Spectral Deferred Correction Methods for Ordinary Differential Equations

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Introduction to Scientific Computing

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Problem Description

Consider an initial value problem:

$$\begin{cases} \varphi'(t) = F(t, \varphi(t)), & t \in [a, b] \\ \varphi(a) = \phi_a \end{cases} \quad -(1.1)$$

The goal is to find the function $\varphi(t)$ satisfy these condition. IVP generally fall into two cases, Non-stiff problem and stiff problem. We will focus our attention on solving stiff problem.
For a stiff problem, standard numerical methods have the following issue:

- **Runge-Kutta**: These method are stable, but can become computationally expensive when requiring very high order
- **Implicit multistep algorithm**: the method can reach very high order convergence, but it have poor stability properties.
  This paper aim to construct a method that can solve **stiff problem** with Arbitrary Order Accuracy. The strategy involves combining Picard Integral with a Gauss-Legendre grid.

1. Notation

1.1 The Picard Integral equation

Integrating equation (1.1) with respect to t, then we have

$$\varphi(t) = \varphi(a) + \int_a^t F(\tau, \varphi(\tau))\, d\tau. \quad -(1.2)$$

which is call Picard Integral equation.

1.2 Residual function

Given an approximate solution $\varphi^0(t)$ to (1.2), then we have

$$\epsilon(t) = \varphi(a) + \int_a^t F(s, \varphi^0(s))\, ds - \varphi^0(t). \quad -(1.3)$$

Remark

Why we choose Picard Integral

Integration is numerically stable, whereas differentiation is numerically unstable.

Now we define the error $\delta(t) = \varphi(t) - \varphi^0(t) - (\star)$, then

$$\begin{aligned} \text{substitue } (\star) \text{ into } (1.2) \ \Rightarrow \quad & \varphi(t) = \varphi(a) + \int_a^t F\left(s, \delta(s) + \varphi^0(s)\right) ds - (1.5) \\ (1.5)\text{-}(1.3) \Rightarrow \quad & \delta(t) = \int_a^t F\left(s, \delta(s) + \varphi^0(s)\right) - F(s, \varphi^0(s))\, ds + \epsilon(t) \\ & \delta(t) = \int_a^t G(s, \delta(s))\, ds + \epsilon(t) \quad - (1.6) \end{aligned}$$

where $G : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ given by $G(t, \delta) = F(t, \varphi^0(t) + \delta(t)) - F(t, \varphi^0(t))$. The form of (1.4) is called Picard-type integral.

Remark

The structure of equation (1.6) is identical to the original Picard integral. Therefore, we can use the same numerical methods to solve for $\delta(t)$, eliminating the need to design a new method for our error terms.

Application to Euler's method

We present an example for how to apply this notation to Euler's method. For the standard forward Euler method applied to φ, we have

$$\varphi_{i+1} = \varphi_i + h_i \cdot F(t_i, \varphi_i), \quad h_i = t_{i+1} - t_i.$$

Then, the forward Euler method for the solution (1.6) is given by

$$\delta_{i+1} = \delta_i + h_i \cdot G(t_i, \delta_i) + (\epsilon(t_{i+1}) - \epsilon(t_i))$$

1.3 Nodes selection

Given $m \in \mathbb{N}$, let $r_1, r_2, \cdots, r_m$ be the Gauss-Legendre nodes on the interval $[-1, 1]$. For an interval $[a, b] \subset \mathbb{R}$, let $s_1, s_2, \cdots, s_m$ be the m Gaussian nodes on the interval $[a, b]$, given by the formula.

$$s_i = \frac{b-a}{2} \cdot r_i + \frac{b+a}{2}$$

Remark

Why Gauss-Legendre Nodes

- **Spectral Accuracy**: Gaussian quadrature provides the highest possible degree of precision for numerical integration, which is essential for the high order convergence rates targeted by the SDC method.
- **Avoiding the Runge Phenomenon**:

1.4 Lagrange Interpolation

Given the nodes $t_1, t_2 \cdots, t_m$ and function values $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m)$, the Lagrange interpolant $L^m(\varphi, t)$ is the unique polynomial of degree $m-1$ defined by

$$L^m(\varphi, t) = \sum_{i=1}^{m} c_i(t) \cdot \varphi_i - (1.7), \quad \text{where } c_i(t) = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}$$

1.5 Spectral Integration and Differential Matrices

Let $f = (f_1, f_2, \cdots, f_m)$ be the vector of function values at the nodes defined by

$$f = \begin{bmatrix} F(t_1) \\ F(t_2) \\ \vdots \\ F(t_m) \end{bmatrix}.$$

Then, the integration matrix S^m is the linear operator that maps the function values f to the integrals of the interpolant from start of the interval to each grid node

$$(S^m(f))_i = \int_a^{t_i} L^m(f, t)\, dt.$$

Also, there can be written in:

$$S^m = \begin{pmatrix} \int_a^{t_1} L_1(f,t)dt & \cdots & \int_a^{t_m} L_1(f,t)\, dt \\ \vdots & & \vdots \\ \int_a^{t_1} L_m(f,t)dt & \cdots & \int_a^{t_m} L_m(f,t)\, dt \end{pmatrix}$$

Remark

- If the underlying function F is a polynomial of degree $m-1$, the operations S^m are exact. For smooth functions, the error decays exponentially with m.
- The matrix S^m depends only on the nodes and is **fixed** for all iterations.

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2. Spectral deferred correction(SDC).

Bases on the derived error equation and spectral matrices, we will construct the full iterative algorithm. The method follows a prediction-correction paradigm

2.1 Step of algorithm

1. Assume we have an approximate solution $\varphi^{[0]}$ on the grip $[a,b]$, By using the standard Gauss-Legendre nodes on $[a,b]$, $\varphi^{[j]}$ will be used to denote the j-th approximate solution.

$$\varphi^{[j]} = (\varphi_1^{[j]}, \varphi_2^{[j]}, \cdots, \varphi_m^{[j]}) = (\varphi(s_1), \varphi(s_2), \cdots, \varphi(s_m))$$

2. The residual function $\epsilon(t)$ will be approximate by the vector $\varphi^{[j]}$ defined by

$$\sigma(\varphi^{[j]}) = S^m \bar{F}(\varphi^{[j]}) - \varphi^{[j]} + \bar{\varphi_a}$$

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pseudocode
1:Input:Initial value, number of nodes m, number of sweeps J
2:Prediction(Initialization):
3:Compute an initial approximate solution on the Gauss-Legendre grid using a
low-order method
4:\\Correction loop
5:for j=1 to J do
    ##Compute Residual
    Calculate the residual vector using the spectral integraion matrix S^m
    \sigma(\varphi^{[j]})=S^m\bar{F}(\varphi^{[j]})-
\varphi^{[j]}+\bar{\varphi_a}

    ##Solve error Equation
    Solve for the correction using a first-order method on the error
equation
    \delta^{[j]}=C_{imp}(G,\sigma(\varphi^{[j-1]}))

    ##Update Solute
    \varphi^{[j]}=\varphi^{[j-1]}+\delta
6:Output:High-order solution \varphi^{[J]}
```

3.Analysis of method

3.1 Behavior at large $|\lambda|$ and extrapolation

- **Explicit Euler**: For stiff problem, the explicit SDC schemes are unstable
- **Implicit Euler**: The implicit SDC schemes are generally stable. However, when analyzing the amplification factor,$Am(\lambda)$ as $|\lambda| \to \infty$, we observe

$$\lim_{|\lambda|\to\infty} = Am(\lambda) = \mu(m, J) \neq 0$$

  This means that the method is Not L-stable.

Therefore, we construct a combinations method.
Composite Schemes

To achieve L-stability, we compute two solutions by using two different number of nodes m_1, m_2 with asymptotic limits μ_1, μ_2. The linear combination

$$\varphi_{comb} = \frac{\mu_2\varphi_1 - \mu_1\varphi_2}{\mu_2 - \mu_1}.$$

3.2 Order of Accuracy

For any sufficiently smooth function F, the approximation convergence with an order of accuracy given by

$$Order = \min(m, J+1)$$

where m is the number of Gauss-Legendre nodes and J is the number of correction sweeps.

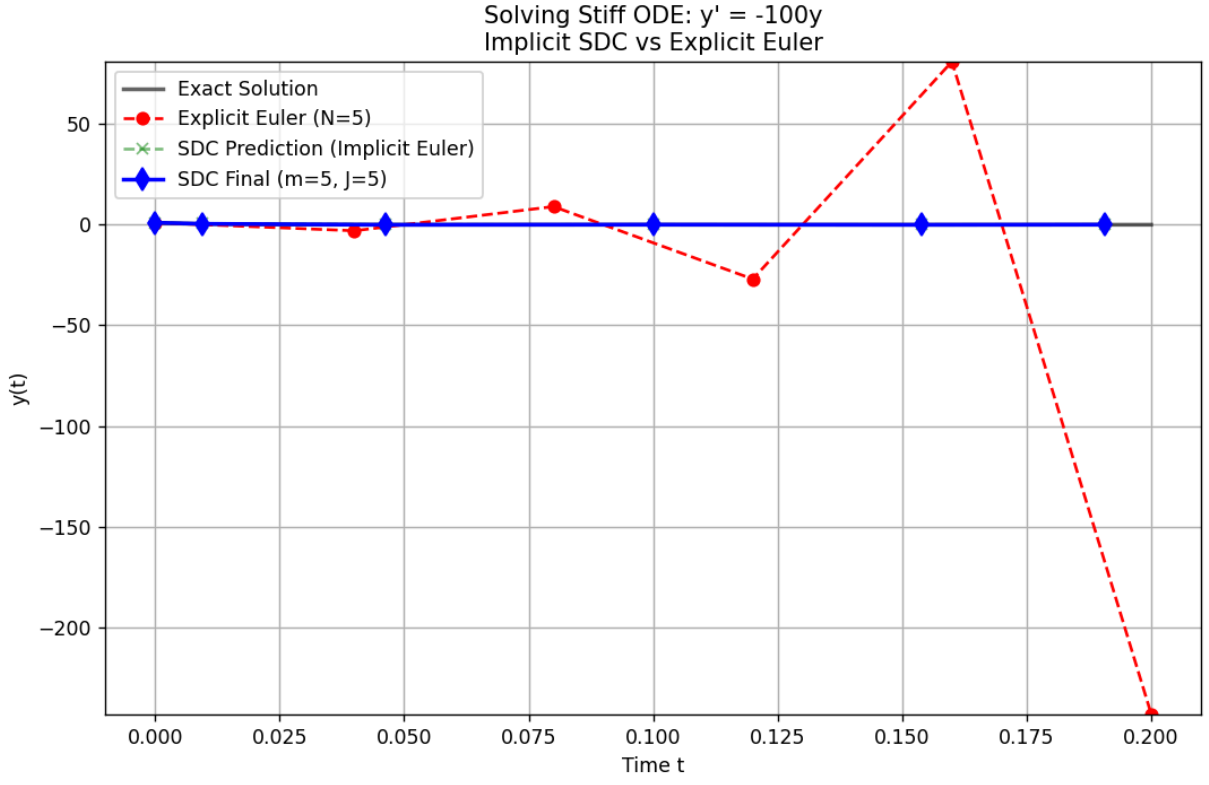
4. Numerical example

Consider

$$\begin{cases} \varphi' = -100\varphi \\ \varphi(0) = 1 \end{cases}$$

4.1 Analysis of result

- **Explicit Euler method**: The Explicit Euler solution (red dashed line) violates the stability condition ($0.04 > 0.02$). Consequently, it instability, oscillating and diverging to values near -250 at $t = 0.2$.
- **SDC**:The SDC result remains bounded and stable despite the large step size. Furthermore, the iterative spectral deferred correction sweeps successfully boost the accuracy, resulting in a solution that is visually indistinguishable from the exact solution



5. Conclusion

We explored Spectral Deferred Correction (SDC) methods for solving stiff ordinary differential equations with arbitrary order accuracy. By using Picard integral equation and Gauss-Legendre quadrature, the SDC approach iteratively corrects a low-order approximation to achieve high-precision results. Consequently, SDC proves to be a powerful and flexible alternative to traditional methods for high-accuracy scientific computing.