1. Prove that $\|\tau\|_h^2 \leq 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2)$.

> Hint: for each internal node $x_j = 1, \cdots, n-1$, integral by parts
> $\tau_h(x_j) = \frac{1}{h^2}(\mathcal{R}_4(x_j+h) + \mathcal{R}_4(x_j-h)) - (12.21)$ to get
> $\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt - \int_{x_j-h}^{x_j}(x_j+h-t)^2\,dt]$.

sol:

For each internal nodes x_j, we have $\mathcal{R}_4(x_j+h) = \int_{x_j}^{x_j+h}(u'''(t) - u'''(x_j))\frac{(x_j+h-t)^2}{2}\,dt - (i)$;

$\mathcal{R}_4(x_j-h) = -\int_{x_j-h}^{x_j}(u'''(t) - u'''(x_j))\frac{(x_j-h-t)^2}{2}\,dt - (ii)$.

From (i), take $z = u''(t) - u''(x_j)$ and so that $dz = u'''(t)\,dt$, and then take $dv = \frac{(x_j+h-t)^2}{2}\,dt$,

which gives $v = -\frac{(x_j+h-t)^3}{6}$ by IBP we have,

$$\mathcal{R}_4(x_j+h) = \left[-(u''(t) - u''(x_j))\frac{(x_j+h-t)^3}{6}\right]_{x_j}^{x_{j+h}} + \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6}$$

$$= -u''(x_j) \cdot \frac{h^3}{6} + \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6} - (iii).$$

Similarly to $\mathcal{R}_4(x_j-h)$, then we have,

$$\mathcal{R}_4(x_j-h) = +u''(x_j) \cdot \frac{h^3}{6} - \int_{x_j-h}^{x_j} u''(t)\frac{(x_j-h-t)^3}{6} - (iv).$$

From $(iii), (iv)$, by (12.21), we get

$$\tau_h(x_j) = \frac{1}{h^2}(\mathcal{R}_4(x_j+h) + \mathcal{R}_4(x_j-h))$$

$$= \frac{1}{h^2}\left[\int_{x_j}^{x_j+h} u''(t)\frac{(x_j-h-t)^3}{6} - \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6}\right].$$

We write $I = \frac{1}{h^2}\left[\int_{x_j}^{x_j+h} u''(t)\frac{(x_j-h-t)^3}{6} - \int_{x_j}^{x_j+h} u''(t)\frac{(x_j+h-t)^3}{6}\right]$, and then $\tau_h(x_j) = f(x_j) - I_j$

$$\|\tau(x_j)\| = h\sum_{j=1}^{n-1}|\tau_h(x_j)|^2$$

by $(a-b)^2 \leq 2a^2 + 2b$ $\leq 2h(\sum_{j=1}^{n-1}|f(x_j)|^2 + \sum_{j=1}^{n-1}|I_j|^2)$

$$= 2\|f\|_h^2 + 2h\sum_{j=1}^{n-1}|I_j|^2.$$

For each $t \in [x_j - h, x_j + h]$ we have $(x_j \pm h - t)^2 \leq h^2$. Then

$$|I_j| \leq \int_{x_j-h}^{x_j+h} |u''(t)|\,dt = \int_{x_j-h}^{x_j+h} |f(t)|\,dt$$

By Cauchy-Schwarz inequality, we have

$$|I_j|^2 \leq (\int_{x_j-h}^{x_j+h} |f(t)|\, dt)^2$$

$$\leq \int_{x_j-h}^{x_j+h} |f(t)|^2\, dt \int_{x_j-h}^{x_j+h} |1|^2\, dt$$

$$= 2h \int_{x_j-h}^{x_j+h} |f(t)|^2\, dt = 2\|f\|_{L^2(0,1)}^2.$$

Hence, we conclude that $\|\tau\|_h^2 \leq 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2)$.

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2. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j(1 - x_j)$

sol:

Note that $T_h g = w_h = \sum_{k=1}^{n-1} g(x_k) G^k$. Since $g = 1$, by $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function, then

$$T_h g = h \left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right]$$

Take $x_j = jh$, for $j = 1, 2, \cdots, n-1$. Then

$$T_h g = h \left[(1 - jh) \sum_{k=1}^{j} kh + jh \sum_{k=j+1}^{n-1} (1 - kh) \right]$$

$$= \frac{1}{2} x_j(1 - x_j)$$

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3. Prove Young's inequality.

$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$, for any $a, b \in \mathbb{R}$, for any $\epsilon > 0$.

sol:

Let $\epsilon > 0$. Let $a, b \in \mathbb{R}$.
By $0 \leq (\sqrt{\epsilon}a - \frac{b}{2\sqrt{\epsilon}})^2$, then

$$a^2\epsilon - ab + \frac{b}{4\epsilon} \geq 0$$

$$ab \leq a^2\epsilon + \frac{b}{4\epsilon}.$$

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4. Show that $\|v_h\|_h \leq \|v_h\|_{h,\infty}$, for any $v_h \in V_h$

sol:

Take $v_j = V_h(x_j)$, for $j = 1, \cdots, n-1$.

Then, we have $\|v_h\|_h = \sqrt{(h\sum_{j=1}^{n-1} v_j^2)}$.

By $|v_j| \le \max_{1 \le k \le n-1} |v_k| = \|v_h\|_{h,\infty}^2$, for any j, then

$$\|v_h\|^2 \le h\sum_{j=1}^{n-1} \|v_h\|_{h,\infty}^2$$
$$= h(n-1)\|v_h\|_{h,\infty}$$

Therefore,

$$\|v_h\|_h = \sqrt{(h\sum_{j=1}^{n-1} v_j^2)}$$
$$\le \|v_h\|_{h,\infty}.$$

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5. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

sol:

By the second order centered finite differences, we have

$$(L_h u_h)(x_j) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \quad , \text{for } j = 1, \cdots, n-1$$
$$(L_h u_h)(x_{j+1}) = -\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2}$$
$$(L_h u_h)(x_{j-1}) = -\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}.$$

Therefore, we have

$$(L_h u_h)(x_j) = -\frac{(L_h u_h)(x_{j+1}) - 2(L_h u_h)(x_j) + (L_h u_h)(x_{j-1})}{h^2}$$
$$= -\frac{u_{i+2} - 4u_{i+1} + 6u_i + 4u_{i-1} - u_{i-2}}{h^4}.$$