Contextual bandits

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The contextual bandit problem

The contextual bandit

A contextual bandit problem proceeds in rounds of the following steps:

1. Observe input/**context** x.
2. Take action a.
3. Receive reward $r \in \mathbb{R}$.

- Action a may depends on x and previously observed (x, a, r) triples.

Example: news article recommendation

- Consider a news website.
- Every day there are 10 top stories.
- We want to highlight one for each user.
    - Choice should be **personalized**.
- What is the action?
    - Selecting one of the top 10 stories
- What is the context?
- What is the reward?

The context

- What is the context?
  - Information about the user, if any (e.g. demographics)
  - Geographic location
  - User identifier (we can learn latent features, collaborative filtering style)
- Can we use things about an individual that may change over the rounds of a bandit?
  - possibly as a result of our actions?
  - e.g. recent reading history; e.g. articles (from previous rounds) shared with friends
- No – this would take us out of the contextual bandit framework.
- Reinforcement learning (RL) is a more general framework that allows for this.
  - i.i.d. contexts are replaced by "states" that may evolve over time.
- We'll give a brief introduction to RL in a later module.

The reward

- What can we use as a reward signal?
- Click (Y/N)
- Spent more than 30 seconds on article page (Y/N)
- More complicated function of time spent reading article
- Was article shared or favorited? (Y/N)
- Figuring out the right reward signal is nontrivial.
  - Requires domain understanding.
  - May need tweaking over time.

Context / reward examples

- User 325. $x = \{\text{Likes sports articles}\}$.
- Actions / rewards
  - Action 1: "Tom Brady retirement" **Reward: 10**
  - Action 2: "Player has meltdown after argument" **Reward: 2**
  - Action 3: "Government considers ban for actor using drugs" **Reward: 3**
- User 823. $x = \{\text{Likes human-interest stories}\}$
- Actions / rewards
  - Action 1: "Tom Brady retirement" **Reward: 1**
  - Action 2: "Player has meltdown after argument" **Reward: 5**
  - Action 3: "Government considers ban for actor using drugs" **Reward: 0**

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- In the terminology of our discussion of causal inference, the reward for each action is a potential outcome.
- We only get to observe the reward corresponding to the action we took (or "treatment" given, in the causal inference terminology).
- Terminology note: Some authors refer to the outcomes we don't observe as "counterfactual" (e.g. [MW15, Ch. 2]).
- Other authors use "counterfactual" to refer to all the potential outcomes that can happen (e.g. [HR20, p. 4]. And one of these counterfactuals, the observed outcome, is also "factual".
- Some authors are careful to avoid the word "counterfactual" because of this ambiguity.
- Just be aware of the different usages – it doesn't matter that much.

The rewards

Conditioned on a context $x \in \mathcal{X}$,

- a reward is generated for each possible action $a \in \mathcal{A} = \{1, \ldots, k\}$.
- These k rewards are represented by **reward vector**

$$R = (R(1), \ldots, R(k)) \in \mathbb{R}^k.$$

- We only observe one entry: $R(A)$, where A is the action we play.

Probabilistic model for contextual bandit

- Context and reward vector are related:
  - The same action will get different rewards in different contexts.

Stochastic contextual k-armed bandit model

- Context and reward vector $(X, R) \in \mathcal{X} \times \mathbb{R}^k$ **drawn jointly** from P.
- Context and reward pairs are i.i.d. over time:

$$(X, R), (X_1, R_1), \ldots, (X_t, R_t) \text{ i.i.d. } \sim P.$$

Action selection

- Action at round t is A_t.
- At beginning of round t, the **history**, or previous **observation sequence** is

$$\mathcal{D}_t = \Big((X_1, A_1, R_1(A_1)), \ldots, (X_{t-1}, A_{t-1}, R_{t-1}(A_{t-1})) \Big).$$

- At round t, action A_t may depend on context X_t and history $\mathcal{D}_t$.
- Note that we **cannot** say $A_t \perp\!\!\!\perp R_t$ – why?
- Because A_t depends on X_t, and R_t depends on X_t.
  - Information about R_t can propagate to A_t through X_t.

Action and reward are conditionally independent given context

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We **can say** that $A_t \perp\!\!\!\perp R_t \mid X_t$ for each t.

- Note that $A \perp\!\!\!\perp R \mid X$ is the exact counterpart to the "ignorability" assumption in causal inference: $(Y(0), Y(1)) \perp\!\!\!\perp W \mid X$. The reward vector $R = (R(1), \ldots, R(k)) \in \mathbb{R}^k$ corresponds to the potential outcome vector $(Y(0), Y(1)) \in \mathbb{R}^2$. The action $A \in \mathcal{A}$ corresponds to the treatment indicator $W \in \{0, 1\}$, and the covariate $X \in \mathcal{X}$ has the same interpretation in each setting.

Stochastic k-armed contextual bandit

Stochastic k-armed contextual bandit

1. Environment samples **context** and **rewards vector** jointly, iid, for each round:

$$(X,R),(X_1,R_1),\ldots,(X_T,R_T)\in\mathcal{X}\times\mathbb{R}^k \text{ i.i.d. from } P,$$

   where $R_t=(R_t(1),\ldots,R_t(k))\in\mathbb{R}^k$.

2. For $t=1,\ldots,T$,
   1. Our algorithm **selects action**/arm $A_t\in\{1,\ldots,k\}$ based on X_t and history

$$\mathcal{D}_t=\Big((X_1,A_1,R_1(A_1)),\ldots,(X_{t-1},A_{t-1},R_{t-1}(A_{t-1}))\Big).$$

   2. Our algorithm **receives reward** $R_t(A_t)$.

- We **never observe** $R_t(a)$ for $a\neq A_t$.
- This is called **stochastic** because the rewards are selected randomly.

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- It might look cleaner to say that at the beginning of every round, the environment generates $(X_t,R_t)\in\mathcal{X}\times\mathbb{R}^k$ from P. But we want to be very clear that $(X_1,R_1),\ldots,(X_T,R_T)$ are
  1. generated i.i.d. and are
  2. generated before any of the actions $A_1,\ldots,A_T$ are generated.

Policies

Policies

- Policies give some structure to action selection.
- A policy at round t
  - gives a conditional distribution over the action A_t to be taken
  - conditioned on the current context X_t and the history $\mathcal{D}_t$.
- We'll denote the policy at round t as $\pi_t(\cdot \mid X_t, \mathcal{D}_t)$.
- Choosing an action according to policy π_t means we choose A_t **randomly** s.t.

$$\mathbb{P}(A_t = a) = \pi(a \mid X_t, \mathcal{D}_t).$$

Optimal policy

- Suppose we knew the function

$$r(a, x) = \mathbb{E}[R \mid A = a, X = x],$$

which gives the expected reward for any action a and context x.

- Then optimal policy would be

$$\pi_t(a \mid X_t, \mathcal{D}_t) = \mathbb{1}\left[a = \arg\max_a r(a, X_t)\right].$$

Example: "direct method"

- We don't know $r(a,x)$, but we can use $\mathcal{D}_t$ as training data:

$$\left(\underbrace{(X_1, A_1)}_{\text{input}}, \underbrace{R_1(A_1)}_{\text{response}}\right), \ldots, \left(\underbrace{(X_{t-1}, A_{t-1})}_{\text{input}}, \underbrace{R_{t-1}(A_{t-1})}_{\text{response}}\right).$$

- Approximating $r(a,x)$ is a regression problem!
- Let $\hat{r}_t(x,a) = \texttt{TrainingAlgorithm}(\mathcal{D}_t)$.
- The policy for the **direct method** is defined as

$$\pi_t(a \mid X_t, \mathcal{D}_t) := \mathbb{1}\left[a = \arg\max_a \hat{r}_t(x,a)\right].$$

- This is a **pure exploitation** method.

Some other approaches

- ε-greedy is an obvious extension of the direct method.
- Thompson sampling: prior is over models $\hat{r}_t(x, a)$
  - equivalently, prior is over model parameters
- Policy gradient: directly optimizing over the policy to improve expected reward
  - we'll return to this in a few weeks as a warm-up for REINFORCE.

References

Resources

- The term contextual bandit was introduced in [LZ07], but the idea has been around much longer.
- A nice history of contextual bandits is given in [TM17], which cites a 1979 paper as the first appearance of contextual bandits.

References I

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