1. Consider a forward SDE,

$$dx_t = f(x_t, t)\, dt + g(x_t, t)dW_t,$$

> $show that the corresponding probability flow ODE is written as$ >

dx_t = \left[f(x_t,t) -\frac{1}{2}\frac{\partial}{\partial x}g^2(x_t, t)- \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)\right],dt.

$sol: By Fokker-Planck function we have$

\frac{\partial p(x,t)}{\partial t}=
-\frac{\partial}{\partial x}\left[f(x,t)p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g^2(x,t)p(x,t)\right]

$Then, we have$

\begin{aligned}
\frac{\partial p}{\partial t}&=-\frac{\partial}{\partial x}\left[fp-\frac{1}{2}\frac{\partial}{\partial x}(g^2p)\right] \
&=-\frac{\partial}{\partial x}[fp-\frac{1}{2}(2p\partial_xg+g^2\partial_xp)] \
&=-\frac{\partial}{\partial x}[(f-\frac{1}{2}\frac{\partial g}{\partial x})p-\frac{1}{2}g^2\frac{\partial p}{\partial x}]
\end{aligned}

$By score function, we have $S(x,\theta) = \nabla_x \log p(x,\theta)$, which can be rewritten into the form $\frac{\partial}{\partial x}\log p(x,$

\begin{aligned}
\frac{\partial p}{\partial t}&=
-\frac{\partial}{\partial x}[(f-\frac{1}{2}\frac{\partial g}{\partial x})p-\frac{1}{2}g^2p\frac{\partial }{\partial x}\log p(x,t)] \
&=-\frac{\partial}{\partial x}[(f-\frac{1}{2}\frac{\partial g}{\partial x}-\frac{1}{2}g^2\frac{\partial }{\partial x}\log p(x,t))p] -(\star\star)\
\end{aligned}

$By the flow ODE$

\frac{\partial p(x,t)}{\partial t}=-\frac{\partial}{\partial x}[v(x,t)p(x,t)]
,$compare to$(⋆⋆) we have $v(x,t) = [f - \frac{1}{2}\frac{\partial g}{\partial x} - \frac{1}{2}g^2\frac{\partial}{\partial x}\log p(x,t)]$ which implies that,

$$dx_t = \left[f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)\right] dt.$$