Problem 9.6

Task 1

(a) The size of the input is the value of n
(b) Let the amount of work done in the base cases be k_0 and k_1, respectively.
(c) Let the amount of work done in the recursive case be l
(d) There are two recursive calls made at each step, on $(n-1)$ and $(n-2)$

The recurrence relation for fib, $B(n)$ is as follows:

$$B(n) = \begin{cases} k_0 & \text{if } n = 0 \\ k_1 & \text{if } n = 1 \\ l + B(n-1) + B(n-2) & \text{if } n > 1 \end{cases}$$

Task 2

I have calculated an exact closed form for $B(n)$ by using a slightly altered tree. Rather than assuming a binary branching tree with some power of 2 nodes on each level as an upper bound, I discovered that if I have input sizes $(n-1)$ and $(n-2)$ as daughers of input size n but consider $(n-1)$ on level 2 and $(n-2)$ on level 3, etc., so that all inputs of the same size are on the same level, then there are $(n+1)$ levels and number of nodes on each level l, not including the last base case is the l^{th} Fibonacci number, i.e. $F(l)$. I understand from reading the hints that this was not the expected solution, but assuming my proof is correect I think it is a more elegant one. I did consult TA Evan Fuller beforehand to check that this approach would be acceptable.

The table is as follows:

level	I/P Size	cost per node	# of nodes	level cost
1	n	l	1	$F(1) * l$
2	$n-1$	l	2	$F(2) * l$
3	$n-2$	l	3	$F(3) * l$
4	$n-3$	l	4	$F(4) * l$
...	...	...	...	...
$n-1$	2	l	$F(n-1)$	$F(n-1) * l$
n	1	k_1	$F(n)$	$F(n) * k_1$
$n+1$	0	k_0	$F(n-1)$	$F(n-1) * k_0$

So the total cost is the sum of all the level costs.

$$
\begin{aligned}
B(n) &= \sum_{i=1}^{n-1} F(i)l + F(n)k_1 + F(n-1)k_0 \\
&= G(n-1)l + F(n)k_1 + F(n-1)k_0
\end{aligned}
$$

where G(n) is the sum of the first n^{th} Fibonacci numbers, as in Problem 5

$$= (F((n+2)-1)-1)l + F(n)k_1 + F(n-1)k_0$$

by the closed form of G(n) as proved in Problem 5

$$= (F(n+1)-1)l + F(n)k_1 + F(n-1)k_0$$

Here I provide the closed form in terms of F(n) for the sake of readability. I could instead express it in terms of the closed form of F(n), $F(n) = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n)$, where $\phi_+^n = \frac{1+\sqrt{5}}{2}$ and $\phi_-^n = \frac{1-\sqrt{5}}{2}$

In the inductive proof I use the recurrence relation F(n) for clarity, but in order to prove B(n)'s order, I will provide a brief proof for the closed form of F(n).

Task 3

Now to prove the closed form is correct by induction. PF:

basis:

$$
\begin{aligned}
B(0) &= k_0 \\
&= 0l + 0k_1 + 1k_0 \\
&= (1-1)l + F(0)k_1 + F(-1)k_0
\end{aligned}
$$

(Note that by the definition of F earlier in the assignment, F is defined for all integers, and its value at -1 is 1. If this step is still a concern to the grader, I could have rewritten this term in the closed form as $(F(n+1) - F(n))k_0$ and this would certainly be equivalent to my current closed form throughout.)

$$
\begin{aligned}
&= (F(1)-1)l + F(0)k_1 + F(-1)k_0 \\
&= (F(0+1)-1)l + F(0)k_1 + F(0-1)k_0
\end{aligned}
$$

$$\begin{aligned}B(1) &= k_1\\ &= 0l + 1k_1 + 0k_0\\ &= (1-1)l + F(1)k_1 + F(0)k_0\\ &= (F(2)-1)l + F(1)k_1 + F(1-1)k_0\\ &= (F(1+1)-1)l + F(1)k_1 + F(1-1)k_0\end{aligned}$$

Step: Assume

$$\begin{aligned}B(n-1) &= (F((n-1)+1)-1)l + F(n-1)k_1 + F((n-1)-1)k_0\\ &= (F(n)-1)l + F(n-1)k_1 + F(n-2)k_0\end{aligned}$$

and

$$\begin{aligned}B(n-2) &= (F((n-2)+1)-1)l + F(n-2)k_1 + F((n-2)-1)k_0\\ &= (F(n-1)-1)l + F(n-2)k_1 + F(n-3)k_0\end{aligned}$$

Then

$$\begin{aligned}B(n) &= l + B(n-1) + B(n-2)\\ &= l + (F(n)-1)l + F(n-1)k_1 + F(n-2)k_0 + (F(n-1)-1)l + F(n-2)k_1 + F(n-3)k_0\\ &= l + (F(n)-1)l + (F(n-1)-1)l + F(n-1)k_1 + F(n-2)k_1 + F(n-2)k_0 + F(n-3)k_0\\ &= (1 + F(n) - 1 + F(n-1) - 1)l + (F(n-1) + F(n-2))k_1 + (F(n-2) + F(n-3))k_0\\ &= (F(n+1)-1)l + F(n)k_1 + F(n-1)k_0\end{aligned}$$

by the recurrence relation F(n)

Q.E.D.

PF of the closed form of F(n), $F(n) = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n)$

basis:

$$\begin{aligned}
F(0) &= 0 \\
&= \frac{1}{\sqrt{5}}(0) \\
&= \frac{1}{\sqrt{5}}(1-1) \\
&= \frac{1}{\sqrt{5}}(\phi_+^0 - \phi_-^0)
\end{aligned}$$

$$\begin{aligned}
F(1) &= 1 \\
&= \frac{1}{\sqrt{5}}(\sqrt{5}) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1-1+\sqrt{5}+\sqrt{5}}{2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2}\right) \\
&= \frac{1}{\sqrt{5}}(\phi_+^1 - \phi_-^1)
\end{aligned}$$

Step: Assume

$$\begin{aligned}
F(n-2) &= \frac{1}{\sqrt{5}}(\phi_+^{n-2} - \phi_-^{n-2}), \\
F(n-1) &= \frac{1}{\sqrt{5}}(\phi_+^{n-1} - \phi_-^{n-1})
\end{aligned}$$

Then

$$\begin{aligned}
F(n) &= F(n-1) + F(n-2) \\
&= \frac{1}{\sqrt{5}}(\phi_+^{n-1} - \phi_-^{n-1}) + \frac{1}{\sqrt{5}}(\phi_+^{n-2} - \phi_-^{n-2}) \\
&= \frac{1}{\sqrt{5}}(\phi_+^{n-1} - \phi_-^{n-1} + \phi_+^{n-2} - \phi_-^{n-2}) \\
&= \frac{1}{\sqrt{5}}(\phi_+^{n-1} + \phi_+^{n-2} - (\phi_-^{n-1} + \phi_-^{n-2})) \\
&= \frac{1}{\sqrt{5}}(\phi_+^{n} - \phi_-^{n})
\end{aligned}$$

by the identity provided in the assignment

Q.E.D.

Finally, I can prove that $B(n) \in O(\phi_+^n)$

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B(n)}{\phi_+^n} \\
&= \lim_{n\to\infty} \frac{(F(n+1)-1)l + F(n)k_1 + F(n-1)k_0}{\phi_+^n} \\
&= \lim_{n\to\infty} \frac{((1/\sqrt{5})(\phi_+^{n+1}-\phi_-^{n+1})-1)l + (1/\sqrt{5})(\phi_+^n-\phi_-^n)k_1 + (1/\sqrt{5})(\phi_+^{n-1}-\phi_-^{n-1})k_0}{\phi_+^n} \\
&= l \lim_{n\to\infty} \frac{(1/\sqrt{5})(\phi_+^{n+1}-\phi_-^{n+1})-1}{\phi_+^n} + \frac{1}{\sqrt{5}}k_1 \lim_{n\to\infty} \frac{\phi_+^n-\phi_-^n}{\phi_+^n} + \frac{1}{\sqrt{5}}k_0 \lim_{n\to\infty} \frac{\phi_+^{n-1}-\phi_-^{n-1}}{\phi_+^n} \\
&= \frac{1}{\sqrt{5}} l \lim_{n\to\infty} \frac{\phi_+^{n+1}-\phi_-^{n+1}}{\phi_+^n} - l \lim_{n\to\infty} \frac{1}{\phi_+^n} + \frac{1}{\sqrt{5}}k_1 \lim_{n\to\infty} \frac{\phi_+^n}{\phi_+^n} - \frac{1}{\sqrt{5}}k_1 \lim_{n\to\infty} \frac{\phi_-^n}{\phi_+^n} + \frac{1}{\sqrt{5}}k_0 \lim_{n\to\infty} \frac{\phi_+^{n-1}}{\phi_+^n} \\
&- \frac{1}{\sqrt{5}}k_0 \lim_{n\to\infty} \frac{\phi_-^{n-1}}{\phi_+^n} \\
&= \frac{1}{\sqrt{5}} l (\lim_{n\to\infty} \left(\phi_+ \left(\frac{\phi_+}{\phi_+}\right)^n - \phi_- \left(\frac{\phi_-}{\phi_+}\right)^n \right) - 0 + \frac{1}{\sqrt{5}}k_1 - 0 + \frac{1}{\sqrt{5}} * \frac{1}{\phi_+}k_0 \lim_{n\to\infty} \frac{\phi_+^n}{\phi_+^n} \\
&- \frac{1}{\sqrt{5}} * \frac{1}{\phi_-}k_0 \lim_{n\to\infty} \frac{\phi_-^n}{\phi_+^n} \\
&= \frac{1}{\sqrt{5}} l\phi_+ + \frac{1}{\sqrt{5}}k_1 + \frac{1}{\sqrt{5}} * \frac{1}{\phi_+}k_0 - 0 \\
&= \frac{1}{\sqrt{5}} \left(l\phi_+ + k_1 + \frac{1}{\phi_+}k_0 \right)
\end{aligned}$$

Since this limit as n approaches infinity of B(n) divided by ϕ_+^n is a constant (l, k_1, k_0, and ϕ_+ are all constants), $B(n) \in O(\phi_+^n)$.