Counting Restricted Compositions of n, An Application of Generating Functions

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Let $n \geq 1$, and recall that a composition of n of length k is an ordered k-tuple of positive integers which sum to n. In class, we showed the number of compositions of n of length k to be

$$\binom{n-1}{k-1} \tag{1}$$

and therefore, the total number c_n of compositions of n is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1} \tag{2}$$

Note that $c_0 = 0$. Also, in equation 9 of assignment 2, you found the generating function of the sequence $\{c_n\}$ to be

$$\frac{x}{1-2x} \tag{3}$$

1

Starting with corollary 43, find functions $g(t)$ and $f(t)$ to derive another proof of equation (2)

$$T_n(fog;0) = \sum_{\pi \in P_n} \binom{l(\pi)}{\delta(\pi)} T_{l(\pi)}(f;g(0)) \prod_{i=1}^{n} (T_i(g;0))^{\pi_i} \tag{4}$$

Let $g(0) = 0$. In the desired function, $T_i(g;0) = 1$ for all $i > 0$ and $T_m(f;0) = 1$ for all $m > 0$. Lets make $f(x)$ equal to the geometric series $\frac{1}{1-x}$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... \tag{5}$$

Note that each coefficient in this equation is 1. Therefore $T_{l(\pi)}(f;g(0)) = 1$. With this we can greatly simplify equation (4) to

$$T_n(fog;0) = \sum_{\pi \in C_n} 1 \times \prod_{i=1}^{n} (T_i(g;0))^{\pi_i} \tag{6}$$

$$g(t) = \frac{t}{1-t} = 0 + t + t^2 + t^3 + ... \tag{7}$$

Because the coefficients are once again all equal to 1, equation (6) simplifies further into

$$T_n(fog;0) = \sum_{\pi \in C_n} 1 \times \prod_{i=1}^{n} 1 = 1 + 1 + 1 + ... = \frac{x}{1-2x} \tag{8}$$

which is the generating function of the sequence $\{c_n\}$. Therefore we have found another proof that the total number c_n of compositions on n is 2^{n-1} by using corollary 43.

2

Let $d(n)$ be the number of compositions of n in which every part is either a 1 or a 2.

$$\text{For } i = 1 \text{ or } 2: g(t) = 0 + T_1(g;0)t + T_2(g;0)t^2 \tag{9}$$

Since we know that $T_i(g;0) = 1$

$$g(t) = t + t^2 \tag{10}$$

$$d(n) = f(g(t)) = \frac{1}{1-(t+t^2)} \tag{11}$$

To find the sequence $d(n)$ we can compute the partitions for the first few n-values

For n = {1}: (1)
$d_1 = 1$
For n = {2}: (1,1),(2)
$d_2 = 2$
For n = {3}: (1,1,1),(1,2),(2,1)
$d_3 = 3$
For n = {4}: (1,1,1,1),(1,1,2),(1,2,1),(2,1,1),(2,2)
$d_4 = 5$
For n = {5}: (1,1,1,1,1),(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1),(2,2,1),(2,1,2),(1,2,2)
$d_5 = 8$

We can see that this function generates the Fibonacci sequence, following the same recursion $T_n = T_{n-1} + T_{n-2}$. Only this sequence skips the first term of the Fibonacci sequence, starting at n=1.

3

Let $g(n)$ be the number of compositions of n in which every part is either a 1 or a 3.

$$\text{For } i = 1 \text{ or } 3 : g(t) = 0 + T_1(g;0)t + T_3(g;0)t^3 \tag{12}$$

$$g(t) = t + t^3 \tag{13}$$

$$g(n) = f(g(t)) = \frac{1}{1-(t+t^3)} \tag{14}$$

Consider again the partitions of $g(n)$

For n = $\{1\}$: (1)
$g_1 = 1$
For n = $\{2\}$: (1,1)
$g_2 = 1$
For n = $\{3\}$: (1,1,1),(3)
$g_3 = 2$
For n = $\{4\}$: (1,1,1,1),(1,3),(3,1)
$g_4 = 3$
For n = $\{5\}$: (1,1,1,1,1),(1,1,3),(1,3,1),(3,1,1)
$g_5 = 4$
For n = $\{6\}$: (1,1,1,1,1,1),(1,1,1,3),(1,1,3,1),(1,3,1,1),(3,1,1,1),(3,3)
$g_6 = 6$

$g(n)$ satisfies the recursion $T_n = T_{n-1} + T_{n-3}$

4

Let $h(n)$ be the number of compositions of n in which every part is a 1 or a j where j is any constant integer $j \geq 2$

$$g(t) = 0 + T_1(g;0)t + T_j(g;0)t^j \tag{15}$$

$$g(t) = t + t^j \tag{16}$$

$$h(n) = f(g(n)) = \frac{1}{1-(t+t^j)} \tag{17}$$

and the recursion is given by $T_n = T_{n-1} + T_{n-j}$

Interestingly enough, the recursion does not take effect until the value of n for $\{n\}$ reaches the constant integer j. Up until this point, the value of T_n will be 1. This is because until the partition allows for a second part to be used, the only possible combination is a series of ones. Take the partitions of $h(n)$ for $j = 4$ for example

For n = $\{1\}$: (1)
  $h_1 = 1$
For n = $\{2\}$: (1,1)
  $h_2 = 1$
For n = $\{3\}$: (1,1,1),
  $h_3 = 1$
For n = $\{4\}$: (1,1,1,1),(4)
  $h_4 = 2$
For n = $\{5\}$: (1,1,1,1,1),(1,4),(4,1)
  $h_5 = 3$
For n = $\{6\}$: (1,1,1,1,1,1),(1,1,4),(1,4,1),(4,1,1)
  $h_6 = 4$
For n = $\{7\}$: (1,1,1,1,1,1,1),(1,1,1,4),(1,1,4,1),(1,4,1,1),(4,1,1,1)
  $h_7 = 5$
For n = $\{8\}$: (1,1,1,1,1,1,1,1),(1,1,1,1,4),(1,1,1,4,1),(1,1,4,1,1),(1,4,1,1,1),(4,1,1,1,1),(4,4)
  $h_8 = 7$

T_n behaves according to the recursion $T_n = T_{n-1} + T_{n-4}$ but only starting at $n = 4$

5

Let $k(n)$ be the number of compositions of n in which every part is a 1, 2, or 3.

$$g(t) = 0 + T_1(g;0)t + T_2(g;0)t^2 + T_3(g;0)t^3 \tag{18}$$

$$g(t) = t + t^2 + t^3 \tag{19}$$

$$k(n) = f(g(n)) = \frac{1}{1 - (t + t^2 + t^3)} \tag{20}$$

For n = $\{1\}$: (1)
  $k_1 = 1$
For n = $\{2\}$: (1,1),(2)
  $k_2 = 2$
For n = $\{3\}$: (1,1,1),(1,2),(2,1),(3)
  $k_3 = 4$
For n = $\{4\}$: (1,1,1,1),(1,1,2),(1,2,1),(2,1,1),(2,2),(3,1),(1,3)
  $k_4 = 7$
For n = $\{5\}$: (1,1,1,1,1),(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1),(1,2,2),(2,1,2),(2,2,1),(1,1,3),(1,3,1),(3,1,1),(2,3),(3,2)
  $k_5 = 13$

$k(n)$ follows the recursion $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, but we have to make the exception rule that $k_0 = 1$ in order for the recursion to start working.

6

The general formula for the generating function (as a composite function) for the number of compositions of n in which every part must come from some set T of positive integers should appear obvious given the previous work on certain sets of integers.

$$g(t) = \sum_{i \in T} t^i \tag{21}$$

$$f(g(t)) = \frac{1}{1 - \sum_{i \in I} t^i} \tag{22}$$

The terms in which the parts of the composition are not in the given set become 0, leaving the composite generating function to be the sum of the remaining terms. And since $g(t) = \frac{t}{1-t}$, the coefficients will all be 1 and the function $g(t)$ is simply the sum of all t^i where i is the integer used in the composition. The recursion for any set T of positive integers is $T_n = \sum_{i \in T} T_{n-i}$

7

The generating function for the number of compositions of n in which every part is odd follows directly from equation (21), substituting $2i - 1$ for i on the right hand side

$$g(t) = \sum_{i \in T} t^{2i-1} \tag{23}$$

$$f(g(t)) = \frac{1}{1 - \sum_{i \in I} t^{2i-1}} \tag{24}$$

When every part is even

$$g(t) = \sum_{i \in T} t^{2i} \tag{25}$$

$$f(g(t)) = \frac{1}{1 - \sum_{i \in I} t^{2i}} \tag{26}$$