Linear Algebra Fall 2022
Homework Problem Set One
Due: 23:59
on 9/19/2022

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Office hour:	make appointments by email or Google Meet

Note: Please do not copy your classmates homework. You have to write your homework on your own. However, you are encouraged to have discussion with your classmates, friends and the instructor. This is very important.

Please hand in Problems A and E.

Problem A: Discuss on the number α such that the following linear system has a solution, no solution or infinitely many solutions ?

$$\begin{cases} 2 = & & y + z, \\ 2 = x + & \alpha y + z, \\ 2 = x + & y & . \end{cases}$$

Problem B: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Please convert the problem of finding the inverse of the matrix A into a problem of solving a linear system of $Bx = b$ where $B \in \mathbb{R}^{9\times 9}$ and $b\mathbb{R}^{9\times 1}$.

(a) What are B and b ?

(b) What is LU factorization of B ?

Problem C: Assume that matrices A and B are both in $\mathbb{R}^{m\times n}$ where m and n are positive integers. Prove or disprove that if $Ax = Bx$ for any $x \in \mathbb{R}^{n\times 1}$, then $A = B$.

Problem D: Let

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}.$$

Please answer the following questions:

(a) Find a matrix B such that $AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$.

(b) Find a matrix C such that $AC = A^2 + A$.

Problem E: Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 1 & 4 & 7 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

Please answer the following questions:

(a) Solve $Ax = b$ by Gauss-Jordan elimination.

(b) Find the LDU factorization of A.

(c) Write A as a product of elementary matrices.

(d) Find the inverse of A.

Problem F: Suppose that $A \in \mathbb{R}^{n\times n}$. If there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $AB = I_n$ where I_n is the identity matrix with size n. Show that A is invertible and $A^{-1} = B$.

Problem G: Let

$$A = \begin{pmatrix} a & 1 & 1 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \text{ where } a \neq 0.$$

Show that A is invertible for any $a \neq 0$ and find its inverse.

Problem H:

(a) Consider the following block matrix M

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are matrices in $\mathbb{R}^{2\times 2}$ and the matrix A is assumed to be invertible. We define the matrix $S = D - CA^{-1}B$. Show that

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}$$

where S and M are assumed to be invertible.

(b) Find the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}.$$

Problem I: Let $A \in \mathbb{R}^{n\times n}$ with $A = [a_{ij}]$ where $a_{ij} = \alpha$, if $i = j$; $a_{ij} = \beta$, if $i \neq j$. If $\alpha \neq \beta$, show that A is invertible and find a formula for A^{-1}.

Problem J: Assume that $A \in \mathbb{R}^{3\times 3}$, the matrix A is invertible and for any vectors $\vec{v}, \vec{w} \in \mathbb{R}^{3\times 1}$ we have $1 + \vec{w}^T A^{-1}\vec{v} \neq 0$. Show that the matrix $A + \vec{v}\vec{w}^T$ is invertible and

$$(A + \vec{v}\vec{w}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{v}\vec{w}^T A^{-1}}{1 + \vec{w}^T A^{-1}\vec{v}},$$

where $\vec{w}^T$ is the transpose of $\vec{w}$.