Optimization and Computational Linear Algebra – Brett Bernstein

Recitation 1

1. Recall that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ can be thought of as the xy-plane. Consider the two vectors $v = (9, 7)$ and $w = (-8, 12)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

   (a) $\text{Span}(v)$

   (b) $\text{Span}(v, w)$

   (c) $\text{Span}(v) \cup \text{Span}(w)$, that is, the vectors in $\text{Span}(v)$ or $\text{Span}(w)$

   (d) $\text{Span}(v) \cap \text{Span}(w)$, that is, the vectors in both $\text{Span}(v)$ and $\text{Span}(w)$

   (e) $\{(1 - t)v + tw : t \in [0, 1]\}$

   (f) $\{(1 - t)v + tw : t \in \mathbb{R}\}$

   (g) $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$

   (h) $\text{Span}(v, w, x)$ where $x = (0, 5)$.

   (i) $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$

   (j) $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

   (k) $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

   (l) $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

   Solution. Below recall that W is a subspace of $\mathbb{R}^n$ if it is nonempty and satisfies

   - $ax \in W$ for all $a \in \mathbb{R}$ and $x \in W$
   - $x + y \in W$ for all $x, y \in W$

   (a) Line through v. It is a subspace.

   (b) All of $\mathbb{R}^2$. It is a subspace.

   (c) 2 lines through v and w. Not a subspace (doesn't contain $v + w$).

   (d) Only the 0 vector. It is a subspace.

   (e) Line segment between v and w. Not a subspace (doesn't contain 0).

   (f) Line through v and w. Not a subspace (doesn't contain 0).

   (g) Infinite wedge between v and w. Not a subspace (doesn't contain $-v$).

   (h) All of $\mathbb{R}^2$. It is a subspace.

   (i) All points within a circle of radius 5 centered at the origin. Not a subspace (doesn't contain $2 \cdot (5, 0)$).

(j) Diagonal line through the origin. It is a subspace.

(k) Parabola. Not a subspace (doesn't contain $2 \cdot (1,1)$).

(l) Horizontal line with y-coordinate equal to 1. Not a subspace (doesn't contain 0).

2. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be distinct and define $C_1 = \{v_1, v_2\}$ and $C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\mathrm{Span}(v_1, v_2, v_3, v_4))$? No proof necessary.

   Solution. Either $C_1 \subset \mathrm{Span}(C_2)$ and the dimension is 2, or the dimension is 3. We will discuss dimension and justify this later in the class.

3. Let $B = \{v_1, \ldots, v_n\}$ be a basis for $\mathbb{R}^n$. Show that for any $x \in \mathbb{R}^n$ there exists a unique $\alpha \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

   Solution. Existence follows immediately from the definition of basis, since it implies that $\mathrm{Span}(B) = \mathbb{R}^n$. For uniqueness, suppose

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$$

   for $\alpha, \beta \in \mathbb{R}^n$. Then we have

$$0 = (\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n.$$

   By linear independence, we must have $\alpha_i = \beta_i$ for all i.

4. True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.

   Solution. False. Consider $B = \{(1,0), (0,1)\}$ and $W = \mathrm{Span}((1,1))$.

5. Suppose $v_1, \ldots, v_m \in \mathbb{R}^n$ are linearly dependent. Prove that if $x \in \mathrm{Span}(v_1, v_2, \ldots, v_m)$ then there are infinitely many $\alpha \in \mathbb{R}^m$ with

$$x = \alpha_1 v_1 + \cdots + \alpha_m v_m.$$

   Solution.

   Proof. By assumption $x \in \mathrm{Span}(v_1, \ldots, v_m)$ so

$$x = a_1 v_1 + \cdots + a_m v_m$$

   for some $a_i \in \mathbb{R}$. Since $v_1, \ldots, v_m$ are linearly dependent, there are $c_1, \ldots, c_m \in \mathbb{R}$ such that

$$c_1 v_1 + \cdots + c_m v_m = 0$$

   where not all $c_i = 0$. Then we have

$$x = a_1 v_1 + \cdots + a_m v_m + r(c_1 v_1 + \cdots + c_m v_m) = (a_1 + rc_1)v_1 + \cdots + (a_m + rc_m)v_m$$

   for all $r \in \mathbb{R}$. This gives infinitely many distinct α where $\alpha_i = a_i + rc_i$ for $r \in \mathbb{R}$. $\square$

6. Find (if they exist) the (global) maximizers and minimizers of the following function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x - 1| + |x - 3| + |x - 10|.$$

Solution. There is no maximizer since $f(x) \to \infty$ as $x \to \infty$. Where f' exists, it is negative when $x < 3$ and positive when $x > 3$ so 3 is the minimizer. [Technically, we are applying the mean value theorem to show f is descending on $(-\infty, 1)$ and $(1, 3)$ and ascending on $(3, 10)$ and $(10, \infty)$.]