8. Determine an interpolating polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, \dots, n,$$

and check that

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides with the *Taylor polynomial.*

Define $T_n(x) := \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$

Then, $T_n^{(k)}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} \frac{j!}{(j-k)!}(x-x_0)^{j-k}$

$\Rightarrow T_n^{(k)}(x_0) = f^{(k)}(x_0).$

We set $Hf := T_n$, then there exists $Hf \in \mathbb{P}$ s.t $(Hf)^{(k)}(x_0) = f^{(k)}(x_0)$

2. Show that, for $n+1$ Chebyshev points of the second kind, the barycentric weights are (after rescaling)

$$w_i = (-1)^i, \quad i = 1, \cdots, n-1,$$

and $w_0 = 1/2$, $w_n = (-1)^n/2$.

1° By Chebyshev points of the second kind is $x_k = \cos(\frac{k\pi}{n})$, for $k=0 \sim n$.

Then $U_{n-1}(x_k) = 0$, for $k = 1, \cdots, n-1$. —— $(*)$

Note that $w_i = \frac{1}{\prod_{j\neq i}(x_j - x_j)}$, $\omega_{n+1}(x) = \prod_{j=0}^{n}(x-x_j)$,

Therefore, $\omega_{n+1}(x) = a(x^2-1)U_{n-1}(x)$ for some constant $a \in \mathbb{R}$.

$\omega'_{n+1}(x) = a(2xU_{n-1}(x) + (x^2-1)U'_{n-1}(x))$

By $(*)$ $\Rightarrow \omega'_{n+1}(x_k) = a(x_k^2-1)U'_{n-1}(x_k)$ —— $(**)$

Set $\theta = \arccos x$, $\frac{d}{dx}U_{n-1}(x) = \frac{d}{d\theta}\frac{\sin(n\theta)}{\sin\theta} \cdot \frac{d\theta}{dx}$

$= \frac{n\cos(n\theta)\sin\theta - \cos\theta(\sin(n\theta))}{\sin^2\theta} \cdot \frac{1}{(-\sin\theta)}$

$x = x_k$, $\theta = \arccos(\cos(\frac{k\pi}{n})) = \frac{k\pi}{n}$

$\Rightarrow U'_{n-1}(x_j) = \frac{n\cos(n\theta_k)}{\sin^2\theta_k} = \frac{n(-1)^j}{1-x_j^2}$ $(***)$

By $(**)$ and $(***)$ $\Rightarrow \omega'_{n+1}(x_j) = an(-1)^j$, (after rescaling)

By previous HW, we have proved $\omega'_{n+1}(x_k) = \frac{1}{w_k(x_k)}$ $\Rightarrow w_i(x_k) = (-1)^j$ for $j = 1, \dots, n-1$

2° For $k=0$, $x_0 = 1$ $\Rightarrow \omega'_{n+1}(x_0) = 2aU_{n-1}(x_0)$

By $\lim_{x_0 \to 1} \frac{\sin(n\arccos x_0)}{\sin(\arccos x_0)} = \lim_{x_0\to 1}\frac{n\cos(n\cos^{-1}x_0)}{\cos(n\cos^{-1}x_0)} = n$

$\Rightarrow \omega'_{n+1}(x_0) = 2na$. Therefore $w_i(x_0) = \frac{1}{2}$ after rescaling.

For $k=n$, $x_n = -1$. Similarly for the case $k=0$, and by $\lim_{x_0\to -1}\frac{\sin(n\cos^{-1}x_0)}{\sin(\cos^{-1}x_0)} = (-1)^{n+1}n$

$\Rightarrow w_i(x_n) = \frac{(-1)^{n-1}}{2}$, after rescaling.

5. Prove that

$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \le |\omega_{n+1}(x)| \le n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = 2/n$.
[*Hint* : let $N = n/2$ and show first that

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\ldots(x+h)x \\ (x-h)\ldots(x-(N-1)h)(x-Nh). \tag{8.74}$$

Then, take $x = rh$ with $N-1 < r < N$.]

Let $N = n/2$. Let $x_j = -1 + jh$
$= -Nh + jh = (j-N)h$

By $w_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$

$= (x-x_0)(x-x_1)\cdots(x-x_n)$

$= (x-Nh)(x-(N-1)h)\cdots x(x+h)\cdots(x+(N-1)h)(x+Nh)$

$= \prod_{j=0}^{2N} (x-(j-N)h)$ (take $k = j-N$)

$\Rightarrow w_{n+1}(x) = \prod_{k=-N}^{N} (x-kh)$

Take $x = rh$, for $N-1 < r < N$

Then $w_{n+1}(x) = h^{2N+1} \prod_{k=-N}^{N} (r-k)$

$= (r-N)(r-(N-1)) \cdot h^{2N+1} \prod_{k=-N}^{N-2} (r-k)$

$|w_{n+1}(x)| = h^{2N+1} |(r-N)(r-(N-1))| \prod_{k=-N}^{N-2} |r-k|$

Hence, $\prod_{m=0}^{N} (N-1+m) < \prod_{m=0}^{N} (r+m) < \prod_{m=0}^{N} (N+m)$ by reindexing $m = -k$.

$\Rightarrow \dfrac{(2N-1)!}{(N-1)!} < \prod_{m=0}^{N} (r+m) < \dfrac{(2N)!}{N!}$ —— (1)

Moreover, $\prod_{m=0}^{N-2} (N-1-m) < \prod_{m=0}^{N-2} (r-m) < \prod_{m=0}^{N-2} (N-m)$ —— (2)

$\Rightarrow (N-1)! < \prod_{m=0}^{N-2} (r-m) < N!$

Multiply (1), (2) $\Rightarrow$ $(2N-1)! < \prod_{k=-N}^{N-2} (r-k) < (2N)!$

$\Rightarrow (n-1)!\, h^{n+1} |(r-N)(r-(N-1))| < |w_{n+1}(x)| < (2N)!\, h^{n+1} |(r-N)(r-(N-1))|$

$\Rightarrow (n-1)!\, h^{n-1} |(x-x_n)(x-x_{n-1})| < |w_{n+1}(x)| < n!\, h^{n-1} |(x-x_n)(x-x_{n-1})|$

Under the assumptions of Exercise 5, show that $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that $|\omega_{n+1}|$ is an even function).
[*Hint* : use (8.74) to prove that $|\omega_{n+1}(x+h)/\omega_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$

Note that $W_{n+1}(x) = \prod_{k=-N}^{N} (x-kh)$. Then $w_{n+1}(x+h) = \prod_{k=-N}^{N} (x-(k-1)h)$

Take $p_n(x) = \dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}$

$$= \frac{\prod_{k=-N}^{N} (x-(k-1)h)}{\prod_{k=-N}^{N} (x-kh)}$$

$$\left(\frac{(x-(-N-1)h)\cancel{(x+Nh)} \cdots (x- \quad (x-(N-1)h)}{\cancel{(x+Nh)}\,(x+(N-1)h) \cdots \quad (x-Nh)} \right)$$

$$= \frac{x+(N+1)h}{x-Nh}$$

For $x \in (0, (N-1)h)$, we have $|p_n(x)| = \left| \dfrac{w_{n+1}(x+h)}{w_{n+1}(x)} \right|$

$$= \frac{x+(N+1)h}{Nh-x}$$

By $x+(N+1)h - Nh - x > 0 \Rightarrow |p_n(x)| > 1$ for $x \in (0, x_{n-1})$.
Therefore $|W_{n+1}(x+h)| > |W_{n+1}(x)|$ for $x \in (0, x_{n+1})$, which implies $|w_{n+1}(x)|$ is strictly increasing at $x \in (0, x_{n-1})$.
Since $|w_{n+1}(x)|$ is an even func., then $|w_{n+1}(x)|$ is strictly decreasing at $x \in (-x_{n-1}, 0)$

Clearly, when $x = x_{n-1}, x_n$, we have $w_{n+1}(x) = 0$.
Hence $|w_{n+1}(x)|$ attains its maximum when $x \in (x_{n-1}, x_n)$.