Binary Numbers

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Numbers

- The need for counting
- A set of symbols is used to represent quantities
    - Tally marks
    - Roman
    - Arab
    - English

Tally Marks

1	|	6	𝍸 |
2	||	7	𝍸 ||
3	|||	8	𝍸 |||
4	||||	9	𝍸 ||||
5	𝍸	10	𝍸𝍸

Arab/Tally Marks/English

1	𝍷𝍷	two
2	𝍷𝍷𝍷	one
3	𝍷	four
4	𝍸 𝍷	three
5	𝍷𝍷𝍷𝍷	six
6	𝍸	eight
7	𝍸 𝍷𝍷𝍷	five
8	𝍸 𝍷𝍷	seven
9	𝍸 𝍸	ten
10	𝍸 𝍷𝍷𝍷𝍷	twelve
11	𝍸 𝍸 𝍷𝍷	nine
12	𝍸 𝍸 𝍷	eleven

Roman

I	1
II	2
III	3
IV	4
V	5
VI	6
VII	7
VIII	8
IX	9
X	10
XI	11
XII	12
XIII	13
XIV	14
XV	15
XVI	16
XVII	17
XVIII	18
XIX	19
XX	20

XXI	21
XXII	22
XXIII	23
XXIV	24
XXV	25
XXVI	26
XXVII	27
XXVIII	28
XXIX	29
XXX	30
XXXI	31
XXXII	32
XXXIII	33
XXXIV	34
XXXV	35
XXXVI	36
XXXVII	37
XXXVIII	38
XXXIX	39
XL	40

XLI	41
XLII	42
XLIII	43
XLIV	44
XLV	45
XLVI	46
XLVII	47
XLVIII	48
XLIX	49
L	50
LI	51
LII	52
LIII	53
LIV	54
LV	55
LVI	56
LVII	57
LVIII	58
LIX	59
LX	60

LXI	61
LXII	62
LXIII	63
LXIV	64
LXV	65
LXVI	66
LXVII	67
LXVIII	68
LXIX	69
LXX	70
LXXI	71
LXXII	72
LXXIII	73
LXXIV	74
LXXV	75
LXXVI	76
LXXVII	77
LXXVIII	78
LXXIX	79
LXXX	80

LXXXI	81
LXXXII	82
LXXXIII	83
LXXXIV	84
LXXXV	85
LXXXVI	86
LXXXVII	87
LXXXVIII	88
LXXXIX	89
XC	90
XCI	91
XCII	92
XCIII	93
XCIV	94
XCV	95
XCVI	96
XCVII	97
XCVIII	98
XCIX	99
C	100
D	500
M	1000

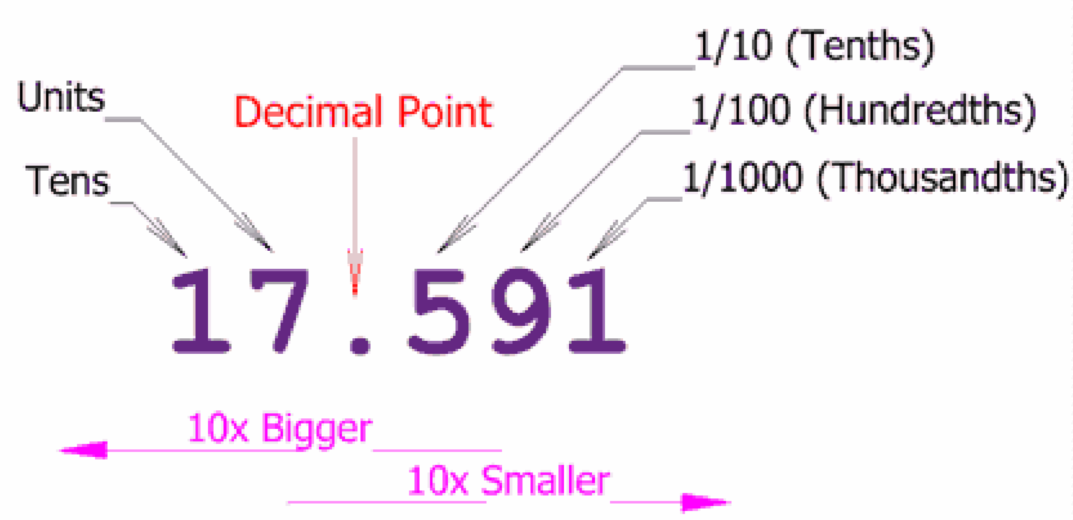
Positional Numeral Systems

- Tally marks and roman numerals use different symbols to represent all the numbers.
- Decimal numeral system is a positional numeral system.
  - The set of symbols is finite (0-9).
  - Their values depend on where they are in a number.
  - For example, the symbol 1 in 123 has a value of 100, but just 1 in 321.

123 ⇨	100
321 ⇨	1

Decimal Numeral System

- Set of symbols: 0, 1, 2, …, 9
- Its base or radix is 10.
- Positional weights:
  - Ones/units digit
  - Tens digit
  - Hundreds digit
  - Thousands digit
  - Etc.



Powers of Ten

- Ten to the powers of x is the number 1 followed by x zeros
- Any number to the power of zero is 1.

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 100$$

$$10^3 = 1000$$

$$10^4 = 10000$$

Decimal Number

Value of the decimal number 1234

$$1234 = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$$

The Zero

- Zero times anything is zero. It does not contribute to the value of a number but it serves as a placeholder.

102	**Tens**
10002	Thousands to tens
10000002	….

Negative Numbers

- A minus sign prefixed to a number

-1234

A General Form for Decimal Numbers

- Base r is 10 for decimals

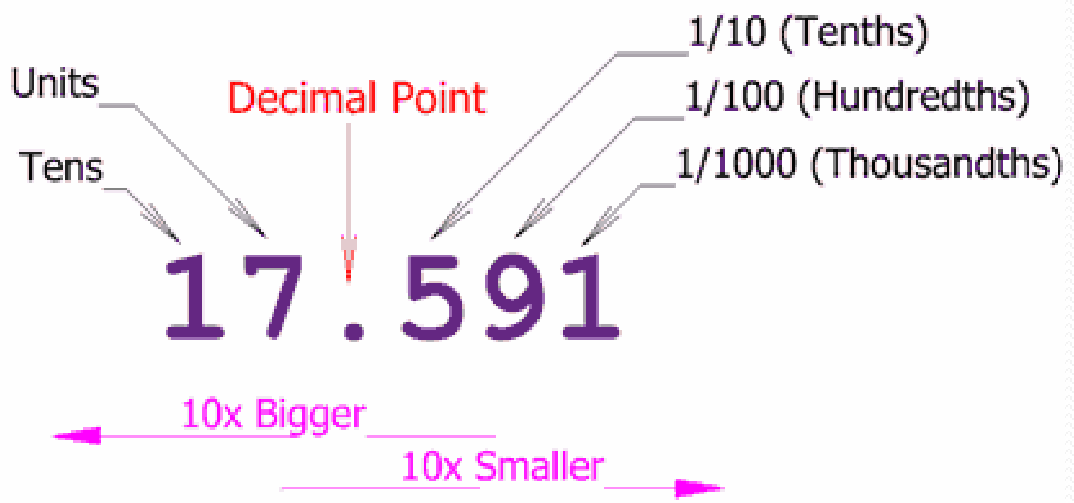
$$a_{n-1}a_{n-2}a_{n-3}\dots a_2a_1a_0 = \sum_{i=0}^{n-1} a_i \times r^i$$

$$= a_0 \times r^0 + a_1 \times r^1 + \cdots + a_{n-1} \times r^{n-1}$$

- Other bases are possible.

Decimal Fractions

- A “decimal point” is used to separated fractions from integrals.



- Repeating decimal like 1/7 may not be represented.

$$\frac{1}{7} = 0.142857242857\ldots$$

Real Numbers

$$a_{n-1}a_{n-2}a_{n-3}\dots a_2a_1a_0.b_0b_1\dots b_{m-1}b_m$$

$$= \sum_{i=0}^{n-1} a_i \times r^i + \sum_{j=0}^{m} b_j \times 10^{-j}$$

$$12.34 = 1 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 4 \times 10^{-2}$$

Binary Numbers

- Base 2
- Set of symbols is {0, 1}
- Powers of 2
  - 1, 2, 4, 8, 16, …

Why Binary?

- Digital circuit components are based on analog transistor circuit.
- Analog circuit is full of uncertainty, i.e., voltage, current are inevitably changing. So logic high is above some threshold, e.g., 2.8 volts. Below 2.8 volts will be logic low.
- Using binary would make a computer system reliable.

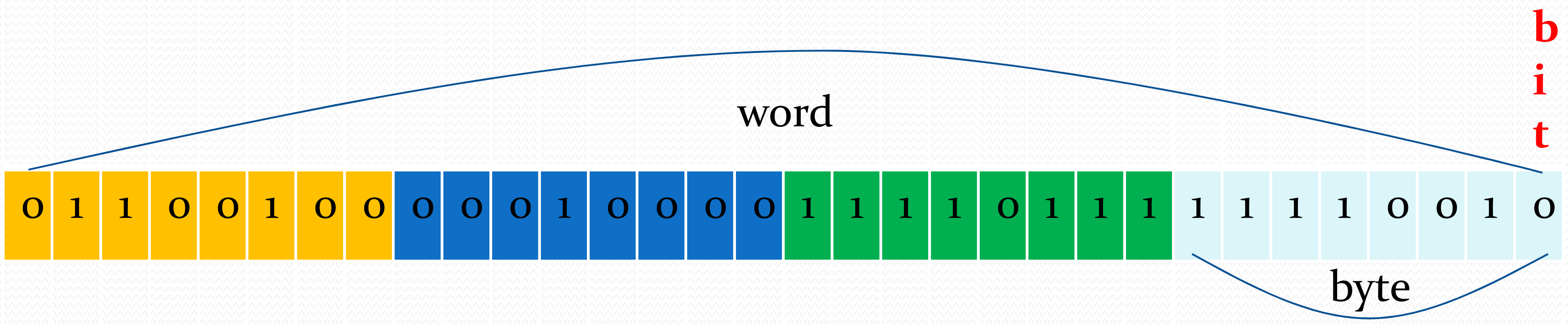
1011

	1	0	1	1
Powers of 2	8	4	2	1
Multiplications	8	0	2	1
Summation	8+2+1=11			

So binary 1011_2 is decimal 11_{10}.

Bits, Bytes, Words

- Units used in binary numeral system.
- Each digit in a binary number is one bit.
- 8 contiguous bits are called one byte.
- Normally, 4 bytes are called one word in Windows.



MSB/LSB

- The fix-sized numbers ease hardware design.
- MSB: most significant bit
- LSB: least significant bit

MSB							LSB
1	0	1	0	1	1	0	0

Powers of 2

2 to the powers of

0	1	
1	2	
2	4	
3	8	
4	16	
5	32	
6	64	
7	128	
8	256	
9	512	
10	1,024	K
20	1,048,576	Mega
30	1,073,741,824	Giga
31	2,147,483,648	
32	4,294,967,296	4 Gigas

Range of Binary Numbers

Number of bits	Range for Unsigned Integers
1	0~1
2	0~3
4	0~15
8	0~255
16	0~65535
32	0~4294967295
64	0~18446744073709551615

Negative Numbers

- One extra bit is reserved for sign. Sing bit zero indicates positive; one denotes negative.
- Sign-and-magnitude: MSB is the sign bit, the rest is the magnitude.
  - Both 0000 and 1000 represent zero in 4-bit system
- Ones' complement: inverse each bits to change the sign.
  - Both 0000 and 1111 represent zero in 4-bit system
- Two's complement: use ones' complement plus one to change sign
  - Only 0000 represents zero. So one more number represented than the other two methods.

Sign-and-Magnitude

- MSB is the sign bit, the rest is the magnitude.
- A natural way
- Use in early computer IBM 7090
- Also used in floating point
- But both 0000 and 1000 represent zero in 4-bit system
- Notations are symmetric to zero

+2	0010
+1	0001
+0	0000
-0	1000
-1	1001
-2	1010

Ones' Complement

- Inverse each bits to change the sign
- Used in early machine PDP-1
- Off by one if sum a positive and a negative
  - 0010+1101=1111, 1111+1=0000
- Both 0000 and 1111 represent zero in 4-bit system

+3	0011
+2	0010
+1	0001
+0	0000
-0	1111
-1	1110
-2	1101
-3	1100

Two's Complement

- Use ones' complement plus one to change sign
- Only 0000 represents zero. So one more number represented than the other two methods.

4	0100
3	0011
2	0010
1	0001
0	0000
-1	1111
-2	1110
-3	1101
-4	1100

Two's Complement Conversion

- Convert a number to binary
- Find ones' complement
- Add one to it
- Example, find two's complement of 4 (0100)

4	decimal
0100	binary
1011	Ones' complement
1011+0001=1100	Plus one

A Quick Way to Find 2's Comp

- Search the first one from right to left
- Flip the rest to the left of the first one

101000110100	**Search the first one from right to left**
101000110100	Flip yellow part
010111001100	1->0, 0->1
010111001100	That's it!

Octal Numerals

- Base r is 8
- Set of symbols is {0, 1, 2, 3, 4, 5, 6, 7}
- Powers of 8
  - 1, 8, 64, 512, …

$$13_8 = 1 \times 8^1 + 3 \times 8^0 = 11_{10}$$

Powers of 8

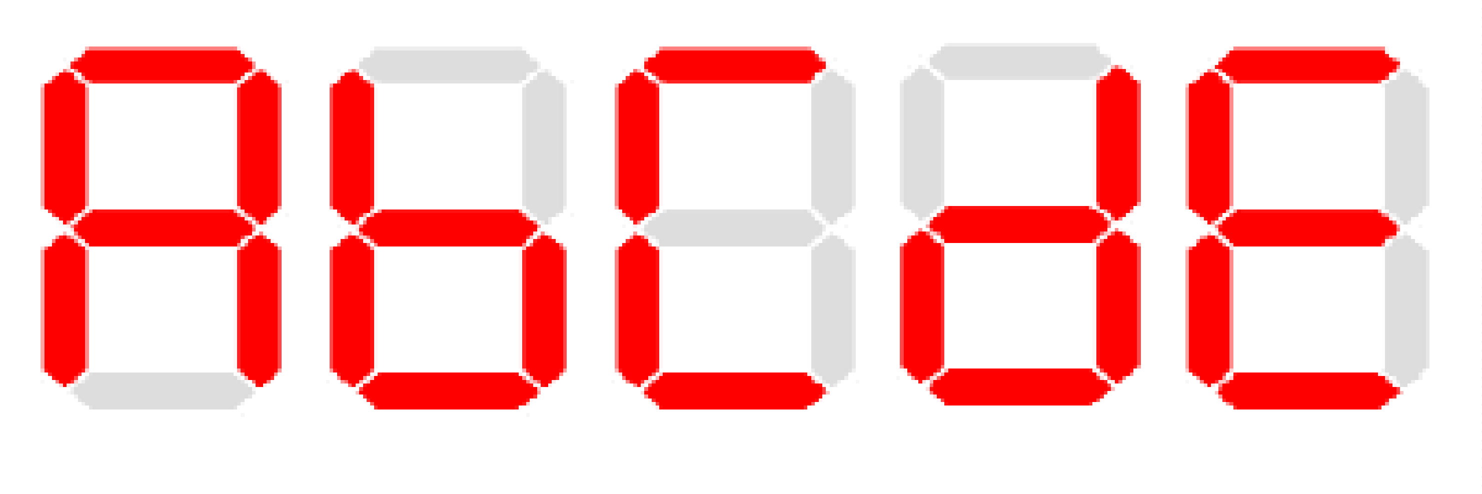
0	1
1	8
2	64
3	512
4	4,096
5	32,768
6	262,144
7	2,097,152
8	16,777,216
9	134,217,728
10	1,073,741,824

Hexadecimal Numerals

- Base r is 16
- Set of symbols is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f}
- Powers of 16
  - 1, 16, 256, 4096,…

$$A1_{16} = A \times 16^1 + 1 \times 16^0 = 10 \times 16^1 + 1 \times 16^0 = 161_{10}$$

7-Segment Display



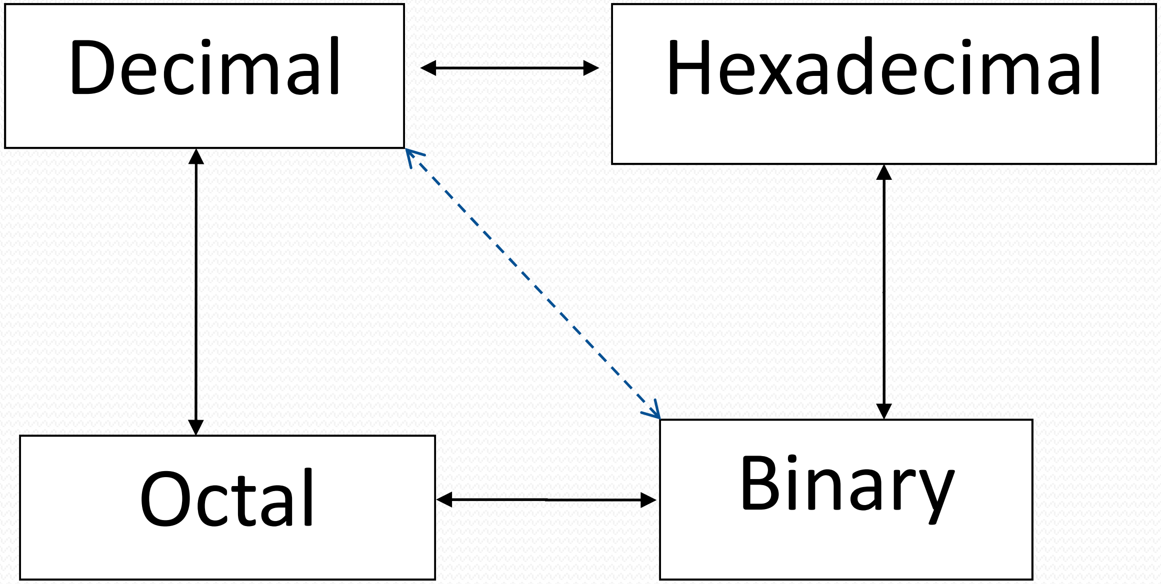
Powers of 16

0	1
1	16
2	256
3	4,096
4	65,536
5	1,048,576
6	16,777,216
7	268,435,456
8	4,294,967,296
9	68,719,476,736
10	1,099,511,627,776

What base should I choose

- A smaller base means a smaller set of symbols is required.
- A larger base will result in more compact representations.

Number Conversion



Binary to Decimal

Convert 1011001010 to decimal (more operations)

- Write down powers of 2 from right to left, starting from 2^0
- Add up the powers for which the binary digits are one

Powers of 2	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Values	512	256	128	64	32	16	8	4	2	1
Binary Digits	1	0	1	1	0	0	1	0	1	0
$a_i \times 2^i$	512	0	128	64	0	0	8	0	2	0
$\sum a_i \times 2^i$	$512 + 128 + 64 + 8 + 2 = 714$									

Why It Works?

- It’s definition.
- When the binary digit is zero, it does not contribute to the value of the number, but it serves as a placeholder.
- When the binary digit is one, the powers of 2 at that position will contribute to the value of the number.

A Better Approach

- Should there are more binary digits, it would require more operations. More operations are error-prone.
- So, a larger radix like 8 or 16 will reduce the total number of calculations.
- From octal or hex to binary is much easier. We will learn it shortly.

Decimal to Hexadecimal

- Convert 1234 to hex
  - Divided by 16 on scratch paper
  - Keep the remainder to the right
  - Keep the quotient to the bottom

16) 1234	2
16) 77	13
4	

  - 1234=4D2

Why It Works?

- 1234 divided by 16, we found there are 77 16's and 2 left. So 2 should be in ones digit.
- Next, 77 divides by 16 (actually 256 from original number), we found 4 256's and 13 left. So 13 should be in 16's digit.
- Since 4 is less than 16, thus, 4 should be just 256's digit.

Hexadecimal to Decimal

- Convert 2A3 to decimal
  - Write down decimals of each digits
  - From right to left, write down powers of 16
  - Multiply each column
  - Sum them together

2	A	3	hex
2	10	3	decimal
256	16	1	Powers of 16
512	160	3	Multiply
512+160+3=675			Sum

Hexadecimal to Binary

- Transcribe each hex digit to 4 binary bits
- Convert 2A3 to binary

2	A	3	hex
0010	1010	0011	Transcribe each digit
0010 1010 0011			Combine them

Hex Digits to Binary

0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

Binary to Hexadecimal

- Convert 1111000000111000 to hex
  - Group 4 digits from right to left, fill in zeros if necessary in the leftmost group
  - Convert each group to a hex digit

0011	1100	0011	1000
3	C	3	8
3C38			

Decimal to Octal

- Convert 1234 to hex
  - Divided by 8 on scratch paper
  - Keep the remainder to the right
  - Keep the quotient to the bottom

8)	1234	2
8)	154	2
8)	19	3
	2	

  - $1234 = 2322_8$

Octal to Decimal

- Convert 2322 to decimal

Powers of 8	8^3	8^2	8^1	8^0
Values	512	64	8	1
Octal Digit	2	3	2	2
$b_i \times 8^i$	1024	192	16	2
$\sum b_i \times 8^i$	$1024 + 192 + 16 + 2 = 1234$			

Octal to Binary

- Convert 3456 to binary

3	4	5	6
↓	↓	↓	↓
011	100	101	110

Octal Digits to Binary

octal	binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Binary to Octal

- Group-of-three approach
  - Group 3 binary digits from right to left
  - Transcribe each group to an octal digit

010	011	111	100
2	3	7	4

Octal to Decimal Real Numbers (Fraction)

- Convert 1.234_8 to decimal (other radix is similar)
  - Each octal digit followed by the period has a negative power of 8, starting from 8^{-1}
  - Other than that the procedure is similar to the integral part

Powers of 8	8^0	.	8^{-1}	8^{-2}	8^{-3}
Fraction	1		$\frac{1}{8}$	$\frac{1}{64}$	$\frac{1}{512}$
Octal Digits	1		2	3	4
$b_j \times 8^{-j}$	1		$\frac{2}{8}$	$\frac{3}{64}$	$\frac{4}{512}$
$\sum a_i + \sum b_j$	$1+\frac{2}{8}+\frac{3}{64}+\frac{4}{512} = 1+\frac{128}{512}+\frac{24}{512}+\frac{4}{512} = 1+\frac{156}{512} = 1.3046875$				

Decimal Real Number to Octal (Fraction)

- Convert decimal 0.3046875 to octal (other radix is similar)
  - Times base/radix and keep the integral digit
  - Until the fraction part is zero

Fraction	Times Radix	Multiplication Results	Integral Digit
0.03046875	$\times 8$	2.4375	2
0.4375	$\times 8$	3.5	3
0.5	$\times 8$	4.0	4

Questions