# Recitation 1

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# Announcements

# Why is Linear Algebra important?

Linear Algebra...

- Appears in ALL applied math, including *data science*
- Is solvable. If you can write it down, you can solve it!
  - (not true for other math, e.g Diff Eq, Integrals)
- Is *fundamental* to understanding tools in machine learning

Relevant applications we will cover in the class:

- Linear Regression
- Principal Component Analysis
- Gradient Descent

First year MSDS students are *highly encouraged* to take this class.

# Concept Review: Vector Spaces

## Definition (Vector space)

$V$ is a vector space over field $F$ when

1. Closure under Addition: $x + y \in V$
2. Sum is commutative $(x + y = y + x)$ and associative $x + (y + z) = (x + y) + z$
3. Additive Identity (in $V$): $0 \in V$ $(x + 0 = x)$
4. Additive Inverse: $(\forall x, \exists - x$ s.t $x + (-x) = 0$
5. Closure under Scalar Multiplication: $\alpha x \in V$
6. Multiplicative Identity (in $F$): $\alpha x \in V$
7. Compatibility in Multiplication: $\alpha(\beta x) = (\alpha\beta)x$
8. Distributivity: $(\alpha + \beta)x = \alpha x + \beta \cdot \vec{y}$ and $\alpha(x + y) = \alpha x + \alpha \cdot y$.

Notice that "vectors" are not explicitly defined.
Optional: prove (after class) that $\mathbb{R}^3$ with standard definitions for addition and scalar multiplication is vector space over field $\mathbb{R}$.

# Concept Review: Vector Spaces

In this class,

Our field is always $\mathbb{R}$. ($\mathbb{C}$ is also a field.)

Standard definitions for vector addition, and scalar multiplication.

But, V is (usually) $\mathbb{R}^n$, or (sometimes) $\mathbb{R}^{n \times n}$.

Everything in linear algebra is in a vector space.

# Concept Review: Subspaces

## Definition (Subspace)

A subset $S$ of a vector space $V$ is a *subspace* if it is closed under addition and scalar multiplication.

1. Closure under Addition: $x + y \in V$
2. Closure under Scalar Multiplication: $\alpha x \in V$

A subspace is also a vector space!
Everything in linear algebra is in a vector space.
Anything *interesting* in linear algebra is in a subspace.
Subspaces are a recurring concept throughout this entire course.

# Questions 1: Subspaces, Span

Recall that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ can be thought of as the $xy$-plane. Consider the two vectors $v = (1, 1)$ and $w = (-1, 2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1. $\text{Span}(v)$
2. $\text{Span}(v, w)$
3. $\text{Span}(v) \cup \text{Span}(w)$, that is, the vectors in $\text{Span}(v)$ or $\text{Span}(w)$
4. $\text{Span}(v) \cap \text{Span}(w)$, that is, the vectors in both $\text{Span}(v)$ and $\text{Span}(w)$
5. $\{(1 - t)v + tw : t \in [0, 1]\}$
6. $\{(1 - t)v + tw : t \in \mathbb{R}\}$
7. $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$
8. $\text{Span}(v, w, x)$ where $x = (0, 5)$.
9. $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$
10. $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
11. $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
12. $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

# Solutions 1: Subspaces, Span

Recall that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ can be thought of as the $xy$-plane. Consider the two vectors $v = (1, 1)$ and $w = (-1, 2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1 $\text{Span}(v)$ True

2 $\text{Span}(v, w)$ True

3 $\text{Span}(v) \cup \text{Span}(w)$, False

4 $\text{Span}(v) \cap \text{Span}(w)$, True

5 $\{(1-t)v + tw : t \in [0, 1]\}$ False

6 $\{(1-t)v + tw : t \in \mathbb{R}\}$ False

7 $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$ False

8 $\text{Span}(v, w, x)$ where $x = (0, 5)$. True

9 $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$ False

10 $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ True

11 $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ False

12 $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ False

# Questions 2: Linear Independence, Bases,

1 Let $v_1, v_2, v_3, v_4$ (all distinct) $\in \mathbb{R}^3$.
Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$.
If $C_1$ and $C_2$ are both linearly independent, what are the possible values for $dim(Span\{v_1, v_2, v_3, v_4\})$? (No formal proof necessary.)

2 Let $v_1, ..., v_n \in \mathbb{R}^n$ be a basis of $\mathbb{R}^n$.
Prove that for $x \in R^n$, there exists unique $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=0}^{n} \alpha_i v_i$.

3 Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
Prove that for $x \in Span(v_1, ..., v_m)$, there exist infinitely many $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $x = \sum_{i=0}^{m} \alpha_i v_i$.

4 True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and $W$ is a subspace of $\mathbb{R}^n$, then some subset of $B$ is a basis for $W$.

1. Let $v_1, v_2, v_3, v_4$ (all distinct) $\in \mathbb{R}^3$ .
   Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$.
   If $C_1$ and $C_2$ are both linearly independent, what are the possible values for $dim(Span\{v_1, v_2, v_3, v_4\})$? (No formal proof necessary.)

**Solution**

*Either* $C_1 \subset Span(C_2)$ *and the dimension is* $2$*, or the dimension is* $3$*.*

# Solutions 2: Linear Independence, Bases

2. Let $v_1, ..., v_n \in \mathbb{R}^n$ be a basis of $\mathbb{R}^n$.
   Prove that for $x \in R^n$, there exists unique $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=0}^{n} \alpha_i v_i$.

## Solution

*By definition of basis, $v_1, ..., v_n$ is a linearly independent set, and spans $\mathbb{R}^n$. Since $x \in \mathbb{R}^n$,*

$$\exists \alpha_1, ..., \alpha_n \ s.t\ x = \sum_{i=0}^{n} \alpha_i v_i.$$

*Let $\beta_1, ..., \beta_n$ s.t $x = \sum_{i=0}^{n} \beta_i v_i$. Then,*

$$x - x = 0 = \sum_{i=0}^{n} (\alpha_i - \beta_i) v_i$$

*Then by definition of linear independence,*

$$\alpha_i - \beta_i = 0 \qquad \forall i \in 1, ..., n$$

*So $\alpha_i = \beta_i, \forall i \in 1, ..., n$*

3. Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
Prove that for $x \in Span(v_1, ..., v_m)$, there exist infinitely many $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $x = \sum_{i=0}^{m} \alpha_i v_i$.

## Solution

*By assumption,* $x \in Span(v_1, \ldots, v_m)$*. So*

$$\exists \beta_1, ..., \beta_m \ \ s.t\ x = \sum_{i=1}^{m} \beta_i v_i$$

*Since* $v_1, \ldots, v_m$ *are linearly dependent, there are* $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$ *such that*

$$\sum_{i=0}^{m} \gamma_i v_i = 0$$

*where not all* $\gamma_i = 0$*.*
*Now, let* $r \in \mathbb{R}$*. Then,*

$$x = x + 0 = \sum_{i=1}^{m} \beta_i v_i + r \sum_{i=0}^{m} \gamma_i v_i = \sum_{i=1}^{m} (\beta_i + r\gamma_i) v_i$$

*This gives infinitely many distinct* $\alpha$ *where* $\alpha_i = \beta_i + r\gamma_i$ *for* $r \in \mathbb{R}$*.*

4. True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and $W$ is a subspace of $\mathbb{R}^n$, then some subset of $B$ is a basis for $W$.

**Solution**

*False. Consider* $B = \{(1,0),(0,1)\}$ *and* $W = Span((1,1))$.

# Questions 3: Bases, Dimension

Let $V$ be the set of functions

$$V \coloneqq \{p : \mathbb{R} \to \mathbb{R} \mid p(x) = \sum_{k=0}^{n} a_k x^k\}$$

where $a_0, \ldots, a_n \in \mathbb{R}$, and $x \in \mathbb{R}$ is *constant*.

1 What kind of function does this set contain?

2 Define an addition operation $+ : V \times V \to V$,
and a scalar multiplication operation $\cdot : \mathbb{R} \times V \to V$,
such that the triple $(V, +, \cdot)$ is a vector space.

3 Find a basis for this vector space.

4 What is the dimension of this vector space?

# Solutions 3: Bases, Dimension

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4 What is the dimension of this vector space?

## Solution

*1 Polynomials evaluated at $x$*

*2 Standard definitions for function addition and scalar multiplication*

*3* $1, x, x^2, ..., x^n$

*4* $n + 1$