Optimization and Computational Linear Algebra – Brett Bernstein

Recitation 3

1. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be treated as column vectors (which is standard). How many rows and columns does xy^T have? What is its rank?

   Solution. $xy^T \in \mathbb{R}^{n\times m}$. It will have rank 0 if $x = 0$ or $y = 0$ and rank 1 otherwise. To see this, note that the ith column is $y_i x$, so assuming $y \neq 0$ we have $\mathrm{Span}(x) = \mathrm{Im}(xy^T)$.

2. Let $x, y \in \mathbb{R}^n$ be treated as column vectors. How many rows and columns does $x^T y$ have? What are its entries?

   Solution. It has 1 row and 1 column (i.e., a real number). The value is given by

$$x^T y = x_1 y_1 + \cdots + x_n y_n.$$

3. Let $A \in \mathbb{R}^{m\times n}$. Show that if $x \in \mathrm{Ker}(A)$ then $v_i^T x = 0$ for $i = 1, \ldots, m$ where $v_i^T \in \mathbb{R}^{1\times n}$ is the ith row of A.

   Solution.

   Proof. By the formula for matrix multiplication we have

$$0 = Ax = \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_m^T & — \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_m^T x \end{bmatrix}.$$

   $\square$

4. Fix $A \in \mathbb{R}^{4\times 5}$ and describe the following set:

$$\left\{ Ax : x = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ c \end{bmatrix},\ a, b, c \in \mathbb{R} \right\}.$$

   Solution. The span of the first, second, and fifth columns of A.

5. Given a matrix $A \in \mathbb{R}^{3\times 3}$, show how to swap the first 2 rows via matrix multiplication.

Solution.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

6. Given a matrix $A \in \mathbb{R}^{3\times 3}$, show how to replace the first row with the sum of the first row and 5 times the third row via matrix multiplication.

   Solution.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

7. Given a matrix $A \in \mathbb{R}^{3\times 3}$, show how to swap the first 2 columns via matrix multiplication.

   Solution.

$$A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

8. ($\star$) Let $A \in \mathbb{R}^{m\times k}$ and let $B \in \mathbb{R}^{k\times n}$. Show that AB can be written as

$$AB = C_1 + C_2 + \cdots + C_k$$

   with $\operatorname{rank}(C_i) \leq 1$ for $i = 1, \ldots, k$.

   Solution. Let a_i denote the ith column of A and let b_j^T denote the jth row of B. Then we have

$$AB = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & b_1^T & — \\ — & b_2^T & — \\ & \vdots & \\ — & b_k^T & — \end{bmatrix} = a_1 b_1^T + a_2 b_2^T + \cdots + a_k b_k^T.$$

9. (Fundamental Theorem) ($\star$) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Prove that $\dim(\operatorname{Ker}(L)) + \operatorname{rank}(L) = n$.

   Solution.

   Proof. Let $v_1, \ldots, v_p \in \mathbb{R}^n$ be a basis for $\operatorname{Ker}(L)$, where $p = \dim(\operatorname{Ker}(L))$. By repeatedly adding vectors not in the span (previous lab) we can grow this list to form a basis for $\mathbb{R}^n$:

$$v_1, \ldots, v_p, v_{p+1}, \ldots, v_n.$$

We claim that $Lv_{p+1}, \dots, Lv_n$ form a basis for $\mathrm{Im}(L)$. Note that if we can show this claim we are done, since this gives

$$\mathrm{rank}(L) = \dim(\mathrm{Im}(L)) = n - p.$$

We first show that we have a spanning set. Let $w \in \mathbb{R}^n$ so that Lw is an arbitrary element of $\mathrm{Im}(L)$. Since $v_1, \dots, v_n$ is a basis for $\mathbb{R}^n$ we can write

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

But then

$$Lw = \alpha_1 Lv_1 + \cdots + \alpha_n Lv_n = \alpha_{p+1} Lv_{p+1} + \cdots + \alpha_n Lv_n$$

since $v_1, \dots, v_p \in \mathrm{Ker}(L)$. Thus $Lw \in \mathrm{Span}(Lv_{p+1}, \dots, Lv_n)$.

Next we show that we have a linearly independent set. Suppose

$$\alpha_1 Lv_{p+1} + \cdots + \alpha_{n-p} Lv_n = 0$$

for some $\alpha \in \mathbb{R}^{n-p}$. Then

$$L(\alpha_1 v_{p+1} + \cdots + \alpha_{n-p} v_n) = 0$$

by linearity. If $\alpha_1 v_{p+1} + \cdots + \alpha_{n-p} v_n \neq 0$ then it is a non-zero vector in the kernel, so we can write

$$\alpha_1 v_{p+1} + \cdots + \alpha_{n-p} v_n = \beta_1 v_1 + \cdots + \beta_p v_p$$

showing that $v_1, \dots, v_n$ are linearly dependent, a contradiction. □

10. ($\star$) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that every $x \in \mathbb{R}^n$ can be written uniquely as $x = u + v$ where $u \in \mathrm{Ker}(L)$ and $v \in \mathrm{Im}(L^T)$ (also called the row space).

Solution.

Proof. Let $v_1, \dots, v_p$ be a basis for $\mathrm{Ker}(L)$ with $p = \dim \mathrm{Ker}(L)$ and let $w_1, \dots, w_q$ be a basis for $\mathrm{Im}(L^T)$ (the row space) with $q = \mathrm{rank}(L)$ (homework shows $\mathrm{rank}(L) = \mathrm{rank}(L^T)$). We claim that

$$v_1, \dots, v_p, w_1, \dots, w_q$$

is a basis for $\mathbb{R}^n$. Note that $p + q = n$ by the fundamental theorem, so we need only prove linear independence. By the previous lab, this reduces to proving that

$$\mathrm{Ker}(L) \cap \mathrm{Im}(L^T) = \{0\}.$$

To that end, let $u \in \mathrm{Ker}(L) \cap \mathrm{Im}(L^T)$. Since $u \in \mathrm{Im}(L^T)$ we can write

$$u = \alpha_1 x_1 + \cdots + \alpha_m x_m$$

where $x_1^T, \ldots, x_m^T$ are the rows of the matrix corresponding to L. Note that

$$u^T u = (\alpha_1 x_1 + \cdots + \alpha_m x_m)^T u = \alpha_1 x_1^T u + \cdots + \alpha_m x_m^T u = 0 + \cdots + 0 = 0$$

since $u \in \mathrm{Ker}(L)$ (apply exercise 3 above). But note that

$$0 = u^T u = u_1^2 + \cdots + u_n^2$$

implies $u_1 = u_2 = \cdots = u_n = 0$. □

11. Determine all solutions $x \in \mathbb{R}^3$ to

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 1 & 3 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution. Suppose x is a solution. By multiplying both sides of the equation by the appropriate matrices (see earlier exercises) we can row-reduce the matrix while maintaining the equality. This gives

$$\begin{aligned}
\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 1 & 3 & 3 \end{bmatrix} x &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \\
\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 3 \end{bmatrix} x &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \\
\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \\
\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

By noting that each row reduction operation is reversible (the correponding matrices are invertible) we see that x is a solution to our original equation iff x is a solution to our final row reduced equation. Writing the above in terms of the coordinates of $x = (x_1, x_2, x_3)$ we have

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving in terms of x_2 (which happens to correspond to a column with no leading non-zero coefficient in it) we obtain

$$\begin{aligned} x_1 &= -3x_2 \\ x_3 &= 0. \end{aligned}$$

Thus the solution set is (letting c be a placeholder for the value of x_2)

$$\left\{ \begin{bmatrix} -3c \\ c \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\} = \left\{ c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right).$$