Recitation 3

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Rank Nullity Theorem

Theorem (Rank-Nullity theorem)

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

One of the most important theorems in linear algebra.
You should be able to state and prove this theorem (with no notes).

Questions: Rank-Nullity Theorem

Let $A \in \mathbb{R}^{l \times h}$ and $B \in \mathbb{R}^{h \times l}$, and $h > l$.

Prove or give a counterexample to the following statements.

1. $\exists A, B$ s.t AB is invertible.
2. $\exists A, B$ s.t.BA is invertible.

Questions: Rank-Nullity Theorem

Let $A \in \mathbb{R}^{l \times h}$ and $B \in \mathbb{R}^{h \times l}$, and $h > l$.
Prove or give a counterexample to the following statements.

Solution

1. $\exists A, B$ *s.t* AB *is invertible.* ***True***
   Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\exists A, B$ *s.t.* BA *is invertible.* ***False****.*
   In order for BA *to be invertible,* $rank(BA) = 3$*. However,* $rank(BA) \le rank(B) = 2$*.*

Symmetric Matrices: That's cute!

- Symmetric Matrices are not just "cute"...
  - They are actually DEEPLY LINKED to many topics in linear algebra.
- Concepts involving Symmetric Matrices
  - Orthogonal Projections (next lecture) are symmetric .
  - Spectral theorem - "eigenvectors of sym matrices are special".
  - PCA: Covariance matrix is symmetric
  - Concavity: Hessian Matrix (matrix of second derivative) is symmetric
- But, we will see most of this later. For now, just trust me!

Questions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.

Prove or give a counter example to the following statements.

1. Show that $\forall x \in \mathbb{R}^n$, $x^T A^T A x \geq 0$
2. When is $x^T A^T A x = 0$?
3. Show that $Ker(A) = Ker(A^T A)$
4. Use this to show $rank(A) = rank(A^T A)$
5. Now, show that $rank(A) = rank(A^T)$

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.
Prove or give a counter example to the following statements.

Solution

1. *Show that* $\forall x \in \mathbb{R}^n$, $x^T A^T Ax \geq 0$
   Let $y = Ax$, *with* $y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$
   Then $x^T A^T Ax = (Ax)^T (Ax) = y^T y = \sum_{i=1}^n {y_i}^2$.
   Since y_i *is a real number,* $\sum_{i=1}^n {y_i}^2 \geq 0$.
2. *When is* $x^T A^T Ax = 0$?
   $\sum_{i=1}^n {y_i}^2 = 0$ *when all* y_i *are 0. So when* $x \in Ker(A)$

Let $A \in \mathbb{R}^{k \times n}$.
Prove or give a counter example to the following statements.

Solution

3. Show that $Ker(A) = Ker(A^T A)$
$Ker(A) \subset Ker(A^T A)$ *is trivial.*

We now show $Ker(A^T A) \subset Ker(A)$.
Let $x \in Ker(A^T A)$.
Then $A^T A x = 0$.
Then $x^T (A^T A x) = x(0) = 0$
By the previous question, then $x \in Ker(A)$.

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.
Prove or give a counter example to the following statements.

Solution

4. *Use this to show* $rank(A) = rank(A^T A)$
By the rank nullity theorem,

$$n = dim(Ker(A)) + rank(A)$$

Now, $A^T A \in \mathbb{R}^{n \times n}$*. So by the rank nullity theorem,*

$$n = dim(Ker(A^T A)) + rank(A^T A)$$

Now,

$$dim(Ker(A)) + rank(A) = dim(Ker(A^T A)) + rank(A^T A)$$

And since $Ker(A) = Ker(A^T A)$*, then*

$$rank(A) = rank(A^T A)$$

Solutions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$.
Prove or give a counter example to the following statements.

Solution

5. *Now show* $rank(A) = rank(A^T)$
   By the previous question,
   $$rank(A) = rank(A^T A).$$
   and similarly,
   $$rank(A^T) = rank(AA^T).$$
   Now, $rank(AB) \leq min(rank(A), rank(B))$*, then we get*
   $$rank(A^T) \geq rank(A^T A) \quad and \quad rank(A) \geq rank(AA^T).$$
   Replacing $rank(A^T A)$ *and* $rank(AA^T)$*, we get that*
   $$rank(A^T) \geq rank(A) \quad and \quad rank(A) \geq rank(A^T).$$
   So $rank(A) = rank(A^T)$

Questions: Matrix Products

Let $x, y \in \mathbb{R}^{n\times 1}$.

1. What is the shape and rank of $x^T y$?
2. What is the shape and rank of xy^T?
3. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Show that the matrix product AB can be expressed as: $AB = C_1 + \cdots C_k$ s.t $rank(C_i) \leq 1$, $\forall i \in \{1, ..., n\}$.
   (Hint, try manually calculating for small values of m, k, n)

Questions: Matrix Products

Let $x, y \in \mathbb{R}^{n\times 1}$ both have rank 1.

Solution

1. *What is the shape and rank of $x^T y$?*
   Shape is 1×1 and rank is 1 (or 0).
2. *What is the shape and rank of xy^T?*
   Shape is $n \times n$ and rank is 1 (or 0).
3. *Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Show that the matrix product AB can be expressed as: $AB = C_1 + \cdots C_k$ s.t $rank(C_i) \leq 1$, $\forall i \in \{1, ..., n\}$.*
   (Hint, try manually calculating for small values of m, k, n)

   Let $A = \begin{bmatrix} | & \dots & | \\ a_1 & \dots & a_k \\ | & \dots & | \end{bmatrix}$ *and* $B = \begin{bmatrix} — & b_1 & — \\ \vdots & \vdots & \vdots \\ — & b_k & — \end{bmatrix}$