and shows a significant speed-up with respect to LAPACK. Note also that, depending on the GEP, LAPACK can show numerical issues, i.e., the pair of positive eigenvalues. Such issues can be shown by other solvers, e.g., ARPACK. With this regard, dual problems appear to be less prone to numerical issues.

Among the libraries that can handle sparse matrices and the full range of primal and dual GEPs, the one that performs the best is Z-PARES. Z-PARES appears to work better with the SI-DAE formulation.

Whenever the maximum number of iterations is reached, the solution provided by FEAST is not accurate, in particular the solutions of the two primal GEPs show high errors. It is interesting to note that, similarly to Z-PARES, FEAST is faster if the SI-DAE

[4]Note that to compute $\boldsymbol{A}_s$ using (10) requires about 0.9 s. This time is not included in the results shown in Table III.
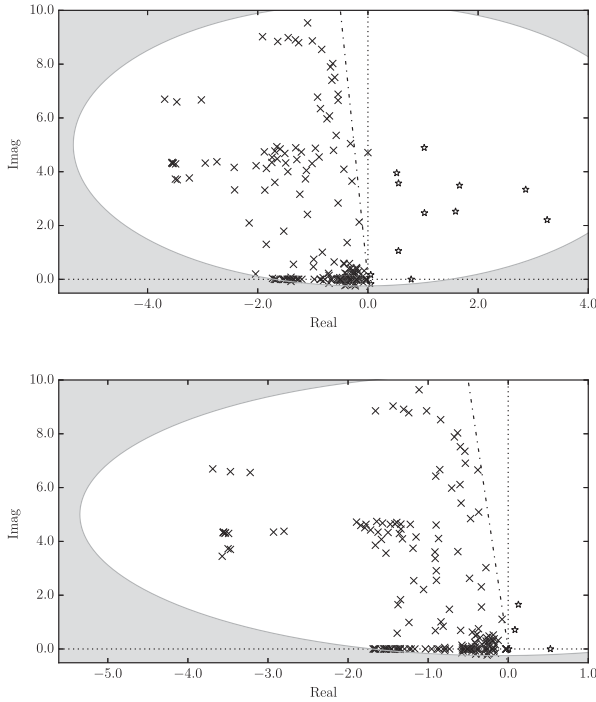
Fig. 3. Rightmost eigenvalues for the all-island Irish system. *Top panel*: solution obtained using Z-PARES and the standard eigenvalue problem (1) with the pencil $s\boldsymbol{I}_p - \boldsymbol{A}_s$. *Bottom panel*: Solution obtained using Z-PARES and the primal GEP problem with the pencil $s\tilde{\boldsymbol{B}}_c - \tilde{\boldsymbol{A}}_c$.

formulation is used and that the only problem that converges is the dual GEP with pencil $\tilde{\boldsymbol{B}}_c - \hat{s}\tilde{\boldsymbol{A}}_c$. FEAST requires generally longer times and a high number of iterations.

The divergence of the SLEPc library depends mostly on the factorization provided by PETSC, which, in this case, prevented to obtain solutions. Note that, with these results, we are not saying that the SLEPc library cannot handle large eigenvalue problems but, rather, that making it work properly for the solution of power system problems is not straightforward and further tuning of the library is required. This is clearly beyond the scope of this paper, whose purpose is to show the performance of existing software tools with *off-the-shelf* parameters and settings.

B. ENTSO-E System

This subsection considers a dynamic model of the ENTSO-E transmission system. The model includes 21,177 buses (1,212 off-line); 30,968 transmission lines and transformers (2,352 off-line); 1,144 coupling devices, i.e., zero-impedance connections (420 off-line); 15,756 loads (364 off-line); and 4,828 power plants. Of these power plants, 1,160 power plants are off-line. The system also includes 364 PSSs. Statistics of the state matrices of the ENTSO-E system are given in Table IV. Note that $\boldsymbol{A}_s$ could not be computed due to a memory error which is likely a consequence of its low sparsity.

Given the huge size of the matrices, even for the simple LEP case, the dense representation required by LAPACK and MAGMA, is not feasible as a memory error is returned when trying to allocate the memory. Among the libraries that support sparse matrices, we test only FEAST and Z-PARES for the primal and

TABLE IV
STATISTICS FOR THE ENTSO-E NETWORK

Matrix	Size	NNZ	NNZ [%]
$\boldsymbol{A}_s$	1,578,631,824	n/a	n/a
$\boldsymbol{A}_c$	21,363,914,896	590,874	0.0027%
$\boldsymbol{B}_\gamma$	21,363,914,896	39,732	0.0002%
$\tilde{\boldsymbol{A}}_c$	21,363,914,896	568,134	0.0026%
$\tilde{\boldsymbol{B}}_c$	21,363,914,896	53,992	0.0003%

dual GEPs problems of obtained with the SI-DAE formulation. These are, in fact, the cases that have shown best performances for the all-island Irish system discussed in the previous section.[5] These two libraries return nearly identical solutions. Also in this case, Z-PARES outperforms FEAST–especially for the primal GEP–from the point of view of computational efficiency. Note that the dual problem is not able to find the eigenvalues closest to the origin of the complex plane, as these lay outside the considered circle. This, however, is not a major issue as critical eigenvalues tend to have a non-negligible imaginary part and, thus, it is not necessary to consider a "huge" search contour for the dual GEP.

C. Concluding Remarks

1) On the Different Solution of the Primal and Dual GEPs: It is relevant to note that the results of the primal and dual GEPs (see Figs. 3 and 4) are different. This is because the primal GEP finds the eigenvalues (say, s) within a given circumference, while the dual GEP finds the inverse of the eigenvalues of the primal GEP (i.e., $\hat{s} = 1/s$) within the same circle. Note that Figs. 3 and 4 show s and $1/\hat{s}$ for the primal and dual GEPs, respectively. This fact also explains why, in the case of dual GEP problem, some eigenvalue falls outside the circumference.

The solutions of the primal and dual problems are complementary as they are able to find different sets eigenvalues. The primal GEP can find eigenvalues very close to the origin, while the dual GEP is able to find eigenvalues with small real part but relatively high imaginary part (i.e., poorly damped eigenvalues). This can be clearly observed in Fig. 3.

2) On the Spuriousness of Positive Eigenvalues: Based on the results shown in Tables III and V, it is clear that the formulation and the solver have a significant impact on the small-signal stability analysis. In particular, it is very important to be able to understand whether a positive eigenvalue is just a spurious solution or actually indicate that the operating point is unstable.

A simple way to double-check the results of the eigenvalue analysis and verify whether the system is stable is to solve a time-domain simulation. If the initial operating point is stable, a small perturbation, e.g., a small variation of a state variable, will lead the system to diverge.[6] In this case, the time-domain

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[5]Note that, in this case, ARPACK, while converging, does not provide a meaningful solution for the pencil $s\boldsymbol{B}_\gamma - \boldsymbol{A}_c$ and takes about 21 hours to complete.

[6]Note that the software tool Dome implements analytical Jacobian matrices, not a numerical differentiation based on DAEs, hence the eigenvalue analysis and the time domain simulation can be considered fairly independent, i.e., no common mode errors.
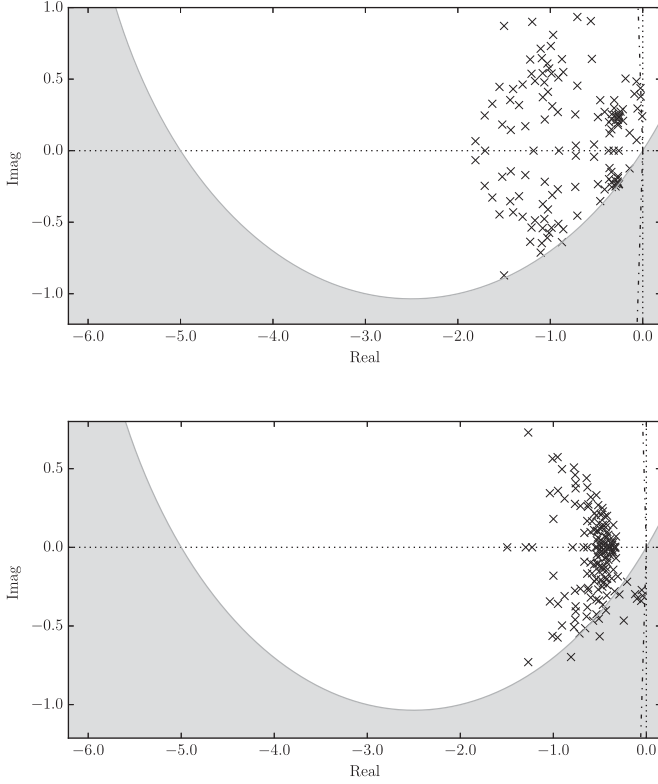
Fig. 4. Eigenvalues in the circle with center $\bar{c} = (-2.5, 2.5)$ and radius $r = \sqrt{2} \cdot 2.5$, and spuriousness indicator 0.05, as found with the library Z-PARES. Upper panel: pencil $s\bar{B}_c - \bar{A}_c$; lower panel: pencil $\bar{B}_c - \hat{s}\tilde{A}_c$.

TABLE V
SMALL-SIGNAL STABILITY ANALYSIS OF THE ENTSO-E NETWORK

Library	Pencil	Time [s]	# of eigs.	Notes
FEAST	$s\tilde{B}_c - \tilde{A}_c$	11,155.69	476	40 eigs. > 0
	$\tilde{B}_c - \hat{s}\tilde{A}_c$	853.44	476	40 eigs. > 0
Z-PARES	$s\tilde{B}_c - \tilde{A}_c$	180.08	381	36 eigs. > 0
	$\tilde{B}_c - \hat{s}\tilde{A}_c$	179.99	476	40 eigs. > 0

$\bar{c} = (-2.5, 2.5)$; $r = \sqrt{2} \cdot 2.5$.

simulation indicates that the operating points of both the all-island Irish system and the ENTSO-E grid are stable, hence positive eigenvalues are certainly spurious.

Among the considered solvers, the only one that allows filtering spurious eigenvalues by means of a spuriousness indicator threshold is Z-PARES. Such an indicator can take values in the interval $[0, 1]$. For the all-island Irish system, if the indicator is set to 0.001, all positive eigenvalues are discarded but so are some genuine ones. Note that the value of the indicator has been defined heuristically. Based on the tests that we have done, we got to the conclusion that the same threshold can be used for different operating points of a given system. In other words, an ISO could use always the same threshold for the same network.

For sake of example, Fig. 4 shows the eigenvalues for the ENTSO-E system obtained using Z-PARES for the primal and dual GEP based on the SI-DAE formulation and considering a

spuriousness operator equal to 0.05. Both problems are solved using a search contour defined by a circumference with center $\bar{c} = (-2.5, 2.5)$ and radius $r = \sqrt{2} \cdot 2.5$. Note that no positive eigenvalue is shown. This is consistent with time domain simulations following a perturbed initial operating point.

Hence, the spuriousness indicator has to be utilized with caution. A good practice is to solve the eigenvalue problem with different solvers (including time-domain analysis) and different settings.

VI. CONCLUSION

The paper discusses the small-signal stability analysis of large power systems considering primal and dual GEPs as well as two formulations of the power system equations, namely, an explicit and a semi-implicit form of DAEs. The equivalence of the primal and dual problems is rigorously proven. The performance of a variety of libraries that are able to solve a fully non-Hermitian GEP are duly discussed. Simulation results show that, for the all-island Irish transmission grid LAPACK and MAGMA libraries perform well. Hence, for networks of the size of the Irish system or smaller, the LEP formulation and QR factorization are viable options. For larger systems, e.g., the ENTSO-E network, results show that available solvers tend to perform better with the SI-DAE formulation than with the Ex-DAE one and that the dual problem appears relatively more numerically reliable than the primal one. However, only two libraries can efficiently solve the non-Hermitian GEP, namely, FEAST and Z-PARES. In particular, the latter appears to be the most efficient and numerically stable solver among those considered in this paper.

Future research will focus on testing the proposed primal/dual formulation and the solvers considered in this paper to eigenvalue problems originated from problems other than power systems.[7]

APPENDIX A
EQUIVALENCE OF THE PRIMAL AND DUAL GEPs

This appendix provides a rigorous proof of the equivalence of the primal and dual GEPs. The main characteristic of this type of equivalence is the inversion of frequency which is defined by the dual role of the different types of eigenvalues. This is the first time such a rigorous mathematical proof is presented.

If we assume that

$$\det(s\boldsymbol{B} - \boldsymbol{A}) = s^{p_0}(s - a_1)^{p_1} \ldots (s - a_\nu s)^{p_\nu},$$

then the finite eigenvalues are the roots of the polynomials,

$$s^{p_0}, (s - a_1)^{p_1}, \ldots, (s - a_\nu)^{p_\nu}.$$

Where $p_i \in \{0, 1, 2, ..., p\}$, $i = 0, 1, ..., \nu$. From the above eigenvalues we can compute the Jordan matrix $\boldsymbol{J}_{p_0}$ of the zero eigenvalue and the Jordan matrix $\boldsymbol{J}_p$ of the finite eigenvalues. The Jordan matrix $\boldsymbol{J}_q$ of the infinite eigenvalues is equal to the Jordan matrix of the zero eigenvalue of the pencil $\boldsymbol{B} - \hat{s}\boldsymbol{A}$. Finally, we will denote with $\boldsymbol{Q}_{p_0}$ the matrix with columns the

[7]With this regard, a large variety of matrices is available at: http://www.cise.ufl.edu/research/sparse/matrices/.

p_0 linear independent eigenvectors of the zero eigenvalue, with $\boldsymbol{Q}_p$ the matrix with columns the p linear independent eigenvectors of the non-zero eigenvalues and with $\boldsymbol{Q}_q$ the matrix with columns the q linear independent eigenvectors of the infinite eigenvalues.

Theorem 1: Consider the system (4). Then its solution is given by

$$z(t) = \boldsymbol{Q}_{p_0} e^{\boldsymbol{J}_{p_0} t} \boldsymbol{c}_{p_0} + \boldsymbol{Q}_p e^{\boldsymbol{J}_p t} \boldsymbol{c}_p \ . \tag{21}$$

Where $\boldsymbol{J}_{p_0}$, $\boldsymbol{J}_p$ are the Jordan matrices of the zero & non-zero eigenvalues of $s\boldsymbol{B} - \boldsymbol{A}$ respectively, $\boldsymbol{Q}_{p_0}$ the matrix with columns the p_0 linear independent eigenvectors of the zero eigenvalue, with $\boldsymbol{Q}_p$ the matrix with columns the p linear independent eigenvectors of the non-zero eigenvalues and $\boldsymbol{c}_{p_0} \in \mathbb{R}^{p_0}$, $\boldsymbol{c}_p \in \mathbb{R}^p$ are constant vectors. Without resorting to further processes of computations, the solution of (5) can be explicitly represented by,

$$\hat{z}(t) = \boldsymbol{Q}_p e^{\boldsymbol{J}_p^{-1} t} \hat{\boldsymbol{c}}_p + \boldsymbol{Q}_q e^{\boldsymbol{J}_q t} \hat{\boldsymbol{c}}_q \ . \tag{22}$$

Where $\boldsymbol{J}_q$, $\boldsymbol{J}_p$ are the Jordan matrices of the infinite & non-zero eigenvalues of $s\boldsymbol{B} - \boldsymbol{A}$ respectively, $\boldsymbol{Q}_q$ the matrix with columns the q linear independent eigenvectors of the infinite eigenvalue, with $\boldsymbol{Q}_p$ the matrix with columns the p linear independent eigenvectors of the non-zero eigenvalues and $\hat{\boldsymbol{c}}_q \in \mathbb{R}^q$, $\hat{\boldsymbol{c}}_p \in \mathbb{R}^p$ are constant vectors.

Proof of Theorem 1: There exist $n \times n$ non-singular matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ such that

$$\boldsymbol{P}\boldsymbol{B}\boldsymbol{Q} = \text{blockdiag}\{ \ \boldsymbol{I}_{p_0} \ \ \boldsymbol{I}_p \ \ \boldsymbol{J}_q \ \} \ ,$$

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{Q} = \text{blockdiag}\{ \ \boldsymbol{J}_{p_0} \ \ \boldsymbol{J}_p \ \ \boldsymbol{I}_q \ \} \ .$$

The matrix $\boldsymbol{Q}$ has columns the $p_0 + p + q$ linear independent eigenvectors of the zero, non-zero and infinite eigenvalues of $s\boldsymbol{B} - \boldsymbol{A}$ and can be written as:

$$\boldsymbol{Q} = [\ \boldsymbol{Q}_{p_0} \ \ \boldsymbol{Q}_p \ \ \boldsymbol{Q}_q \ ] \ .$$

Then by considering the transformation $z(t) = \boldsymbol{Q}w(t)$ and substituting it into (4) we obtain

$$\boldsymbol{B}\boldsymbol{Q}w'(t) = \boldsymbol{A}\boldsymbol{Q}w(t),$$

Whereby multiplying by $\boldsymbol{P}$ and setting

$$w(t) = [\ \boldsymbol{w}_{p_0}^T \ \ \boldsymbol{w}_p^T \ \ \boldsymbol{w}_q^T \ ]^T,$$

we obtain three subsystems of (4):

$$\boldsymbol{w}'_{p_0}(t) = \boldsymbol{J}_{p_0} \boldsymbol{w}_{p_0}(t) \tag{23}$$

$$\boldsymbol{w}'_p(t) = \boldsymbol{J}_p \boldsymbol{w}_p(t) \tag{24}$$

$$\boldsymbol{J}_q \boldsymbol{w}'_q(t) = \boldsymbol{w}_q(t) \ . \tag{25}$$

The subsystems (23) and (24) have the solutions

$$\boldsymbol{w}_{p_0}(t) = e^{\boldsymbol{J}_{p_0} t} \boldsymbol{c}_{p_0} \ ,$$

$$\boldsymbol{w}_p(t) = e^{\boldsymbol{J}_p t} \boldsymbol{c}_p \ .$$

For the subsystem (25), since $\boldsymbol{J}_q$ is the Jordan matrix of the zero eigenvalue of $\boldsymbol{B} - \hat{s}\boldsymbol{A}$, we have a nil-potent matrix, i.e. there

exists q_* such that $\boldsymbol{J}_q^{q_*} = \boldsymbol{0}_{q,q}$. In order to find the solution of the system (24), we obtain the following equations

$$\boldsymbol{J}_q \boldsymbol{w}'_q(t) = \boldsymbol{w}_q(t),$$

$$\boldsymbol{J}_q^2 \boldsymbol{w}''_q(t) = \boldsymbol{J}_q \boldsymbol{w}'_q(t),$$

$$\vdots$$

$$\boldsymbol{J}_q^{q_*-1} \boldsymbol{w}_q^{(q_*-1)}(t) = \boldsymbol{J}_q^{q_*-2} \boldsymbol{w}_q^{(q_*-2)}(t),$$

$$\boldsymbol{J}_q^{q_*} \boldsymbol{w}_q^{(q_*)}(t) = \boldsymbol{J}_q^{q_*-1} \boldsymbol{w}_q^{(q_*-1)}(t),$$

whereby taking the sum of the above equations and by using the fact that $\boldsymbol{J}_q^{q_*} = \boldsymbol{0}_{q,q}$ we arrive at the solution of (25)

$$\boldsymbol{w}_q(t) = \boldsymbol{0}_{q,1}. \tag{26}$$

By using the above solutions of the three subsystems, we arrive at the solution of (4):

$$z(t) = \boldsymbol{Q}_{p_0} e^{\boldsymbol{J}_{p_0} t} \boldsymbol{c}_{p_0} + \boldsymbol{Q}_p e^{\boldsymbol{J}_p t} \boldsymbol{c}_p \ .$$

For the dual system (5), we consider the transformation $z\hat{(t)} = \boldsymbol{Q}w\hat{(t)}$ and substituting it into (5) we obtain.

$$\boldsymbol{B}\boldsymbol{Q}\hat{w}(t) = \boldsymbol{A}\boldsymbol{Q}\hat{w}'(t).$$

Whereby multiplying by $\boldsymbol{P}$ and setting

$$\hat{w}(t) = [\ \hat{\boldsymbol{w}}_{p_0}^T \ \ \hat{\boldsymbol{w}}_p^T \ \ \hat{\boldsymbol{w}}_q^T \ ]^T,$$

we arrive at three subsystems:

$$\hat{\boldsymbol{w}}_{p_0} = \boldsymbol{J}_{p_0} \hat{\boldsymbol{w}}'_{p_0}, \quad \hat{\boldsymbol{w}}_p = \boldsymbol{J}_p \hat{\boldsymbol{w}}'_p, \quad \boldsymbol{J}_q \hat{\boldsymbol{w}}_q = \hat{\boldsymbol{w}}'_q.$$

with solutions:

$$\hat{\boldsymbol{w}}_{p_0} = \boldsymbol{0}_{p_0,1}, \quad \hat{\boldsymbol{w}}_p = e^{\boldsymbol{J}_p^{-1} t} \hat{\boldsymbol{c}}_p, \quad \hat{\boldsymbol{w}}_q = e^{\boldsymbol{J}_q t} \hat{\boldsymbol{c}}_q.$$

By using the above solutions of the three subsystems, we arrive at the solution of (4):

$$z(t) = \boldsymbol{Q}_p e^{\boldsymbol{J}_p^{-1} t} \hat{\boldsymbol{c}}_p + \boldsymbol{Q}_q e^{\boldsymbol{J}_q t} \hat{\boldsymbol{c}}_q \ .$$

The proof is completed. ∎

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Federico Milano (S'02–M'04–SM'09–F'16) received the M.E. and Ph.D. degrees in electrical engineering from the University of Genoa, Genoa, Italy, in 1999 and 2003, respectively. From 2001 to 2002, he was with the University of Waterloo, Waterloo, ON, Canada, as a Visiting Scholar. From 2003 to 2013, he was with the University of Castilla-La Mancha, Ciudad Real, Spain. In 2013, he joined the University College Dublin, Dublin, Ireland, where he is currently a Professor of Power Systems Control and Protections. His research interests include power system modelling, control, and stability analysis.

Ioannis Dassios received the Ph.D. degree in applied mathematics from the Department of Mathematics, University of Athens, Athens, Greece, in 2012. He worked for 16 months as a Postdoctoral Research and Teaching Fellow in optimization at the School of Mathematics, University of Edinburgh, Edinburgh, U.K. He also worked for 6 months as a Research Associate at the School of Mechanical, Aerospace and Civil Engineering, University. of Manchester, Manchester, U.K. He is currently a Research Fellow at MACSI, University of Limerick, Limerick, Ireland, and a Senior Researcher at ERC, University College Dublin, Dublin, Ireland.