Problem 1

Let $u(x,t)=1$ be the exact solution of (1). We assume that $f\equiv 0$. Explain why the energy estimate (2) does not hold here.

Reference:

$$u_t=\nu u_{xx}+f(x,t) \tag{1}$$

$$E(t)\le e^{-\gamma t}E(0)+\frac{1}{\gamma}\int_0^t e^{\gamma(s-t)}F(s)\,ds \tag{2}$$

sol: During the process of the proof, we have used the Poincare inequality,

$$\int_0^1 u^2\,dx\le (C_p)^2\int_0^1 (u_x)^2\,dx.$$

By using the exact solution $u(x,t)=1$, then we have

$$\begin{aligned}LHS&=\int_0^1 u^2\,dx=\int_0^1 1\,dx=1\\ RHS&=(C_p)^2\int_0^1 (u_x)^2\,dx=(C_p)^2\cdot\int_0^1 0\,dx=0\quad \text{since } u(x,t) \text{ is a constant.}\end{aligned}$$

Hence, the inequality is clearly wrong, then $u(x,t)=C$, where $C\neq 0$ is a constant will make the estimate (2) dose not hold. That is the energy estimate will hold only in Dirchlet's boundary condition.

Problem 2

Consider the diffusion equation:

$$u_t=u_{xx},\quad 0\le x\le 1,\quad t\ge 0$$

with initial and boundary conditions

$$u(x,0)=\sin\left(\frac{1}{2}\pi x\right)+\frac{1}{2}\sin(2\pi x),\quad u(0,t)=0,\quad u(1,t)=e^{-\pi^2 t/4},\quad t\ge 0.$$

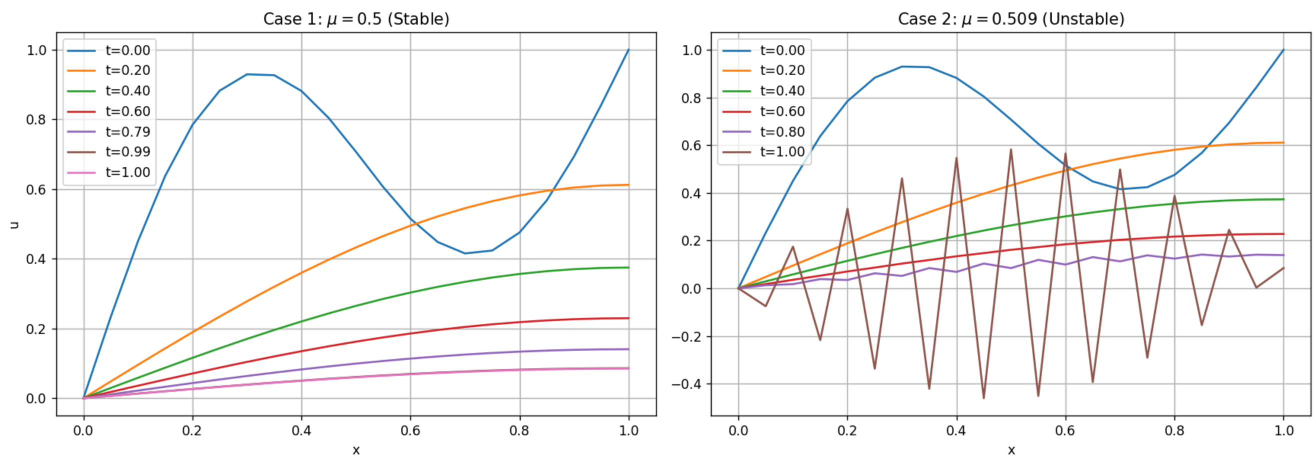
Solve the problem using Forward Euler finite difference method with $\mu=0.5$ and $\mu=0.509$.

sol:

```
Python
Setting function
def solve_diffusion(mu, N=20, T_end=1.0):
    L = 1.0
    dx = L / N
    x = np.linspace(0, L, N+1)

    \\define function
    dt = mu * dx**2
    time_steps = int(T_end / dt)

    \\define initial condition
    u = np.sin(0.5 * np.pi * x) + 0.5 * np.sin(2 * np.pi * x)
```



Analysis:

Forward Euler finite difference method give us $\frac{u_k^{n+1}-u_k^n}{\Delta t} = \nu\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2}$. By taking $\nu = 1$, hence we have

$$\begin{aligned} u_k^{n+1} &= u_k^n + \mu(u_{k+1}^n - 2u_k^n + u_{k-1}^n) = L[u^n], \quad \mu = \frac{\Delta t}{\Delta x^2}. \\ &= \mu u_{k+1}^n + (1-2\mu)u_k^n + \mu_{k-1}^n \end{aligned}$$

Observe that the numerical solution is stable when $\mu < \frac{1}{2}$. Hence the result of $\mu = 0.509$ become unstable.

Problem 3

Consider the diffusion equation:

$$u_t = u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$$

with initial and boundary conditions

$$u(x,0) = \sin(2\pi x)e^x, \quad u(0,t) = u(1,t) = 0, \quad t \ge 0.$$

With the aid of Fast Sine transform, solve the problem using (a) finite difference method and (b) method of line approach to $T = 1$. Find the order of convergence.

sol:

a) By using Fast Sine transform on FDM, we have $\frac{\hat{u}_k^{n+1} - \hat{u}_n^k}{\Delta t} = \lambda_k \hat{u}_K^{n+1}$, where λ_k is the eigenvalue of $-\frac{1}{h^2}A$, and A is defined by

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & & -1 & 2 \end{pmatrix}.$$

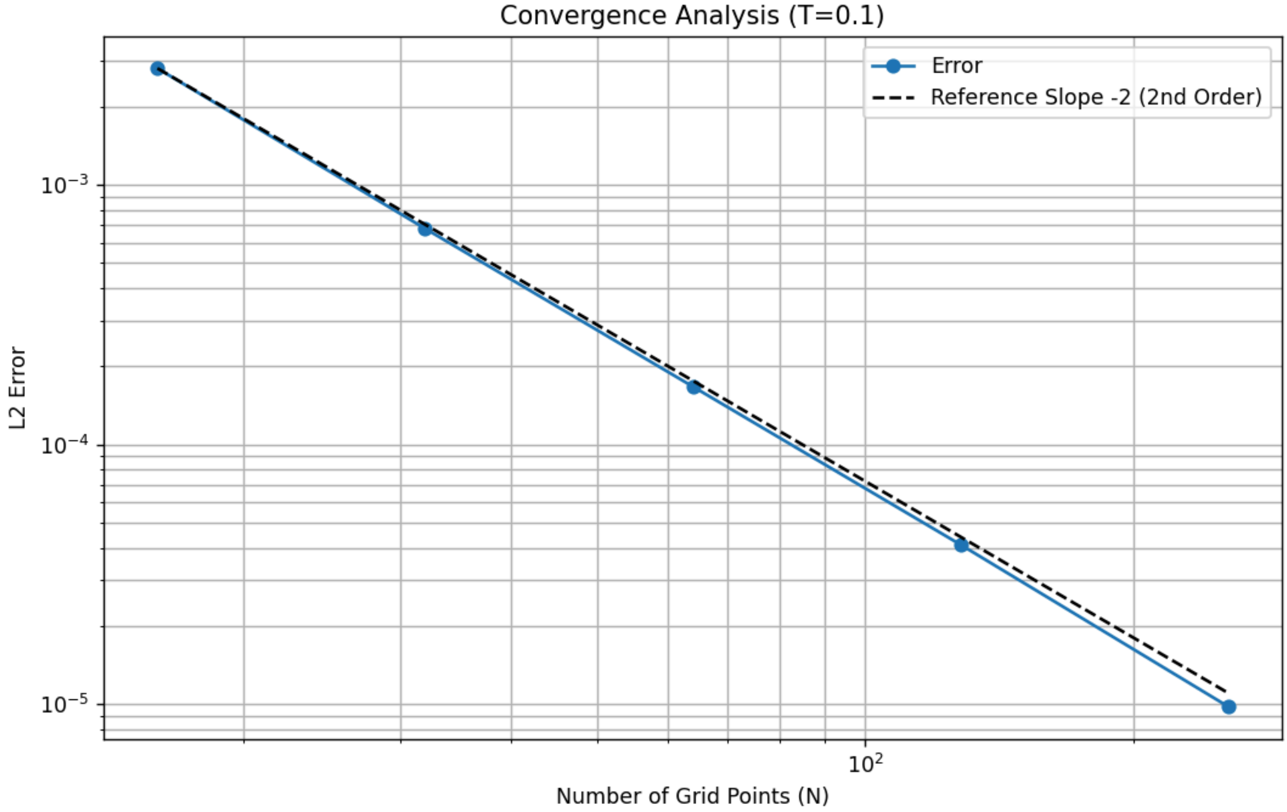
Hence, we want to solve

$$\hat{u}_k^{n+1} = \frac{\hat{u}_k^n}{1 - \Delta t \lambda_k}$$

```
python
def solve_diffusion_fst(N, T_end):
    dx = 1.0 / N
    dt = 0.2 * dx**2
    x_interior = np.linspace(dx, 1-dx, N-1)
    \\initial condition
    u = np.sin(2 * np.pi * x_interior) * np.exp(x_interior)

    k = np.arange(1, N)
    lambda_k = - (2 - 2 * np.cos(k * np.pi / N)) / (dx**2)
```

```
Computing reference solution (N=1024)...
N          | L2 Error         | Order
----------------------------------------
16         | 2.82e-03         | 0.00
32         | 6.79e-04         | 2.05
64         | 1.67e-04         | 2.02
128        | 4.11e-05         | 2.02
256        | 9.76e-06         | 2.07
```



b)

By MOL, we have the discrete system $\frac{du}{dt} = -\frac{1}{h^2}Au$, where A is defined by the previous part. Define

$$\hat{u_k}(T) = \hat{u}_k(0) \cdot e^{\lambda_k T}.$$

```
Computing reference solution at T=1.0 (N=2048)...
N          | L2 Error         | Order
----------------------------------------
16         | 3.65e-07         | 0.00
32         | 8.88e-08         | 2.04
64         | 2.20e-08         | 2.02
128        | 5.45e-09         | 2.01
256        | 1.34e-09         | 2.02
```