1. Differentiation and Integration

1.1. Derivatives and Differentiation

$\frac{d}{dx}$ - *this little cute sign is the differential sign. It is quite useful. It is a pure pleasure to compute. You can differentiate practically any function without efforts – the formulae are easy to remember and will save you many times. Definitely,* ***the differential is your bro****.*

Definition: $f'(x) = \frac{df}{dx} = \frac{d}{dx} f \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Geometric meaning: The rate of change of the function $f(x)$ in response to change of the argument x. Slope of the of function at given point.

Basic differentiation formulas:

F1. $C' = 0$, where $C = const$ is a constant. Note: this can still be a function, but depending on a different argument (e.g. y).

F2. $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$

F3a. $\frac{d}{dx}\left[a^x\right] = a^x \ln a$; F3b. $\frac{d}{dx}\left[e^x\right] = e^x$

F4a. $\frac{d}{dx}\left[\log_a(x)\right] = \frac{\log_a e}{x}$; F4b. $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$

F5a. $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$; F5b. $\frac{d}{dx}\left[sh(x)\right] = ch(x)$

F6a. $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$; F6b. $\frac{d}{dx}\left[ch(x)\right] = sh(x)$

F7. $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$

F8. $\frac{d}{dx}\left[\arccos(x)\right] = -\frac{1}{\sqrt{1-x^2}}$

F9. $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$

F10. $\frac{d}{dx}[arcctg(x)] = -\frac{1}{1+x^2}$

Rules of differentiation:

Assume functions u and v depend on argument x, and C is a constant.

R1. $\frac{d}{dx}[u+v] = u' + v'$

R2a. $\frac{d}{dx}[u \cdot v] = u'v + v'u$; R2b. $\frac{d}{dx}[C \cdot u] = C \cdot u'$

R3. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - v'u}{v^2}$

R4. $\frac{d}{dx}[u(v(x))] = \frac{du}{dv}(v(x)) \cdot \frac{dv}{dx}(x)$ (chain rule)

Examples:

Example1. Compute derivative of $f(r) = \left(\frac{\sigma}{r}\right)^{12}$ with respect to (w.r.t) r.

Solution A: Representing the function as a negative power of r: $f(r) = \sigma^{12} r^{-12}$. Using R2b and then F2, we obtain: $\frac{d}{dr} f(r) = \frac{d}{dr}\left(\sigma^{12} r^{-12}\right) = \sigma^{12} \frac{d}{dr}\left(r^{-12}\right) = \sigma^{12}\left(-12 r^{-13}\right) = -12\sigma^{12} r^{-13} = -\frac{12}{r}\left(\frac{\sigma}{r}\right)^{12}$

Solution B: Use the intermediate function $x(r) = \frac{\sigma}{r}$, the function is $f(r) = (x(r))^{12}$. Then, using the chain rule, R4, and then R2b and formula F2, we obtain:

$$\frac{d}{dr} f(r) = \frac{d}{dx}(x(r))^{12} \frac{dx(r)}{dr} = 12(x(r))^{11} \sigma\left(-r^{-2}\right) = -12\left(\frac{\sigma}{r}\right)^{11} \frac{\sigma}{r^2} = -\frac{12}{r}\left(\frac{\sigma}{r}\right)^{12}$$

Example 2. Compute derivative of $g(x) = (x - x_0)\exp\left(-A(x-x_0)^2\right)$ with respect to (w.r.t) x.

Solution: We will first apply rule R2a:

$$g'(x) = \left[\frac{d}{dx}(x - x_0)\right]\exp\left(-A(x-x_0)^2\right) + (x - x_0)\left[\frac{d}{dx}\exp\left(-A(x-x_0)^2\right)\right]$$

The derivative in the second term is computed using chain rule R4, with the intermediate function $v(x) = -A(x-x_0)^2$:

$$\frac{d}{dx}\exp\left(-A(x-x_0)^2\right) = \mid v(x) \equiv -A(x-x_0)^2 \mid = \frac{d}{dx}\exp(v) = \left(\frac{d}{dv}\exp(v)\right)\cdot\left(\frac{dv}{dx}\right) =$$
$$= \exp(v)\frac{d}{dx}\left(-A(x-x_0)^2\right) = \mid y \equiv (x-x_0) \mid = -A\exp(v)\frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = -A\exp(v)\cdot 2y =$$
$$= -2A(x-x_0)\exp\left(-A(x-x_0)^2\right)$$

So, combining two parts together, we will find:

$$g'(x) = \left[1-2A(x-x_0)^2\right]\cdot\exp\left(-A(x-x_0)^2\right)$$

Example 3. Compute derivative of $\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$ w.r.t. x_2

Solution: Again, we rely on the chain rule, R4, and use it several times:

$$\frac{d}{dx_2}\frac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}} = \mid s = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 \mid =$$
$$= \frac{d}{dx_2}s^{-1/2} = \frac{d}{ds}s^{-1/2}\frac{ds}{dx_2} = -\frac{1}{2}s^{-3/2}\frac{d\left((x_1-x_2)^2+const\right)}{dx_2} = -\frac{1}{2}s^{-3/2}\frac{d\left((x_1-x_2)^2\right)}{dx_2} = \mid t = (x_1-x_2) \mid$$
$$= -\frac{1}{2}s^{-3/2}\frac{d}{dx_2}\left(t^2\right) = -\frac{1}{2}s^{-3/2}\frac{d}{dt}\left(t^2\right)\frac{dt}{dx_2} = -\frac{1}{2}s^{-3/2}\cdot 2t\cdot(-1) = t\cdot s^{-3/2} =$$
$$= \frac{(x_1-x_2)}{\left[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right]^{3/2}}$$

1.2. Integrals and Integration

$\int\cdot\, dx$ *- this concentrated evil, right from the middle of hell, will troll you all your life. The calculations will take priceless years of your life and kilometers of paper. is quite useful. The sign looks like a torture tool, which it is. It is a pure pleasure to compute. Run away from it in fear –* ***it is not your bro****. But still, don't be afraid.*

Definition: If $I \subset R$, $f : I \to R$ then $F : I \to R$ is called an **antiderivative** of f if $F'(x) = f(x), \forall x \in I$

Definition: If $\forall x \in I, \exists F$, then the function f is called **integrable on the interval** I

Definition: If $f : I \to R$ is integrable on I, then $\int f(x)dx = \{F(x)+C\}$ is called an **indefinite integral** of f

Definition: If $I=[a,b]\subset R$, $f:I\to R$ and $F:I\to R$ is the antiderivative of f, then $\int_a^b f(x)dx = F(b)-F(a)$ is called a definite integral of f.

Geometric meaning of definite integral: The "area" (or volume or hypervolume, in higher dimensions).

Basic integration formulas:

F1. $\int 0dx = C$

F2. $\int x^n dx = \frac{x^{n+1}}{n+1}+C$, $n\in Z, n\neq -1$

F3. $\int \frac{dx}{x} = \ln|x| + C$, $\forall x\in R\setminus\{0\}$

F4a. $\int \cos(x)dx = \sin(x)+C$

F4b. $\int \sin(x)dx = -\cos(x)+C$

F5a. $\int \frac{dx}{\cos^2(x)} = tg(x)+C$

F5b. $\int \frac{dx}{\sin^2(x)} = -ctg(x)+C$

F6a. $\int e^x dx = e^x + C$

F6b. $\int a^x dx = \frac{a^x}{\ln a}+C$

F7a. $\int ch(x)dx = sh(x)+C$

F7b. $\int sh(x)dx = ch(x)+C$

F8a. $\int \frac{dx}{ch^2(x)} = th(x)+C$

F8b. $\int \frac{dx}{sh^2(x)} = -cth(x)+C$

F9. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} arctg\left(\frac{x}{a}\right)+C$

F10. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C,\ \ a\neq 0$ ("high" logarithm)

F11. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)+C$

F12. $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|+C$ ("long" logarithm)

Rules of integration:

Assume functions u and v depend on argument x, and a,b,C are the constants.

R1. $\int [u(x)+v(x)]dx = \int u(x)dx + \int v(x)dx$

R2. $\int Cu(x)dx = C\int u(x)dx$

R3. If $\int u(x)dx = U(x)+C$, then $\int u(ax)dx = \frac{1}{a}U(ax)+C$

R4. If $\int u(x)dx = U(x)+C$, then $\int u(x+b)dx = U(x+b)+C$

R5. (integration by part) $\int u \cdot dv = u \cdot v - \int v \cdot du$

Integration Methods - Examples:

A. Integration using the change of variable (this is very powerful method, which is the most useful in many situations)

Example 1: Compute $I = \int \sqrt{\sin(x)}\cos(x)dx$

Solution: Note that $\frac{d\sin x}{dx} = \cos x \Rightarrow \cos(x)dx = d\sin(x)$, so we can re-write our integral as:

$I = \int \sqrt{\sin(x)}\cos(x)dx = \int \sqrt{\sin(x)}d\sin(x)$. Lets introduce a new variable $t = \sin(x)$, the we get:

$I = \int t^{1/2}dt = \frac{2t^{3/2}}{3} + C$. Now, return to the original variables, we obtain: $I = \frac{2\sin^{3/2}(x)}{3} + C$

Example 2: Compute $I = \int \frac{xdx}{1+x^4}$

Solution: Note that $\frac{d[x^2]}{dx} = 2x \Rightarrow xdx = \frac{1}{2}d[x^2]$, so we can re-write our integral as:

$I = \int \frac{\frac{1}{2}d[x^2]}{1+x^4}$. Lets introduce a new variable $t = x^2$, the we get:

$I = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}arctg(t) + C$. Now, return to the original variables, we obtain: $I = \frac{1}{2}arctg(x^2) + C$

Example 3: Compute $I = \int \frac{arctg(x)dx}{1+x^2}$

Solution: Note that $\frac{d[arctg(x)]}{dx} = \frac{1}{1+x^2} \Rightarrow \frac{dx}{1+x^2} = d[arctg(x)]$, so we can re-write our integral as:

$I = \int arctg(x)d[arctg(x)]$. Lets introduce a new variable $t = arctg(x)$, the we get:

$I = \int t \cdot dt = \frac{t^2}{2} + C$. Now, return to the original variables, we obtain: $I = \frac{1}{2}[arctg(x)]^2 + C$

B. Integrals of type $I = \int \frac{dx}{ax^2 + bx + c}$

Note that $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 \pm k^2\right]$, where $\pm k^2 = \frac{c}{a} - \frac{b^2}{4a^2}$

Here, the sign in front of k^2 is defined by the sign of the expression $a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$: just make sure k^2 is non-negative. Then, the overall expression reduces to the integral of type $I = \int \frac{dx}{a\left(x + \frac{b}{2a}\right)^2 \pm k^2}$. With the change of variable $t = x + \frac{b}{2a} \Rightarrow dt = dx$, the integral reduces further to $I = \frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$, which can be computed using either F9 or F10.

Example 1: Compute $I = \int \frac{dx}{2x^2 + 4x + 4}$.

Solution: Note that

$2x^2 + 4x + 4 = 2(x^2 + 2x + 2) = 2((x^2 + 2x + 1) - 1 + 2) = 2((x+1)^2 + 1)$, so that the integral can be represented as:

$$I = \int \frac{dx}{2[(x+1)^2 + 1]} = \frac{1}{2}\int \frac{dx}{(x+1)^2 + 1} = | t = x + 1 | = \frac{1}{2}\int \frac{dt}{t^2 + 1} = \frac{1}{2}[arctg(t) + C] =$$

$$\frac{1}{2}arctg(x+1) + C'$$

Example 2: Compute $I = \int \frac{dx}{2x^2 + 4x - 4}$.

Solution: Similar to the example above,

$2x^2 + 4x - 4 = 2\left(x^2 + 2x - 2\right) = 2\left(\left(x^2 + 2x + 1\right) - 1 - 2\right) = 2\left(\left(x+1\right)^2 - 3\right)$, so:

$$I = \int \frac{dx}{2x^2 + 4x - 4} = \frac{1}{2}\int \frac{dx}{(x+1)^2 - 3} = \mid t = x+1, dt = dx \mid = \frac{1}{2}\int \frac{dt}{t^2 - 3} =$$

$$= \frac{1}{2}\left[\frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C\right] = \frac{1}{4\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C' = \frac{1}{4\sqrt{3}}\ln\left|\frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}\right| + C'$$

C. Integrals of type $I = \int \frac{(Ax+B)dx}{ax^2 + bx + c}$

The idea here is analogous to that for integrals of the type $I = \int \frac{dx}{ax^2 + bx + c}$ - we start with the denominator: and represent is as $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 \pm k^2\right]$, where $\pm k^2 = \frac{c}{a} - \frac{b^2}{4a^2}$. Then, we use the change of variables:

$$I = \int \frac{(Ax+B)dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{(Ax+B)dx}{\left(x + \frac{b}{2a}\right)^2 \pm k^2} = \mid t = x + \frac{b}{2a} \mid = \frac{1}{a}\int \frac{\left(A\left(t - \frac{b}{2a}\right) + B\right)dt}{t^2 \pm k^2} =$$

$$= \frac{A}{a}\int \frac{tdt}{t^2 \pm k^2} + \frac{B - \frac{Ab}{2a}}{a}\int \frac{dt}{t^2 \pm k^2}$$

The second integral was already discussed above. To solve the first integral, we use one more change of variables: $y = t^2, dy = 2tdt$, so:

$$\int \frac{tdt}{t^2 \pm k^2} = \frac{1}{2}\int \frac{dy}{y \pm k^2} = \frac{1}{2}\ln\left|y \pm k^2\right| + C = \frac{1}{2}\ln\left|t^2 \pm k^2\right| + C.$$

Example 1: Compute $I = \int \frac{(2x-1)dx}{-2x^2 + 4x + 4}$.

Solution: Note that

$-2x^2+4x+4=-2\left(x^2-2x-2\right)=-2\left(\left(x^2+2x+1\right)-3\right)=-2\left((x+1)^2-3\right)$, so that the integral can be represented as:

$$I=\int\frac{(2x-1)dx}{-2x^2+4x+4}=-\frac{1}{2}\int\frac{(2x-1)dx}{(x+1)^2-3}=\mid t=x+1, dt=dx, x=t-1\mid=$$

$$=-\frac{1}{2}\int\frac{(2t-3)dt}{t^2-3}=-\int\frac{tdt}{t^2-3}+\frac{3}{2}\int\frac{dt}{t^2-3}=\mid y=t^2, dy=2tdt\mid=-\frac{1}{2}\int\frac{dy}{y-3}+\frac{3}{2}\int\frac{dt}{t^2-3}=$$

$$=-\frac{1}{2}\ln|y-3|+\frac{3}{2}\frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C=-\frac{1}{2}\ln\left|(x+1)^2-3\right|+\frac{\sqrt{3}}{4}\ln\left|\frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}\right|+C$$

D. Integrals of type $I=\int\frac{dx}{\sqrt{ax^2+bx+c}}$ are computed using the same idea as above and then, using formulas F11, F12. Just lets consider an example.

Example 1: Compute $I=\int\frac{dx}{\sqrt{3x^2+3x+1}}$

Solution: Note that

$$3x^2+3x+1=3\left(x^2+2\cdot\frac{1}{2}x+\frac{1}{3}\right)=3\left(x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+\frac{1}{3}\right)=3\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{12}\right)$$, so

$$I=\int\frac{dx}{\sqrt{3x^2+3x+1}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{12}}}=\mid t=x+\frac{1}{2}, dx=dt\mid=\frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{t^2+\frac{1}{12}}}=$$

$$=\frac{1}{\sqrt{3}}\ln\left|t+\sqrt{t^2+\frac{1}{12}}\right|+C=\frac{1}{\sqrt{3}}\ln\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{12}}\right|+C$$