1. Prove that $\|\tau\|_h^2 \le 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2)$.

> Hint: for each internal node $x_j = 1, \cdots, n-1$, integral by parts
> $\tau_h(x_j) = \frac{1}{h^2}(\mathcal{R}_4(x_j+h) + \mathcal{R}_4(x_j-h)) - (12.21)$ to get
> $\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt - \int_{x_j-h}^{x_j}(x_j+h-t)^2\,dt]$.

sol:

For each internal nodes x_j, we have $\mathcal{R}_4(x_j+h) = \int_{x_j}^{x_j+h}(u'''(t) - u'''(x_j))\frac{(x_j+h-t)^2}{2}\,dt - (i)$;
$\mathcal{R}_4(x_j-h) = -\int_{x_j-h}^{x_j}(u'''(t) - u'''(x_j))\frac{(x_j-h-t)^2}{2}\,dt - (ii)$.
From (i), take $z = u''(t) - u''(x_j)$ and so that $dz = u'''(t)\,dt$, and then take $dv = \frac{(x_j+h-t)^2}{2}\,dt$, which gives $v = -\frac{(x_j+h-t)^3}{6}$ by IBP we have,

$$\begin{aligned}
\mathcal{R}_4(x_j+h) &= \left[-(u''(t) - u''(x_j))\frac{(x_j+h-t)^3}{6}\right]_{x_j}^{x_{j+h}} + \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6} \\
&= -u''(x_j)\cdot\frac{h^3}{6} + \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6} - (iii).
\end{aligned}$$

Similarly to $\mathcal{R}_4(x_j-h)$, then we have,

$$\mathcal{R}_4(x_j-h) = +u''(x_j)\cdot\frac{h^3}{6} - \int_{x_j-h}^{x_j} u''(t)\frac{(x_j-h-t)^3}{6} - (iv).$$

From $(iii), (iv)$, by (12.21), we get

$$\begin{aligned}
\tau_h(x_j) &= \frac{1}{h^2}(\mathcal{R}_4(x_j+h) + \mathcal{R}_4(x_j-h)) \\
&= \frac{1}{h^2}\left[\int_{x_j}^{x_j+h} u''(t)\frac{(x_j-h-t)^3}{6} - \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6}\right].
\end{aligned}$$

We write $I = \frac{1}{h^2}\left[\int_{x_j}^{x_j+h} u''(t)\frac{(x_j-h-t)^3}{6} - \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6}\right]$, and then $\tau_h(x_j) = f(x_j) - I_j$

$$\begin{aligned}
\|\tau(x_j)\| &= h\sum_{j=1}^{n-1}|\tau_h(x_j)|^2 \\
\text{by } (a-b)^2 \le 2a^2 + 2b \quad &\le 2h(\sum_{j=1}^{n-1}|f(x_j)|^2 + \sum_{j=1}^{n-1}|I_j|^2) \\
&= 2\|f\|_h^2 + 2h\sum_{j=1}^{n-1}|I_j|^2.
\end{aligned}$$

For each $t \in [x_j - h, x_j + h]$ we have $(x_j \pm h - t)^2 \le h^2$. Then

$$|I_j| \le \int_{x_j-h}^{x_j+h}|u''(t)|\,dt = \int_{x_j-h}^{x_j+h}|f(t)|\,dt$$

By Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|I_j|^2 &\le (\int_{x_j-h}^{x_j+h} |f(t)|\, dt)^2 \\
&\le \int_{x_j-h}^{x_j+h} |f(t)|^2\, dt \int_{x_j-h}^{x_j+h} |1|^2\, dt \\
&= 2h \int_{x_j-h}^{x_j+h} |f(t)|^2\, dt = 2\|f\|^2_{L^2(0,1)}.
\end{aligned}
$$

Hence, we conclude that $\|\tau\|_h^2 \le 3(\|f\|_h^2 + \|f\|^2_{L^2(0,1)})$.

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2. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j(1 - x_j)$

sol:
Note that $T_h g = w_h = \sum_{k=1}^{n-1} g(x_k) G^k$. Since $g = 1$, by $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function, then

$$
T_h g = h\left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k)\right]
$$

Take $x_j = jh$, for $j = 1, 2, \cdots, n-1$. Then

$$
\begin{aligned}
T_h g &= h\left[(1 - jh)\sum_{k=1}^{j} kh + jh \sum_{k=j+1}^{n-1} (1 - kh)\right] \\
&= \frac{1}{2} x_j(1 - x_j)
\end{aligned}
$$

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3. Prove Young's inequality.

> $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$, for any $a, b \in \mathbb{R}$, for any $\epsilon > 0$.

sol:
Let $\epsilon > 0$. Let $a, b \in \mathbb{R}$.
By $0 \le (\sqrt{\epsilon} a - \frac{b}{2\sqrt{\epsilon}})^2$, then

$$
\begin{aligned}
a^2\epsilon - ab + \frac{b}{4\epsilon} \ge 0 \\
ab \le a^2\epsilon + \frac{b}{4\epsilon}.
\end{aligned}
$$

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4. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty}$, for any $v_h \in V_h$

sol:

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5. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

sol:

By the second order centered finite differences, we have

$$
\begin{aligned}
(L_h u_h)(x_j) &= -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \quad \text{, for j} = 1, \cdots, n-1 \\
(L_h u_h)(x_{j+1}) &= -\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2} \\
(L_h u_h)(x_{j-1}) &= -\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(L_h u_h)(x_j) &= -\frac{(L_h u_h)(x_{j+1}) - 2(L_h u_h)(x_j) + (L_h u_h)(x_{j-1})}{h^2} \\
&= -\frac{u_{i+2} - 4u_{i+1} + 6u_i + 4u_{i-1} - u_{i-2}}{h^4}.
\end{aligned}
$$