Recitation 2

Alex Dong

CDS, NYU

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Some Etymology...

Definition (Linearity: Wikipedia)

The property of a mathematical relationship (function) that can be graphically represented as a straight line.

Definition (Algebra: Wikipedia)

The study of mathematical symbols and the rules for manipulating these symbols.

Linear algebra is the study of manipulating letters/symbols which are used to represent linear transformations.

Concept Review: Linear Transformations

Definition (Linear Transformation)

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

One of the most important (but boring) theorems in Linear Algebra.

Theorem (Matrix Representation Theorem)

All linear transformations can be represented as matrices; all matrices represent linear transformations.

Think about which framework to use in your proofs!

A Note about Gaussian Elimination

Gaussian elimination is a procedure to calculate the solutions of a matrix equation.

We will not cover this in the course, but you should at least be familiar with it.

If you've already studied it in previous courses, that should be enough.

If this is the first time you've heard this, then please do some light studying to familiarize yourself with the process.

Matrix Notation

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a $m \times n$ matrix which is an element of $\mathbb{R}^{m\times n}$. (!!Note the order!!)

$$\mathrm{T} = \begin{array}{cc} & n \\ m & \begin{pmatrix} T_{1,1} & ... & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{pmatrix} \end{array}$$

This matrix has m rows and n columns.
$T_{i,j}$ represents the value of the entry in the ith row and jth column.

Linear Transformations and Subspaces

Linear transformations are fundamentally connected to subspaces. How? We will spend a lot of time on investigating the *action* of a linear transformation on subspaces.

Key questions in linear algebra:

- What does a linear transformation do to 1-dimensional subspaces?
- What does a linear transformation to do a n-dimensional subspace?

Questions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solutions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solution

1. *Linear, Kernel is* $\{0\}$.
2. *Linear, Kernel is* $\{(c, -c) : c \in \mathbb{R}\}$.
3. *Not linear,* $f_3(0, 0) = (0, 0, 1)$.
4. *Not linear,* $f_4(1, 0) + f_4(0, 1) = 2$ *and* $f_4(1, 1) = \sqrt{2}$.
5. *Not linear,* $f_5(0) = 3$.

Questions 2: Matrix Manipulation

Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

1. Calculate AB
2. Calculate BC
3. What does A do to B?
4. What does C do to B?

Solutions 2: Matrix Manipulation

Solution

1. $AB = \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$
2. $BC = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$
3. *Five times first row, switch second and third row*
4. *First column becomes twice the second column plus one times third column, second column stays the same, switch 3rd and fourth columns.*

Questions 1: Invertibility

Let $S \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times k}$ and $U \in \mathbb{R}^{k\times k}$.
Let S and U be invertible.

1. Prove that $Ker(S) = 0$.

Now, prove or give a counter example to the following statements:

2. $Ker(T) = Ker(TU)$
3. $Ker(ST) = Ker(T)$

Solutions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.
Let S and U be invertible.

Solution

1. *Prove that* $Ker(S) = 0$.
   We prove by contradiction.
   Suppose that $Ker(S) \neq 0$. *Then* $\exists x \neq 0$ *s.t* $Sx = 0$.
   Now, consider $S^{-1}Sx$.
   $(S^{-1}S)x = Ix = x$,
   and $S^{-1}(Sx) = 0$.
   We have reached a contradiction, so $Ker(S) = 0$

Solutions 3: Invertibility

Solution

2. $Ker(T) = Ker(TU)$. ***False***

   $$Consider\ T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   $$Ker(T) = \{\begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R}\}.$$

   $$\mathrm{Ker}(TU) = \{\begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R}\}$$

3. $Ker(ST) = Ker(T)$. ***True***

   We'll show that $Ker(ST) \subset Ker(T)$.
   Let $x \in Ker(ST)$.
   So, $STx = 0$.
   Since S *is invertible, then* $Ker(S) = 0$.
   Therefore, $Tx = 0$, *and* $x \in Ker(T)$.
   $Ker(T) \subset Ker(ST)$ *is straightforward.*

Matrix Multiplication Mechanics: Inner Products

(We haven't defined inner product yet)
Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$
Rows of first matrix "line up" with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{k} a_{1,i} b_{i,1} & \dots & \dots \\ \sum_{i=0}^{k} a_{2,i} b_{i,1} & \dots & \dots \\ \vdots & \dots & \dots \\ \sum_{i=0}^{k} a_{n-1,i} b_{i,1} & \dots & \dots \\ \sum_{i=0}^{k} a_{n,i} b_{i,1} & \dots & \dots \end{bmatrix}$$

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$$= \begin{bmatrix} \dots & \sum_{i=0}^{k} a_{1,i} b_{i,2} & \dots \\ \dots & \sum_{i=0}^{k} a_{2,i} b_{i,2} & \dots \\ \dots & \vdots & \dots \\ \dots & \sum_{i=0}^{k} a_{n-1,i} b_{i,2} & \dots \\ \dots & \sum_{i=0}^{k} a_{n,i} b_{i,2} & \dots \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

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Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$
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$$= \begin{bmatrix} \dots & \dots & \sum_{i=0}^{k} a_{1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^{k} a_{2,i} b_{i,m} \\ \dots & \dots & \vdots \\ \dots & \dots & \sum_{i=0}^{k} a_{n-1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^{k} a_{n,i} b_{i,m} \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Inner Product method is the exact definition of matrix multiplication. Each entry of the resultant matrix is an inner product of a row of the first matrix and a column of the second matrix.

More M.M.M: Linear Combination of Columns

Each column of the result is a linear combination of the columns of A.

$$\begin{bmatrix} | & | & \dots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \dots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} | & \dots & | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} & \dots & \sum_{i=1}^{k} \mathbf{a_i} b_{k,m} \\ | & \dots & | \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

Each column of the result is a linear combination of the columns of A.

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} | & \cdots & | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} & \cdots & \sum_{i=1}^{k} \mathbf{a_i} b_{k,m} \\ | & \cdots & | \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

One dimensional case:

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{k-1,1} \\ b_{k,1} \end{bmatrix} = \begin{bmatrix} | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} \\ | \end{bmatrix}$$

Result is in the span of columns of A! (Keep this in mind for later).