1. Consider the boundary value problem, $-u'' = e^{\sin(x)}, u(0) = 0, u(1) = 0.$

- Solve the problem and check the accuracy of your solutions.
- Explore the limitation of accuracy: identify the smallest error you can achieve and explain what prevents further improvement.

sol: The exact solution is given by

$$u(x) = \int_0^1 G(x,s) f(s)\, ds$$

where $G(x,s)$ is Green's function, $f(s) = e^{\sin(s)}$. Then, we have

$$u(x) = (1-x)\int_0^x s e^{\sin(s)}\, ds \; + x\int_x^1 (1-s)e^{\sin(s)}\, ds.$$

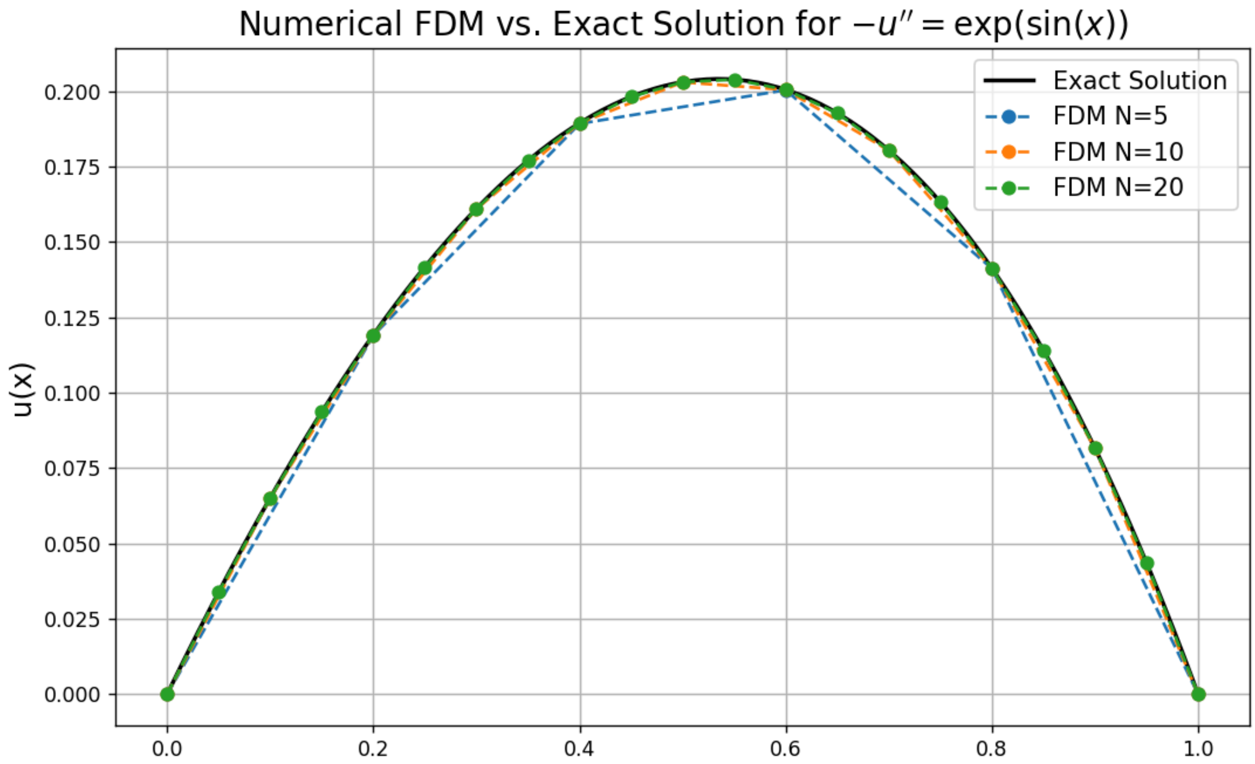
Now, we want to use FDM to its numerical solution.
Given $P = \{0 = x_0 < \cdots < x_n = 1\}, x_j = jh$, denote $f(x_j) = f_j$. Then

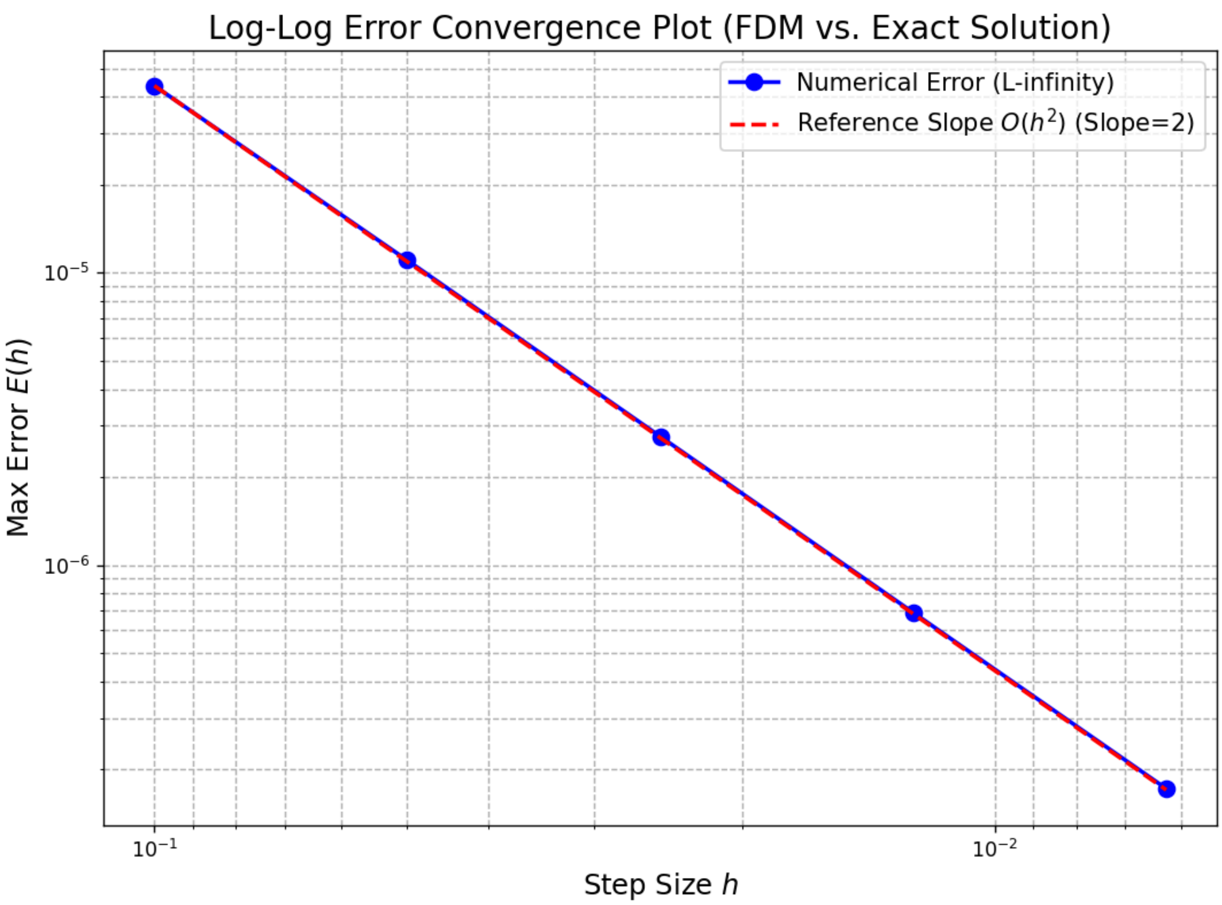
$$\frac{u_{j+1-2u_j+u_{j-1}}}{h^2} = f_j, \quad j = 1, \cdots, n-1$$

Hence, we can solve the Linear system $L_h u_n = f$ to get numerical solution.
The strategy to verify the accuracy of the solution is we want to use the difference of the N to get the numerical solution. Since FDM is $O(h^2)$ then we have

$$\text{Error of convergence} = \log_2(\frac{E_1}{E_2}) = 2$$

where $E_1 = \max|u_h(x_i) - u_{\frac{h}{2}}(x_i)|$, $E_2 = \max|u_{\frac{h}{2}}(x_i) - u_{\frac{h}{4}}(x_i)|$, the following result satisfy my verify.





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2. Consider the nonlinear boundary value problem: $-u'' + \sin(u) = 0, \quad u(0) = 1, u(1) = 1$. Solve the problem and check the accuracy of your solutions.

sol: By using the method which has show in (1), then we have