Optimization and Computational Linear Algebra – Brett Bernstein

Recitation 4

It may be helpful if after recitation you try to re-solve these problems by yourself, and use them as additional study problems for the class.

1. Let $A, B \in \mathbb{R}^{n\times n}$. Prove that AB is invertible iff A and B are invertible.

   Solution.

   Proof. First suppose A and B are invertible. We claim that $(AB)^{-1} = B^{-1}A^{-1}$. To see this note that

   $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

   Conversely, suppose that AB is invertible. Since $Bx = 0$ implies $ABx = 0$ we must have $\ker(B) = \{0\}$ (which is equivalent to $\text{rank}(B) = n$ by the fundamental theorem). Furthermore, since $\text{rank}(AB) \leq \text{rank}(A)$ (from homework 3) we must have $\text{rank}(A) = n$. But an $n \times n$ matrix is invertible iff it has $\text{rank}(n)$, thus the proof is complete. $\square$

2. Let $A \in \mathbb{R}^{m\times n}$. Under what conditions is there a matrix B such that $AB = I$ where I is the $m \times m$ identity matrix?

   Solution. The condition is $\text{rank}(A) = m$. To see this note that the ith column of AB is equal to Ab_i where b_i is the ith column of b. Thus the question can be reformulated as: "When can the equation $Ax = e_i$ be solved for each i?" This is the same as saying $e_i \in \text{im}(A)$ for $i = 1, \ldots, m$, which is the same as saying $\text{im}(A) = \mathbb{R}^m$.

3. Let $A \in \mathbb{R}^{m\times n}$. Under what conditions is there a matrix B such that $BA = I$ where I is the $n \times n$ identity matrix?

   Solution. When $\text{rank}(A) = n$, or equivalently, when $\ker(A) = \{0\}$. To see note that $BA = I$ iff $A^TB^T = I$, and apply the previous exercise.

4. Determine which of the following statements are equivalent. Below $A \in \mathbb{R}^{m\times n}$.

   (a) The columns of A are linearly independent.

   (b) The rows of A are linearly independent.

   (c) $\text{rank}(A) = n$.

   (d) $\text{rank}(A) = m$.

   (e) The equation $Ax = 0$ has exactly one solution $x \in \mathbb{R}^n$.

   (f) $Ax = b$ has at least one solution $x \in \mathbb{R}^n$ for every $b \in \mathbb{R}^m$.

(g) $\ker(A) = \{0\}$.

(h) The span of the columns of A is $\mathbb{R}^m$.

(i) $\operatorname{im}(A) = \mathbb{R}^m$.

(j) $\operatorname{im}(A^T) = \mathbb{R}^n$.

(k) The linear transformation corresponding to A is one-to-one.

(l) The linear transformation corresponding to A is onto.

Solution. The following groups of statements are equivalent.

- a, c, e, g, j, k
- b, d, f, h, i, l

5. Explain why each of the following functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is not an inner product.

(a) $f(x, y) = x_1y_2 + x_2y_3 + x_3y_1$

(b) $f(x, y) = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$

(c) $f(x, y) = x_1y_1 + x_2y_2$

Solution.

(a) $f(e_1, e_1) = 0$ (must be > 0)

(b) $f(2e_1, e_1) = 2^2 f(e_1, e_1)$ (instead of $2f(e_1, e_1)$)

(c) $f(e_3, e_3) = 0$ (must be > 0)

6. Let $x = (\cos\theta_1, \sin\theta_1) \in \mathbb{R}^2$ and $y = (\cos\theta_2, \sin\theta_2) \in \mathbb{R}^2$ be two vectors on the unit circle (i.e., $\|x\| = 1 = \|y\|$). Explain the phrase "$x^T y$ gives a measure of the angle between x and y."

Solution. Note that

$$x^T y = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) = \cos(\theta_1 - \theta_2),$$

so for vectors on the unit circle, the Euclidean inner product gives the cosine of the angle between them.

In general we have $x^T y = \|x\|\|y\|\cos(\theta)$ where θ is the angle between x and y measured in the plane $\operatorname{Span}(x, y)$.

7. Let $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. For $A \in \mathbb{R}^{m\times n}$ compute $x^T Ay$ in terms of the entries of x, y, and A.

Solution.

$$x^T Ay = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_i y_j$$

8. Suppose you have the equation $LUx = b$ where $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix, $U \in \mathbb{R}^{n\times n}$ is an upper triangular matrix, and $b \in \mathbb{R}^n$ is a fixed vector. Give a procedure for how you could solve for x using only "substitution" as described in class.

   Solution. First solve $Ly = b$ for y. Then solve $Ux = y$ for x.

9. Fix $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and suppose the set

$$S = \{x \in \mathbb{R}^n : Ax = b\}$$

   is non-empty.

   (a) Under what conditions is S a subspace of $\mathbb{R}^n$?

   (b) Suppose S is not a subspace of $\mathbb{R}^n$. Show there is a $v \in \mathbb{R}^n$ such that

$$S - v := \{x - v : x \in S\}$$

   is a subspace. What does this say geometrically?

   Solution.

   (a) S is a subspace iff $b = 0$ (so that $S = \ker(A)$). Otherwise $0 \notin S$.

   (b) Let v be any element of S. Then we claim that $S - v = \ker(A)$. To see this note that for any $x \in S$ we have

$$A(x - v) = Ax - Av = b - b = 0.$$

   Geometrically, this says that the solution set to a linear system, if it is non-empty, has the form $\ker(A) + v$ where v is any solution.

10. Prove or disprove: Let $n > 1$. Every $A \in \mathbb{R}^{n\times n}$ can be written as $A = LU$ where L is lower triangular and U is upper triangular.

   Solution. False. Let $M \in \mathbb{R}^{2\times 2}$ be defined by

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Suppose $M = LU$. Since $\operatorname{rank}(M) = 2$ it must be that

$$L = \begin{bmatrix} L_{1,1} & 0 \\ L_{2,1} & L_{2,2} \end{bmatrix}$$

   with $L_{1,1} \neq 0$. But since $M_{1,1} = 0$ we must have

$$U = \begin{bmatrix} U_{1,1} & U_{1,2} \\ 0 & U_{2,2} \end{bmatrix}$$

   with $U_{1,1} = 0$. But then $\operatorname{rank}(U) < 2$ a contradiction (since then $\ker(U) \neq \{0\}$ so $\ker(LU) \neq \{0\}$).

11. Suppose you are given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. How do you solve the equation $Ax = b$ using Python?

Solution. Use

```
numpy.linalg.lstsq
```

and check if the returned vector x has small residual: $\|Ax - b\|$.