Spectral Deferred Correction Methods for Ordinary Differential Equations
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> ✎ **Problem Description**
>
> Consider an initial value problem:
>
> $$\begin{cases} \varphi'(t) = F(t, \varphi(t)), & t \in [a, b] \\ \varphi(a) = \phi_a \end{cases} - (1.1)$$
>
> The goal is to find the function $\varphi(t)$ satisfy these condition. IVP generally fall into two cases, Non-stiff problem and stiff problem. We will focus our attention on solving stiff problem.
>
> For a stiff problem, standard numerical methods have the following issue:
>
> - **Runge-Kutta**: These method are stable, but can become computationally expensive when requiring very high order
> - **Implicit multistep algorithm**: the method can reach very high order convergence, but it have poor stability properties.
>   This paper aim to construct a method that can solve **stiff problem** with Arbitrary Order Accuracy. The strategy involves combining Picard Integral with a Gauss-Legendre grid.

1. Notation

1.1 The Picard Integral equation

Integrating equation (1.1) with respect to t, then we have

$$\varphi(t) = \varphi(a) + \int_a^t F(\tau, \varphi(\tau)) \, d\tau. \quad -(1.2)$$

which is call Picard Integral equation.

1.2 Residual function

Given an approximate solution $\varphi^0(t)$ to (1.2), then we have

$$\epsilon(t) = \varphi(a) + \int_a^t F(s, \varphi^0(s)) \, ds - \varphi^0(t). \quad -(1.3)$$

> ✎ **Remark**

Now we define the error $\delta(t) = \varphi(t) - \varphi^0(t) - (\star)$, then

$$\text{substitue } (\star) \text{ into } (1.2) \implies \varphi(t) = \varphi(a) + \int_a^t F\left(s, \delta(s) + \varphi^0(s)\right) ds - (1.5)$$

$$(1.5)\text{-}(1.3) \implies \delta(t) = \int_a^t F\left(s, \delta(s) + \varphi^0(s)\right) - F(s, \varphi^0(s)) \, ds + \epsilon(t)$$

$$\delta(t) = \int_a^t G(s, \delta(s)) \, ds + \epsilon(t) \quad - (1.6)$$

where $G : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ given by $G(t, \delta) = F(t, \varphi^0(t) + \delta(t)) - F(t, \varphi^0(t))$. The form of (1.4) is called Picard-type integral.

> ✏️ **Remark**
>
> The structure of equation (1.6) is identical to the original Picard integral. Therefore, we can use the same numerical methods to solve for $\delta(t)$, eliminating the need to design a new method for our error terms.

> ≣ **Application to Euler's method**
>
> We present an example for how to apply this notation to Euler's method. For the standard forward Euler method applied to φ, we have
>
> $$\varphi_{i+1} = \varphi_i + h_i \cdot F(t_i, \varphi_i), \quad h_i = t_{i+1} - t_i.$$
>
> Then, the forward Euler method for the solution (1.6) is given by
>
> $$\delta_{i+1} = \delta_i + h_i \cdot G(t_i, \delta_i) + (\epsilon(t_{i+1}) - \epsilon(t_i))$$

1.3 Nodes selection

Given $m \in \mathbb{N}$, let $r_1, r_2, \cdots, r_m$ be the Gauss-Legendre nodes on the interval $[-1, 1]$. For an interval $[a, b] \subset \mathbb{R}$, let $s_1, s_2, \cdots, s_m$ be the m Gaussian nodes on the interval $[a, b]$, given by the formula.

$$s_i = \frac{b - a}{2} \cdot r_i + \frac{b + a}{2}$$

> ✏️ **Remark**
>
> Why Gauss-Legendre Nodes

- **Spectral Accuracy**: Gaussian quadrature provides the highest possible degree of precision for numerical integration, which is essential for the high order convergence rates targeted by the SDC method.
- **Avoiding the Runge Phenomenon**:

1.4 Lagrange Interpolation

Given the nodes $t_1, t_2 \cdots, t_m$ and function values $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m)$, the Lagrange interpolant $L^m(\varphi, t)$ is the unique polynomial of degree $m - 1$ defined by

$$L^m(\varphi, t) = \sum_{i=1}^{m} c_i(t) \cdot \varphi_i - (1.7), \quad \text{where } c_i(t) = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}$$

1.5 Spectral Integration and Differential Matrices

Let $f = (f_1, f_2, \cdots, f_m)$ be the vector of function values at the nodes defined by

$$f = \begin{bmatrix} F(t_1) \\ F(t_2) \\ \vdots \\ F(t_m) \end{bmatrix}.$$

Then, we have a differential matrix D^m is the linear operator that maps the function values f to the derivatives of the interpolant at the grid nodes

$$(D^m(f))_i = \frac{d}{dt} L^m(f, t_i).$$

Moreover, the integration matrix S^m is the linear operator that maps the function values f to the integrals of the interpolant from start of the interval to each grid node

$$(S^m(f))_i = \int_a^{t_i} L^m(f, t) \, dt.$$

> ✏️ **Remark**
>
> If the underlying function F is a polynomial of degree $m - 1$, the operations D^m and S^m are exact. For smooth functions, the error decays exponentially with m.

2. Spectral deferred correction(SDC).

Bases on the derived error equation and spectral matrices, we will construct the full iterative algorithm. The method follows a prediction-correction paradigm

```pseudocode
1:Input:Initial value, number of nodes m, number of sweeps J
```

```
2:Prediction(Initialization):
3: Compute an initial approximate solution on the Gauss-Legendre grid using
a low-order method
4:\\Correction loop
5:for j=1 to J do
    ##Compute Residual
    Calculate the residual vector using the spectral integraion matrix S^m
    ##Solve error Equation
    Solve for the correction using a first-order method on the error
equation
    ##Update Solute
    \varphi^{[j]}=\varphi^{[j-1]}+\delta
6:Output:High-order solution \varphi^{[J]}
```

3.Analysis of method

3.1 Behavior at large $|\lambda|$ and extrapolation

- **Explicit Euler**: For stiff problem, the explicit SDC schemes are unstable
- **Implicit Euler**: The implicit SDC schemes are generally stable. However, when analyzing the amplification factor, $Am(\lambda)$ as $|\lambda| \to \infty$, we observe

$$\lim_{|\lambda| \to \infty} = Am(\lambda) = \mu(m, J) \neq 0$$

This means that the method is Not L-stable.

Therefore, we construct a combinations method.

Composite Schemes

To achieve L-stability, we compute two solutions by using two different number of nodes m_1, m_2 with asymptotic limits μ_1, μ_2. The linear combination

$$\varphi_{comb} = \frac{\mu_2 \varphi_1 - \mu_1 \varphi_2}{\mu_2 - \mu_1}.$$

3.2 Order of Accuracy

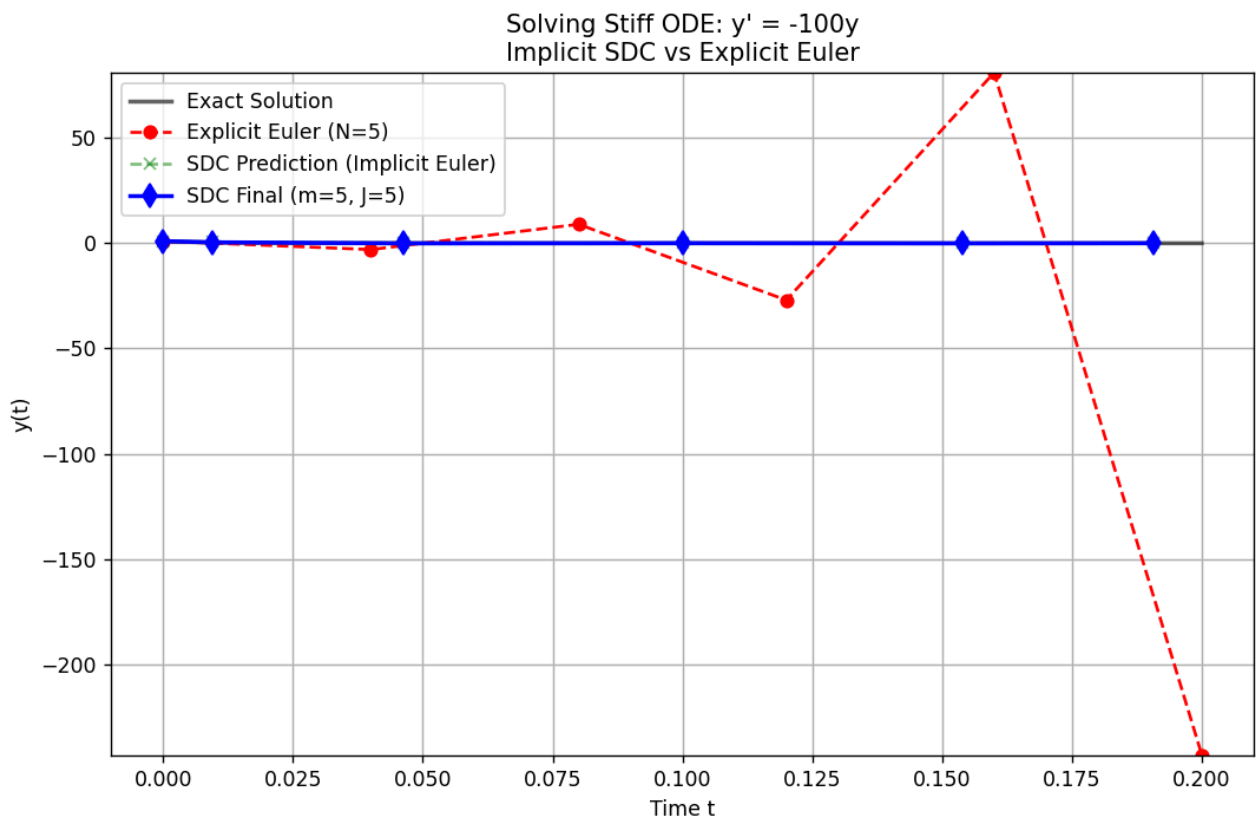For any sufficiently smooth function F, the approximation convergence with an order of accuracy given by

$$Order = \min(m, J + 1)$$

where m is the number of Gauss-Legendre nodes and J is the number of correction sweeps.

4. Numerical example

Consider

$$\begin{cases} \varphi' = -100\varphi \\ \varphi(0) = 1 \end{cases}$$

Solving Stiff ODE: y' = -100y
Implicit SDC vs Explicit Euler

5. Conclusion

We explored Spectral Deferred Correction (SDC) methods for solving stiff ordinary differential equations with arbitrary order accuracy. By using Picard integral equation and Gauss-Legendre quadrature, the SDC approach iteratively corrects a low-order approximation to achieve high-precision results. Consequently, SDC proves to be a powerful and flexible alternative to traditional methods for high-accuracy scientific computing.