

Recitation 2

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Some Etymology...

Definition (Linearity: Wikipedia)

The property of a mathematical relationship (function) that can be graphically represented as a straight line.

Definition (Algebra: Wikipedia)

The study of mathematical symbols and the rules for manipulating these symbols.

- ▶ Linear algebra is the study of *manipulating* letters/symbols which are used to represent linear transformations.
- ▶ Two types of manipulation....

Type 1: Linear Transformations as Letters

Definition (Linear Transformation)

A function $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

- ▶ Our linear transformation here is represented by the *letter* L .
- ▶ We will examine the rules behind manipulating L from an *algebraic* perspective, such as...
 - ▶ Associative? Commutative?
 - ▶ Invertible?
 - ▶ Derivatives? (Homework 9)

Type 2: Linear Transformations as Matrices

Theorem (Matrix Representation Theorem)

*All linear transformations represent matrices;
all matrices represent linear transformations.*

- ▶ Important, but boring theorem.
- ▶ Linear transformations can also be represented by matrices

$$L = \begin{bmatrix} L_{1,1} & \dots & L_{1,n} \\ \vdots & \ddots & \vdots \\ L_{m,1} & \dots & L_{m,n} \end{bmatrix}$$

- ▶ We will also examine the *mechanical* perspective of linear transformations, such as...
 - ▶ How to actually multiply?
 - ▶ Interpretation of multiplication
 - ▶ Using matrix multiplication simply for calculations.
(Removing the notion of a transformation)
- ▶ (!) Think about which framework to use in your proofs!

A Note about Gaussian Elimination

- ▶ Gaussian Elimination is a procedure to calculate the solutions of a matrix equation.
- ▶ Not covered in this course, but you should be familiar with it.
- ▶ If this is the first time you've heard this, then please do some light studying to familiarize yourself with the process.
- ▶ Just know this at the high school/undergrad level
- ▶ **If you've already studied it in previous courses, that should be enough.**

Questions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solutions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solution

1. *Linear, Kernel is $\{0\}$.*
2. *Linear, Kernel is $\{(c, -c) : c \in \mathbb{R}\}$.*
3. *Not linear, $f_3(0, 0) = (0, 0, 1)$.*
4. *Not linear, $f_4(1, 0) + f_4(0, 1) = 2$ and $f_4(1, 1) = \sqrt{2}$.*
5. *Not linear, $f_5(0) = 3$.*

Matrix Notation

- ▶ A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is represented by a $m \times n$ matrix which is an element of $\mathbb{R}^{m \times n}$. (!! Note the order !!)

$$T = \begin{matrix} & \begin{matrix} n \end{matrix} \\ \begin{matrix} m \end{matrix} & \left(\begin{array}{ccc} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{array} \right) \end{matrix}$$

- ▶ This matrix has m rows and n columns.
- ▶ $T_{i,j}$ represents the entry in the i th row and j th column.

Matrix Multiplication Mechanics: Inner Products

- ▶ Next few slides go over “Inner Product Method” of matrix multiplication.
 - ▶ (This is a term I made up....)
 - ▶ We haven’t covered inner products yet
- ▶ Each entry of the resultant matrix is an inner product of a row of the first matrix and a column of the second matrix
- ▶ This is the *exact* definition of matrix multiplication.
- ▶ Most straightforward way to calculate a matrix product

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix}
 a_{1,1} & \dots & a_{1,k} \\
 a_{2,1} & \dots & a_{2,k} \\
 \vdots & \dots & \vdots \\
 a_{n-1,1} & \dots & a_{n-1,k} \\
 a_{n,1} & \dots & a_{n,k}
 \end{bmatrix}
 \begin{bmatrix}
 b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\
 \vdots & \vdots & \dots & \vdots & \vdots \\
 b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m}
 \end{bmatrix}$$

$$= \begin{bmatrix}
 \sum_{i=0}^k a_{1,i} b_{i,1} & \dots & \dots \\
 \sum_{i=0}^k a_{2,i} b_{i,1} & \dots & \dots \\
 \vdots & \dots & \dots \\
 \sum_{i=0}^k a_{n-1,i} b_{i,1} & \dots & \dots \\
 \sum_{i=0}^k a_{n,i} b_{i,1} & \dots & \dots
 \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix}
 a_{1,1} & \dots & a_{1,k} \\
 a_{2,1} & \dots & a_{2,k} \\
 \vdots & \dots & \vdots \\
 a_{n-1,1} & \dots & a_{n-1,k} \\
 a_{n,1} & \dots & a_{n,k}
 \end{bmatrix}
 \begin{bmatrix}
 b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\
 \vdots & \vdots & \dots & \vdots & \vdots \\
 b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m}
 \end{bmatrix}$$

$$= \begin{bmatrix}
 \dots & \sum_{i=0}^k a_{1,i} b_{i,2} & \dots \\
 \dots & \sum_{i=0}^k a_{2,i} b_{i,2} & \dots \\
 \dots & \vdots & \dots \\
 \dots & \sum_{i=0}^k a_{n-1,i} b_{i,2} & \dots \\
 \dots & \sum_{i=0}^k a_{n,i} b_{i,2} & \dots
 \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$
$$= \begin{bmatrix} \dots & \dots & \sum_{i=0}^k a_{1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^k a_{2,i} b_{i,m} \\ \vdots & \vdots & \vdots \\ \dots & \dots & \sum_{i=0}^k a_{n-1,i} b_{i,m} \\ \dots & \dots & \sum_{i=0}^k a_{n,i} b_{i,m} \end{bmatrix}$$

- This is the *exact* definition of matrix multiplication.
- Most straightforward way to calculate a matrix product

More M.M.M: Linear Combination of Columns

- ▶ Next few slides go over “Linear Combination of Columns” method of matrix multiplication.
 - ▶ (Also a term I made up....heh...)
 - ▶ We *have* covered linear combinations :)
- ▶ Each *column* of the result is a *linear combination of the columns* of the first matrix.
- ▶ Much more interpretable!
- ▶ (!) Keep an eye out for this
- ▶ Less straightforward way of calculating

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A .

$$\begin{bmatrix} \begin{array}{|c|c|} \hline \mathbf{a}_1 & \mathbf{a}_2 \\ \hline \end{array} & \dots & \begin{array}{|c|c|} \hline \mathbf{a}_{k-1} & \mathbf{a}_k \\ \hline \end{array} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^k \mathbf{a}_i b_{i,1} & \dots & \sum_{i=1}^k \mathbf{a}_i b_{i,m} \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A .

$$\begin{bmatrix} \begin{array}{|c|c|} \hline \mathbf{a}_1 & \mathbf{a}_2 \\ \hline \end{array} & \dots & \begin{array}{|c|c|} \hline \mathbf{a}_{k-1} & \mathbf{a}_k \\ \hline \end{array} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix} \\
 = \begin{bmatrix} \sum_{i=1}^k \mathbf{a}_i b_{i,1} & \dots & \sum_{i=1}^k \mathbf{a}_i b_{i,m} \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

One dimensional case (for B):

$$\left[\begin{array}{cc|cc|c} \text{a}_1 & \text{a}_2 & \dots & \text{a}_{k-1} & \text{a}_k \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{k-1,1} \\ b_{k,1} \end{bmatrix} = \left[\begin{array}{c} \sum_{i=1}^k \text{a}_i b_{i,1} \\ \hline \vdots \end{array} \right]$$

- ▶ Result is in the span of columns of A !
- ▶ Much more interpretable!
- ▶ (!) Keep an eye out for this, especially if columns of A have meaning.

Questions 2: Matrix Manipulation

$$\text{Let } A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1. Calculate AB
2. Calculate BC
3. What does A do to B ?
4. What does C do to B ?

5. Can you find an x s.t $Cx = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

Solutions 2: Matrix Manipulation

Solution

$$1. AB = \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$$2. BC = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

3. Five times first row, switch second and third row

4. First column becomes twice the second column plus one times third column, second column stays the same, switch 3rd and fourth columns.

5. No, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the span of the columns of C

Linear Transformations and Subspaces

- ▶ Linear transformations are *fundamentally connected* to subspaces.
- ▶ We will spend a lot of time on investigating the *action* of a linear transformation *from* subspaces, and *to* subspaces
- ▶ Key questions in linear algebra:
 - ▶ What does a linear transformation do to 1-dimensional (and *by linearity*) n -dimensional subspaces?
 - ▶ What are “nice” combinations of 1-dimensional subspaces?
 - ▶ How do linear transformations cut up vector spaces?
 - ▶ For a given linear transformation, are there certain, *special* subspaces? (Lec 6,7)

Questions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.

Let S and U be invertible.

1. Prove that $\text{Ker}(S) = \{0\}$.

Now, prove or give a counter example to the following statements:

2. $\text{Ker}(T) = \text{Ker}(TU)$
3. $\text{Ker}(ST) = \text{Ker}(T)$

Solutions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.

Let S and U be invertible.

1. Prove that $\text{Ker}(S) = \{0\}$.

Solution

We prove by contradiction.

Suppose that $\text{Ker}(S) \neq 0$. Then $\exists x \neq 0$ s.t $Sx = 0$.

Now, consider $S^{-1}Sx$.

$$(S^{-1}S)x = Ix = x,$$

$$\text{and } S^{-1}(Sx) = 0.$$

We have reached a contradiction, so $\text{Ker}(S) = 0$

Solutions 3: Invertibility

Solution

2. $\text{Ker}(T) = \text{Ker}(TU)$. **False**

$$\text{Consider } T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Ker}(T) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}.$$

$$\text{Ker}(TU) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

3. $\text{Ker}(ST) = \text{Ker}(T)$. **True**

We'll show that $\text{Ker}(ST) \subset \text{Ker}(T)$.

Let $x \in \text{Ker}(ST)$.

So, $STx = 0$.

Since S is invertible, then $\text{Ker}(S) = 0$.

Therefore, $Tx = 0$, and $x \in \text{Ker}(T)$.

$\text{Ker}(T) \subset \text{Ker}(ST)$ is straightforward.

Question 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$. Show that:

$$\text{Ker}(T) \cap \text{Im}(T) = \{0\} \iff \text{If } T^2 v = 0, \text{ then } Tv = 0$$

Solution 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$. Show that:

$$\text{Ker}(T) \cap \text{Im}(T) = \{0\} \iff \text{If } T^2v = 0, \text{ then } Tv = 0$$

Solution

(\implies)

Assume that $\text{Ker}(T) \cap \text{Im}(T) = \{0\}$.

Assume that $T^2v = 0$. We will show that $Tv = 0$

Since $T^2v = 0$, then $T(Tv) = 0$, so $Tv \in \text{Ker}(T)$.

Now, by definition, $Tv \in \text{Im}(T)$, so $Tv \in \text{Ker}(T) \cap \text{Im}(T)$, and $Tv = 0$

(\impliedby)

Assume that $T^2v = 0 \implies Tv = 0$

Let $y \in \text{Ker}(T)$, and $y \in \text{Im}(T)$. We show that $y = 0$.

Since $y \in \text{Ker}(T)$, then $Ty = 0$.

Since $y \in \text{Im}(T)$, then $\exists x$ s.t $Tx = y$.

Then $0 = Ty = T(Tx) = T^2x$. Since $T^2x = 0$, then $Tx = 0$. So $y = Tx = 0$.