TP n°1: Computational Statistics

$\boxed{\text{Exo 1}}$ R rv with Rayleigh distrib. (1) ~~(?)~~ $\Theta \sim \mathcal{U}([0, 2\pi])$
R, Θ indep

(1) We use the transfer theorem that states that

$\forall h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous bounded, $E(h(X,Y)) = \int_{\mathbb{R}\times\mathbb{R}} h(x,y) f(x) f(y) dx dy$
where f is the density of $\mathcal{N}(0,1)$

Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous bounded.

$$E(h(X,Y)) = E(R\cos\Theta, R\sin\Theta) = \int_{\mathbb{R}\times\mathbb{R}} h(r\cos\theta, r\sin\theta) \underbrace{f_R(r) f_\Theta(\theta)}_{\text{independancy}} dr d\theta$$

$$= \int_{\mathbb{R}_+ \times [0, 2\pi]} h(r\cos\theta, r\sin\theta)\, r e^{-r^2/2} \times \frac{1}{2\pi} dr d\theta$$

we then change variables by considering $g : \begin{cases} \mathbb{R}_+ \times [0, 2\pi] \to \mathbb{R} \times \mathbb{R} \\ (r, \theta) \longmapsto (r\cos\theta, r\sin\theta) \end{cases}$
and we have

$$|\mathrm{Jac}(g)(r,\theta)| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r = \sqrt{x^2+y^2} \text{ where } x = r\cos\theta ; y = r\sin\theta$$

Consequently, by considering $x = r\cos\theta$, $y = r\sin\theta$, we have

$$E_{X,Y}[h(X,Y)] = \int_{\mathbb{R}^2} h(x,y) \sqrt{x^2+y^2}\, e^{-\frac{x^2+y^2}{2}} \times \frac{1}{2\pi} \times \frac{1}{\sqrt{x^2+y^2}} dx dy$$

$$= \int_{\mathbb{R}^2} h(x,y) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dx dy$$

Cl $E_{X,Y}[h(X,Y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dx dy$,

which shows that $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$

(2) we have to find how to sample R. Let $\begin{cases} r \in \mathbb{R}_+ \\ u \in [0,1] \end{cases}$

$$F_R(r) = u \iff \underbrace{\int_0^r x e^{-x^2/2} dx}_{\left[-e^{-r^2/2}\right]_0^r} = u \iff 1 - e^{-r^2/2} = u \iff r = \sqrt{-2\ln(1-u)}$$

we thus only need to sample from a uniform distribution in $[0,1]$ and then take $\sqrt{-2\ln(1-u)}$ to sample r.

$\mathcal{C}$ · To sample $\mathcal{N}(0,1)$, we

$u =$ uniform $(0,1)$

$\theta =$ uniform $(0, 2\pi)$

$x = \sqrt{-2\ln(1-u)} \cos(\theta)$

return x

(3) a) we have $\left.\begin{matrix} 2U_1 - 1 \sim \mathcal{U}([-1,1]) \\ 2U_2 - 1 \sim \mathcal{U}([-1,1]) \end{matrix}\right\}$ indep.

At the end of the loop, we know that $V_1^2 + V_2^2 \leq 1$, which is equivalent to $\|V\|_2^2 \leq 1$ where $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$. we know

that this means that V has to be in a disk of $r = 1$. Also, as U_1 and U_2 are distributed uniformly, we can conclude that (V_1, V_2) is distributed uniformly on the disk of radius 1.

b) With the same reasoning as before, we have

$$\mathbb{P}(\text{ending the loop at stage 1}) = \mathbb{P}\left(\text{be in the disk of radius 1} \;\middle|\; \begin{matrix} U_1 \sim \mathcal{U}([0,1]) \\ U_2 \sim \mathcal{U}([0,1]) \end{matrix}\right)$$

$$= \frac{\text{area of disk } r=1}{\text{area of square 1}} = \frac{\pi \times 1^2}{2 \times 2} = \frac{\pi}{4}$$

$$\mathbb{P}\left(\substack{\text{ending the loop} \\ \text{at stage } n}\right) = \mathbb{P}\left(\substack{\text{loop reaches} \\ \text{stage } n} \cap \substack{\text{loop gets out} \\ \text{at stage } n}\right)$$

$$= \mathbb{P}\left(\substack{\text{get out at} \\ \text{stage } n} \,\middle|\, \substack{\text{didn't get out} \\ \text{at stage } 1 \dots n-1}\right) \underbrace{\mathbb{P}\left(\substack{\text{didn't get out on} \\ \text{stage } 1 \dots n-1}\right)}_{\text{independancy between stages}}$$

$$= \frac{\pi}{4} \times \left(1 - \frac{\pi}{4}\right)^{n-1}$$

(b) This follows a geometric law of parameter $\pi/4$; we know that the expected number of steps is $\boxed{\frac{4}{\pi}}$

c) we use the transfer theorem exactly as in (1). Let $h : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuous bounded.

$$E\, h(T,V) = E_{V_1,V_2}\left[h\left(\frac{V_1}{\sqrt{V_1^2+V_2^2}}, V_1^2 + V_2^2\right)\right]$$

$$= \int_{\mathbb{R}\times\mathbb{R}} h\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, v_1^2+v_2^2\right) \underbrace{f_{(V_1,V_2)}(v_1,v_2)}_{\substack{\text{uniform distribution on the} \\ \text{disk } r=1 \,\to\, \frac{1}{\pi}\mathbb{1}_{D_1}(V_1,V_2)}} dv_1\, dv_2$$

$$= \int_{D_1^+} h\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, v_1^2+v_2^2\right) \times \frac{1}{\pi} dv_1\, dv_2 + \int_{D_1^-} h\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, v_1^2+v_2^2\right) \times \frac{1}{\pi} dv_1\, dv_2$$

On pose $\Phi : \begin{cases} [-1,1] \times [-1,1] \longrightarrow [-1,1] \times [0,1] \\ (v_1, v_2) \longmapsto \left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, v_1^2+v_2^2\right) \end{cases}$ $\underbrace{\text{sur } D_1^+}$ bijective d'inverse

$$\Phi_1^{-1} : \begin{cases} [-1,1] \times [0,1] \longrightarrow D_1^+ \\ (\theta, r) \longmapsto \left(\theta\sqrt{r}, \sqrt{r(1-\theta^2)}\right) \end{cases}$$

On a

$$\mathrm{Jac}(\Phi_1^{-1})(\theta,r) = \begin{bmatrix} \sqrt{r} & \frac{\theta}{2\sqrt{r}} \\ \frac{-\theta\sqrt{r}}{\sqrt{1-\theta^2}} & \frac{\sqrt{1-\theta^2}}{2\sqrt{r}} \end{bmatrix}$$

D'où $\det \mathrm{Jac}(\Phi_1^{-1})(\theta,r) = \frac{\sqrt{1-\theta^2}}{2} + \frac{\theta^2}{2\sqrt{1-\theta^2}} = \frac{1}{2} \times \frac{1}{\sqrt{1-\theta^2}}$

De même

$$\Phi_2^{-1} : \begin{cases} D_1^- \longrightarrow [-1,1] \times [-1,1] \\ (\theta, r) \longmapsto \left(\theta\sqrt{r}, -\sqrt{r(1-\theta^2)}\right) \end{cases}$$ bijection sur D_1^- et

$$\det \mathrm{Jac}(\Phi_2^{-1})(\theta,r) = \sqrt{r} \times \frac{-\sqrt{1-\theta^2}}{2\sqrt{r}} - \frac{\theta}{2\sqrt{r}} \times \frac{\sqrt{r}\times\theta}{\sqrt{1-\theta^2}}.$$

$$= -\frac{\sqrt{1-\theta^2}}{2} - \frac{\theta^2}{2\sqrt{1-\theta^2}}$$

$$= \frac{-1}{2\sqrt{1-\theta^2}}$$

D'où

$$E\,h(T_1,V) = \int_{[-1,1]\times[0,1]} h(t,v) \times \frac{1}{2\sqrt{1-t^2}} \times \frac{1}{\pi}\,dt\,dv + \int_{[-1,1]\times[0,1]} h(t,v) \frac{1}{2\sqrt{1-t^2}} \frac{1}{\pi}\,dt\,dv$$

$$= \int_{[-1,1]\times[0,1]} h(t,v) \frac{1}{\sqrt{1-t^2}} \times \frac{1}{\pi}\,dt\,dv$$

Let's take $\theta \sim \mathcal{U}([0,2\pi])$ and take $Y = \cos\theta$.

$F_Y(y) = P(\cos\theta \le y) = P(\theta \le \arccos y) + P(\theta \ge 2\pi - \arccos y)$

$= 2\arccos y$

$$f_Y(y) = \frac{-2}{\sqrt{1-y^2}} \mathbb{1}_{[-1,1]}(y)$$

By taking $f_Y(y) = \frac{1}{\sqrt{1-y^2}} \mathbb{1}_{[-1,1]}(y)$ and $f_T(t) = \frac{\mathbb{1}_{[-1,1]}(t)}{\sqrt{1-t^2}}$ we have

$$E\,h(T_1,V) = \int_{[-1,1]\times[0,1]} h(t,v) \frac{\mathbb{1}_{[-1,1]}(t)}{\sqrt{1-t^2}} \times \mathbb{1}_{[0,1]}(v)\,dv.$$

(c) T_1, V are independant, $T_1 \sim \cos\theta$ and $V \sim \mathcal{U}([0,1])$

$T_2 = \pm\sqrt{1-\cos^2\theta} = \pm|\sin\theta| = \sin\theta.$

$(T_1, T_2) \sim (\cos\theta, \sin\theta)$ and $\theta \sim \mathcal{U}([0,2\pi])$

d) The distribution of the output (X, Y) is $(R\cos\Theta, R\sin\Theta)$ and S is a realisation of R. We know the distribution of $(\cos\Theta, \sin\Theta)$ with precedent questions

$\boxed{\text{Exo 2}}$

(1) Let $x \in [0,1]$

- if $x \neq \frac{1}{k}$, we have $\begin{cases} X_{n+1} = \frac{1}{k+1} \text{ with proba } 0 \\ X_{n+1} \sim \mathcal{U}([0,1]) \text{ with proba } 1 \end{cases}$,

Consequently, in this case, $P(x,A) = \mathbb{P}(X_{n+1} \in A \mid X_{n+1} \sim \mathcal{U}([0,1]))$

$$= |A \cap [0,1]|$$
$$= \int_{A \cap [0,1]} dt$$

- if $x = \frac{1}{k}$, we have $\begin{cases} X_{n+1} = \frac{1}{k+1} \text{ with proba } 1 - X_n^2 \\ X_{n+1} \sim \mathcal{U}([0,1]) \text{ ——— } X_n^2 \end{cases}$

consequently,

$$P(x,A) = (1-x^2)\,\mathbb{P}(X_{n+1} \in A \mid X_{n+1} = \tfrac{1}{k+1}) + x^2\,\mathbb{P}(X_{n+1} \in A \mid X_{n+1} \sim \mathcal{U}([0,1]))$$

$$= (1-x^2)\,\delta_{\frac{1}{k+1}}(A) + x^2 \times \int_{A \cap [0,1]} dt$$

(2) we will show that $\pi P = \pi$

Let $x \in [0,1]$. we have $\pi(A) = \int_A \mathbb{1}_{[0,1]}(t)\,dt$

we want to prove that $\int_0^1 P(x,A)\,dx = \int_{A \cap [0,1]} dx$

we have

$$\int_0^1 P(x,A)\,dx = \int_{[0,1] \setminus \{\frac{1}{k}, k \in \mathbb{N}^*\}} P(x,A)\,dx + \underbrace{\int_{\{\frac{1}{k}, k \in \mathbb{N}^*\}} P(x,A)\,dx}_{=0}$$

" 0 because discrete values and as x is uniformly distributed on $[0,1]$, the probability of having $\frac{1}{k}$ is 0.

$$= \int_{A \cap [0,1] \setminus \{\frac{1}{k}, k \in \mathbb{N}^*\}} P(x,A)\,dx = \int_{A \cap [0,1]} dx$$

② we have $\pi P = \pi$, which shows that if we take $X_0 \sim \pi$,

$X_n \sim \pi$ $\forall n \in \mathbb{N}^*$, and it shows the property.

(3) Let $x \notin \{\frac{1}{k}, k \in \mathbb{N}^*\}$

f bounded measurable

$$Pf(x) = \mathbb{E}(f(X_1) \mid X_0 = x) = \int_{[0,1]} f(y) \times \underbrace{f_{X_1 \mid X_0 = x}(y)}_{X_1 \sim \mathcal{U}([0,1]) \to 1} dy$$

$$= \int_{[0,1]} f(y) \times 1 \, dy = \boxed{\int_{[0,1]} f(y) \, dy}$$

$$= \int_{[0,1]} f(y) \pi(y) \, dy$$

Let's show by a recursive reasoning that $P^n f(x) = \int_{[0,1]} f(y) \pi(y) \, dy$ for $n \geq 1$

* for $n = 1$, ok (above)
* suppose the property for $n \in \mathbb{N}^*$. we have

$$P^{n+1} f(x) = \mathbb{E}(P^n f(X_1) \mid X_0 = x)$$

$$= \int_{[0,1]} \underbrace{P^n f(z)}_{\text{hyp} : \int_{[0,1]} f(y)\pi(y) dy} \times \overbrace{f_{X_1 \mid X_0 = x}(z)}^{= \pi(z) = 1} dz$$

$$= \int_{[0,1]} \int_{[0,1]} f(y) \pi(y) \, dy \, \pi(z) dz$$

$$= \boxed{\int_{[0,1]} f(y) \pi(y) \, dy}$$

cc

$$P^n f(x) = \int f(y) \pi(y) \, dy \quad \text{for all } n \geq 1$$

(4) a. Let's show with recursivity on $n \in \mathbb{N}^*$ that

$$P^n\left(x, \frac{1}{n+k}\right) = \prod_{m=1}^{n} \left(1 - \frac{1}{(m+k-1)^2}\right)$$

* for $n=1$, $P\left(x, \frac{1}{k+1}\right) = \left(1 - \frac{1}{k^2}\right) \times \delta_{\frac{1}{k+1}}\left(\frac{1}{k+1}\right) + \frac{1}{k^2} \times \underset{\text{uniform distrib}}{0}$

$$= 1 - \frac{1}{k^2}$$

* Suppose it true for a certain $n \in \mathbb{N}^*$. we have

$$P^{n+1}\left(x, \frac{1}{n+k+1}\right) = \int_{[0,1]} P^n\left(\frac{1}{k}, z\right) P\left(z, \frac{1}{n+k+1}\right) dz$$

if $z \neq \frac{1}{n+k}$, we have $P\left(z, \frac{1}{n+k+1}\right) = 0$ as integrating on a single value equals 0.

if $z = \frac{1}{n+k}$, $P^{n+1}\left(x, \frac{1}{n+k+1}\right) = \underbrace{P^n\left(\frac{1}{k}, \frac{1}{n+k}\right)}_{\text{hyp.}} \underbrace{P\left(\frac{1}{n+k}, \frac{1}{n+k+1}\right)}_{1 - \frac{1}{(n+k)^2} \text{ as shown initially}}$

$$= \prod_{m=1}^{n} \left(1 - \frac{1}{(m+k-1)^2}\right) \times \left(1 - \frac{1}{(n+k)^2}\right)$$

$$= \prod_{m=1}^{n+1} \left(1 - \frac{1}{(m+k-1)^2}\right)$$

Cl $P^n\left(x, \frac{1}{n+k}\right) = \prod_{m=1}^{n} \left(1 - \frac{1}{(m+k-1)^2}\right)$

b. we have $\pi(A) = 0$ as A is a discrete set.

moreover,

$$P^n\left(\frac{1}{k}, A\right) = \sum_{q=0}^{+\infty} \underbrace{P^n\left(\frac{1}{k}, \frac{1}{k+1+q}\right)}_{0 \text{ if } q \neq n-1} = \underbrace{P^n\left(\frac{1}{k}, \frac{1}{n+k}\right)}_{\text{expression on (a)}} \xrightarrow[n \to +\infty]{} 0$$

Cl. $\boxed{P^n(x, A) \xrightarrow[n \to +\infty]{} \pi(A)}$