1. Consider the boundary value problem $u''(x) = f(x), x < 0 < 1$ and $u(0) = u(1) = 0$ with $f(x) = \frac{1}{x}$. Using $u(x) = \int_0^1 G(s,x)f(s)\,ds - (\star)$ prove that $u(x) = -x\log(x)$. This show that $u \in C^2(0,1)$ but $u(0)$ is $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$.

sol: Take

$$G(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1. \end{cases} - (\star\star)$$

Then, by $(\star)$ we have

$$\begin{aligned} u(x) &= \int_0^x s(1-x)\frac{1}{s}\,ds + \int_x^1 x(1-s)\frac{1}{s}\,ds \\ &= (1-x)x + x\left(\int_x^1 \frac{1}{s} - 1\right) \\ &= x(1 - x - \ln x - 1 + x) \\ &= -x\ln x. \end{aligned}$$

Hence, we have concluded that $u(x) = -x\ln x$. From $u(1) = 0$ and $\lim_{x\to 0} u(x) = 0$ which can proved by L'Hopital's rule, then $u \in C^0([0,1])$.

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2. Clerify the summation by parts formula $\sum_{j=0}^{n-1}(w_{j+1} - w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1} - v_j)w_{j+1}$, and show that for $v_h \in V_h^0$, $(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1} - v_j)^2$.

sol:

$$\begin{aligned} RHS &= \sum_{j=0}^{n-1}(w_{j+1}v_j - w_j v_j) \\ &= (w_1 v_0 + w_2 v_1 + \cdots + w_n v_{n-1}) \; - \; (w_0 v_0 + w_1 v_1 + \cdots + w_{n-1}v_{n-1}) \\ &= -w_0 v_0 + w_1(v_0 - v_1) + w_2(v_1 - v_2) + \cdots + w_{n-1}(v_{n-2} - v_{n-1}) + w_n(v_{n-1} - v_n) + w_n v_n \\ &= w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} w_{j+1}(v_{j+1} - v_j). \end{aligned}$$

Define $(L_h v)_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$. Then

$$
\begin{aligned}
(L_h, v_h)_h = h\sum_{j=1}^{n-1}(L_h v)\cdot v_j &= h\sum_{j=1}^{n-1}\frac{v_{j+1}-2v_j+v_{j-1}}{h^2}v_j \\
&= \sum_{j=1}^{n-1}\frac{v_{j+1}-v_j}{h}v_j - \frac{v_j-v_{j-1}}{h}v_j \\
\text{by taking } \mathrm{u}_{j+1} = v_{j+1}-v_j &= \sum_{j=1}^{n-1}(u)\frac{u_{j+1}-u_j}{h}v_j \\
\text{by the previous part} &= h^{-1}(u_n v_n - u_0 v_0 - \sum_{j=1}^{n-1}u_{j+1}(v_{j+1}-v_j)) \\
\text{by } u_n = u_0 = 0, &= \sum_{j=1}^{n-1}(v_{j+1}-v_j)^2.
\end{aligned}
$$

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3. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in $(\star\star)$ and G^k is its corresponding discrete counterpart solution of $(\star\star)$.

sol:

Goal: We want to show that $L_h(hG) = e^k$, and so by $L_h(G^k) = e^k$ which have completed the prove.

Fix x_k is a point. By h is a step as constant number, then $L_h(hG) = h(L_hG)$.

Note that,

$$
L_hG(x_k) = \frac{G(x_k+1, x_k) - 2G(x_k, x_k) + G(x_k-1, x_k)}{h^2} - (i)
$$

For $j \neq k$, by $(\star\star)$, we can see that $G(x_j, x_k)$ is a line, and so for each $l = 0, 1, \cdots, k-1$ and $l = k+1, k+2, \cdots, n+1$ of the nodes x_l, we have $L_hG = 0$.

For $j = k$, by (i) and define $x_k = kh$ for $k = 0, 1, \cdots, n$, then we have the following case:

$$
\begin{aligned}
&G(x_k, x_k) = x_k(1-x_k) = kh(1-kh) - (ii) \\
&G(x_{k+1}, x_k) = G(x_k+h, x_k) = x_k(1-x_k-h) = kh(1-(k+1)h) - (iii) \\
&G(x_k-1, x_k) = (1-x_k)x_{k-1} = (1-kh)(k-1)h - (iv)
\end{aligned}
$$

By $(i) \sim (iv)$, then we have

$$
L_hG(x_k) = \frac{k(1-kh+h-1+kh)+(1-kh)(k-1-k)}{h} = \frac{1}{h}.
$$

Hence, we have $L_hG(hx_j) = 1$.

Therefore we conclude that,

$$
L_h(hG) = \begin{cases} 0, & \text{if j} \neq k, \\ 1, & \text{if j} = k. \end{cases}
$$