

Recitation 2

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Some Etymology...

Definition (Linearity: Wikipedia)

The property of a mathematical relationship (function) that can be graphically represented as a straight line.

Definition (Algebra: Wikipedia)

The study of mathematical symbols and the rules for manipulating these symbols.

- ▶ Linear algebra is the study of *manipulating* letters/symbols which are used to represent linear transformations.
- ▶ Two types of manipulation....

Type 1: Linear Transformations as Letters

Definition (Linear Transformation)

A function $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

- ▶ Our linear transformation here is represented by the *letter* L .
- ▶ We will examine the rules behind manipulating L from an *algebraic* perspective, such as...
 - ▶ Associative? Commutative?
 - ▶ Invertible?
 - ▶ Derivatives? (Homework 9)

Type 2: Linear Transformations as Matrices

Theorem (Matrix Representation Theorem)

*All linear transformations represent matrices;
all matrices represent linear transformations.*

- ▶ Important, but boring theorem.
- ▶ Linear transformations can also be represented by matrices

$$L = \begin{bmatrix} L_{1,1} & \dots & L_{1,n} \\ \vdots & \ddots & \vdots \\ L_{m,1} & \dots & L_{m,n} \end{bmatrix}$$

- ▶ We will also examine the *mechanical* perspective of linear transformations, such as...
 - ▶ How to actually multiply?
 - ▶ Interpretation of multiplication
 - ▶ Using matrix multiplication simply for calculations.
(Removing the notion of a transformation)
- ▶ (!) Think about which framework to use in your proofs!

A Note about Gaussian Elimination

- ▶ Gaussian Elimination is a procedure to calculate the solutions of a matrix equation.
- ▶ Not covered in this course, but you should be familiar with it.
- ▶ If this is the first time you've heard this, then please do some light studying to familiarize yourself with the process.
- ▶ Just know this at the high school/undergrad level
- ▶ **If you've already studied it in previous courses, that should be enough.**

Questions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solutions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solution

1. *Linear, Kernel is $\{0\}$.*
2. *Linear, Kernel is $\{(c, -c) : c \in \mathbb{R}\}$.*
3. *Not linear, $f_3(0, 0) = (0, 0, 1)$.*
4. *Not linear, $f_4(1, 0) + f_4(0, 1) = 2$ and $f_4(1, 1) = \sqrt{2}$.*
5. *Not linear, $f_5(0) = 3$.*

Matrix Notation

- ▶ A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is represented by a $m \times n$ matrix which is an element of $\mathbb{R}^{m \times n}$. (!! Note the order !!)

$$T = \begin{matrix} & \begin{matrix} n \\ T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{matrix} \\ \begin{matrix} m \\ \end{matrix} \left(\right. \end{matrix}$$

- ▶ This matrix has m rows and n columns.
- ▶ $T_{i,j}$ represents the entry in the i th row and j th column.

Matrix Multiplication Mechanics: Inner Products

- ▶ Next few slides go over “Inner Product Method” of matrix multiplication.
 - ▶ (This is a term I made up....)
 - ▶ We haven’t covered inner products yet
- ▶ Each entry of the resultant matrix is an inner product of a row of the first matrix and a column of the second matrix
- ▶ This is the *exact* definition of matrix multiplication.
- ▶ Most straightforward way to calculate a matrix product

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix}
 a_{1,1} & \dots & a_{1,k} \\
 a_{2,1} & \dots & a_{2,k} \\
 \vdots & \dots & \vdots \\
 a_{n-1,1} & \dots & a_{n-1,k} \\
 a_{n,1} & \dots & a_{n,k}
 \end{bmatrix}
 \begin{bmatrix}
 b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\
 \vdots & \vdots & \dots & \vdots & \vdots \\
 b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m}
 \end{bmatrix}$$

$$= \begin{bmatrix}
 \sum_{i=0}^k a_{1,i} b_{i,1} & \dots & \dots \\
 \sum_{i=0}^k a_{2,i} b_{i,1} & \dots & \dots \\
 \vdots & \dots & \dots \\
 \sum_{i=0}^k a_{n-1,i} b_{i,1} & \dots & \dots \\
 \sum_{i=0}^k a_{n,i} b_{i,1} & \dots & \dots
 \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix}
 a_{1,1} & \dots & a_{1,k} \\
 a_{2,1} & \dots & a_{2,k} \\
 \vdots & \dots & \vdots \\
 a_{n-1,1} & \dots & a_{n-1,k} \\
 a_{n,1} & \dots & a_{n,k}
 \end{bmatrix}
 \begin{bmatrix}
 b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\
 \vdots & \vdots & \dots & \vdots & \vdots \\
 b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m}
 \end{bmatrix}$$

$$= \begin{bmatrix}
 \dots & \sum_{i=0}^k a_{1,i} b_{i,2} & \dots \\
 \dots & \sum_{i=0}^k a_{2,i} b_{i,2} & \dots \\
 \dots & \vdots & \dots \\
 \dots & \sum_{i=0}^k a_{n-1,i} b_{i,2} & \dots \\
 \dots & \sum_{i=0}^k a_{n,i} b_{i,2} & \dots
 \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$
$$= \begin{bmatrix} \dots & \dots & \sum_{i=0}^k a_{1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^k a_{2,i} b_{i,m} \\ \vdots & \vdots & \vdots \\ \dots & \dots & \sum_{i=0}^k a_{n-1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^k a_{n,i} b_{i,m} \end{bmatrix}$$

- This is the *exact* definition of matrix multiplication.
- Most straightforward way to calculate a matrix product

More M.M.M: Linear Combination of Columns

- ▶ Next few slides go over “Linear Combination of Columns” method of matrix multiplication.
 - ▶ (Also a term I made up....heh...)
 - ▶ We *have* covered linear combinations :)
- ▶ Each *column* of the result is a *linear combination of the columns* of the first matrix.
- ▶ Much more interpretable!
- ▶ (!) Keep an eye out for this
- ▶ Less straightforward way of calculating

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A .

$$\begin{bmatrix} \begin{array}{|c|c|} \hline \mathbf{a}_1 & \mathbf{a}_2 \\ \hline \end{array} & \dots & \begin{array}{|c|c|} \hline \mathbf{a}_{k-1} & \mathbf{a}_k \\ \hline \end{array} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^k \mathbf{a}_i b_{i,1} & \dots & \sum_{i=1}^k \mathbf{a}_i b_{i,m} \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A .

$$\begin{bmatrix} \begin{array}{|c|c|} \hline \mathbf{a}_1 & \mathbf{a}_2 \\ \hline \end{array} & \dots & \begin{array}{|c|c|} \hline \mathbf{a}_{k-1} & \mathbf{a}_k \\ \hline \end{array} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix} \\
 = \begin{bmatrix} \sum_{i=1}^k \mathbf{a}_i b_{i,1} & \dots & \sum_{i=1}^k \mathbf{a}_i b_{i,m} \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

One dimensional case (for B):

$$\left[\begin{array}{cc|c|c} \text{a}_1 & \text{a}_2 & \dots & \text{a}_{k-1} & \text{a}_k \\ \hline \end{array} \right] \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{k-1,1} \\ b_{k,1} \end{bmatrix} = \left[\begin{array}{c} \sum_{i=1}^k \text{a}_i b_{i,1} \\ \hline \end{array} \right]$$

- ▶ Result is in the span of columns of A !
- ▶ Much more interpretable!
- ▶ (!) Keep an eye out for this, especially if columns of A have meaning.

Questions 2: Matrix Manipulation

$$\text{Let } A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Calculate AB
2. Calculate BC
3. What does A do to B ?
4. What does C do to B ?

5. Can you find an x s.t $Cx = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

Solutions 2: Matrix Manipulation

Solution

$$1. AB = \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$$2. BC = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

3. *Five times first row, switch second and third row*

4. *First column becomes twice the second column plus one times third column, second column stays the same, switch 3rd and fourth columns.*

5. No, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the span of the columns of C

Linear Transformations and Subspaces

- ▶ Linear transformations are *fundamentally connected* to subspaces.
- ▶ We will spend a lot of time on investigating the *action* of a linear transformation *from* subspaces, and *to* subspaces
- ▶ Key questions in linear algebra:
 - ▶ What does a linear transformation do to 1-dimensional (and *by linearity*) n -dimensional subspaces?
 - ▶ What are “nice” combinations of 1-dimensional subspaces?
 - ▶ How do linear transformations cut up vector spaces?
 - ▶ For a given linear transformation, are there certain, *special* subspaces? (Lec 6,7)

Questions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.

Let S and U be invertible.

1. Prove that $\text{Ker}(S) = \{0\}$.

Now, prove or give a counter example to the following statements:

2. $\text{Ker}(T) = \text{Ker}(TU)$
3. $\text{Ker}(ST) = \text{Ker}(T)$

Solutions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.

Let S and U be invertible.

1. Prove that $\text{Ker}(S) = \{0\}$.

Solution

We prove by contradiction.

Suppose that $\text{Ker}(S) \neq 0$. Then $\exists x \neq 0$ s.t $Sx = 0$.

Now, consider $S^{-1}Sx$.

$$(S^{-1}S)x = Ix = x,$$

$$\text{and } S^{-1}(Sx) = 0.$$

We have reached a contradiction, so $\text{Ker}(S) = 0$

Solutions 3: Invertibility

Solution

2. $\text{Ker}(T) = \text{Ker}(TU)$. **False**

$$\text{Consider } T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Ker}(T) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}.$$

$$\text{Ker}(TU) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

3. $\text{Ker}(ST) = \text{Ker}(T)$. **True**

We'll show that $\text{Ker}(ST) \subset \text{Ker}(T)$.

Let $x \in \text{Ker}(ST)$.

So, $STx = 0$.

Since S is invertible, then $\text{Ker}(S) = 0$.

Therefore, $Tx = 0$, and $x \in \text{Ker}(T)$.

$\text{Ker}(T) \subset \text{Ker}(ST)$ is straightforward.

Question 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$. Show that:

$$\text{Ker}(T) \cap \text{Im}(T) = \{0\} \iff \text{If } T^2v = 0, \text{ then } Tv = 0$$

Solution 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$. Show that:

$$\text{Ker}(T) \cap \text{Im}(T) = \{0\} \iff \text{If } T^2v = 0, \text{ then } Tv = 0$$

Solution

(\implies)

Assume that $\text{Ker}(T) \cap \text{Im}(T) = \{0\}$.

Assume that $T^2v = 0$. We will show that $Tv = 0$

Since $T^2v = 0$, then $T(Tv) = 0$, so $Tv \in \text{Ker}(T)$.

Now, by definition, $Tv \in \text{Im}(T)$, so $Tv \in \text{Ker}(T) \cap \text{Im}(T)$, and $Tv = 0$

(\impliedby)

Assume that $T^2v = 0 \implies Tv = 0$

Let $y \in \text{Ker}(T)$, and $y \in \text{Im}(T)$. We show that $y = 0$.

Since $y \in \text{Ker}(T)$, then $Ty = 0$.

Since $y \in \text{Im}(T)$, then $\exists x$ s.t $Tx = y$.

Then $0 = Ty = T(Tx) = T^2x$. Since $T^2x = 0$, then $Tx = 0$. So $y = Tx = 0$.