Inverse Propensity Weighting

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Contents

Recap and Warmup

Recap: Missing at random (MAR) setting

- Full data: $(X_1, Y_1), \ldots, (X_n, Y_n)$
- Observed data: $(X_1, R_1, R_1 Y_1), \ldots, (X_n, R_n, R_n Y_n)$
  - where $R_1, \ldots, R_n \in \{0, 1\}$ is the response indicator.
- In missing at random (MAR) setting, $R_i \perp\!\!\!\perp Y_i \mid X_i$
- Probability of response is given by the **propensity score function**:

$$\pi(x) = \mathbb{P}(R_i = 1 \mid X_i = x) \quad \forall i.$$

Exercise: $\mathbb{E}[R_i \mid X_i] = \pi(X_i)$

Show $\mathbb{E}[R_i \mid X_i] = \pi(X_i)$

- Let $f(x) = \mathbb{E}[R_i \mid X_i = x]$. (So $\mathbb{E}[R_i \mid X_i] = f(X_i)$.)
- Then

$$\begin{aligned} f(x) &= 1 \cdot \mathbb{P}(R_i = 1 \mid X_i = x) + 0 \cdot \mathbb{P}(R_i = 0 \mid X_i = x) \\ &= \pi(x) \end{aligned}$$

- So $\mathbb{E}[R_i \mid X_i] = \pi(X_i)$.

Inverse propensity weighting (IPW): Introduction

Observed responses represent multiple unobserved responses

- Suppose $\mathcal{X} = \{0, 1\}$ corresponding to two types of people.
- e.g. $X_i = \mathbb{1}$[individual i has previously responded to a survey]
- When $X_i = 1$, response probability $\pi(1) = \mathbb{P}(R_i = 1 \mid X_i = 1) = 0.2$
- When $X_i = 0$, response probability $\pi(0) = \mathbb{P}(R_i = 1 \mid X_i = 0) = 0.1$
- If we observe Y_i for an individual with $X_i = 1$,
  - That individual represents roughly 5 people from the full data.
- If we observe Y_i for an individual with $X_i = 0$,
  - That individual represents roughly 10 people from the full data.

Inverse propensity weighted (IPW) Mean

- When[1] $\pi(x) > 0 \ \forall x \in \mathcal{X}$,
  we can define the **IPW mean estimator** for $\mathbb{E}Y$:

$$\begin{aligned} \hat{\mu}_{\text{ipw}} &= \frac{1}{n} \sum_{i:R_i=1} \frac{Y_i}{\pi(X_i)} \\ &= \frac{1}{n} \sum_{i=1}^{n} \frac{R_i Y_i}{\pi(X_i)} \end{aligned}$$

- We'll show that this is unbiased and asymptotically does the right thing.
- Though it has some issues, which we'll also explore

[1] We assume here and everywhere below that $\pi(x) > 0 \ \forall x \in \mathcal{X}$.

IPW: Lemmas

- In the MAR setting, where $Y_i \perp\!\!\!\perp R_i \mid X_i$, we have for a generic term in the mean:

$$
\begin{aligned}
\mathbb{E}\left[\frac{RY}{\pi(X)}\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{RY}{\pi(X)} \mid X\right]\right] \quad \text{Adam's Law} \\
&= \mathbb{E}\left[\frac{1}{\pi(X)}\mathbb{E}[RY \mid X]\right] \quad \text{Taking out what is known} \\
&= \mathbb{E}\left[\frac{1}{\pi(X)}\mathbb{E}[R \mid X]\mathbb{E}[Y \mid X]\right] \quad \text{By MAR assumption} \\
&= \mathbb{E}[\mathbb{E}[Y \mid X]] \quad \text{since } \mathbb{E}[R \mid X] = \mathbb{P}(R = 1 \mid X) = \pi(X) \\
&= \mathbb{E}Y \quad \text{Adam's Law}
\end{aligned}
$$

IPW: Unbiased and Consistent

- The estimator $\hat{\mu}_{\text{ipw}} = \frac{1}{n}\sum_{i=1}^{n} \frac{R_i Y_i}{\pi(X_i)}$ is **unbiased** for $\mathbb{E}Y$, since

$$\mathbb{E}\hat{\mu}_{\text{ipw}} = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{R_i Y_i}{\pi(X_i)}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{R_i Y_i}{\pi(X_i)}\right] = \mathbb{E}Y.$$

- The estimator $\hat{\mu}_{\text{ipw}}$ is "**consistent**" That is,

$$\hat{\mu}_{\text{ipw}} \xrightarrow{P} \mathbb{E}Y \text{ as } n \to \infty$$

  by the Law of Large Numbers, since $\left(\frac{R_i Y_i}{p(R_i|X_i)}\right)_{i=1}^{n}$ are i.i.d. with expectation μ.

- Unbiased and consistent are nice properties to have,
  - but does NOT mean it's a great estimator....

IPW on MCAR: example and simulation

Recap: MAR, MCAR, and IPW estimators

- We started by talking about the MCAR setting
  - i.e. Response indicator R_i independent of response Y_i
- We discussed the complete case estimator for $\mathbb{E}Y$:

$$\hat{\mu}_{\text{cc}} = \frac{\sum_{i=1}^n R_i Y_i}{\sum_{i=1}^n R_i}$$

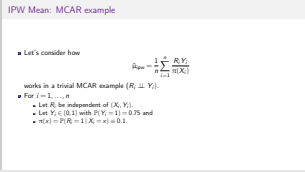
- In the MAR setting, R_i and Y_i can be dependent, so $\hat{\mu}_{\text{cc}}$ may be quite biased.
- We introduced $\hat{\mu}_{\text{ipw}}$ and showed it's **unbiased** in the MAR setting.
- But how good is $\hat{\mu}_{\text{ipw}}$?
- Let's see how it compares to $\hat{\mu}_{\text{cc}}$ in the simple MCAR setting.

IPW Mean: MCAR example

- Let's consider how

$$\hat{\mu}_{\text{ipw}} = \frac{1}{n}\sum_{i=1}^{n}\frac{R_i Y_i}{\pi(X_i)}$$

  works in a trivial MCAR example ($R_i \perp\!\!\!\perp Y_i$).
- For $i = 1, \ldots, n$
  - Let R_i be independent of (X_i, Y_i).
  - Let $Y_i \in \{0, 1\}$ with $\mathbb{P}(Y_i = 1) = 0.75$ and
  - $\pi(x) = \mathbb{P}(R_i = 1 \mid X_i = x) \equiv 0.1$.



- In MCAR, we **could** have R_i depend on X_i. If that were the case, then X_i and Y_i would need to be independent, since for MCAR we need $R_i \perp\!\!\!\perp Y_i$.

IPW for MCAR example: $n = 1$ case

- Suppose $n = 1$:
- If $Y_1 = 0$ or $R_1 = 0$, then $\hat{\mu}_{\text{ipw}} = \frac{R_1 Y_1}{0.1} = 0$.
- If $Y_1 = 1$ and $R_1 = 1$, then $\hat{\mu}_{\text{ipw}} = \frac{R_1 Y_1}{0.1} = \frac{1}{0.1} = 10$.
- The good: $\mathbb{E}\hat{\mu}_{\text{ipw}} = 10 \cdot \mathbb{P}(Y_i = 1)\mathbb{P}(R_i = 1) = 10 \cdot \frac{3}{4} \cdot \frac{1}{10} = 0.75$, so unbiased.
- The bad: We can get $\hat{\mu}_{\text{ipw}} = 10$ even though $Y_i \in \{0, 1\}$.

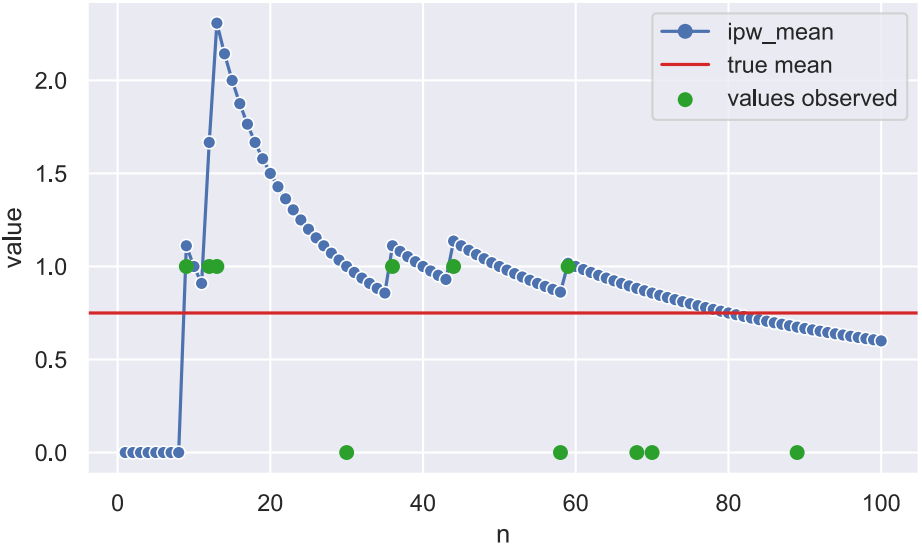
IPW for MCAR example, continued

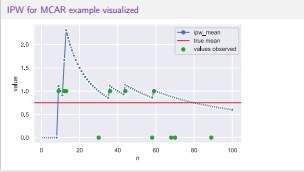
- Now let n grow.
- Suppose $R_1 = Y_1 = 1$, but $R_2 = \cdots = R_n = 0$.
- Then

$$\hat{\mu}_{\text{ipw}} = \frac{1}{n}\sum_{i=1}^{n}\frac{R_i Y_i}{\pi(X_i)} = \frac{1}{n}\left(\frac{1\cdot 1}{0.1} + \underbrace{0+\cdots+0}_{(n-1)}\right) = \frac{10}{n}$$

- So the IPW estimate starts at 10, and then decreases like $O(1/n)$ towards 0.
- Our estimate keeps changing even though we don't have new observations of Y.

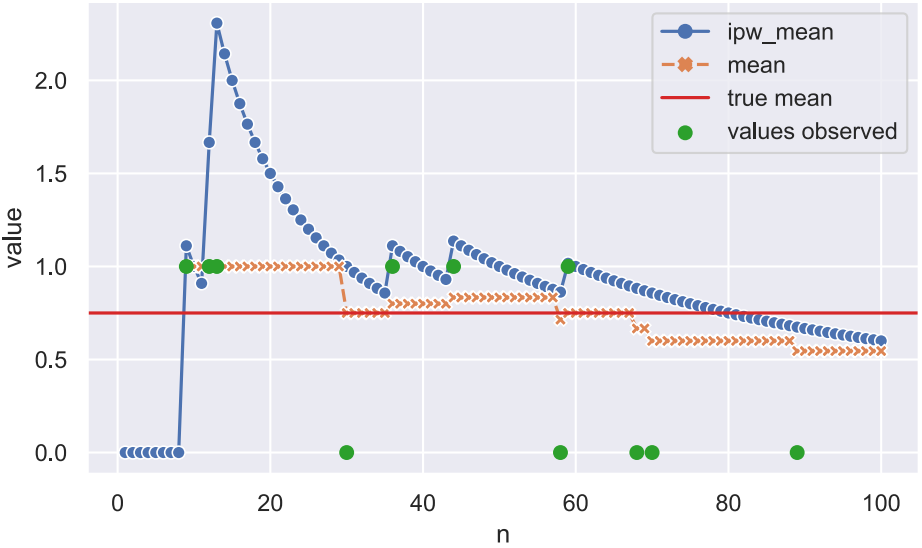
IPW for MCAR example visualized

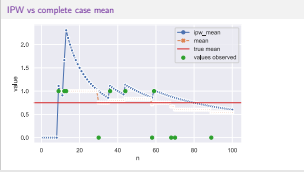




- The green dots represent observed values of Y_i.
- We can see that we had no observations of Y until $Y_9 = 1$.
- The horizontal red line shows the true mean of the Y_i's.
- The blue dots show $\hat{\mu}_{\text{ipw}}$ as n increases.
- Note the large jumps in $\hat{\mu}_{\text{ipw}}$ whenever we get an observation with $Y_i = 1$. This is because each observed Y_i is scaled up by an inverse propensity weight of 10.
- Also note that between observations with $Y_i = 1$, ipw_mean decays like $1/n$ towards 0.

IPW vs complete case mean





- We've added in the orange line here, which represent the mean of the complete cases.
- Note that the orange line is constant between observations, and jumps whenever we get a new observation.

IPW vs complete case mean, 5000x

- We repeat the experiment[2] above 5000 times ($n = 100$ samples in each) and get the following.
- Recall that the true mean is $\mu = 0.75$.

estimator	mean	SD	SE	bias	RMSE
mean	0.751654	0.143943	0.002036	0.001654	0.143953
ipw_mean	0.752540	0.262224	0.003708	0.002540	0.262237

[2] In the very rare event that the sample in a particular repetition has 0 complete cases, we take $\hat{\mu}_{cc} = 0$.

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To generate this table, we repeated the following 5000 times:

1. Get a sample of size $n = 100$ from the distribution described previously.
2. Evaluate the estimators on this sample.

- The table then displays various summary statistics of the performance of these estimators across the 5000 repetitions.
- Mean and SD are just the mean and SD of the estimate across the repeats.
- The SE is the uncertainty of the mean as an estimator $\mathbb{E}\hat{\mu}_{\text{ipw}}$ and $\mathbb{E}\hat{\mu}_{\text{cc}}$ (depending on the row) – so SE = SD / sqrt(num_repeats=5000).
- "Bias" is computed as the difference between the mean of the estimators and the true mean μ .
- The SE is also the uncertainty of this bias.
- For each repeat we compute the RMS difference between the estimate and μ.

IPW vs Complete case mean, 5000x (continued)

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- Both estimators have
  - bias within 1 SE of 0 (i.e. indistinguishable from 0).
  - SD is orders of magnitude larger than bias (and essentially equals the RMS error).
- **Complete case mean is better than ipw_mean in RMSE for this MCAR distribution.**

IPW vs Complete case mean, 5000x (continued)

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- The bias of mean (the mean of the observations) and ipw_mean are both within 1 SE of 0, so we can't conclude from this data that there's bias.
- Indeed, we know theoretically that the IPW estimator is unbiased.
- In both cases, the bias is orders of magnitude smaller than the SD, which is what drives the RMSE.
- For this dataset, it seems like the complete case mean is the clear winner, between the two.
- Recall that $\text{RMSE}^2 = \text{SD}^2 + \text{bias}^2$.
- This explains why the contribution of the bias to the RMSE is even less than one might expect from looking at the relative magnitudes of SD and bias.