Optimization and Computational Linear Algebra – Brett Bernstein

Recitation 2

1. Which of the following functions are linear? If the function is linear, what is the kernel?

    (a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$

    (b) $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$

    (c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$

    (d) $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$

    (e) $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

    Solution.

    (a) Linear:
    $$f_1(ca, cb) = (2ca, ca + cb) = c(2a, a + b) = cf_1(a, b)$$
    for all $c \in \mathbb{R}$ and
    $$\begin{aligned} f_1(a_1 + a_2, b_1 + b_2) &= (2(a_1 + a_2), (a_1 + a_2) + (b_1 + b_2)) \\ &= (2a_1, a_1 + b_1) + (2a_2, a_2 + b_2) \\ &= f_1(a_1, b_1) + f_1(a_2, b_2). \end{aligned}$$
    Kernel is $\{0\}$.

    (b) Linear, similar proof. Kernel is $\{(c, -c) : c \in \mathbb{R}\}$.

    (c) Not linear, $f_3(0, 0) = (0, 0, 1)$.

    (d) Not linear, $f_4(1, 0) + f_4(0, 1) = 2$ and $f_4(1, 1) = \sqrt{2}$.

    (e) Not linear, $f_5(0) = 3$.

2. If v, w are linearly independent, must $v, v + w$ also be linearly independent? Must it have the same span?

    Solution. Proven more generally in next question.

3. Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{R}^n$. Prove that each of the following are linearly independent and have the same span as $v_1, \ldots, v_m$.

    (a) $T = \{v_1, \ldots, v_{i-1}, av_i, v_{i+1}, \ldots, v_m\}$ where $a \neq 0$ and $1 \leq i \leq m$ is any index.

    (b) $U = \{v_1, \ldots, v_{i-1}, v_i + bv_j, v_{i+1}, \ldots, v_m\}$ where $j \neq i$ are indices, and $b \in \mathbb{R}$.

    Solution.

(a) *Proof.* First we prove linear independence. Suppose

$$\sum_{k \neq i} \alpha_k v_k + \alpha_i (a v_i) = 0$$

for some $\alpha \in \mathbb{R}^m$. Regrouping we have

$$\sum_{k \neq i} \alpha_k v_k + (\alpha_i a) v_i = 0$$

showing $\alpha = 0$ by the linear independence of $v_1, \ldots, v_m$ and using the fact that $a \neq 0$.

Next we prove it has the same span. Suppose $x \in \operatorname{Span}(v_1, \ldots, v_m)$ so that

$$x = \sum_{k=1}^{m} \alpha_k v_k = \sum_{k \neq i} \alpha_k v_k + (\alpha_i / a) a v_k$$

for some $\alpha \in \mathbb{R}^m$. This shows that $x \in \operatorname{Span}(T)$. Thus we have proven $\operatorname{Span}(v_1, \ldots, v_m) \subseteq \operatorname{Span}(T)$. To get the other direction, same argument can be applied in reverse using $1/a$ in place of a. $\square$

(b) *Proof.* First we prove linear independence. Suppose

$$\sum_{k \neq i} \alpha_k v_k + \alpha_i (v_i + b v_j) = 0$$

for some $\alpha \in \mathbb{R}^m$. Regrouping we have

$$0 = \sum_{k \neq i,j} \alpha_k v_k + \alpha_i v_i + (\alpha_j + \alpha_i b) v_j.$$

By the linear independence of B we have $\alpha_k = 0$ for $k \neq j$ and $\alpha_j + \alpha_i b = 0$. But $\alpha_i = 0$ so $\alpha_j = 0$ as well.

Next we prove it has the same span. Suppose $x \in \operatorname{Span}(v_1, \ldots, v_m)$ so that

$$x = \sum_{k=1}^{m} \alpha_k v_k = \sum_{k \neq i,j} \alpha_k v_k + \alpha_i (v_i + b v_j) + (\alpha_j - b \alpha_i) v_j$$

for some $\alpha \in \mathbb{R}^m$. This shows that $x \in \operatorname{Span}(U)$. The same argument can be applied again by replacing b with $-b$ to obtain the other direction. $\square$

4. Let $v_1, \ldots, v_m$ be linearly independent vectors in a subspace V of $\mathbb{R}^n$. Show that if they do not span V then there is a vector $w \in V$ such that $v_1, \ldots, v_m, w$ are linearly independent.

   Solution.

Proof. Choose $w \in V \setminus \mathrm{Span}(v_1, \ldots, v_m)$. Suppose

$$\alpha_1 v_1 + \cdots + \alpha_m v_m + bw = 0$$

for some $\alpha \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Rearranging we have

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = -bw.$$

If $b = 0$ then we must have $\alpha = 0$ by the linear independence of $v_1, \ldots, v_m$. If $b \neq 0$ then we divide to obtain

$$(-\alpha_1/b)v_1 + \cdots + (-\alpha_m/b)v_m = w$$

showing $w \in \mathrm{Span}(v_1, \ldots, v_m)$, a contradiction. □

5. Let $v_1, \ldots, v_m$ span $\mathbb{R}^n$ and suppose $w_1, \ldots, w_p$ is a subset of $\mathbb{R}^n$ with $p > m$. Then $w_1, \ldots, w_p$ are linearly dependent.

   Solution.

   Proof. Assume, for contradiction, that $w_1, \ldots, w_p$ are linearly independent. Since $v_1, \ldots, v_m$ spans $\mathbb{R}^n$, we can represent each w_i as a linear combination of the v_j:

$$\begin{array}{rcl} w_1 & = & \alpha_{1,1}v_1 + \cdots + \alpha_{1,m}v_m \\ w_2 & = & \alpha_{2,1}v_1 + \cdots + \alpha_{2,m}v_m \\ \vdots & \vdots & \vdots \\ w_p & = & \alpha_{p,1}v_1 + \cdots + \alpha_{p,m}v_m. \end{array}$$

   The earlier exercise shows we can perform row-reduction (also called Gaussian elimination) on the $p \times m$ matrix with entries $\alpha_{i,j}$ and preserve linear independence. Since $p > m$ row reduction will yield at least one row that is entirely zero. But a linearly independent set cannot have a zero vector giving us a contradiction. (See slides for more details on row reduction.) □

   Note, the proof only requires that $w_1, \ldots, w_p \in \mathrm{Span}(v_1, \ldots, v_m)$, so we actually obtain a more general result for arbitrary vector spaces.

6. Prove that any basis for $\mathbb{R}^n$ has length n.

   Solution.

   Proof. The standard basis has length n. If another basis B has a different length, then applying the previous exercise shows whichever of B or the standard basis is larger must be linearly dependent, a contradiction. □

7. Let V be a subspace of $\mathbb{R}^n$. Prove that V has a basis. [Recall that a basis for V is a linearly independent set of vectors that spans V.]

   Solution.

   Proof. Our plan is to begin with $S_0 = \emptyset$ and grow a basis for V one vector at a time. By exercise 4, if $V \neq \{0\}$ we can add a non-zero vector $v_1 \in V$ to obtain $S_1 = \{v_1\}$. Then we choose a vector $v_2 \in V \setminus \mathrm{Span}(v_1)$ and obtain $S_2 = \{v_1, v_2\}$. By exercise 4, we can always add a vector to S_k and maintain linear independence as long as $\mathrm{Span}(S_k) \neq V$. Thus, we are done if we can show this process of adding vectors must terminate in a finite number of steps. But we cannot add more than n vectors, since the earlier result shows the resulting set would be linearly dependent. □

8. Let $B = \{v_1, \ldots, v_m\}$ and $C = \{w_1, \ldots, w_p\}$ be disjoint linearly independent subsets of $\mathbb{R}^n$ (with m and p elements, respectively). Show that $B \cup C$ is linearly independent if $\mathrm{Span}(B) \cap \mathrm{Span}(C) = \{0\}$. [Disjointness is not necessary, but we include it to avoid confusion.]

   Solution.

   Proof. Suppose we have

$$\sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{p} \beta_j w_j = 0$$

   for some $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^p$. Then we can write

$$\sum_{i=1}^{m} \alpha_i v_i = \sum_{j=1}^{p} -\beta_j w_j.$$

   If $\alpha \neq 0$ then we must have $\sum_{i=1}^{m} \alpha_i v_i \neq 0$ since B is linearly independent. But then $\sum_{i=1}^{m} \alpha_i v_i \in \mathrm{Span}(B)$ and $\sum_{i=1}^{m} \alpha_i v_i \in \mathrm{Span}(C)$ contradicting our assumption that $\mathrm{Span}(B) \cap \mathrm{Span}(C) = \{0\}$. If $\alpha = 0$ then we have

$$0 = \sum_{j=1}^{p} -\beta_j w_j$$

   which forces $\beta = 0$ by the linear independence of C. Thus we have shown $\alpha = \beta = 0$ completing the proof of linear independence. □