Recitation 3

Alex Dong

CDS, NYU

Fall 2020

Rank Nullity Theorem

Theorem (Rank-Nullity Theorem (!!!))

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

- **One of the most important theorems in linear algebra**.
- You should be able to state and prove this theorem (with no notes).
- The ‘conservation of dimension’ theorem
    - The main rank inequality:
    - $rank(AB) \leq min(rank(A), rank(B))$
- If $n > m$, then the $max(dim(Im(L))) = m$

Questions: Rank-Nullity Theorem

Let $A \in \mathbb{R}^{l \times h}$ and $B \in \mathbb{R}^{h \times l}$, and $h > l$.
Prove or give a counterexample to the following statements.

1. $\exists A, B$ s.t AB is invertible.
2. $\exists A, B$ s.t.BA is invertible.

Questions: Rank-Nullity Theorem

Let $A \in \mathbb{R}^{l \times h}$ and $B \in \mathbb{R}^{h \times l}$, and $h > l$.
Prove or give a counterexample to the following statements.

Solution

1. *$\exists A, B$ s.t AB is invertible.* ***True***

   $$Consider\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.\ AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. *$\exists A, B$ s.t.BA is invertible.* ***False.***

   In order for BA to be invertible, $rank(BA) = 3$. However, $rank(BA) \leq rank(B) = 2$.

Symmetric Matrices: That's cute!

- Symmetric Matrices are not just "cute"...
  - They are actually DEEPLY LINKED to many topics in linear algebra.
- Concepts involving Symmetric Matrices
  - Orthogonal Projections (Lec 4) are symmetric.
  - Spectral Theorem (Lec 7) "eigenvectors of symmetric matrices are orthogonal".
  - PCA: Covariance matrix is symmetric
  - Concavity: Hessian Matrix (matrix of second derivative) is symmetric
- But, we will see most of this later. For now, just trust me!

Questions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

Prove/answer the following statements.

1. Show that $\forall x \in \mathbb{R}^n$, $x^T A^T A x \geq 0$
2. When is $x^T A^T A x = 0$?
3. Show that $Ker(A) = Ker(A^T A)$
4. Use this to show $rank(A) = rank(A^T A)$
5. Now, show that $rank(A) = rank(A^T)$

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

Solution

1. *Show that* $\forall x \in \mathbb{R}^n$, $x^T A^T A x \geq 0$
   Let $y = Ax$, *with* $y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$
   Then $x^T A^T A x = (Ax)^T (Ax) = y^T y = \sum_{i=1}^n {y_i}^2$.
   Since y_i *is a real number,* $\sum_{i=1}^n {y_i}^2 \geq 0$.
2. *What happens when* $x^T A^T A x = 0$?
   $\sum_{i=1}^n {y_i}^2 = 0 \iff y_i = 0 \quad \forall i \in \{1, ..., n\}$
   So, $x^T A^T A x = 0 \iff x \in Ker(A)$

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

3. Show that $Ker(A) = Ker(A^T A)$

Solution

$Ker(A) \subset Ker(A^T A)$ is trivial.

We now show $Ker(A^T A) \subset Ker(A)$.
Let $x \in Ker(A^T A)$.
Then $A^T Ax = 0$.
Then $x^T(A^T Ax) = x(0) = 0$
By the previous question, then $x \in Ker(A)$.

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

Prove or give a counter example to the following statements.

4. Use this to show $rank(A) = rank(A^T A)$

Solution

By the rank nullity theorem,

$$n = dim(Ker(A)) + rank(A)$$

Now, $A^T A \in \mathbb{R}^{n \times n}$*. So by the rank nullity theorem,*

$$n = dim(Ker(A^T A)) + rank(A^T A)$$

Setting these equations equal to each other yields:

$$dim(Ker(A)) + rank(A) = dim(Ker(A^T A)) + rank(A^T A)$$

And since $Ker(A) = Ker(A^T A)$*, then*

$$rank(A) = rank(A^T A)$$

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

5. Now show $rank(A) = rank(A^T)$

Solution

By the previous question,

$$rank(A) = rank(A^T A).$$

and similarly,

$$rank(A^T) = rank(AA^T).$$

Recall that $rank(T_1 T_2) \leq min(rank(T_1), rank(T_2))$, *and therefore,*

$$rank(T_1 T_2) \leq rank(T_1).$$

Applying this to $rank(A^T A)$ *and* $rank(AA^T)$ *yields:*

$$rank(A^T) \geq rank(A^T A) \quad \text{and} \quad rank(A) \geq rank(AA^T).$$

Replacing $rank(A^T A)$ *and* $rank(AA^T)$, *we get that*

$$rank(A^T) \geq rank(A) \quad \text{and} \quad rank(A) \geq rank(A^T).$$

So $rank(A) = rank(A^T)$

Solutions: Matrix Products

Let $x, y \in \mathbb{R}^{n\times 1}$.

1. What is the shape and rank of $x^T y$?
2. What is the shape and rank of xy^T?
3. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Show that the matrix product AB can be expressed as: $AB = C_1 + \cdots C_k$ s.t $rank(C_i) \leq 1$, $\forall i \in \{1, ..., n\}$.
   (Hint, try manually calculating for small values of m, k, n)

Questions: Matrix Products

Let $x, y \in \mathbb{R}^{n\times 1}$ both have rank 1.

Solution

1. *What is the shape and rank of $x^T y$?*
   Shape is 1×1 and rank is 1 (or 0).
2. *What is the shape and rank of xy^T?*
   Shape is $n \times n$ and rank is 1 (or 0).
3. *Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Show that the matrix product AB can be expressed as: $AB = C_1 + \cdots C_k$ s.t rank$(C_i) \leq 1$, $\forall i \in \{1, ..., n\}$.*
   (Hint, use 2. and manually calculating for small values of m, k, n)

   $$\text{Let } A = \begin{bmatrix} | & \dots & | \\ a_1 & \dots & a_k \\ | & \dots & | \end{bmatrix} \text{ and } B = \begin{bmatrix} — & b_1 & — \\ \vdots & \vdots & \vdots \\ — & b_k & — \end{bmatrix}$$