Math 421-HW01

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On this week's homework, I collaborated on all the problems with Douglas Marquette. The only resources I used for problem 1 were chapters 1 and 2 of the Spivak textbook and the sample homework document in LaTeX. For the second problem, I used the forum (https:// www.physicsforums.com /threads/ prove-if-x-y-are-reals-that-xy-x-y.925572/) to get inspiration for a proof by cases. On problem 3, I used chapters 1 and 2 of the Spivak textbook as well as this forum (https://math.stackexchange.com/questions/ 1947451/prove-that-if-n-is-an-odd-integer-then-n2-has-a-remainder-of-1-when-divided-by) to inspire my direct proof.

Prove the following statements:

1. If b and c are odd integers and a is an integer, then $ab + ac$ is an even integer

   Proof. If b and c are odd integers and a is an integer, we want to show that $ab + ac$ is an even integer, that is, we want to show $ab + ac = 2k$ for some integer k. By the definition of odd, there exist integers a, b, and c such that $a = x$, $b = 2y + 1$, and $c = 2z + 1$. Using algebra, we obtain:

$$\begin{aligned} ab + bc &= x \cdot (2y+1) + x \cdot (2z+1) \\ &= 2xy + 2xz + 2x \\ &= 2(xy + xz + x) \end{aligned}$$

   $\therefore$ by the closure of the integers under addition and multiplication, $xy + xz + x$ is an integer, call it k. Thus $ab + ac = 2k$ and is therefore an even integer, which was to be proved. □

2. $|x \cdot y| = |x| \cdot |y|$ where x and y are numbers

   Proof. We will use proof by cases.

   Case 1: Both x and y are non-negative. The product of two non-negative numbers, in this case x and y, is non-negative. From this, we can deduce $|x \cdot y| = x \cdot y$ from the left-hand side (LHS). Concerning the right-hand side (RHS), we can similarly deduce in the context of x and y, $|x| = x$ and $|y| = y$, so $|x| \cdot |y| = x \cdot y$. Thus,

$$\begin{aligned} |x \cdot y| &= |x| \cdot |y| \\ RHS &= LHS \end{aligned}$$

   $\therefore |x \cdot y| = |x| \cdot |y|$ when x and y are non-negative

   Case 2: Both x and y are negative. In this case, when both x and y are negative:

$$\begin{aligned} |-x \cdot -y| &= |x \cdot y| \\ |-x| \cdot |-y| &= |x| \cdot |y| \\ |x \cdot y| &= |x| \cdot |y| \end{aligned}$$

Case 3: In this case just one of x or y is negative; Therefore, we get:

$$\begin{aligned}
|x \cdot -y| &= |-xy| = |x \cdot y| \\
|x| \cdot |-y| &= |x| \cdot |y| \\
|-x \cdot y| &= |-xy| = |x \cdot y| \\
|-x| \cdot |y| &= |x| \cdot |y| \\
|x \cdot y| &= |x| \cdot |y|
\end{aligned}$$

$\therefore$ we have proven by cases that the statement $|x \cdot y| = |x| \cdot |y|$ holds for all negative and non-negative cases of x and y. □

3. If x is an odd integer, then x^2 has a remainder of 1 when divided by 8, that is $x^2 = 8c + 1$ for some integer c.

Proof. We will prove this statement via direct proof.

Let x be an odd integer, by definition, x can be written as $2k + 1$ for some integer k. Furthermore, x^2 can be written as:

$$\begin{aligned}
x^2 &= (2k+1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 4k(k+1) + 1
\end{aligned}$$

Let $4k(k + 1) = 8c$ where c is an integer. Through substitution, we get:

$$x^2 = 4k(k+1) + 1 = 8c + 1$$

We know c is an integer because $4k(k + 1)$ is an integer by closure of the integers under addition.
$\therefore$ we have proven directly if x is an odd integer, then x^2 has a remainder of 1 when divided by 8, that is $x^2 = 8c + 1$ for some integer c.

□