UNIVERSITY OF LINCOLN

CMP9780, EGR3031 & BME3002

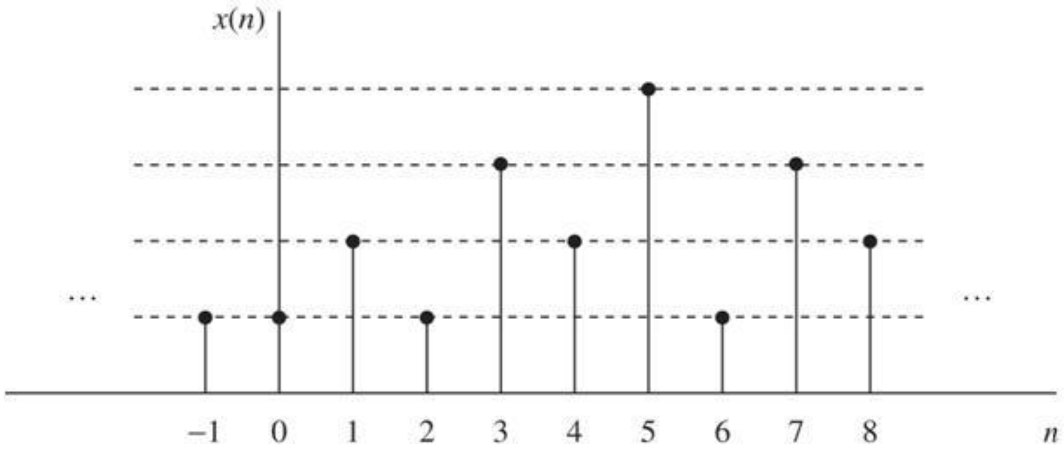
Correlation & Convolution

The Story so far…

- Introduction to Signals
- Sampling
- Fourier Transform
- Discrete and Fast Fourier Transform

Discrete-Time Signal

- A discrete-time signal is a function of independent *integer* variables.
- If x(n) is discrete-time signal, then
  - $-\infty < n < \infty$
- If, needed, then x(n) was obtained from sampling an analogue signal $x_a(t)$, then $x(n) = x_a(nT)$
  - Where T is the sampling period.
- **Beside the graphical representation there are other alternatives**
  - Functional representation
  - Tabular representation
  - Sequence representation



$$x(n) = \begin{cases} 1, & n = 0 \\ 0, & elsewhere \end{cases}$$

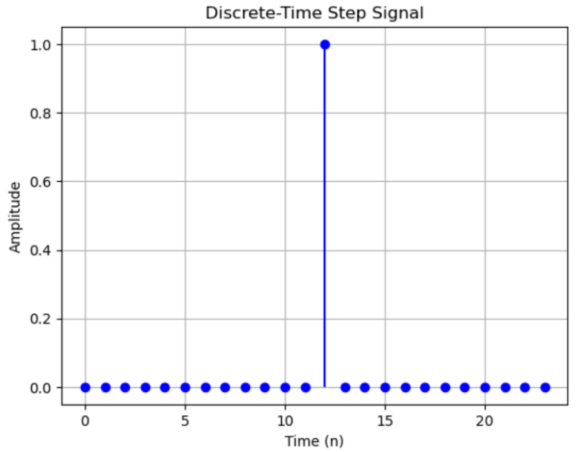
n	$\cdots$	−2	−1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

$$x(n) = \{\cdots 1, \underset{\uparrow}{1}, 2, 1, 3, \ldots\}$$

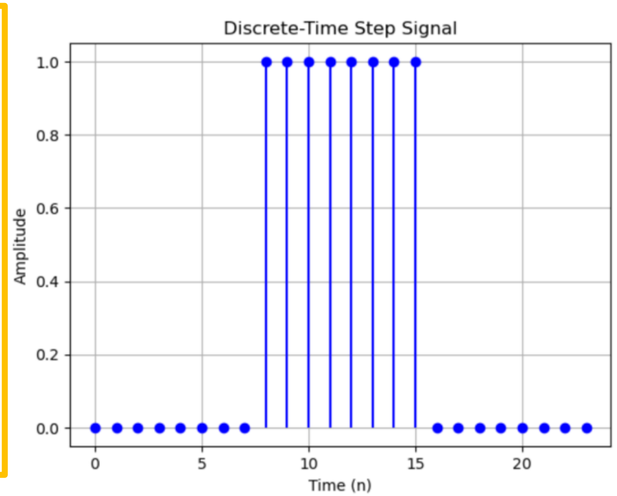
Discrete-Time Signals

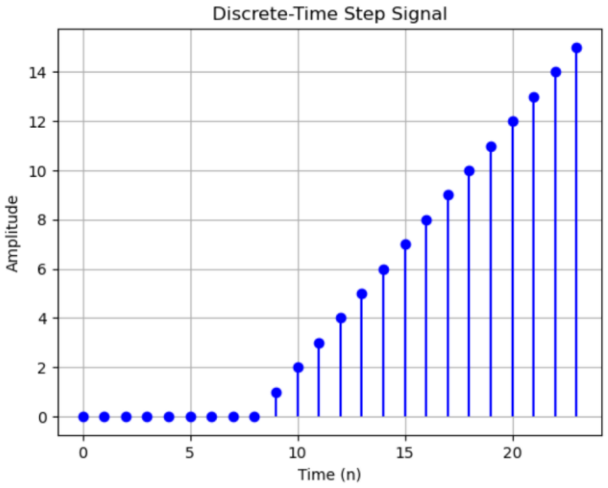
- They are good for modelling some physical phenomenon
  - Exponential Impulse also exists.
    - Decay and growth
    - Guitar note
  - Step/Ramp
    - Sudden change or switch
  - Unit Impulse
    - Analysis the response of Linear time invariant (LTI) systems

Unit Impulse

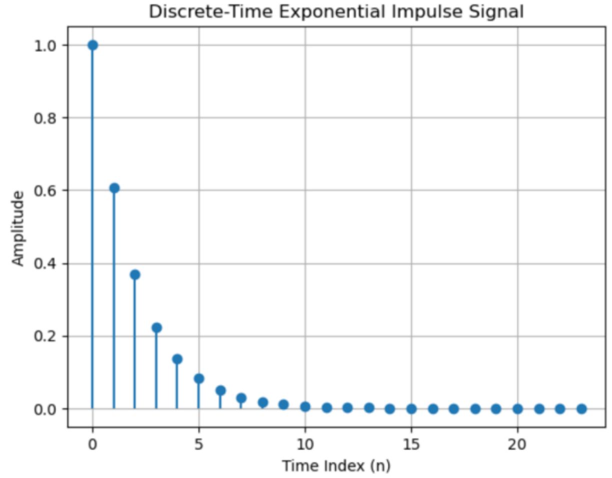


Unit Step





Unit Ramp

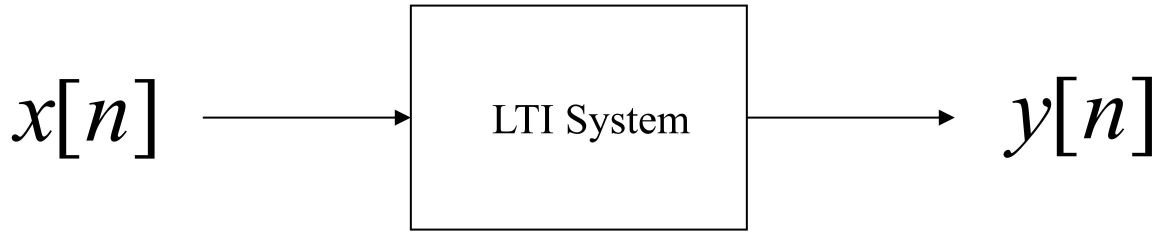


Exponential Impulse

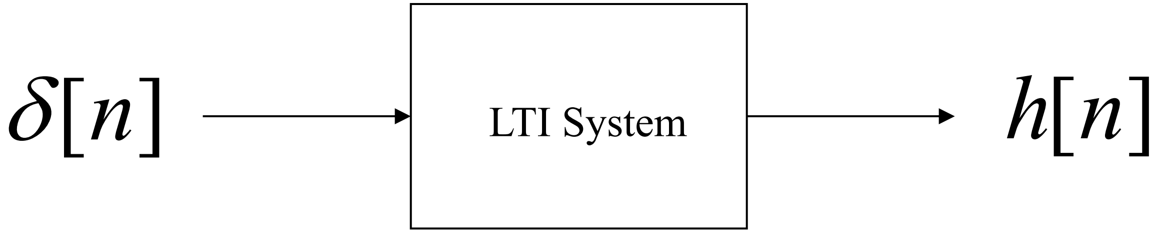
CONVOLUTION

DT Unit-Impulse Response

- Consider the DT SISO system:



- If the input signal is x[n] = δ[n] and the system has no energy at n = 0, the output y[n] = h[n] is called the impulse response of the system



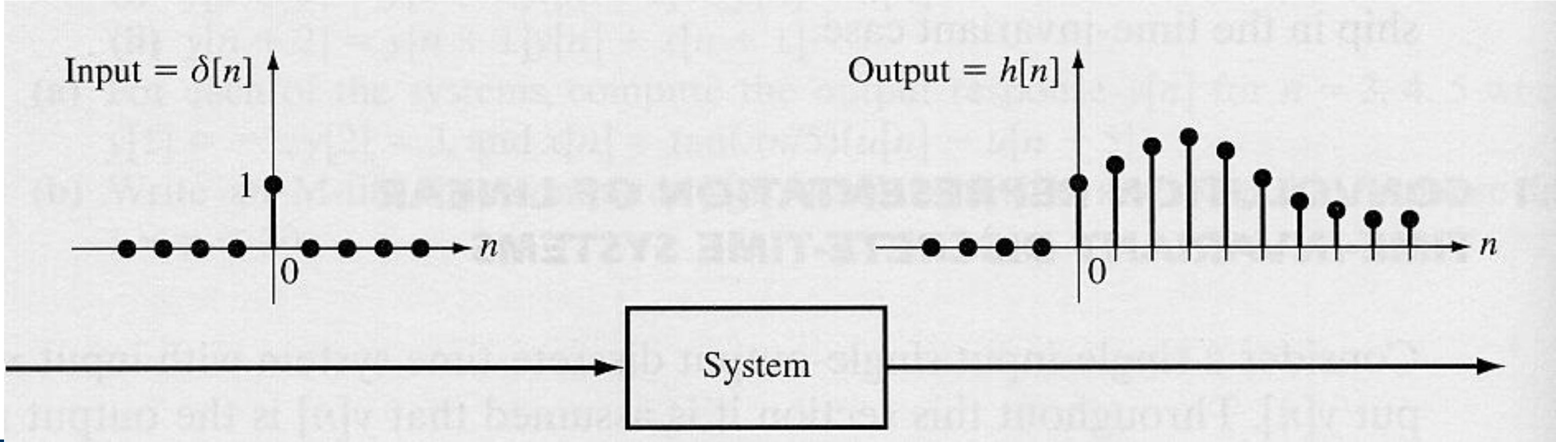
Example

- Consider the DT system described by

$$y[n]+ay[n-1]=bx[n]$$

- Its impulse response could be found to be

$$h[n]=\begin{cases}(-a)^n b, & n=0,1,2,\ldots \\ 0, & n=-1,-2,-3,\ldots\end{cases}$$



Representing Signals in Terms of Shifted and Scaled Impulses

- Many physical systems can be modelled as linear time-invariant (LTI) systems
- By the principle of superposition, the response y[n] of a discrete-time LTI system is the sum of the responses to the individual shifted impulses making up the input signal x[n].

Representing Signals in Terms of Shifted and Scaled Impulses

- Let x[n] be an arbitrary input signal to a DT LTI system
- Suppose that $x[n] = 0$ for $n = -1, -2, \ldots$
- This signal can be represented as a **weighted sum of shifted impulses**

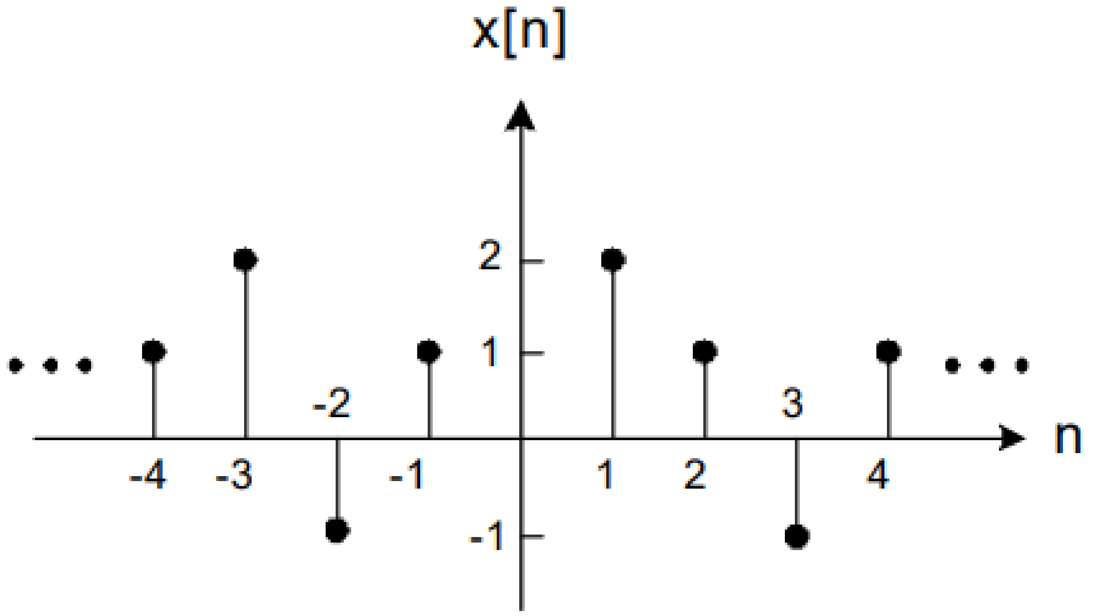
$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{i=0}^{\infty} x[i]\delta[n-i], \quad n = 0, 1, 2, \ldots$$

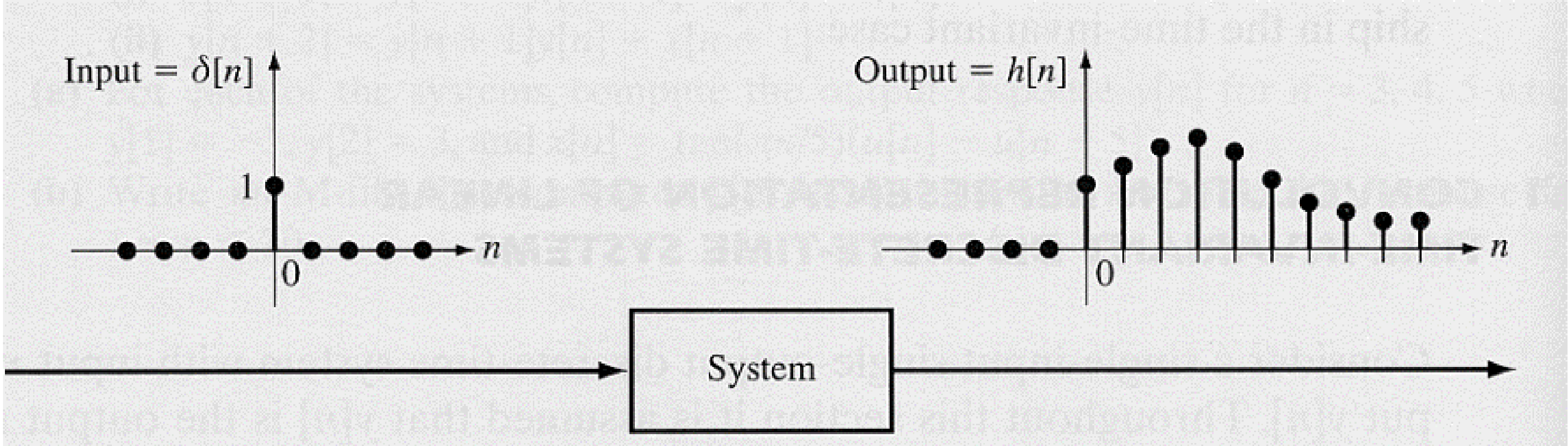
Representing Signals in Terms of Shifted and Scaled Impulses

- **Example,** consider the signal on the right:
- It can be expressed as a sum of the shifted impulses:
  - x[n] = ...+ x[-3]d[n + 3]+ x[-2]d[n + 2]+ x[-1]d[n +1] + x[0]d[n]+ x[1]d[n -1] + x[2]d[n - 2] + ..
- More generally this can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$



Exploiting Time-Invariance and Linearity



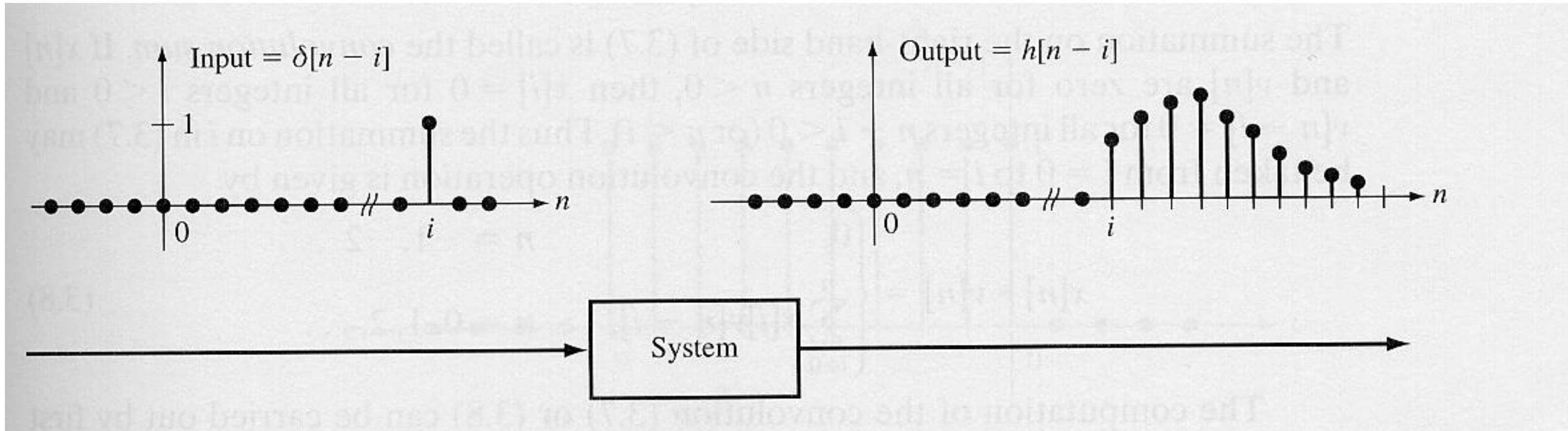


Figure 3.2 The shifted unit-pulse response $h[n-i]$.

The original signal shifted in time is still the same response but also shifted.

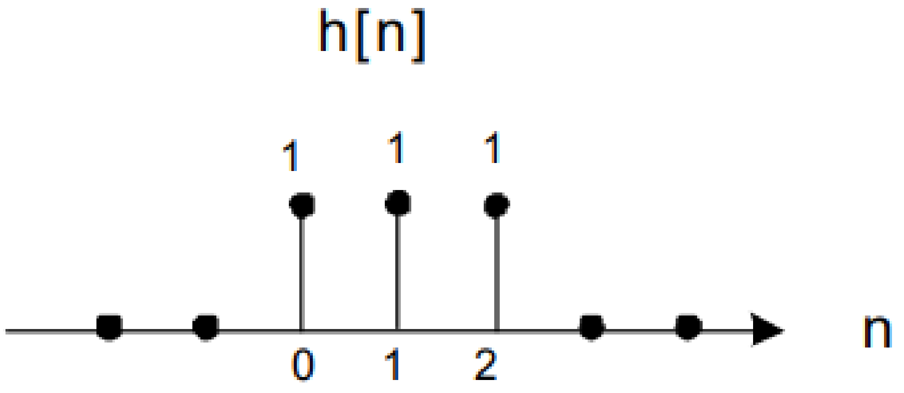
$$y[n] = \sum_{i=0}^{\infty} x[i]h[n-i], \quad n \geq 0$$

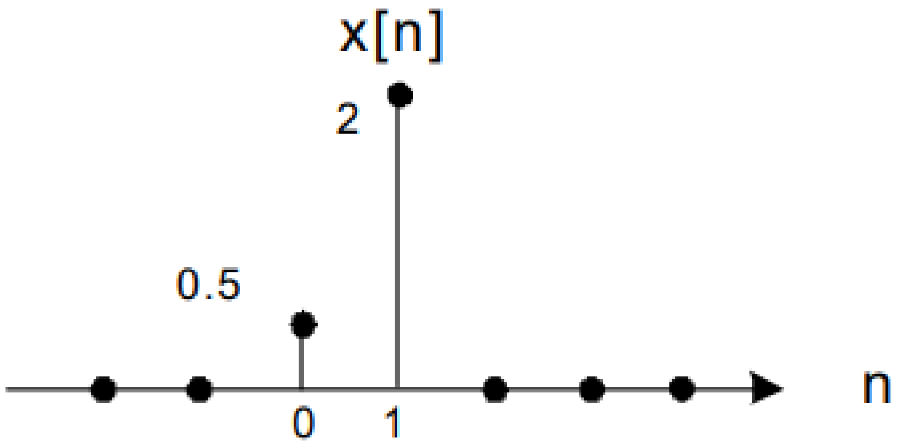
Demo: Example#1

- Consider the LTI system with impulse response h[n] and input x[n], as illustrated on right of slide.

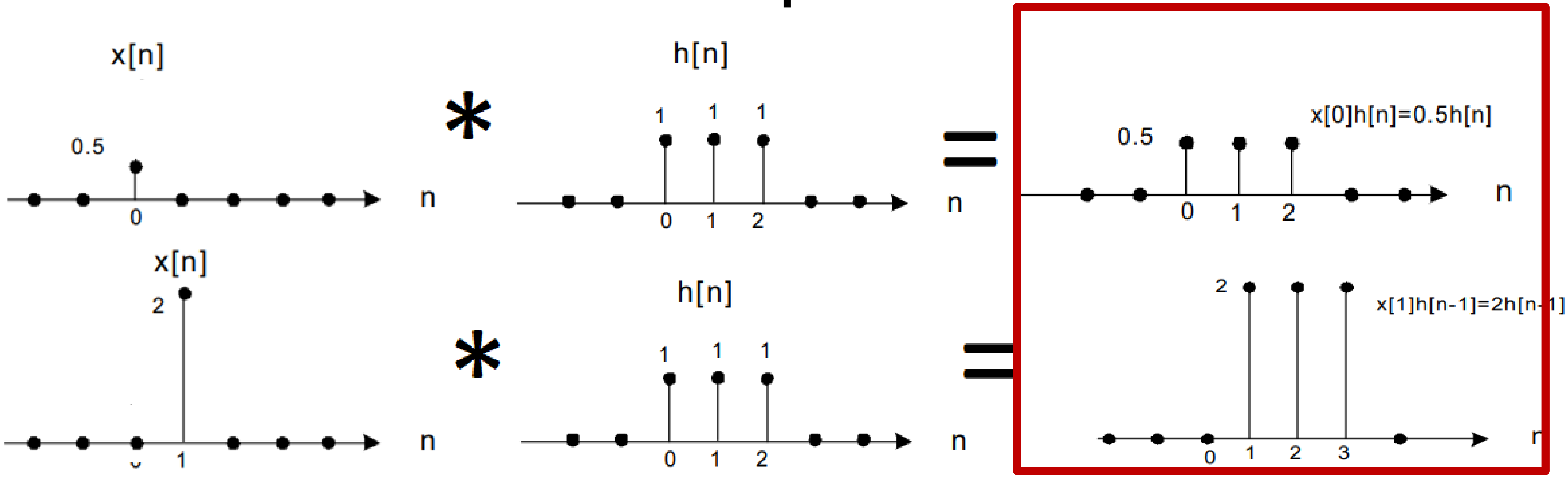
- What is the overall response of the system?

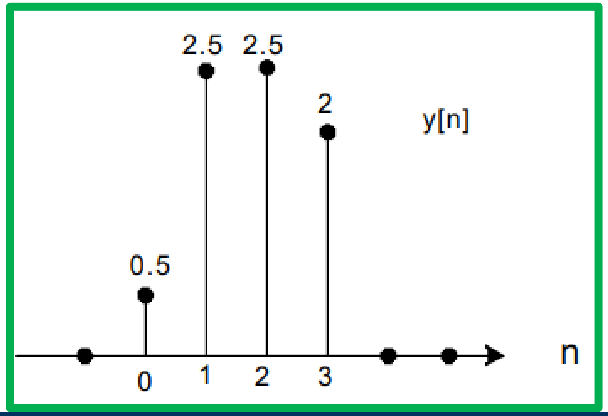
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$





Example





Convolution Operation

The Convolution Sum

- This summation is called the convolution sum

$$y[n] = \underbrace{\sum_{\boxed{i=0}}^{\infty} x[i]h[n-i]}_{x[n]*h[n]}$$

- Equation $y[n] = x[n] * h[n]$ is called the *convolution representation of the system*
- **Remark:** a DT LTI system is completely described by its impulse response *h*[*n*]

The Convolution Sum for Noncausal Signals

- Suppose that we have two signals x[n] and v[n] that are **not zero for negative times** (noncausal signals)
  - The signal exists for both negative and positive times. (example exponential, sine and cosine signals)
- Then, their convolution is expressed by the two-sided series

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i]$$

Example: Convolution of Two Rectangular Pulses

- Suppose that both x[n] and v[n] are equal to the rectangular pulse p[n] (causal signal) depicted below

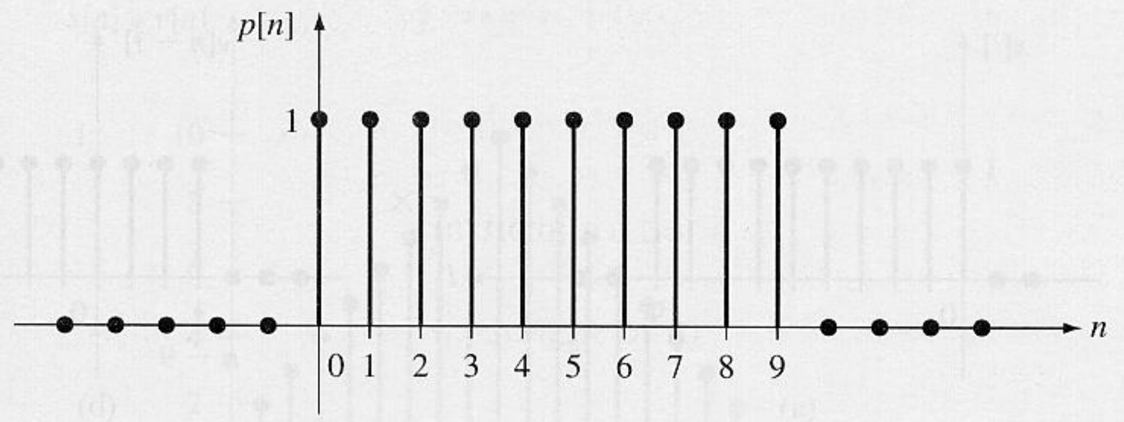


Figure 3.3 The pulse $p[n]$.

Is this the signal causal or non-causal?

The Folded Pulse

- The signal $v[-i]$ is equal to the pulse p[i] folded about the vertical axis

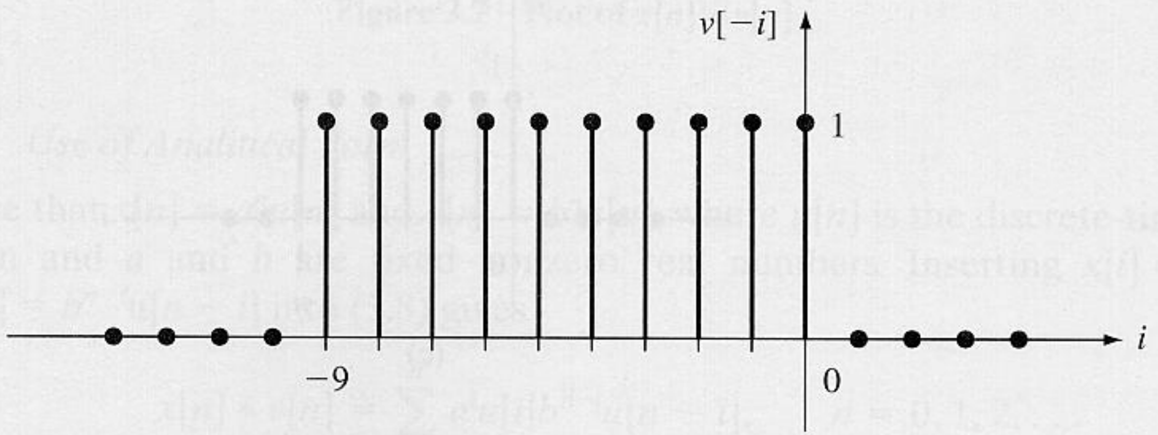


Figure 3.4 The folded pulse.

Is this the signal causal or non-causal?

Sliding $v[n-i]$ over $x[i]$

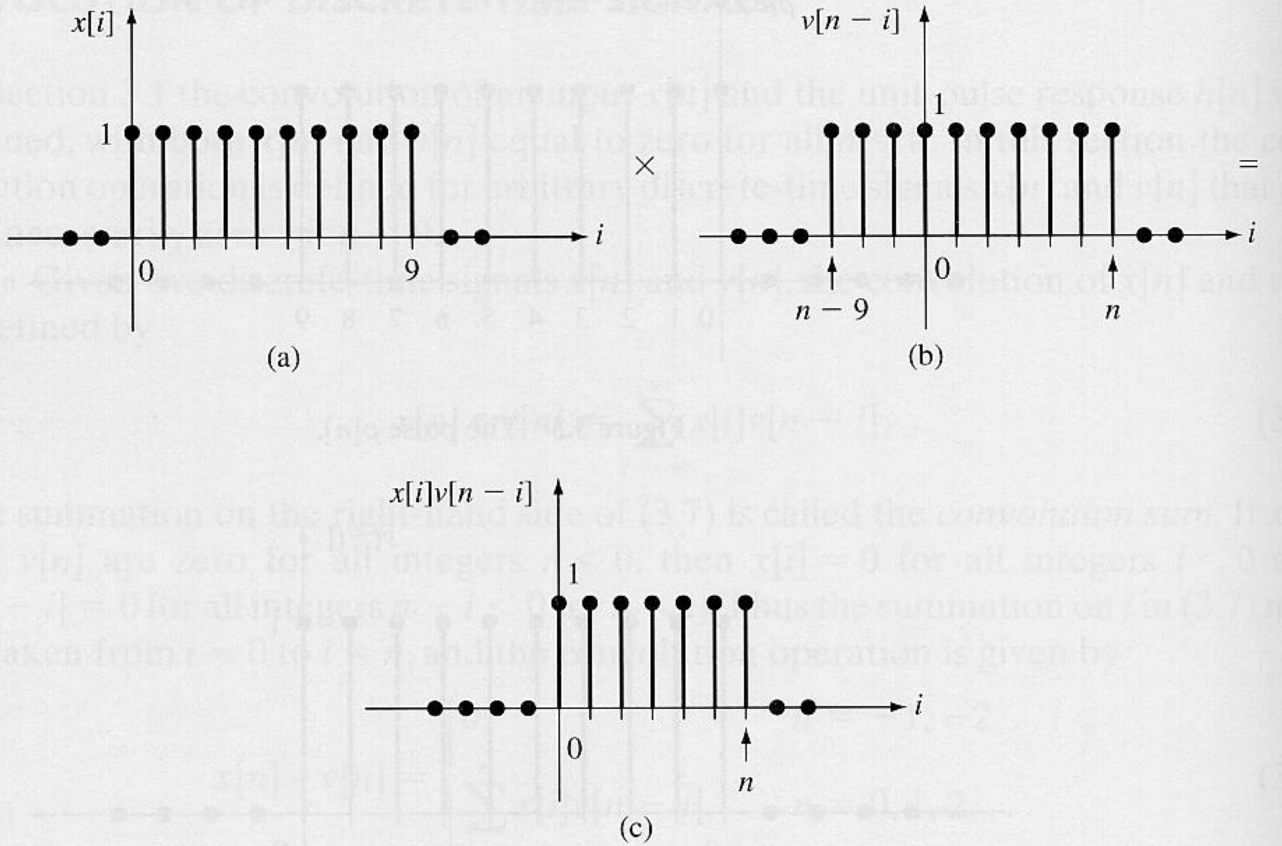


Figure 3.5 Plots of (a) $x[i]$; (b) $v[n-i]$; (c) $x[i]v[n-i]$ for $0 \le n \le 9$.

Sliding $v[n-i]$ over $x[i]$ - Cont'd

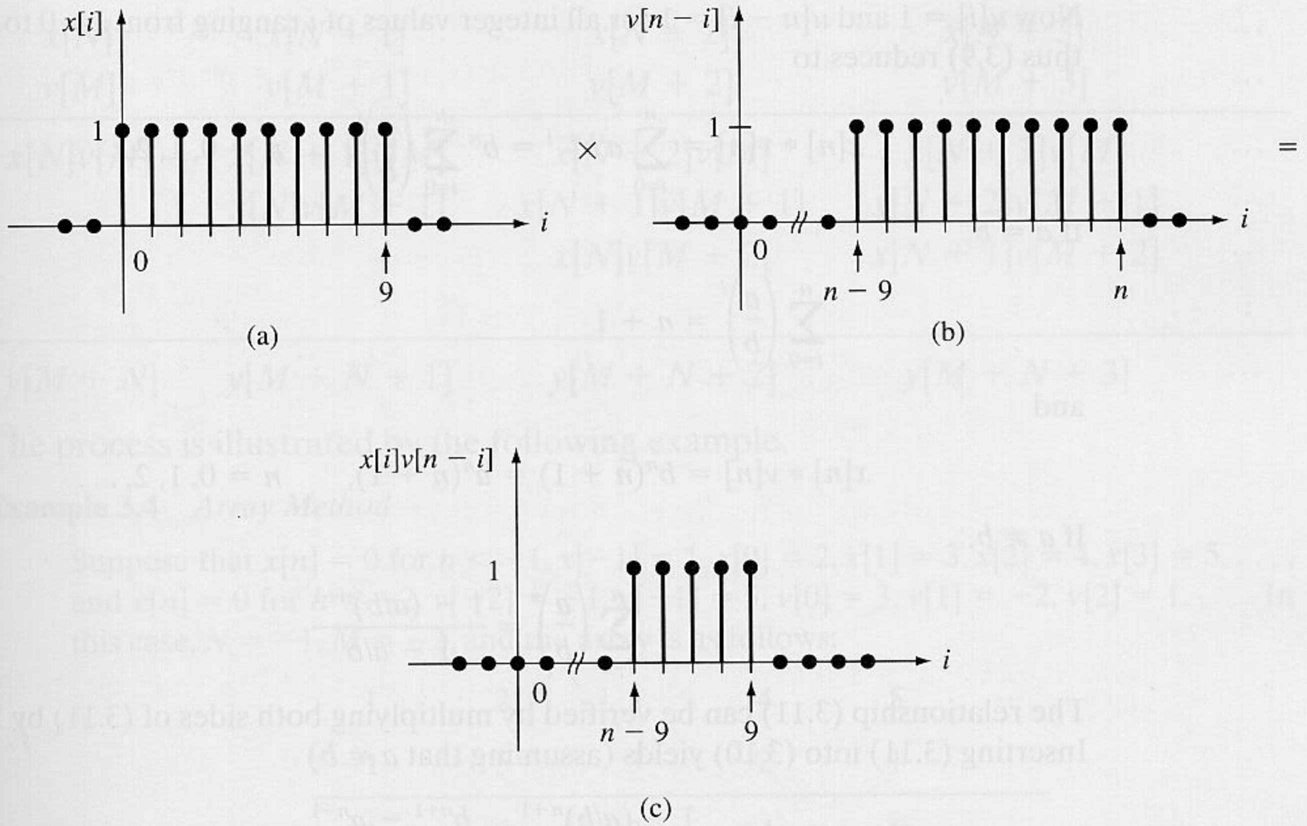


Figure 3.6 Plots of (a) $x[i]$; (b) $v[n-i]$; (c) $x[i]v[n-i]$ for $10 \le n \le 18$.

Plot of $x[n]*v[n]$

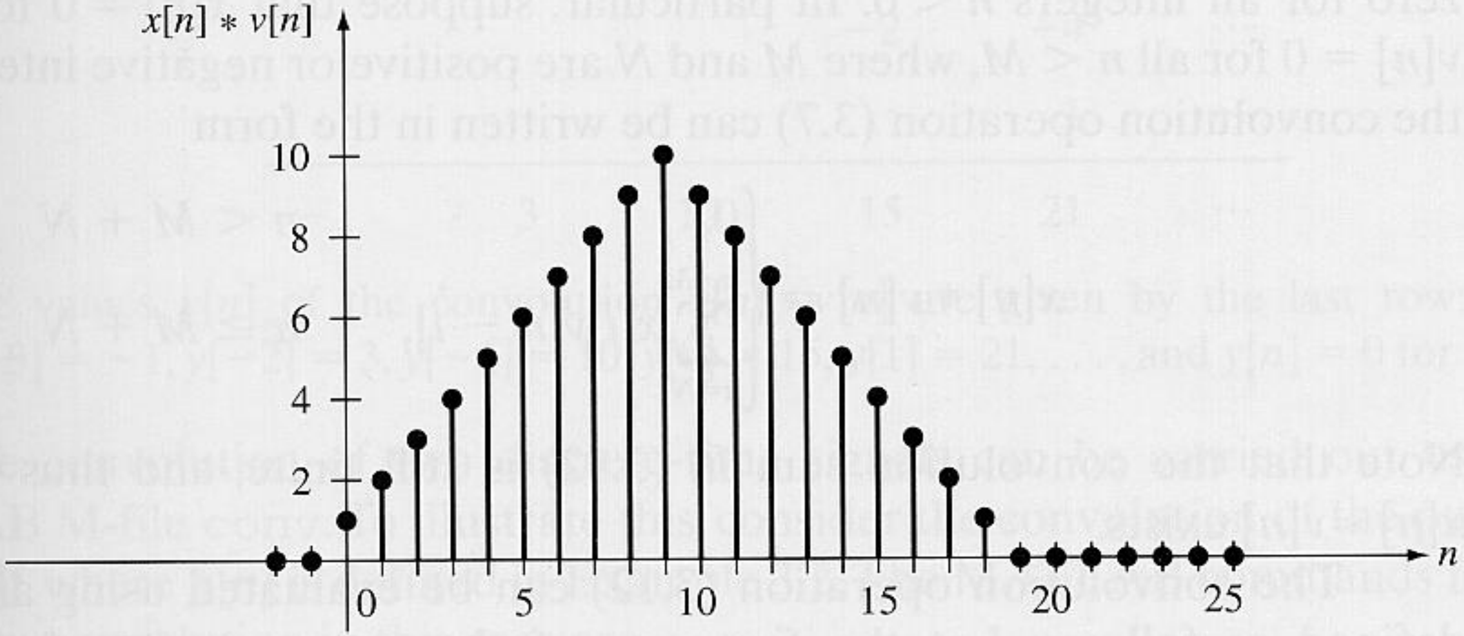


Figure 3.7 Plot of $x[n] * v[n]$.

Properties of the Convolution Sum

- Associativity

$$x[n]*(v[n]*w[n])=(x[n]*v[n])*w[n]$$

- Commutativity

$$x[n]*v[n]=v[n]*x[n]$$

- Distributivity w.r.t. addition

$$x[n]*(v[n]+w[n])=x[n]*v[n]+x[n]*w[n]$$

Properties of the Convolution Sum - Cont'd

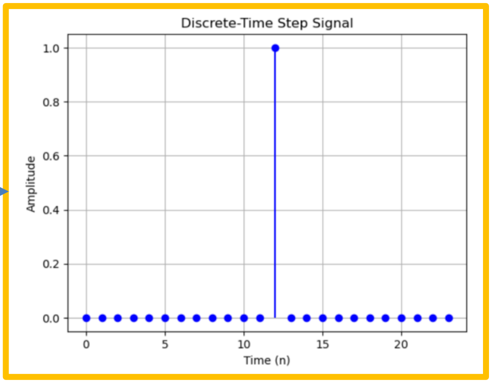
- Shift property: define

$$\begin{cases} x_q[n] = x[n-q] \\ v_q[n] = v[n-q] \\ w[n] = x[n] * v[n] \end{cases}$$

then $$w[n-q] = x_q[n] * v[n] = x[n] * v_q[n]$$

- Convolution with the unit impulse

$$x[n] * \boxed{\delta[n]} = x[n]$$



- Convolution with the shifted unit impulse

$$x[n] * \delta_q[n] = x[n-q]$$

Computing Convolution

- Consider the DT LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- impulse response:

$$h[n] = \sin(0.5n), \quad n \geq 0$$

- input signal:

$$x[n] = \sin(0.2n), \quad n \geq 0$$

Computing Convolution

- Suppose we want to compute y[n] for n = 0,1, …, 40

```python
import numpy as np
import matplotlib.pyplot as plt
```

```python
# Define the range of 'n' values from 0 to 40
t = np.arange(0, 41)
```

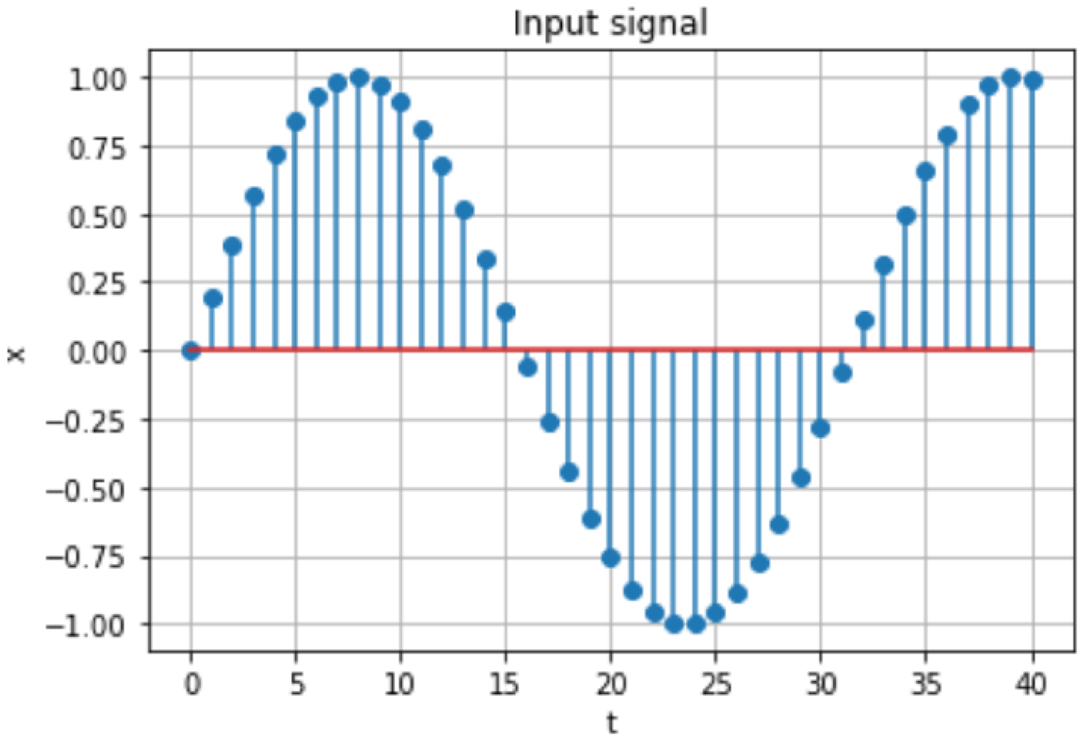
```python
# Define the signals 'x' and 'h'
x = np.sin(0.2 * t)
h = np.sin(0.5 * t)
```

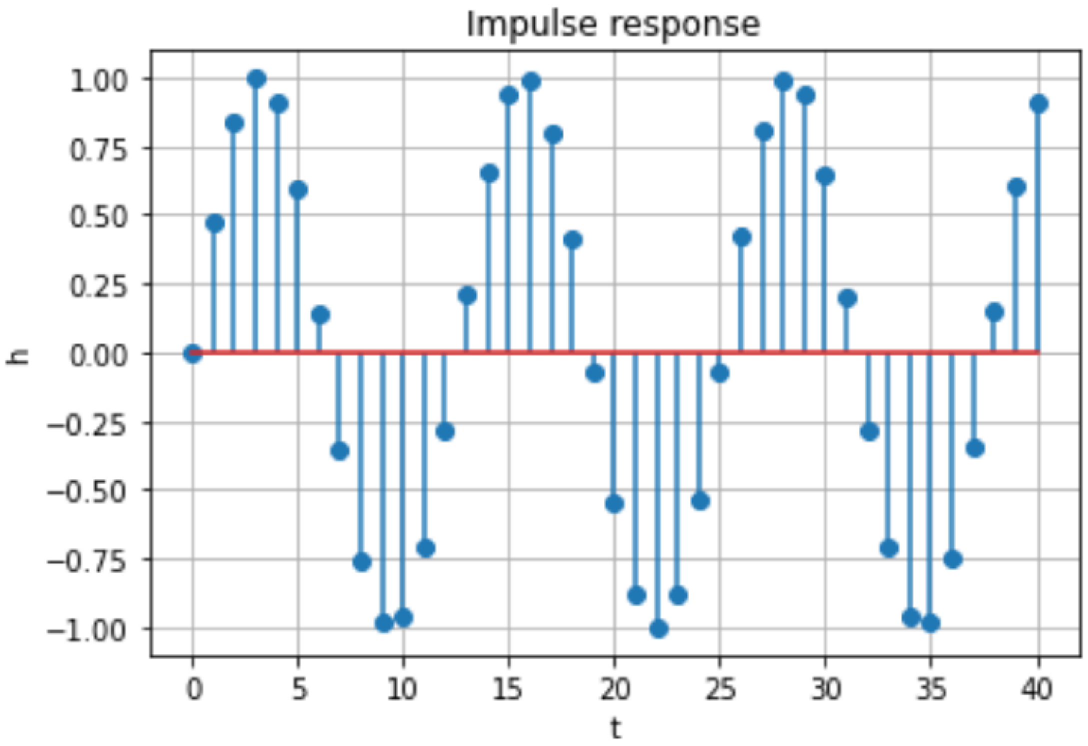
```python
# Perform convolution
y = np.convolve(x, h, 'full')
```

```python
# Trim 'y' to match the length of 'n'
y = y[:len(t)]
```

```python
# Plot the result using stem plot
plt.stem(t, y)
plt.xlabel('t')
plt.ylabel('y')
plt.title('Convolution of x and h')
plt.grid(True)
plt.show()
```

Computing Convolution



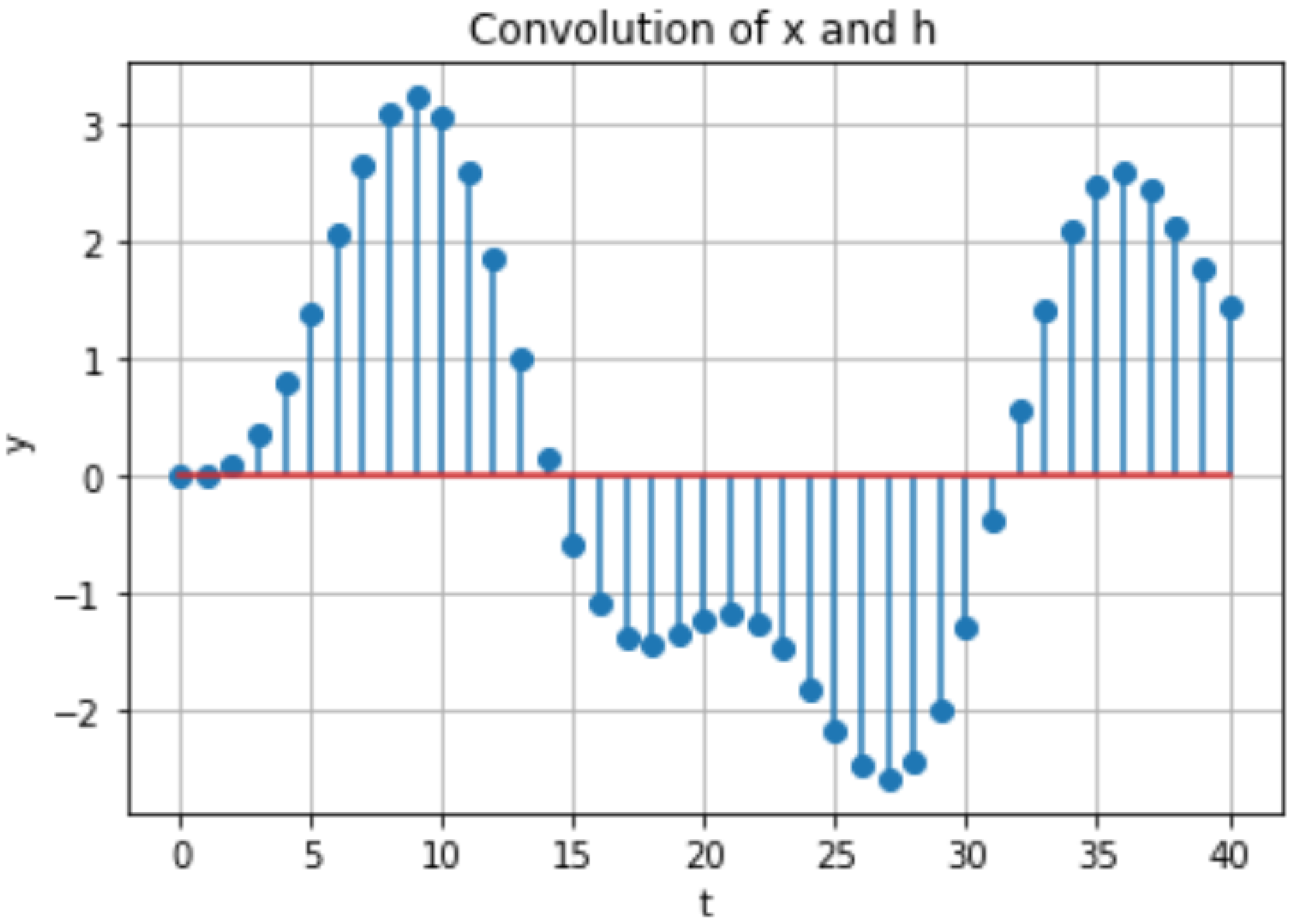


$$h[n] = \sin(0.5n), \quad n \geq 0$$

$$x[n] = \sin(0.2n), \quad n \geq 0$$

Computing Convolution

$$y[n] = x[n] * h[n]$$



CT Unit-Impulse Response

- Consider the CT SISO system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

- If the input signal is $x(t) = \delta(t)$ and the system has no energy at $t = 0^-$, the output $y(t) = h(t)$ is called the impulse response of the system

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$$

Exploiting Time-Invariance

- Let x[n] be an arbitrary input signal with

$$x(t) = 0, \text{ for } t < 0$$

- Using the sifting property of $\delta(t)$, we may write

$$x(t) = \int_{0^-}^{\infty} x(\tau)\delta(t-\tau)d\tau, \quad t \geq 0$$

- Exploiting time-invariance, it is

$$\delta(t-\tau) \longrightarrow \boxed{\text{System}} \longrightarrow h(t-\tau)$$

Exploiting Time-Invariance

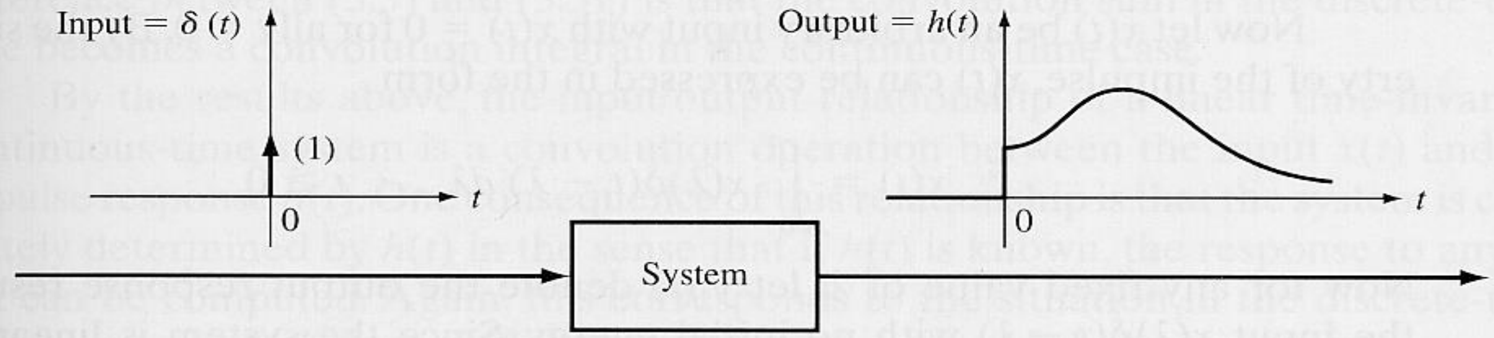


Figure 3.10 Generation of the impulse response.

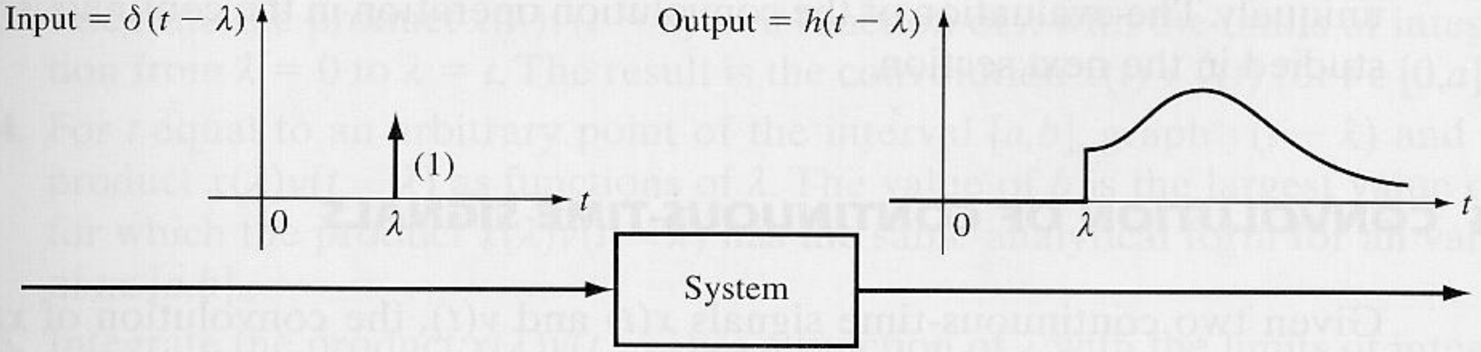


Figure 3.12 Time-shifted impulse response $h(t-\lambda)$.

Exploiting Linearity

- Exploiting linearity, it is

$$y(t) = \int_{0^-}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

- If the integrand $x(\tau)h(t-\tau)$ does not contain an impulse located at $\tau = 0$, the lower limit of the integral can be taken to be 0,i.e.,

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

The Convolution Integral

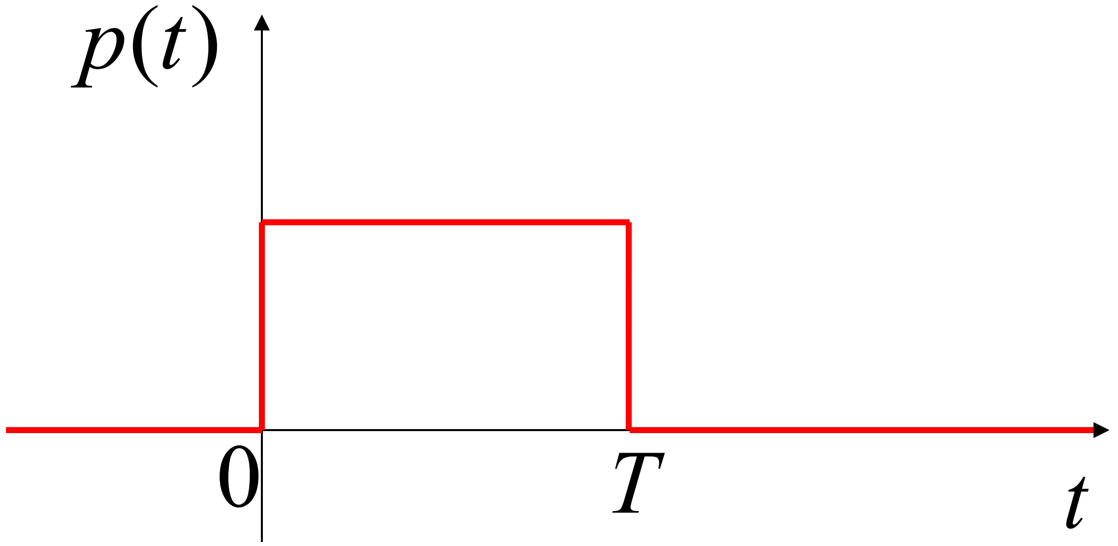
- This particular integration is called the convolution integral

$$y(t) = \underbrace{\int_0^\infty x(\tau)h(t-\tau)d\tau}_{x(t)*h(t)}, \quad t \geq 0$$

- Equation $y(t) = x(t) * h(t)$ is called the convolution representation of the system
- Remark: a CT LTI system is completely described by its impulse response h(t)

Example: Analytical Computation of the Convolution Integral

- Suppose that $x(t) = h(t) = p(t),$ where p(t) is the rectangular pulse depicted in figure



Example – Cont'd

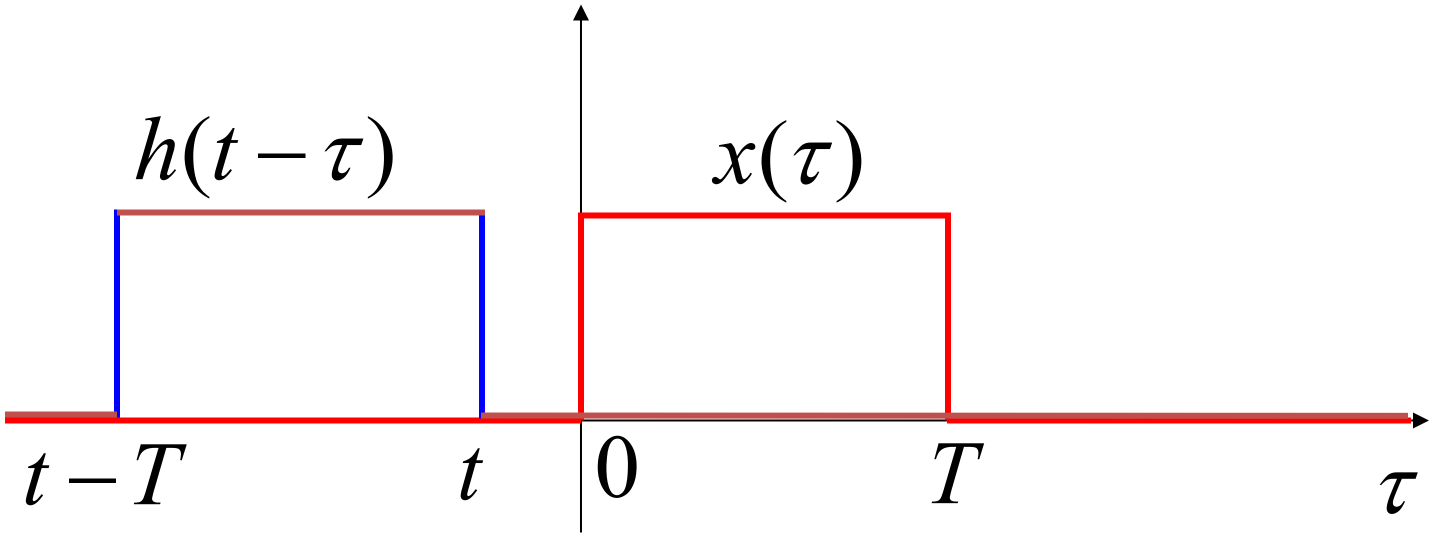
- In order to compute the convolution integral

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

- we have to consider four cases:

Example – Cont'd

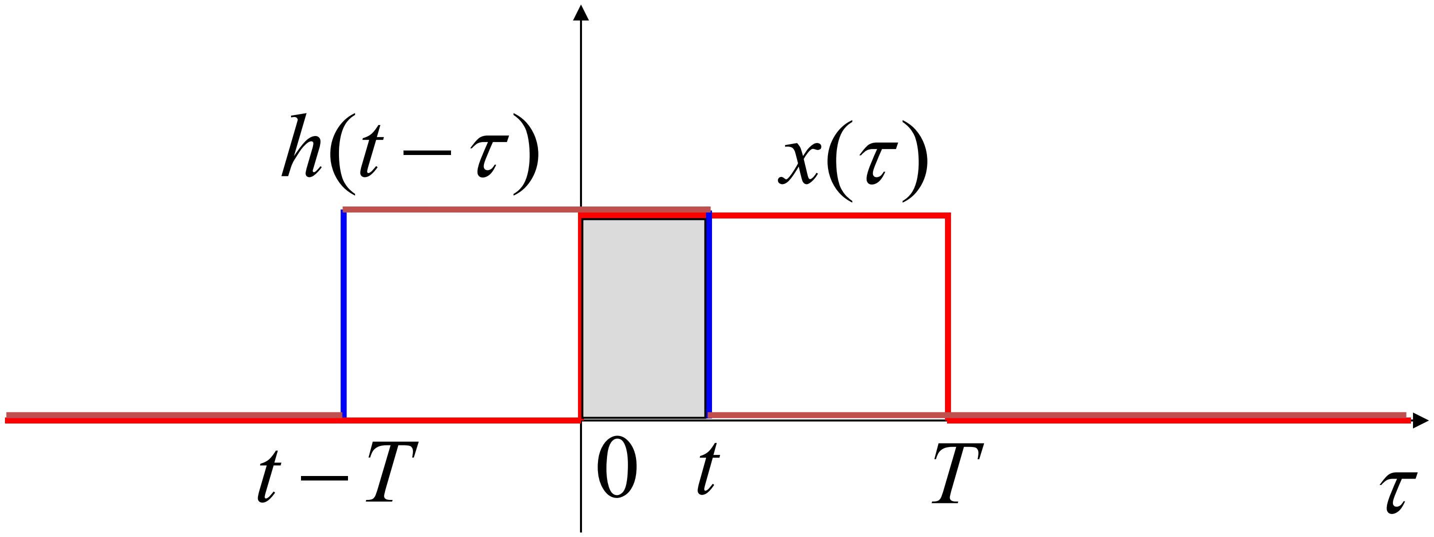
- Case 1: $t \leq 0$



$$y(t) = 0$$

Example – Cont'd

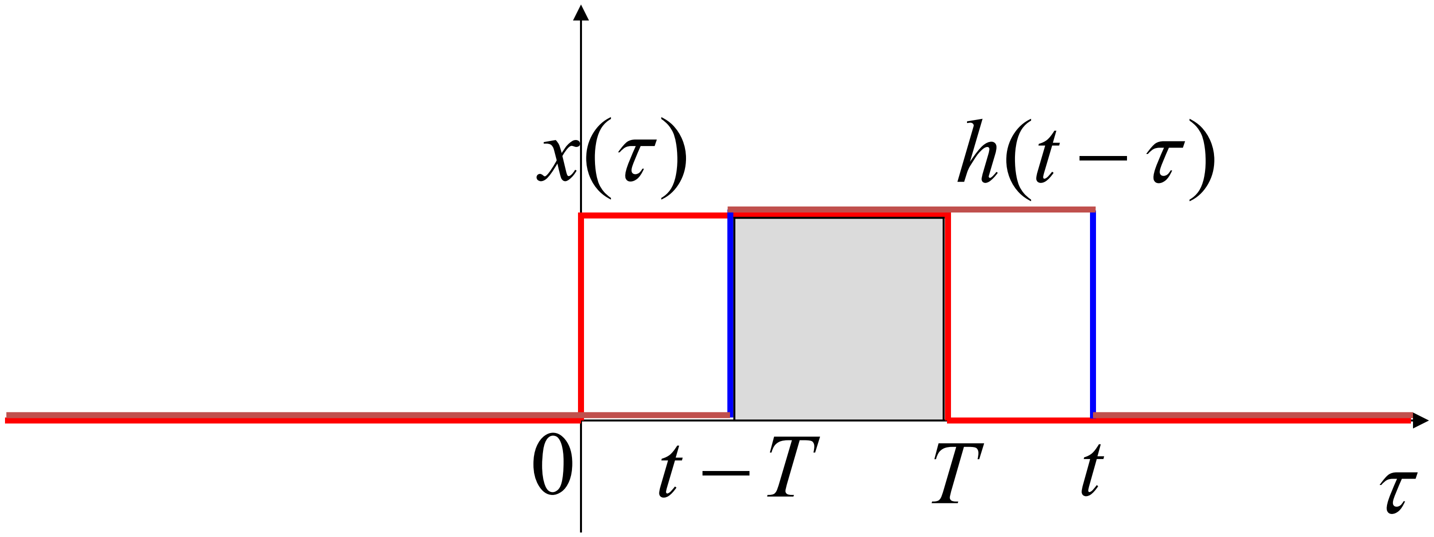
- Case 2: $0 \le t \le T$



$$y(t) = \int_0^t d\tau = t$$

Example – Cont'd

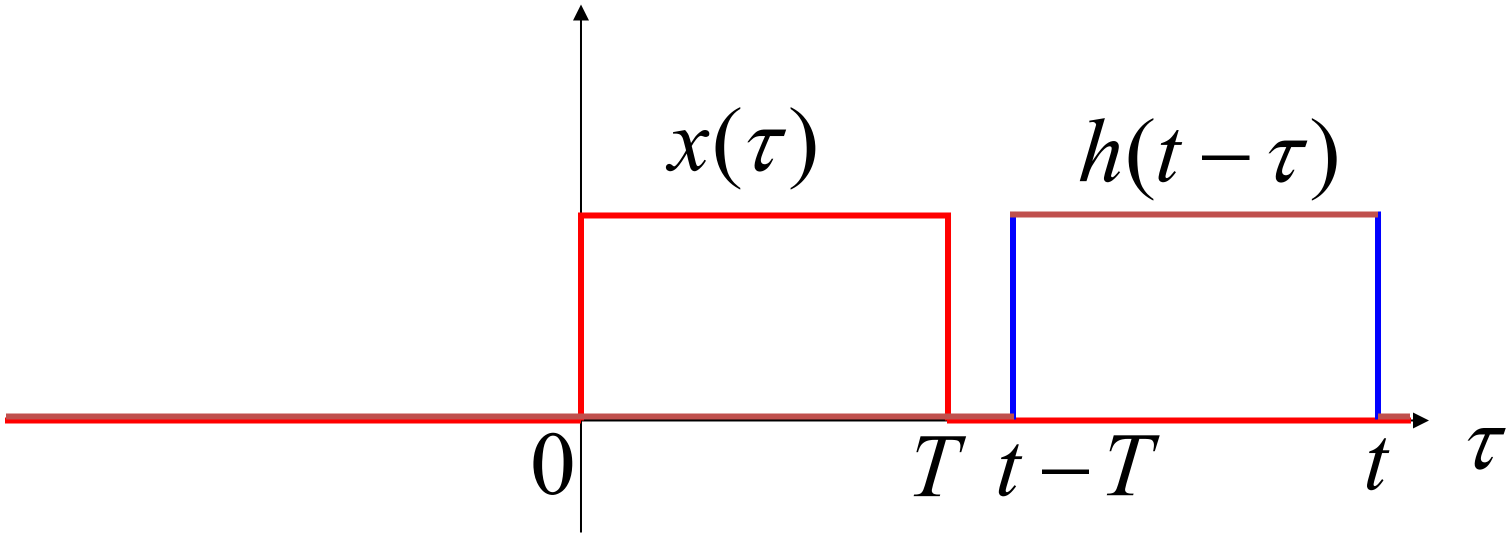
- Case 3: $0 \le t - T \le T \quad \rightarrow \quad T \le t \le 2T$



$$y(t) = \int_{t-T}^{T} d\tau = T - (t - T) = 2T - t$$

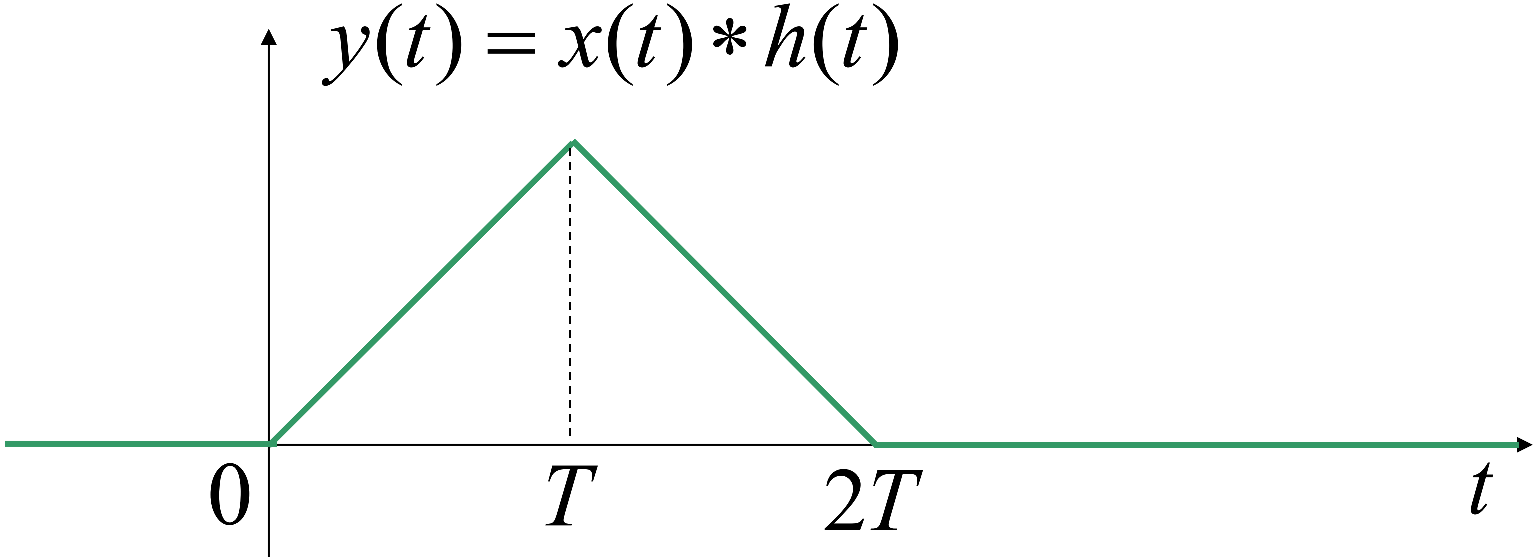
Example – Cont'd

- Case 4: $T \le t - T \quad \rightarrow \quad 2T \le t$



$$y(t) = 0$$

Example – Cont'd



Continuous Convolution

- $y(t) = x(t) * h(t)$
- $= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)dt$
  - Or
- $= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)dt$

Discrete Convolution

- $y(n) = x(n) * h(n)$

- $= \sum_{k=-\infty}^{k=\infty} x(k)h(n-k)$

  – Or

- $= \sum_{k=-\infty}^{k=\infty} x(n-k)h(k)$

Properties of Convolution

- Commutative
- $x_1(t) * x_2(t) = x_2(t) * \ x_1(t)$
- Distributive
- $x_1(t) * \ [x_2(t) + x_3(t)] =$
  $$[x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$
- Associative
- $x_1(t) * \ [x_2(t) * x_3(t)] =$
  $$[x_1(t) * x_2(t)] * x_3(t)$$

Properties of Convolution

- Shifting
- $x_1(t) * \ x_2(t) = y(t)$
- $x_1(t) * \ x_2(t - t_0) = y(t - t_0)$
- $x_1(t - t_0) * \ x_2(t) = y(t - t_0)$
- $x_1(t - t_0) * \ x_2(t - t_1) = y(t - t_0 - t_1)$

CORRELATION

Correlation of Discrete-Time Signals

Transmitted Signal, $x(n)$

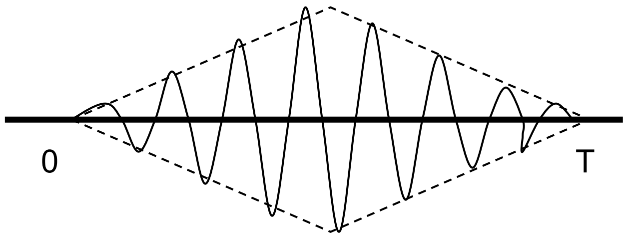
- A mathematical operation that closely resembles convolution.
- It involves two signals just like convolution
- However, it objective is to measure the degree to which the signals are similar
- Usually encountered in radar, digital communication, image processing and other areas in science and engineering

Reflected Signal,

$y(n) = \boldsymbol{\alpha} x(n-D) + w(n)$

Where α is the attenuation factor

Measures similarity between 2 signals



Cross-Correlation

- Cross-correlation of x(n) and y(n) is a sequence, $r_{xy}(l)$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n-l)\,y(n) \qquad l = 0, \pm 1, \pm 2, \ldots$$

- Reversing the order, $r_{yx}(l)$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n-l)\,x(n) \qquad l = 0, \pm 1, \pm 2, \ldots$$

- => $\boxed{r_{xy}(l) = r_{yx}(-l)}$

Example

- *Determine the cross correlation of the two signals*
  - $x(n) = \{\dots, 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0, \dots\}$
  - $y(n) = \{\dots, 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0, \dots\}$

- *Using this equation*
  - $r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$
- When l=0 we have
  - $r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$
- Therefore the sequence $x(n)y(n)$ becomes
  - $\{\dots, 0, 0, 2, 1, 6, -14, 4, 2, 6, 0, 0, \dots\}$
- Hence the overall sum becomes
  - $r_{xy}(0) = 7$

- Similarly,
  - $r_{xy}(1) =$ 13, $r_{xy}(2) =$ -18, $r_{xy}(3) =$ 16, $r_{xy}(4) =$ -7

Similarity to Convolution

- It is apparent through observation in the calculations that cross correlation is
  - Shifting one of the signals
  - Multiplying the two signals
  - Summing over all values of the product signal
- There is no folding (time-reversal)
- Therefore, **the convolution of x(n) with y(-n) yields the cross correlation $\boldsymbol{r_{xy}(l)}$**

$$r_{xy}(l) = x(l) * y(-l)$$

$$r_{yx}(l) = y(l) * x(-l)$$

- ..
  - ..

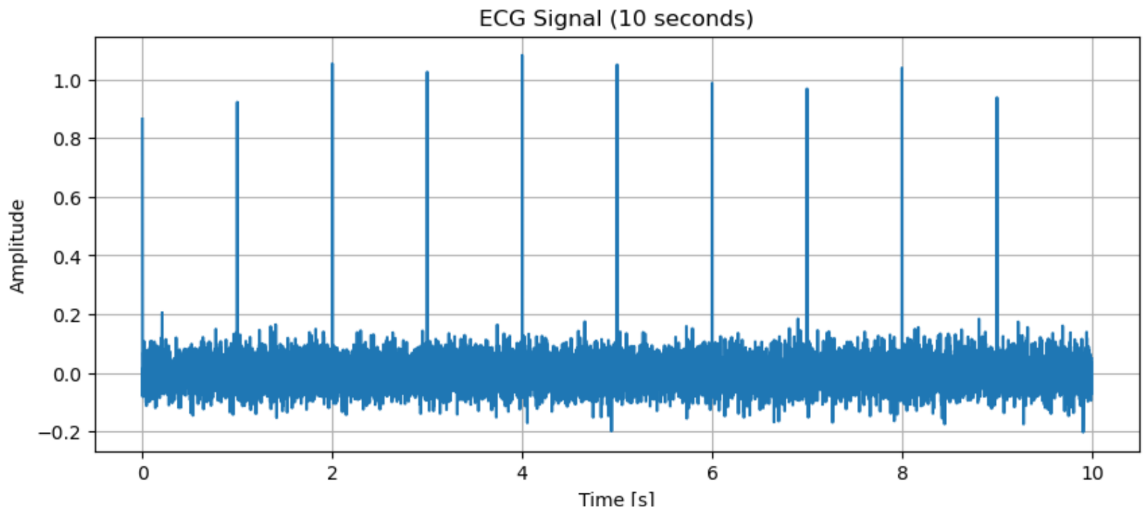
Auto-Correlation

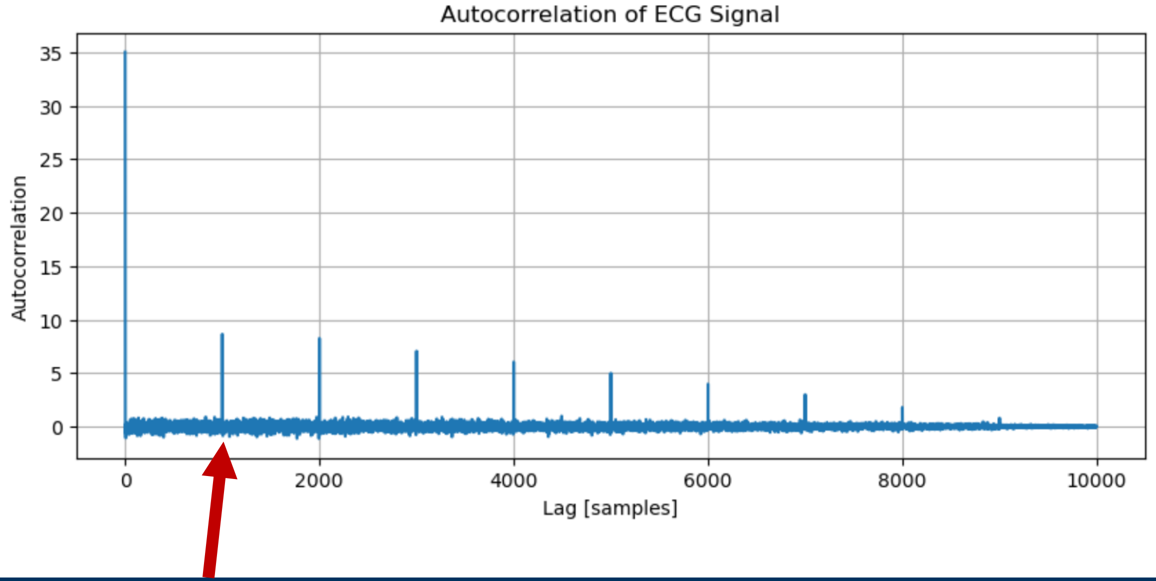
- Correlation of a signal with itself

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = r_{xx}(-l) \qquad l = 0, \pm 1, \pm 2, \dots$$

- Used to differentiate the presence of a like-signal, e.g., zero or one
- Auto-correlation function is an even function
  - $r_{xx}(l) = r_{xx}(-l)$

Periodicity using Autocorrelation



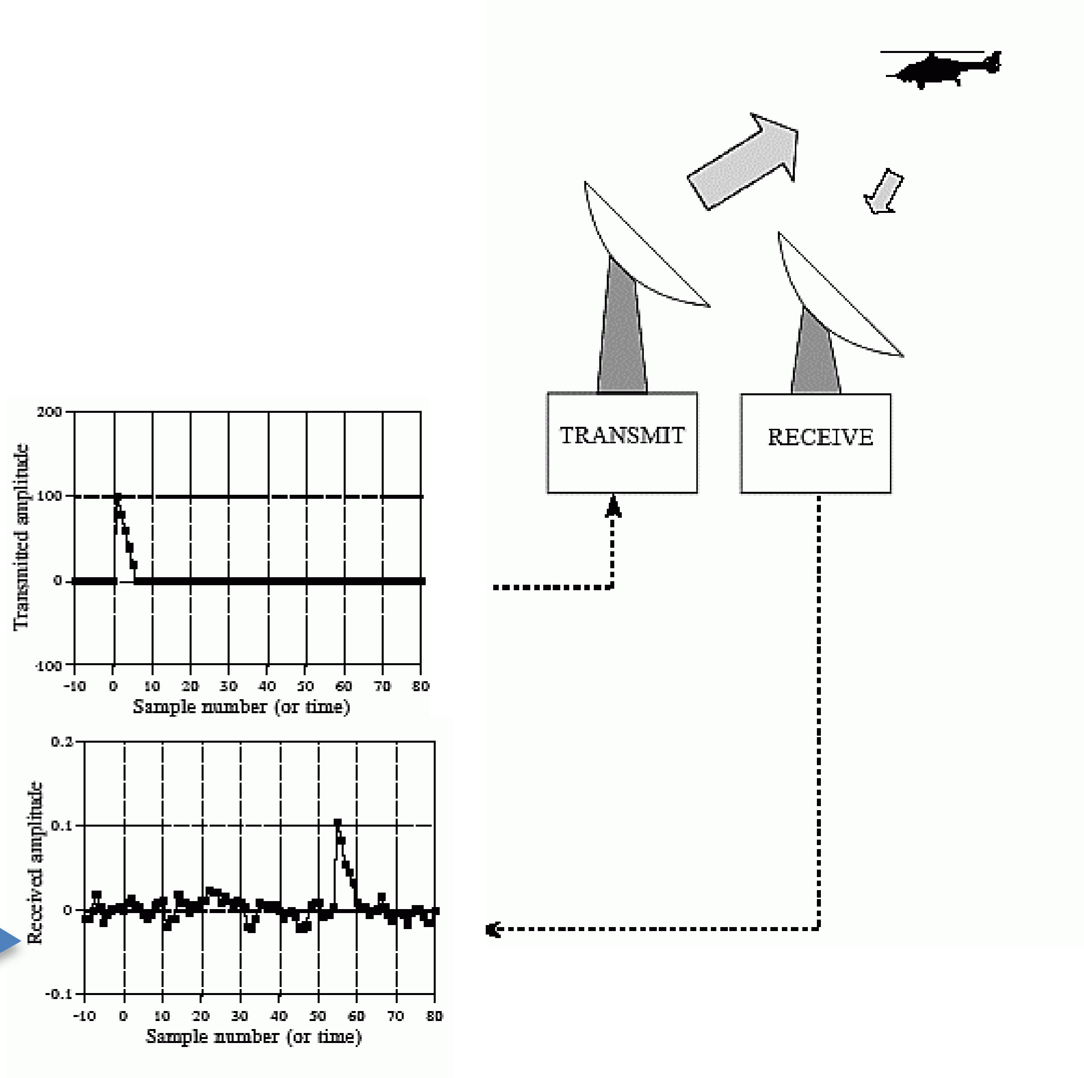


- ECG signal (above) and its autocorrelation (below)
- Find the peak of the autocorrelation.
  - The first peak after lag zero gives the period between the heartbeats
  - 1000 lags
- In seconds ➔ 1000/fs = 1 sec
- Change to beats per minute = 60/1 = 60 BPM

Cross-Correlation Concept

Transmitted waveform

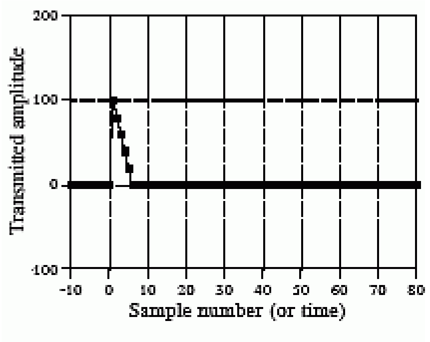
Shifted version of the
transmitted waveform + noise

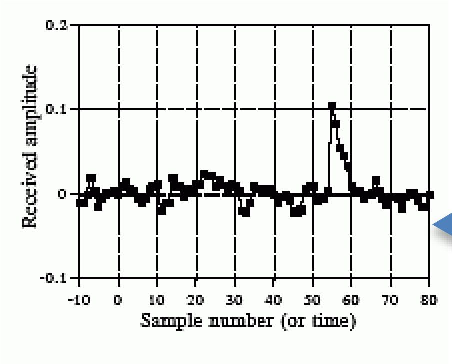


Cross-Correlation of Discrete-time Signals

- Cross-correlation is an efficient way to measure the degree to which two signals (one template and the other the test signal) are similar to each other.

Template





Shifted version of the template+ noise

- *Cross-Correlation* is a mathematical operation that resembles convolution. It measures the degree of similarity between two signals.

Mathematical Definition

$x(n)$ Transmitted/Desired Signal

$$y(n) = \alpha x(n - D) + \omega(n)$$

Received/Test Signal

Attenuation factor

Delayed version of the input

Additive noise

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n), \quad l = 0, \pm 1, \pm 2, \ldots$$

- $r_{yx}(l)$ is thus the *folded* version of $r_{xy}(l)$ around $l = 0$:

$$r_{xy}(l) = r_{yx}(-l)$$

Calculation of cross-correlation

- Cross-correlation involves the same sequence of steps as in convolution *except* the folding part, so basically the cross-correlation of two signals involves:
    1. Shifting one of the sequences
    2. Multiplication of the two sequences
    3. Summing over all values of the product

Cross-correlation vs. Convolution

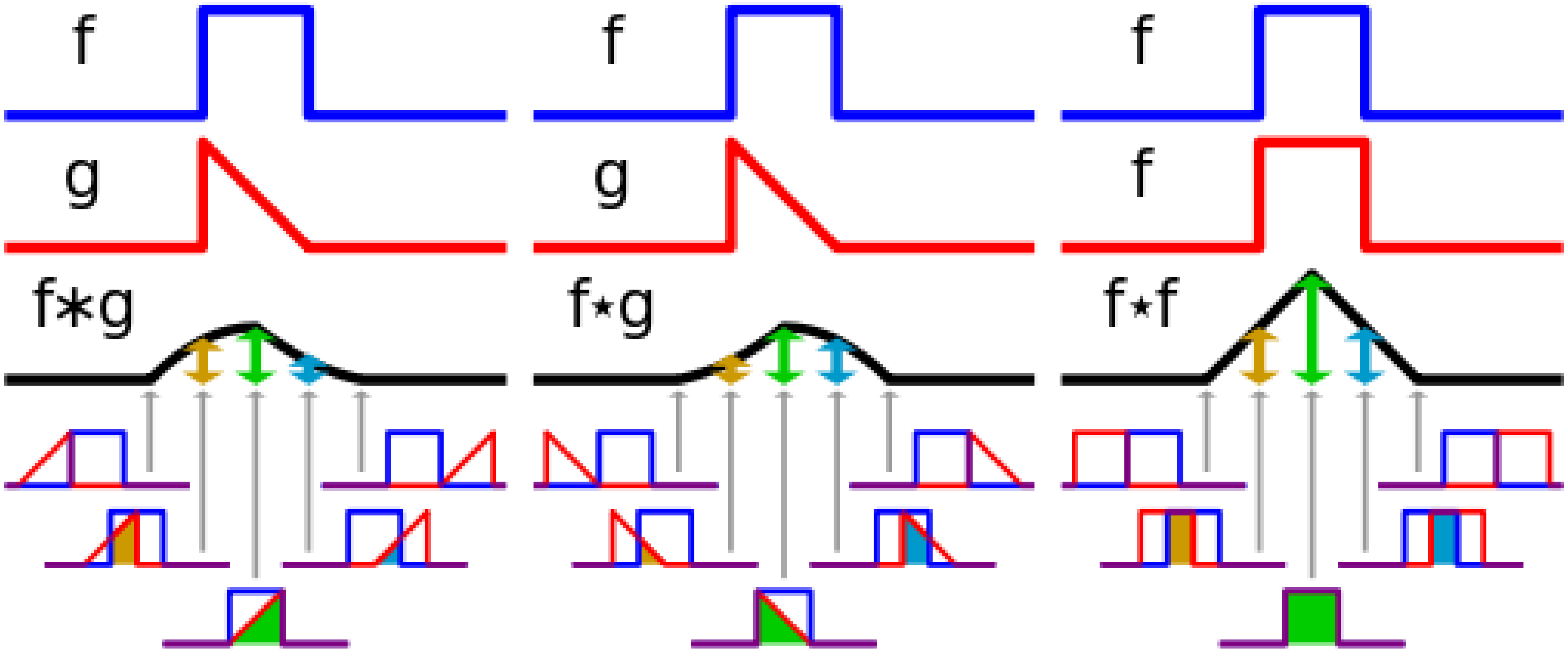
- The cross-correlation machine and convolution machine are identical, except that in the correlation machine this flip doesn't take place, and the samples run in the normal direction.

$$r_{xy}(l) = x(l) * y(-l)$$

Cross-correlation is non-commutative.

- Convolution is the relationship between a system's input signal, output signal, and the impulse response. Correlation is a way to detect a known waveform in a noisy background.
- The similar mathematics is only a convenient coincidence.

Convolution, Cross-correlation, and Autocorrelation



Convolution describes the response of a linear and time-invariant system to an input signal.

The inverse Fourier transform of the pointwise product in frequency space.

Cross-correlation is a measure of similarity of two signals.

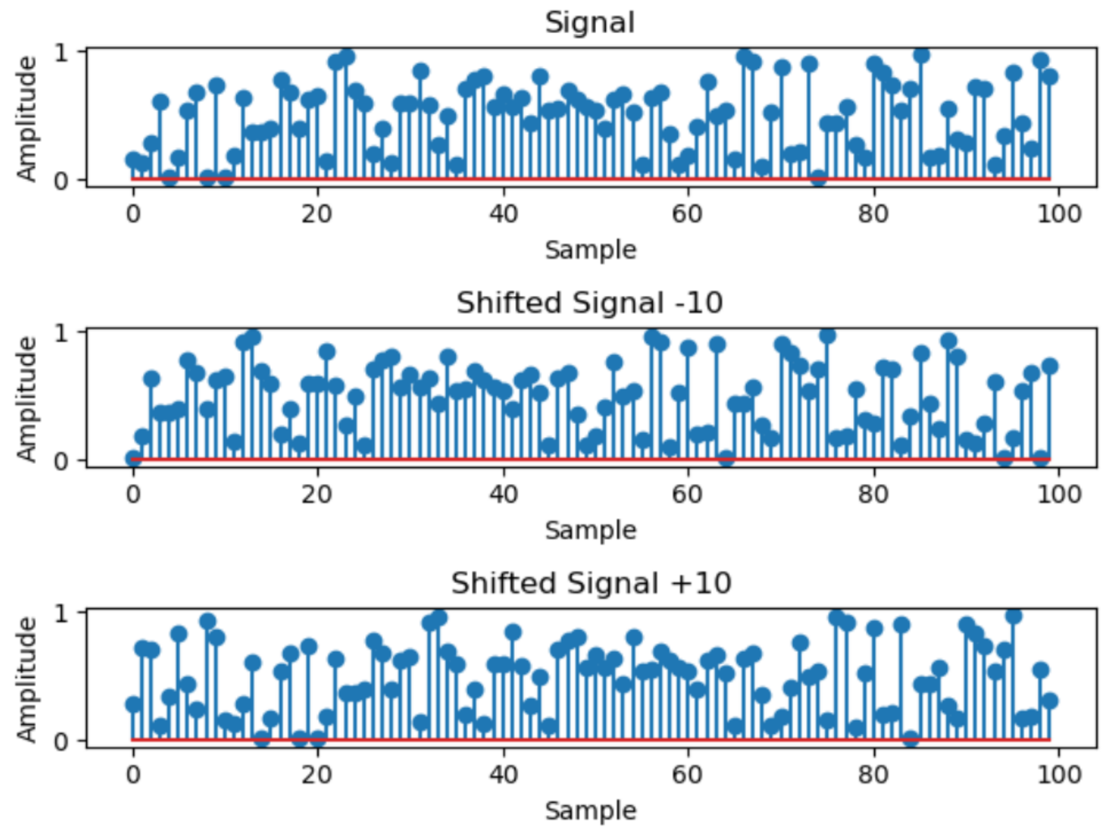
It can be used for finding a shift between two signals.

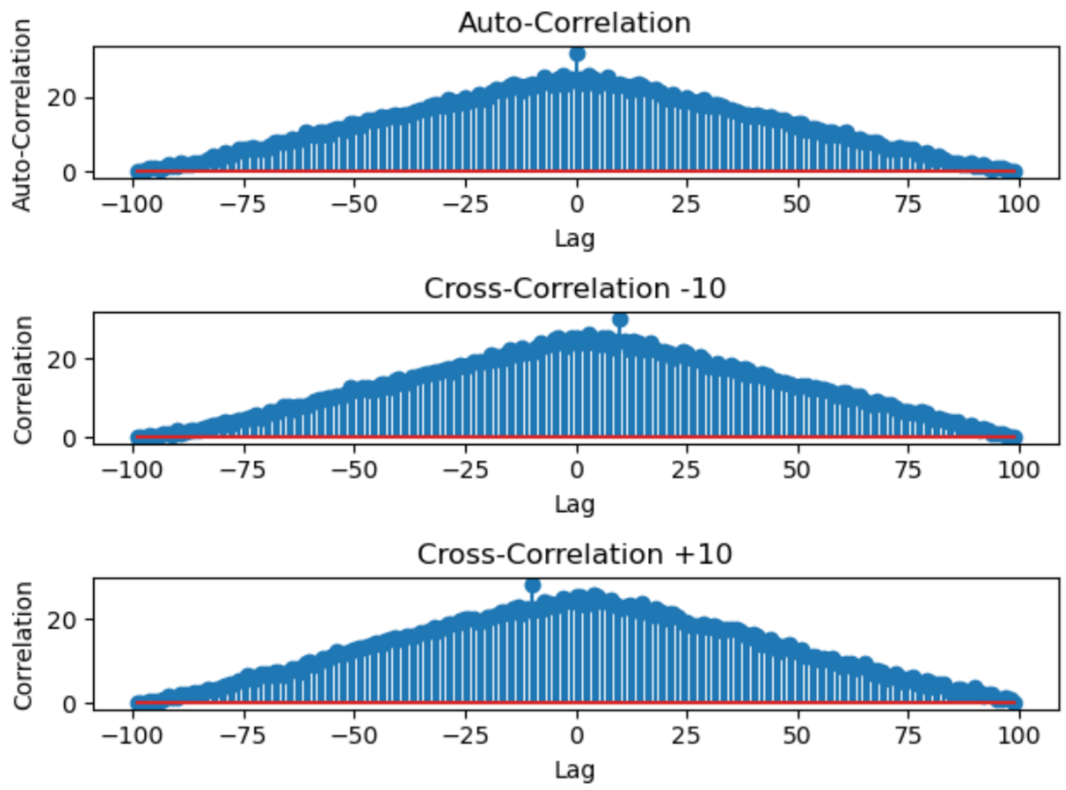
Auto-correlation is the cross-correlation of a signal with itself.

It can be used for finding periodic signals obscured by noise.

http://en.wikipedia.org/wiki/Convolution

Correlation Illustration





Top: Autocorrelation
Middle and Bottom: Correlation with signal delayed by -10 and 10 respectively.

Summary

- Convolution
- Convolution Examples
- Convolution Theorem
- Correlation
- Correlation Examples

Further Reading

- Sign in with University of Lincoln and Read Online
  - **Title:** Digital Signal Processing : Pearson New International Edition
  - **Authors:** John Proakis and Dimitris Manolakis
  - **Online Link:** https://ebookcentral.proquest.com/lib/ulinc/detail.action?docID=5174771
  - What to read:
    - Discrete Time Signals and Systems (**Chapter 2**)

Questions

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