$\boxed{\text{Exo 1}}$ for $c \in \mathbb{R}^d$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times d}$, we want to solve

$\min_x \; c^T x$ st $\begin{matrix} Ax = b \\ x \geq 0 \end{matrix}$ (P) and $\min_y \; b^T y$ st $A^T y \leq c$ (D)

(1) The dual problem can be written as

$L(\lambda, \nu) = \min_x L(x, \lambda, \nu) = \min_x \left[c^T x + \nu^T (Ax - b) - \lambda^T x \right]$

however, we have $\dfrac{\partial L(x,\lambda,\nu)}{\partial x} = c^T + \nu^T A - \lambda^T$

$\dfrac{\partial L(x,\lambda,\nu)}{\partial x} = 0 \iff c^T + \nu^T A - \lambda^T = 0$

As L is linear in x, we have

$L(\lambda, \nu) = \begin{cases} -\nu^T b & \text{if } A^T \nu - \lambda + c = 0 \\ -\infty & \text{if } A^T \nu - \lambda + c \neq 0 \end{cases}$

(1) we want to maximise $-\nu^T b$ with $\begin{matrix} A^T \nu - \lambda + c = 0 \\ \lambda \succeq 0 \end{matrix}$

ie $\boxed{\max_\nu \; -b^T \nu \text{ st } A^T \nu + c \succeq 0}$

(2) The dual problem can be written as

$L(\lambda) = \min_y L(\lambda, y) = \min_y \left[-b^T y + \lambda^T (A^T y - c) \right]$

$= \min_y (-b^T + \lambda^T A^T) y - \lambda^T c$

$= \begin{cases} -\lambda^T c & \text{if } -b^T + \lambda^T A^T = 0 \\ -\infty & \text{otherwise} \end{cases}$ with $\lambda \succeq 0$

$L(\lambda, y)$ linear in y

(2) we want to minimise $+\lambda^T c$ st $\begin{cases} -b^T + \lambda^T A^T = 0 \\ \lambda \succeq 0 \end{cases}$ ie $\boxed{\min_\lambda \; + c^T \lambda \text{ st } \begin{cases} -b + A\lambda = 0 \\ \lambda \succeq 0 \end{cases}}$

(3) The dual problem can be written as

$$L(\lambda_1, \lambda_2, \nu) = \min_{x,y} \; c^T x - b^T y + \nu^T (Ax - b) - \lambda_1^T x + \lambda_2^T (A^T y - c)$$

$$= \min_{x,y} \; (c^T + \nu^T A - \lambda_1^T) x + (\lambda_2^T A^T - b^T) y - \nu^T b - \lambda_2^T c$$

$$\underset{L \text{ linear in } x \text{ and } y}{=} \begin{cases} -\nu^T b - \lambda_2^T c & \text{if } c^T + \nu^T A = \lambda_1^T \text{ and } b^T = \lambda_2^T A^T \\ -\infty & \text{otherwise} \end{cases}$$

(l) we want to maximise

$$\min_{\nu, \lambda_1, \lambda_2} \; -\nu^T b - \lambda_2^T c \quad \text{st} \begin{cases} c^T + \nu^T A = \lambda_1^T \\ \lambda_1 \succcurlyeq 0 \quad \lambda_2^T \succcurlyeq 0 \\ A \lambda_2 = b \end{cases}$$

The condition on λ_1 can be forgotten as λ_1 does not appear in the min, we just have to keep $A^T \nu + c \succcurlyeq 0$. Finally, maximising on ν is the same as maximising on $-\nu$. We can thus write that our problem is equivalent to

$$\boxed{\min_{\nu, \lambda_2} \; b^T \nu - c^T \lambda_2 \quad \text{st} \begin{cases} c \succcurlyeq A^T \nu & \lambda_2 \succcurlyeq 0 \\ A \lambda_2 = b \end{cases}}$$

(l The problem is self-dual.

(4) The problem (Self-Dual) is separable in x and y. Indeed, as the problem is bounded and feasible, we can separate the conditions in (Self-Dual), giving

$$(\text{Self-Dual}) \iff \min_x c^T x + \min_y -b^T y \quad \text{st} \begin{cases} Ax = b \\ x \succcurlyeq 0 \\ A^T y \preccurlyeq c \end{cases}$$

; which shows that

(Self-Dual) is equivalent to solving (P) and (D)

As the problem is feasible, we have shown during the course that strong duality holds for (D) we thus have

$$\max_{y} b^T y = \min_{\lambda} +c^T\lambda \ \text{st} \begin{cases} -b + A\lambda = 0 \\ \lambda \geq 0 \end{cases}$$
st $A^T y \leq c$

which concludes

(c) the optimal value for (Self-Dual) is 0

$\boxed{\text{Exo 2}}$ For $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, we want to solve $\min_x \|Ax-b\|_2^2 + \|x\|_1$

(1) Let $x, y \in \mathbb{R}^d$, we want to find $f^*(y) = \sup_z (y^T z - \sum |z_i|)$
we have

$$y^T x - \sum_{i=1}^d |x_i| = \sum_{i=1}^d x_i y_i - \sum_{i=1}^d |x_i|$$

* Case n°1 : $\exists i_0 \in [\![1,d]\!] : y_{i_0} > 1$. Then we take $x_i = \begin{cases} n \text{ if } i = i_0 \\ 0 \text{ otherwise} \end{cases}$
and we have

$$y^T x - \sum_{i=1}^d |x_i| = y_{i_0} x_{i_0} - x_{i_0} = n(\underbrace{y_{i_0} - 1}_{>0}) \xrightarrow[n \to +\infty]{} +\infty$$

* Case n°2 : $\exists i_0 \in [\![1,d]\!] : y_{i_0} < -1$. We take $x_i = \begin{cases} -n \text{ if } i = i_0 \\ 0 \text{ otherwise} \end{cases}$
and we have

$$y^T x - \sum_{i=1}^d |x_i| = -n y_{i_0} - n = -n(\underbrace{y_{i_0} + 1}_{<0}) \xrightarrow[n \to +\infty]{} +\infty$$

* Case n°3 : $\forall i \in [\![1,d]\!]$, $|y_i| \leq 1$ then

$$\sum_{i=1}^d x_i y_i - \sum_{i=1}^d |x_i| = \sum_{i=1}^d \underbrace{\overbrace{x_i y_i}^{\leq |x_i|} - |x_i|}_{\leq 0} \leq \sum_{i=1}^d 0 \leq 0$$
. As 0 is reached by choosing $x = 0$, we have $f^*(y) = 0$ in this case

Cl. $f^*(y) = \begin{cases} +\infty \text{ if } \|y\|_\infty > 1 \\ 0 \text{ if } \|y\|_\infty \leq 1 \end{cases}$

(2) Let's compute the dual of (RLS)
we have

$$(RLS) \iff \inf_{x,y} \|y\|_2^2 + \|x\|_1 \quad st \quad y = Ax - b.$$

Now

$$L(\nu) = \inf_{x,y} \|y\|_2^2 + \|x\|_1 + \nu^T(y - Ax + b)$$

$$= \inf_x (\|x\|_1 - \nu^T A x) + \inf_y (\|y\|_2^2 + \nu^T y) + \nu^T b$$

$$= \underbrace{-\sup_x (\nu^T A x - \|x\|_1)}_{= 0 \text{ if } \|\nu^T A\|_\infty \leq 1 \text{ ; } +\infty \text{ otherwise with } (1)} + \inf_y (\|y\|_2^2 + \nu^T y) + \nu^T b.$$

The function $g : y \mapsto \|y\|_2^2 + \nu^T y$ is convex and differentiable

$$\nabla g(y) = 2y + \nu = 0 \iff y = -\frac{1}{2}\nu$$

we thus have

$$L(\nu) = \begin{cases} \frac{1}{4}\|\nu\|_2^2 + (-\frac{1}{2})\|\nu\|_2^2 + \nu^T b & \text{if } \|\nu^T A\|_\infty \leq 1 \\ +\infty & \text{if } \|\nu^T A\|_\infty > 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{4}\|\nu\|_2^2 + \nu^T b & \text{if } \|A^T \nu\|_\infty \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

The dual problem is thus

$$\boxed{\max_\nu \; -\frac{1}{4}\|\nu\|_2^2 + \nu^T b \quad \text{if } \|A^T\nu\|_\infty \leq 1}$$

Exo3

(1) we consider $\min_{w,z} \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2}\|w\|_2^2$ st $\begin{matrix} z_i \geq 1 - y_i (w^T x_i) \ \forall i \\ z \geq 0 \end{matrix}$

$$\min_w \frac{1}{n}\sum_{i=1}^n \ell(w, x_i, y_i) + \frac{\tau}{2}\|w\|_2^2 = \tau \min_{w,z} \frac{1}{\tau n}\sum_{i=1}^n \max\{0, 1 - y_i w^T x_i\} + \frac{1}{2}\|w\|_2^2$$

$$= \tau \min_{w,z} \frac{1}{\tau n}\sum_{i=1}^n z_i + \frac{1}{2}\|w\|_2^2 \text{ st } z_i = \max\{0, 1 - y_i w^T x_i\}$$

$$= \tau \min_{w,z} \frac{1}{\tau n} \mathbb{1}^T z + \frac{1}{2}\|w\|_2^2 \text{ st } z_i = \max\{0, 1 - y_i w^T x_i\}$$

As we have $z_i \geq 0 \ \forall i \in [1, n]$, if we consider a such that

$\exists i_0 : a_{i_0} > \max\{0, 1 - y_{i_0} w^T x_{i_0}\}$, then we will have

$\frac{1}{\tau n} \mathbb{1}^T a < \frac{1}{\tau n} \mathbb{1}^T b$ where $\begin{cases} b \text{ respects the conditions} \\ b = \begin{cases} a_i & \text{for } i \neq i_0 \\ \max\{0, 1 - y_i w^T x_i\} & \text{if } i = i_0 \end{cases} \end{cases}$

we thus have

$$\min_w \frac{1}{n}\sum \ell(w, x_i, y_i) + \frac{\tau}{2}\|w\|_2^2 = \tau \min_{w,z} \frac{1}{\tau n}\mathbb{1}^T z + \frac{1}{2}\|w\|_2^2 \text{ st } \begin{cases} z \geq 0 \\ z_i \geq 1 - y_i w^T x_i \end{cases}$$

(2) Sep 2 is equivalent to Sep 1, we just have a τ multiplicator added.

(2) we consider

$$L((\lambda_i)_{i \in [1,n]}, \pi) = \min_{w,z} \frac{1}{n\tau}\mathbb{1}^T z + \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^n \lambda_i (1 - y_i w^T x_i - z_i) - \pi^T z$$

$$= \min_{w,z} \left[\frac{1}{n\tau}\mathbb{1} - \lambda - \pi\right]^T z + \min_w \left[\frac{1}{2}\|w\|_2^2 - \sum_i \lambda_i y_i w^T x_i\right] + \sum_i \lambda_i$$

$$= \begin{cases} \min_w \frac{1}{2}\|w\|_2^2 - \sum_{i=1}^n \lambda_i y_i w^T x_i + \sum \lambda_i & \text{if } \frac{1}{n\tau}\mathbb{1} - \lambda - \pi = 0 \\ -\infty & \text{otherwise} \end{cases}$$

we consider $g : w \mapsto \frac{1}{2}\|w\|_2^2 - \sum_i \lambda_i y_i x_i^T w$

g is differentiable and convex and we have

$\nabla g(w) = w - \sum_i \lambda_i y_i x_i = 0 \Leftrightarrow w = \sum_i \lambda_i y_i x_i$.

Consequently, we have

$$g(w_{min}) = \frac{1}{2}\Big(\sum_i \lambda_i y_i x_i\Big)^T\Big(\sum_j \lambda_j y_j x_j\Big) - \sum_i \lambda_i y_i \Big(\sum_j \lambda_j y_j x_j\Big)^T x_i$$

$$= \frac{1}{2}\sum_{1\leq i,j\leq n} \lambda_i y_i \lambda_j y_j x_i^T x_j - \sum_{1\leq i,j\leq n} \lambda_i \lambda_j y_i y_j x_j^T x_i$$

$$= -\frac{1}{2}\sum_{1\leq i,j\leq n} \lambda_i \lambda_j y_i y_j x_i^T x_j .$$

(c) The dual problem of (Sep 2) is

$$\max_{\lambda,\pi} \sum_{i=1}^n \lambda_i - \frac{1}{2}\sum_{1\leq i,j\leq n} \lambda_i \lambda_j y_i y_j x_i^T x_j \quad \text{st} \begin{cases} \frac{1}{n\tau}\mathbb{1} - \lambda - \pi = 0 \\ \lambda \succcurlyeq 0 \\ \pi \succcurlyeq 0 \end{cases}$$

ie

$$\max_{\lambda} \sum_{1\leq i\leq n} \lambda_i - \frac{1}{2}\sum_{1\leq i,j\leq n} \lambda_i \lambda_j y_i y_j x_i^T x_j \quad \text{st} \quad 0 \preccurlyeq \lambda \preccurlyeq \frac{1}{n\tau}\mathbb{1}$$

$\boxed{\text{Exo 4}}$ we consider (K) $\min_x c^T x$ st $\sup_{a \in S} a^T x \leq b$ with $S = \{a : C^T a \preceq d\}$

As shown by the hint, we will first focus on $\sup_a a^T x$ st $C^T a \preceq d$ (Q)

In this case, as we suppose the problem feasible, we have $p^* = d^*$ (strong duality) as shown during class.

The dual function of (Q) is $\left[\text{as } \sup_a a^T x = \inf_a -a^T x\right]$

$$g(\lambda) = \inf_a (-x^T a + \lambda^T C^T a - \lambda^T d) \underset{\substack{\uparrow \\ \text{linear in } a}}{=} \inf_a (-x^T + \lambda^T C^T) a - \lambda^T d$$

$$= \begin{cases} -\lambda^T d & \text{if } C\lambda = x \\ -\infty & \text{otherwise} \end{cases}$$

dual problem: maximize $-d^T \lambda$ st $C\lambda = x$, $\lambda \succeq 0$

Q. dual problem: minimize $d^T z$ st $Cz = x$, $z \succeq 0$ (D_x)

we have

$$(K) \iff \min_x c^T x \quad \text{st} \quad \inf_z d^T z \leq b, \; Cz = x, \; z \succeq 0$$

Let $x \in \mathbb{R}^n$ ~~such that~~ and $\exists z \in \mathbb{R}^m$: $\begin{cases} \cancel{d^T z = \inf_{t \in D_x} d^T t} \\ d^T z \leq b \end{cases}$ $Cz = x$, $z \succeq 0$ $D_x := \left\{\begin{array}{l}\text{feasible points} \\ \text{of } D_x\end{array}\right\}$

$$d^T z = \inf_{t \in D_x} d^T t \iff d^T z = \sup_{a \in S} a^T x \iff d^T z = \sup_{a \in S} (C^T a) z$$

$$\iff d^T z = d^T z \quad \text{which is true.}$$

Consequently, we have

$$(K) \iff \min_x c^T x \quad \text{st} \quad \begin{array}{l} d^T z \leq b \\ C^T z = x \\ z \succeq 0. \end{array}$$

Exo 5

1. The Lagrangian can be written

$$L(x,\lambda,\nu) = c^T x + \lambda^T(Ax-b) - \nu^T x + x^T \operatorname{diag}(\nu)\, x$$
$$= x^T \operatorname{diag}(\nu) x + (c^T + \lambda^T A - \nu^T) x - \lambda^T b$$

Let's minimize this over x.

$$\frac{\partial L(x,\lambda,\nu)}{\partial x} = 2x^T \operatorname{diag}(\nu) + c^T + \lambda^T A - \nu^T = 0 \Leftrightarrow 2\operatorname{diag}(\nu) x = \nu - A$$

$$\Leftrightarrow x = \frac{1}{2}\operatorname{diag}\left(\frac{1}{\nu}\right)(\nu - A^T\lambda - c)$$

$\nu_i \geq 0$ since DV (si $\nu_i < 0$ $\inf_x x^T \operatorname{diag}(\nu) x = -$

Thus, we take $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, we have

$$y^T \operatorname{diag}(\nu) y = (y_1 \cdots y_n) \begin{pmatrix} \nu_1 y_1 \\ \vdots \\ \nu_n y_n \end{pmatrix} = \sum_i \nu_i y_i^2$$

$$L(\lambda,\nu) = \frac{1}{4}(\nu^T - \lambda^T A - c^T)\operatorname{diag}\left(\frac{1}{\nu}\right)(\nu - A^T\lambda - c) + \frac{1}{2}(c^T + \lambda^T A - \nu^T)\operatorname{diag}\left(\frac{1}{\nu}\right)(\nu - A^T\lambda - c) - \lambda^T b$$

$$= -\frac{1}{4}\sum_i \frac{(c_i + (A^T\lambda)_i - \nu_i)^2}{\nu_i} - \lambda^T b$$

$y = (\nu - A^T\lambda - c)$

Dual problem:

$$\sup_{\lambda,\nu} -\frac{1}{4}\sum_i \frac{(c_i + (A^T\lambda)_i - \nu_i)^2}{\nu_i} - \lambda^T b \quad \text{st} \quad \begin{matrix}\nu \geq 0 \\ \lambda \geq 0\end{matrix}$$

$$\Leftrightarrow \sup_\lambda \sum_i \sup_{\nu_i \geq 0} -\frac{(c_i + (A^T\lambda)_i - \nu_i)^2}{4\nu_i} - \lambda^T b \quad \text{st } \lambda \geq 0$$

$$\boxed{\Leftrightarrow \sup_\lambda -b^T\lambda + \sum_{i=1}^n \min\{0, c_i + A_i^T\lambda\} \quad \text{st } \lambda \geq 0}$$

2. The Lagrangian of the LP relaxation is

$$L(x,\lambda_1,\lambda_2,\lambda_3) = c^T x + \lambda_1^T(Ax-b) + \lambda_2^T(x - \mathbb{1}) - \lambda_3^T x$$
$$= (c^T + \lambda_1^T A + \lambda_2^T - \lambda_3^T) x - \lambda_1^T b - \lambda_2^T \mathbb{1}$$

$$L(\lambda_1,\lambda_2,\lambda_3) = \begin{cases} -\lambda_1^T b - \lambda_2^T \mathbb{1} & \text{if } c^T + \lambda_1^T A + \lambda_2^T - \lambda_3^T = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is thus

$\sup \; -5^T \lambda_1 - \mathbb{1}^T \lambda_2$ st $A^T \lambda_1 + c + \lambda_2 - \lambda_3 = 0$, which is equivalent

$\lambda_{1,2,3} \succeq 0$

to the Lagrange relaxation problem.

(e) The lower bounds for both problems are the same