Welfare guarantees for proportional allocations

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The proportional allocation mechanism

- First introduced by Kelly (1997).
- Used for the allocation of divisible resources (e.g. bandwidth).

The mechanism

- User i submits a bid b_i to the mechanism.
- The mechanism allocates to user i a fraction d_i of the resource equal to her bid over the total bids submitted by all users, i.e.,

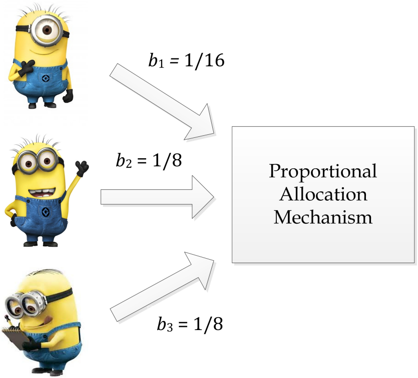
$$d_i = \frac{b_i}{\sum_j b_j}.$$

- The mechanism collects from user i an amount equal to the bid she submitted.

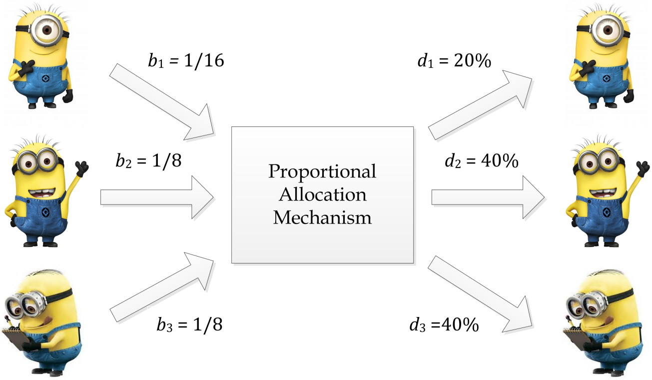
An example

Proportional
Allocation
Mechanism

An example



An example



$b_1 = 1/16$
$b_2 = 1/8$
$b_3 = 1/8$
Proportional Allocation Mechanism
$d_1 = 20\%$
$d_2 = 40\%$
$d_3 = 40\%$

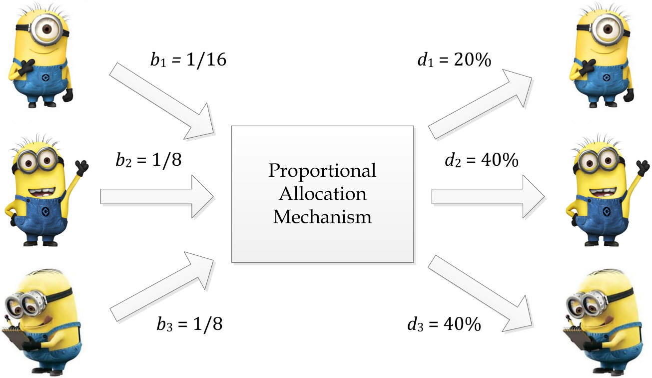
Proportional allocation games

- The mechanism defines a game among the users who act as players (bidders).
- Bidder i has a valuation function v_i for the share of the resource she is allocated.
  - v_i is concave, non-negative, and non-decreasing
- $\mathbf{b} = (b_1, ..., b_n)$ is a bid vector, representing a state of the game.
- B_{-i} is the sum of bids of all bidders except i, $B = B_{-i} + b_i$.
- The utility of bidder i is the value for the resource share she receives minus her payment, i.e.,

$$u_i(\mathbf{b}) = v_i(d_i) - b_i.$$

- Each bidder aims to maximize her utility.

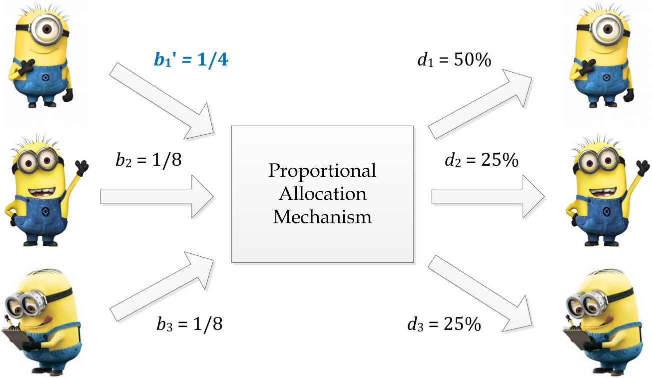
An example



$$v_1(x) = x \qquad\qquad v_2(x) = v_3(x) = \tfrac{2}{3}x$$

$$u_1(\mathbf{b}) = 20\% - 1/16 \simeq 0.138 \qquad u_2(\mathbf{b}) = u_3(\mathbf{b}) = \tfrac{2}{3} \cdot 40\% - 1/8 \simeq 0.142$$

An example



$$v_1(x) = x \qquad v_2(x) = v_3(x) = \frac{2}{3}x$$

$$u_1(\mathbf{b}') = 50\% - 1/4 = 0.25 \qquad u_2(\mathbf{b}') = u_3(\mathbf{b}') = \frac{2}{3} \cdot 25\% - 1/8 \simeq 0.042$$

Equilibrium concepts — Full information

Pure Nash Equilibrium (PNE)

A bid vector $\mathbf{b}$ is a pure Nash equilibrium if no bidder i has any incentive to unilaterally deviate to any other bid strategy b_i' in order to increase her utility, i.e.,

$$u_i(\mathbf{b}) \geq u_i(b_i', \mathbf{b}_{-i}).$$

- In any proportional allocation game, there exists a unique PNE (Hajek and Gopalakrishnan, 2002).

Equilibrium concepts — Full information

- The bid strategy of a bidder can also be randomized.
- In this case, each bidder aims to maximize her expected utility $\mathbb{E}\left[u_i(\mathbf{b})\right]$.
- Bid strategies may be independent (mixed) or correlated.
  - Informally, a correlated bid vector defines a probability distribution over all possible bid vectors.

Coarse-Correlated Equilibrium (CCE)

A correlated bid vector $\mathbf{b}$ is a coarse-correlated equilibrium if no bidder i has any incentive to unilaterally deviate to any other deterministic bid b'_i in order to increase her expected utility, i.e.,

$$\mathbb{E}\left[u_i(\mathbf{b})\right] \geq \mathbb{E}\left[u_i(b'_i, \mathbf{b}_{-i})\right].$$

Equilibrium concepts — Incomplete information

- Bidder i draws (independently from the others) her valuation function $\mathbf{v}_i$ from a probability distribution $\mathbf{F}_i$.
  - These distributions are common knowledge.
- Goal: maximize the expected utility for every possible valuation function.
- The bid strategy $b_i(\mathbf{v}_i)$ of bidder i depends on the exact valuation and $\mathbf{F}_{-i}$.

Bayes-Nash Equilibrium (BNE)

A vector $\mathbf{b}$ of bid functions is a Bayes-Nash equilibrium if no bidder i has any incentive to unilaterally deviate to any other bid $b_i'(\mathbf{v}_i)$ in order to increase her expected utility for any valuation function drawn from $\mathbf{F}_i$, i.e.,

$$\mathbb{E}_{\mathbf{v}_i \sim \mathbf{F}_i}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{v}_i \sim \mathbf{F}_i}[u_i(b_i'(\mathbf{v}_i), \mathbf{b}_{-i})].$$

Price of Anarchy

- The (expected) social welfare of an allocation $d(\mathbf{b}) = (d_1, ..., d_n)$ is the total (expected) value of the bidders, i.e.,

$$SW(d) = \sum_i \mathbb{E}\left[v_i(d_i)\right].$$

- SW^* is the maximum value of the social welfare over all possible allocations.
- The social welfare at equilibria is, in general, suboptimal.

Price of Anarchy (PoA)

Consider any equilibrium concept EQ $\in$ {PNE, CCE, BNE}. The price of anarchy over EQ is the minimum value of the (expected) social welfare among all equilibria over the optimal social welfare, i.e.,

$$\text{PoA} = \min_{\mathbf{b}\in \text{EQ}} \frac{SW(d(\mathbf{b}))}{SW^*}.$$

Related work

- Johari and Tsitsiklis (2004)
  - Proved a tight bound of 3/4 for PoA over PNE.
  - Key observation: the worst case occurs when bidders have *linear* valuation functions.
- Syrgkanis and Tardos (2013)
  - Presented a general analysis framework for the class of smooth mechanisms.
  - Proved a lower bound of 26.8% for PoA over CCE and BNE.
  - Technique: Bound the utility of each bidder by the utility she would have by deviating to a *randomized* bid.

Unilateral deviations

- By definition, at equilibrium the utility of each bidder is maximized.
- So, the utility at equilibrium is bounded by the utility a bidder would have at any other bid strategy.
- We use unilateral deviations to specific *deterministic* bid strategies that depend on the bids at equilibrium.

Lemma (*Deviation lemma*)

Consider a bidder with a concave and non-decreasing valuation function $v : [0, 1] \to \mathbb{R}^+$ and let Γ be the random variable denoting the sum of bids of the other bidders. Then, for every $z \in [0, 1]$ and for every $\mu > 0$, the expected utility the bidder would have by deviating to the deterministic bid $\mu z \mathbb{E}[\Gamma]$ is at least $\frac{3\mu-1}{4\mu} v(z) - \mu z \mathbb{E}[\Gamma]$.

Bounding the social welfare

Theorem (*SW-PoA theorem*)

The price of anarchy of proportional allocation games over coarse-correlated equilibria or Bayes-Nash equilibria is at least $1/2$.

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- $x_i =$ fraction of resource bidder i gets in optimal allocation.

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$$\mathbb{E}\left[u_i(\mathbf{b})\right] \geq \mathbb{E}\left[u_i(x_i \mathbb{E}\left[B_{-i}\right], \mathbf{b}_{-i})\right] \geq \frac{1}{2} v_i(x_i) - x_i \mathbb{E}\left[B_{-i}\right].$$

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- Sum over all bidders and use the facts $B_{-i} \leq B$ and $\sum_i x_i = 1$ to get

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$$\begin{aligned} \sum_i \mathbb{E}\left[u_i(\mathbf{b})\right] &\geq \frac{1}{2} \sum_i v_i(x_i) - \sum_i x_i \mathbb{E}\left[B_{-i}\right] \qquad (1) \\ &\geq \frac{1}{2} SW^* - \mathbb{E}\left[B\right]. \end{aligned}$$

□

Limitations of our techniques

- Equation (1) has the generic form:

$$\sum_i \mathbb{E}\left[u_i(\mathbf{b})\right] \geq \lambda SW^* - \mu \sum_i x_i \mathbb{E}\left[B_{-i}\right]$$

- The smoothness arguments of Syrgkanis and Tardos lead to a version of this inequality with $\lambda = 2 - \sqrt{3}$ and $\mu = 1$.
- We improve these parameters to $\lambda = 1/2$ and $\mu = 1$.
- They cannot get any better!

Lemma (*Limitations of smoothness arguments*)

For every $\epsilon > 0$, there exists a proportional allocation game such that for every λ and μ satisfying

$$\sum_i u_i(\mathbf{b}) \geq \lambda SW^* - \mu \sum_i x_i B_{-i},$$

it holds that $\frac{\lambda}{\max\{1,\mu\}} \leq \frac{1}{2} + \epsilon$.

Budget-constrained bidders

- In this setting, bidder i also has a non-negative budget c_i.
- The effective welfare of a deterministic allocation d is defined as

$$EW(d) = \sum_i \min\{v_i(d_i), c_i\}.$$

  - Intuition: Each bidder's willingness-to-pay is capped by her ability-to-pay.
- Incomplete information setting: $\mathbf{c}_i$ and $\mathbf{v}_i$ are drawn randomly from a distribution $\mathbf{F}_i$.
- Refinement of the effective welfare:

$$EW(d) = \sum_i \mathbb{E}_{(\mathbf{v}_i, \mathbf{c}_i) \sim \mathbf{F}_i} \left[\min\{\mathbb{E}_{(\mathbf{v}_{-i}, \mathbf{c}_{-i}) \sim \mathbf{F}_{-i}} \left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\right]$$

Bounding the effective welfare

- Using the deviation lemma in our analysis, we obtain:

Theorem (*EW-PoA theorem*)

The price of anarchy of proportional allocation games with budget-constrained bidders over coarse-correlated or Bayes-Nash equilibria is at least 0.3596.

- The JT-bound of $3/4$ does not extend to more complicated settings.

Theorem (*Negative result*)

For every $\epsilon > 0$*, there exists a proportional allocation game among budget-constrained bidders with price of anarchy at most* $1/2 + \epsilon$ *over pure Nash equilibria, with respect to the effective welfare benchmark.*

Open problems

- Main question: Is 3/4 the tight bound for all equilibrium concepts?

Lemma (*No-mixed lemma*)

The set of mixed Bayes-Nash equilibria in any proportional allocation game (possibly with budget constraints) coincides with that of pure Bayes-Nash equilibria.

- Do coarse-correlated equilibria coincide with pure ones?
- What is the PoA in the case of coarse-correlated Bayesian equilibria?

Lemma (*CCBE lemma*)

There exists a proportional allocation game that has price of anarchy at most 0.7154 *over coarse-correlated Bayesian equilibria.*

The end...