Recitation 3

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Norms

Norms measure distances! Think about all the properties of distance that make sense.

- distance $= 0$ means at the same point
- distance is always non-negative
- distance follows triangle inequality (well... at least in euclidean space)

Norms

Shorthand way to remember what the properties do.

Definition (Norm)

A norm $\|\cdot\|$ on V verifies the following points:

1. *Triangular inequality*: $\|u+v\| \leq \|u\| + \|v\|$ "Euclidean space"
2. *Homogeneity*: $\|\alpha v\| = |\alpha| \times \|v\|$ "farther actually means farther"
3. *Positive definiteness*: if $\|v\| = 0 \implies v = 0$. "Non-negative"

Inner Products

Inner products measure angles! ... but not directly.

$$cos(\theta) = \frac{< u, v >}{\|u\|\|v\|}$$

or

$$< u, v > = \ cos(\theta)\|u\|\|v\|$$

When u, v are unit vectors, inner product gives a measure for angle between vectors.

Inner Products in Machine Learning

- Angles can be used as a measure of similarity
- Kernel Tricks - Increase data complexity
  - Sometimes you have to calculate $x_{old}^T x_{new}$, equivalently $< x_{old}, x_{new} >$
  - You can replace the inner product with a inner product in a higher dimensional space
  - Instead of calculating $< x_{old}, x_{new} >$, define a function K and calculate $< K(x_{old}), K(x_{new}) >$
  - if you pick "the right" higher dimensional space, your data can be a lot easier to work with

Orthogonality

- Angles can be used as a measure of similarity
- Vectors u, v are orthogonal if and only if $< u, v >= 0$
- Vectors are orthogonal $\implies$ vectors are as dissimilar as possible
- Orthogonal coordinate systems are good because we can view each coordinate independently.

Orthogonal Projections

- Projections form an important part of linear algebra.
- One of the key ideas in linear algebra is to decompose a matrix. Specifically, to build the original matrix by using smaller, *easier to understand* pieces. By examining these *simpler* pieces, we can understand what the matrix does. We can even tinker with them.

1 Projections

Idempotence

In class, we saw any orthogonal projection P_S takes the form

$$P_S = VV^T$$

where V is a matrix of orthonormal columns that form a basis for S. There is a more general definition of a projection - known as *idempotence*.

Definition (Idempotence)

An matrix P is idempotent when

$$P^2 = P$$

Orthgonal projections vs Idempotence

Definition (Idempotence)

An matrix P is idempotent when

$$P^2 = P$$

1. Show that all orthogonal projections are idempotent.
2. Give an example of an idempotent matrix that is not an orthogonal projection.
   (Hint: Show that your matrix does not minimize the distance to subspace it projects onto.)

Questions: Orthogonal Projections

1. Let $v_1, ..., v_k$ be a list of orthogonal vectors. Show that $v_1, ..., v_k$ are linearly independent.
2. Let U be the subspace of $\mathbb{R}^n$ with orthonormal basis $u_1, ..., u_k$.
    1. Prove that the orthogonal projection of $v \in \mathbb{R}^n$ onto U can be expressed as $P_U = \sum_{i=0}^{k} < v, u_i >$. (Use the fact that the orthonormal basis for a subspace of $\mathbb{R}$ can be extended to obtain an orthonormal basis for $\mathbb{R}$)
    2. Prove that $P_U(v) \leq \|v\|$
    3. Prove that $v - P_U(v)$ is orthogonal to $P_U(v)$

Questions: Norms and Inner Products

1. Which of the following functions are inner products for $x, y \in \mathbb{R}^3$?
   i. $f(x,y) = x_1y_2 + x_2y_3 + x_3y_1$
   ii. $f(x,y) = x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_1^2$
   iii. $f(x,y) = x_1y_1 + x_3y_3$
1. For $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, prove that

$$\|Ax\| \leq \|x\| \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j}^2}$$