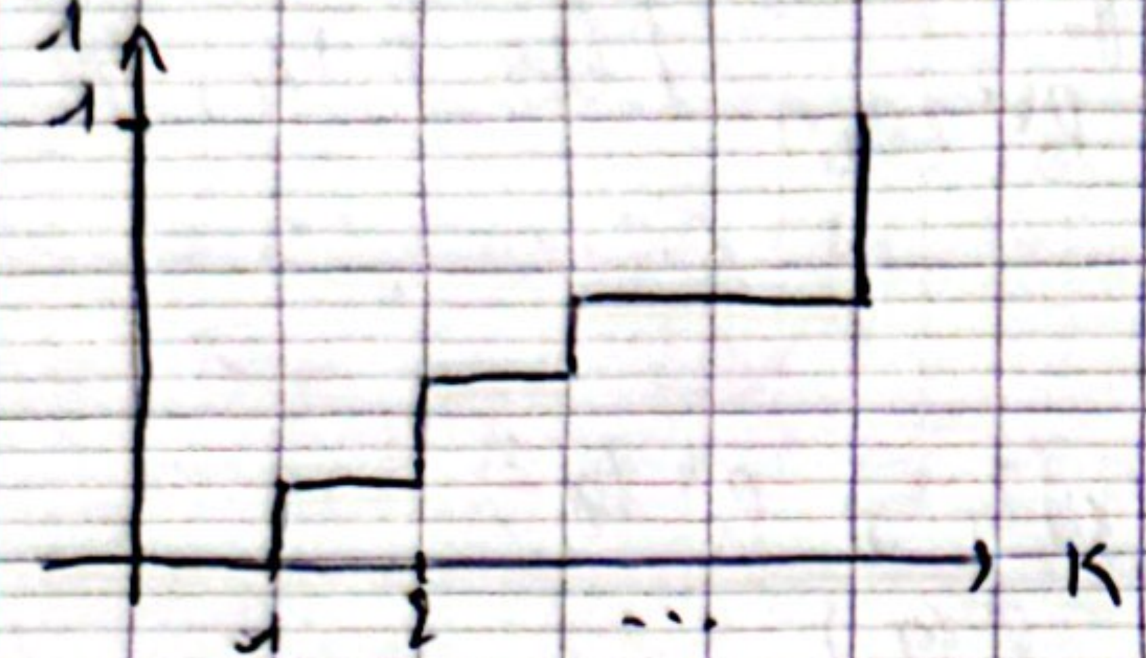
TP n°2 : Computational Statistics.

$\boxed{\text{Exo 1}}$

(1) we consider X r.v. discrete such that $\forall i \in [\![1,n]\!]$, $\mathbb{P}(X=x_i) = p_i$



I am looking for F st $\mathbb{P}(F(U) \leq K) = \mathbb{P}(X \leq$

$$\mathbb{P}(X \leq K) = \sum_{i=0}^{K} p_i = \mathbb{P}\left(U \leq \sum_{i=0}^{K} p_i\right)$$

$$\Rightarrow F = \min K \quad \text{st} \quad U \leq \sum_{i=0}^{K} p_i \text{ and } K \leq n$$

(2) (3) view notebook

$\boxed{\text{Exo 2}}$

(1) The ensemble of parameters is $\theta = \{\{\alpha_i\}_{1\leq i\leq n}, \{\mu_i\}_{1\leq i\leq n}, \{\Sigma_i\}_{1\leq i\leq}$

We can thus obtain that
$p_\theta(x) = \sum_{j=1}^{p} p_\theta(x|z=j)\, p_\theta(z=j) = \sum_{j=1}^{p} \alpha_j \varphi_{\mu_j,\Sigma_j}(x)$ where φ is the density of an multinomial gaussian of parameters (μ_j, Σ_j)

The (x_i) are iid, so
$$\mathcal{L}(x_1 \text{---} x_n, \theta) = \prod_{i=1}^{n} \sum_{j=1}^{p} \alpha_j \varphi_{\mu_j,\Sigma_j}(x_i)$$

(2) view notebook

(3) we want to maximize $\mathcal{L} = p_\theta(x)$ where p_θ can be rewritten
$p_\theta(x) = \mathbb{E}_{q(z)}\left[\dfrac{p_\theta(x,z)}{q(z)}\right]$ with q a density function.

With Jensen's inequality, we deduce

$$\log p_\theta(x) \geq \underbrace{E_{q(z)}\left[\log \frac{p_\theta(x,z)}{q(z)}\right]}_{\text{to maximise}}$$

As $q(z)$ does not depend on θ, we maximise $E_{q(z)}[\log p_\theta(x,z)]$ and we have

$$\theta^{t+1} = \arg\max_\theta \sum_{i=1}^{n} E_{p_\theta^t(z_i|x_i)}[\log p_\theta(x_i, z_i)]$$

we have

$$\tau_{j,i} = p_\theta^t(z_i=j|x_i) = \frac{p_\theta^t(x_i|z_i=j)\, p_\theta^t(z_i=j)}{p_\theta^t(x_i)}$$

$$= \frac{p_{\theta^t}(x_i|z_i=j)\,\alpha_j^t}{\sum_{k=1}^{p} p_{\theta^t}(x_i|z_i=k)\,\alpha_k^t}$$

Then,

$$\sum_{i=1}^{n} E_{\tau_{j,i}}(\log p_\theta(x_i, z_i)) = \underbrace{\sum_i \sum_j \log p_\theta(x_i|z_i=j) + \log \alpha_j)\,\tau_{j,i}}_{f(\theta)}$$

we have $L(\theta, \lambda) = f(\theta) + \lambda(1 - \sum_j \alpha_j)$

By maximising on θ, we have

$$\mu_j^{t+1} = \frac{\sum_{i=1}^{n} \tau_{j,i}\, x_i}{\sum_{i=1}^{n} \tau_{j,i}^t} \qquad \alpha_j^{t+1} = \frac{\sum_{i=1}^{n} \tau_{j,i}^t}{\sum_{j=1}^{p} \sum_{i=1}^{n} \tau_{j,i}^t}$$

$$\Sigma_j^{t+1} = \frac{\sum_{i=1}^{n} \tau_{j,i}^t (x_i - \mu_j)(x_i - \mu_j)^T}{\sum_{i=1}^{n} \tau_{j,i}^t}$$

Exo 3

(Q4) we want to find as shown on the previous page

$$\theta^{t+1} = \arg\max_\theta \sum_{i=1}^{n} \tilde{\omega}_i^t \log g_\theta(x_i^t)$$

The new parameter $\tilde{\omega}$ is going to vary with θ^{t+1} at each time step. We have

$$\theta^{t+1} = \arg\max_\theta \sum_{i=1}^{n} \tilde{\omega}_i^t \, E_{g_\theta^t(z|x_i)} \left[\log g_\theta(x_i, z)\right]$$

we have, with the same reasoning as before (see Q2)

$$\tau_{j,i} := \frac{\varphi_{\mu_j,\Sigma_j}(x_i)\, \alpha_j^t}{g_\theta(x_i)} \quad ; \quad f(\theta) = \sum_i \tilde{\omega}_i^t E_{\tau_{j,i}} \log g_\theta(x_i, z_i)$$

$$= \sum_i \sum_j \log g_\theta(x_i | z_i = j) + \log \alpha_j \, \tau_{j,i} \, \tilde{\omega}_i$$

After maximisation,

$$\mu_j^{t+1} = \frac{\sum_i \tilde{\omega}_i^t \tau_{j,i}^t x_i^t}{\sum_{i=1}^{n} \tilde{\omega}_i^t \tau_{j,i}^t} \qquad \alpha_j^{t+1} = \frac{\sum_i \tilde{\omega}_i^t \tau_{j,i}^t}{\sum_j \sum_i \tilde{\omega}_i^t \tau_{j,i}^t}$$

$$\Sigma_j^{t+1} = \frac{\sum_{i=1}^{n} \tilde{\omega}_i \tau_{j,i}^t (x_i^t - \mu_j)(x_i^t - \mu_j)^T}{\sum_{i=1}^{n} \tilde{\omega}_i^t \tau_{j,i}^t}.$$