> 1. Consider the boundary value problem

$$u'' = f, \quad u(0) = 0, \quad u(1) = 0,$$

> where

$$f(x) = \begin{cases} 1 & 0.4 \le x \le 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

> - Find the exact solution of this problem.
> - Solve the problem using finite difference method and check the accuracy of your solutions.

sol:
Guess

$$u(x) = \begin{cases} ax, & \text{, if x} \le 0.4, \\ c + dx + \frac{1}{2}x^2 & \text{, if } 0.4 \le x \le 0.6, \\ b(1-x) & \text{, if } x \ge 0.6. \end{cases}$$

When $x = 0.4$, by continuity of u, we have

$$0.4a = c + 0.4d + \frac{1}{2}0.4^2 \quad (1)$$

Moreover, by continuity of u', we have

$$a = d + 0.4 \quad (2)$$

Similarly for $x = 0.6$, then we have the linear system

$$\begin{cases} 0.4a = c + 0.4d + \frac{1}{2}0.4^2 \quad (1) \\ a = d + 0.4 \quad (2) \\ 0.4b = c + 0.6d + \frac{1}{2}0.6^2 \quad (3) \\ -b = d + 0.6 \quad (4) \end{cases}$$

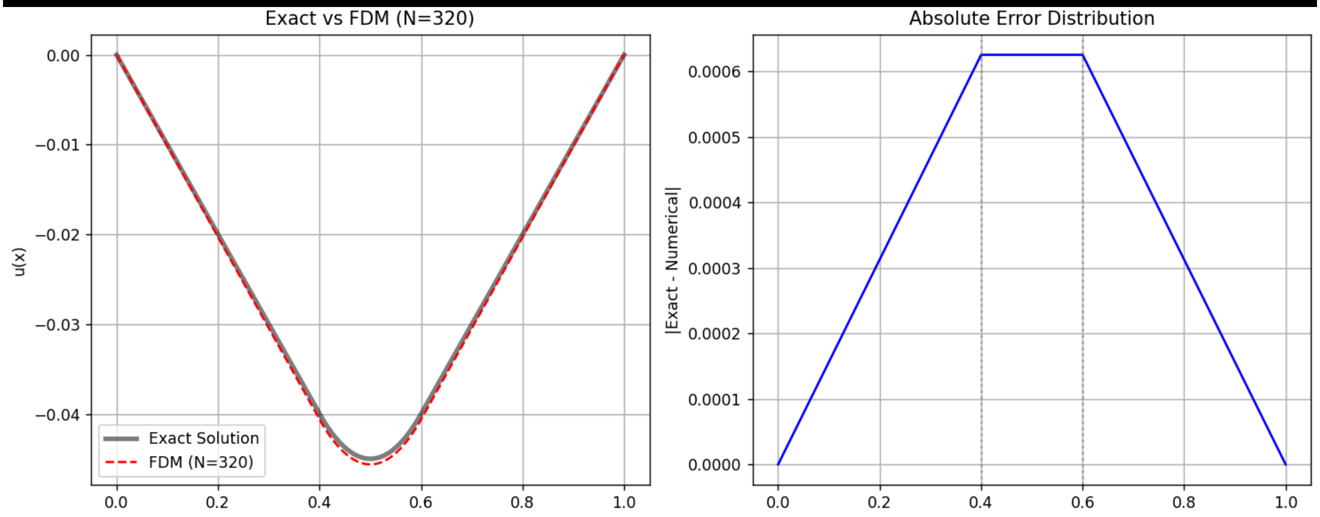
Substitute (2) into (1), then we get $c = 0.08$.
Substitute (4) into (3), then we get $d = -0.5$.
Therefore, we conclude that $a = -0.1, b = -0.1$, and so the exact solution is

$$u(x) = \begin{cases} -0.1x, & \text{, if x} \le 0.4, \\ 0.08 - 0.5x + 0.5x^2 & \text{, if } 0.4 \le x \le 0.6, \\ -0.1(1-x) & \text{, if } x \ge 0.6. \end{cases}$$

Next, we want to find the numerical solution by the Finite Difference Method. That is we want to solve $Au = b$, where A is a matrix has the form

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & & -1 & 2 \end{pmatrix}. -(\beta)$$

N	h	Max Error	Ratio (Err_old/Err_new)
10	0.1000	2.0000e-02	N/A
20	0.0500	1.0000e-02	2.0000
40	0.0250	5.0000e-03	2.0000
80	0.0125	2.5000e-03	2.0000
160	0.0063	1.2500e-03	2.0000
320	0.0031	6.2500e-04	2.0000



> 2. Consider the boundary value problem:
>
> $$u'' - 2u' + u = 1, \quad u(0) = 0, \quad u'(1) = 1.$$

> - Show that the solution is unique by considering the homogeneous problem.
> - Develop a 2nd-order finite difference method.
> - Solve the problem and check the accuracy of your solutions.

sol:
By contradiction, assume $w(t) = u_1(t) - u_2(t)$ is a solution of the boundary value problem, for any solution u_1, u_2
Since we have the boundary condition,

$$\begin{cases} w'' - 2w' + w = 0 & -(i) \\ w(0) = u_1(0) - u_2(0) = 0 & -(ii) \\ w'(1) = u_1'(1) - u_2'(1) = 0 & -(iii) \end{cases}$$

From the next part, we will show that $w(t) = c_1 e^t + c_2 t e^t$. By the boundary condition, we have conclude that, $c_1 = 0; c_2 = 0$, which implies that $u_1(t) = u_2(t)$

Next, we assume the the exact solution has the form $u(t) = u_h(t) + u_p(t)$.
Characteristic polynomial $u'' - 2u' + u = 0$. Then, we have specify function, $r^2 - 2r + 1 = 0$, which implies that $r = 1$. Thus we have the exact solution $u_h(t) = C_1 e^t + C_2 t e^t$.
For particular solution, we want to find $u_p(t)$ such that $\mathcal{L}[u_p] = 1$. Clearly $u_p(t) = 1$. Hence, $u(t) = C_1 e^t + C_2 t e^t + 1$, and so that $u'(t) = C_1 e^t + C_2(e^t + te^t)$.
From $u(0) = 0 \Rightarrow C_1 = -1$. From

$$\begin{aligned} u'(1) = 1 \Rightarrow -e + C_2(e+e) &= 1 \\ (-1 + 2C_2)e &= 1 \\ C_2 &= \frac{1+e^{-1}}{2}. \end{aligned}$$

Hence, $u(t) = 1 - e^t + \frac{1+e^{-1}}{2} t e^t$.

For the numerical solution, by central Difference, we have

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h}$$

Substitute it into BDV, then we have

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - 2\frac{u_{i+1} - u_{i-1}}{2h} + u_i &= 1 \\ u_{i+1} - 2u_i + u_{i-1} - h(u_{i+1} - u_{i-1}) + h^2 u_i &= h^2 \\ (1+h)u_{i-1} + (h^2 - 2)u_i + (1-h)u_{i+1} &= h^2 \quad \text{for i} = 1, \cdots, N-1 \end{aligned}$$

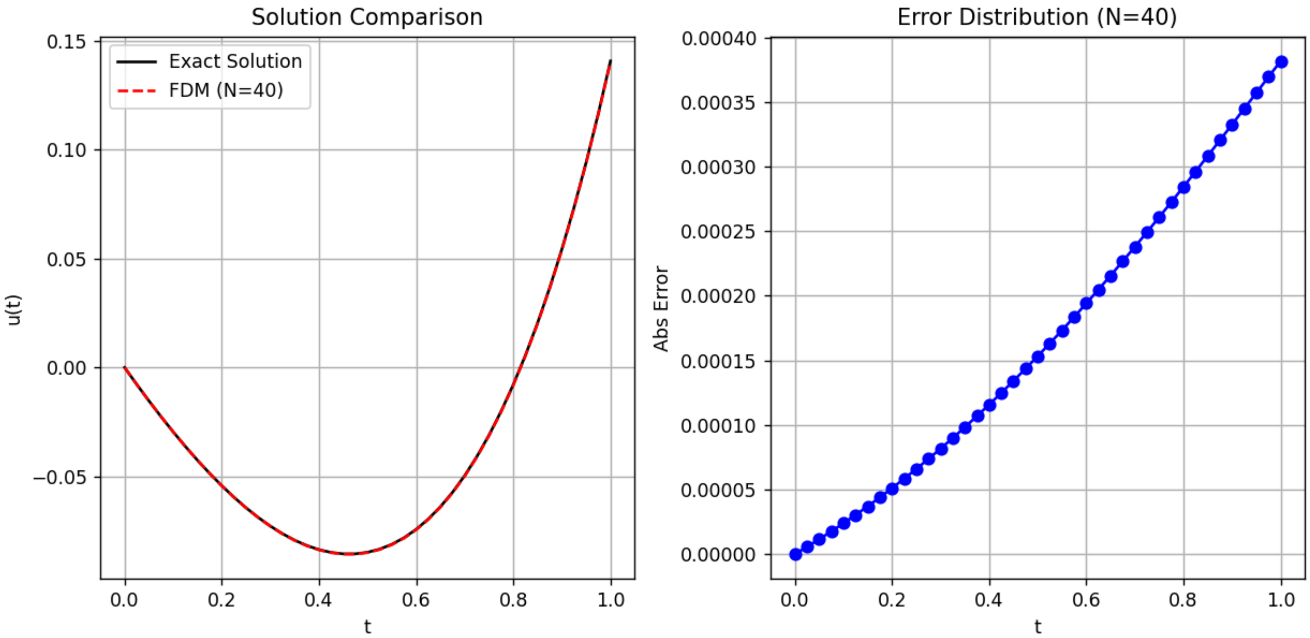
For $i = 0$, we have $u(0) = 0$. For $i = N$, then we have $u'(1) = \frac{u_{N+1} - u_{N-1}}{2h} = 1$. Substitute it into the previous function, then

$$\begin{aligned} (1+h)u_{N-1} + (h^2-2)u_N + (1-h)u_{N+1} &= h^2 \\ 2u_{N-1} + (h^2 - 2)u_N &= h^2 - 2h(1-h) \end{aligned}$$

That is the matrix has the form that

$$A = \begin{pmatrix} h^2 - 2 & 1-h & & & \\ 1+h & h^2-2 & 1-h & & \\ & 1+h & h^2-2 & 1-h & \\ & & & \ddots & \\ & & & 2 & h^2-2 \end{pmatrix} \quad b = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ 1 - \frac{2(1-h)}{h} \end{pmatrix}$$

N	h	Max Error	Ratio
10	0.1000	6.1323e-03	N/A
20	0.0500	1.5282e-03	4.01
40	0.0250	3.8174e-04	4.00
80	0.0125	9.5416e-05	4.00
160	0.0063	2.3853e-05	4.00



> 3. Consider the boundary value problem:

$$u'' = \sin(2\pi x), \quad u'(0) = 0, \quad u'(1) = 0.$$

> - Show that the consistency condition is satisfied so that the solution of the problem exists.
> - Develop a 2nd-order finite difference method.
> - Solve the problem and check the accuracy of your solutions.

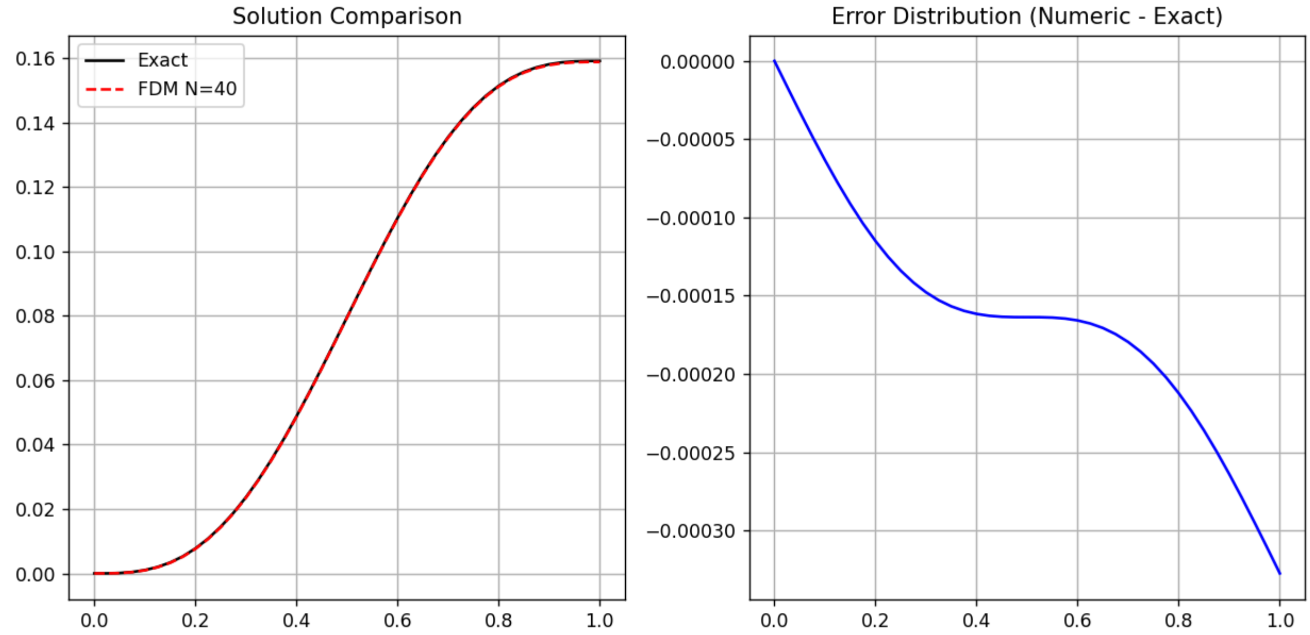
sol:
By FTC, we have $u'(1) - u'(0) = 0 = \int_0^1 \sin(2\pi x)\,dx$, that is we want to claim that $\int_0^1 \sin(2\pi x)\,dx = 0$.

$$\begin{aligned}\int_0^1 \sin(2\pi x)\,dx &= \left[-\frac{1}{2\pi}\cos(2\pi x)\right]_0^1 \\ &= \frac{1}{2\pi}(1-1) = 0\end{aligned}$$

By integrating $u'' = \sin(2\pi x)$, we have $u'(x) = -\frac{1}{2\pi}\cos(2\pi x) + C_1 - (\star)$. By $u'(0) = 0$, then we have $C_1 = \frac{1}{2\pi}$
Integrate $(\star)$, then we have $u(x) = \frac{1}{2\pi}(x - \frac{1}{2\pi}\sin(2\pi x)) + C_2$. Set $C_2 = 0$

By FDM, we want to solve $Au = b$, where A is a matrix has the form

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{pmatrix} \quad u = ()$$



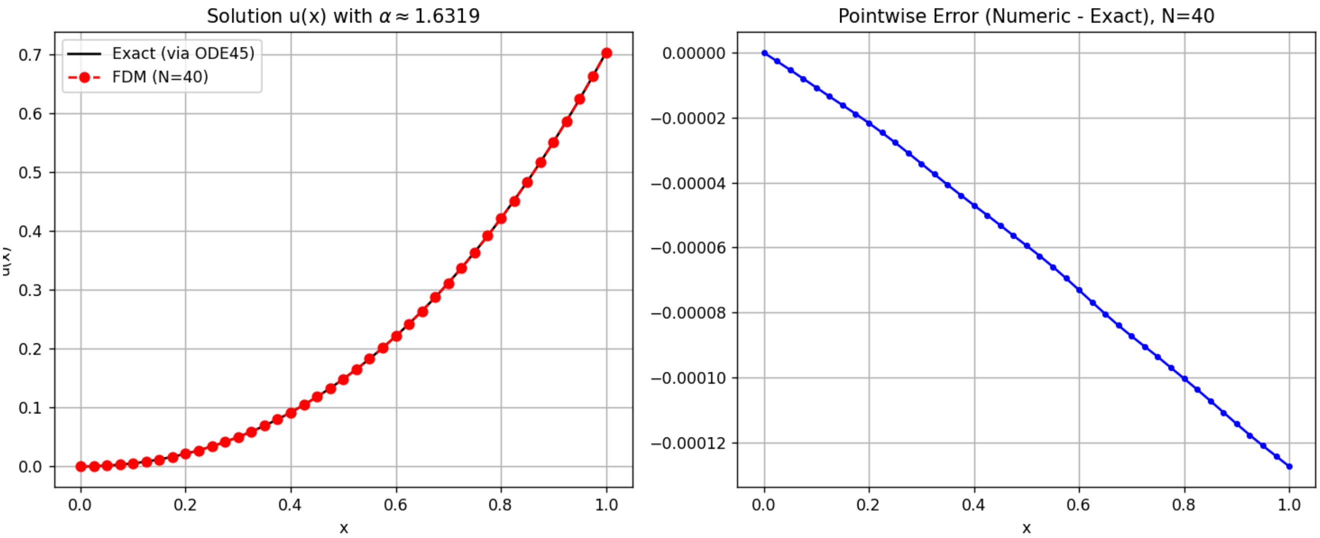
> 4. Consider the boundary value problem:
>
> $$u'' = e^{\sin(x)}, \quad u'(0) = 0, \quad u'(1) = \alpha.$$

> - Determine α such that the problem has at least one solution.
> - Solve the problem by finding one of its solution.

sol:
Similar to problem 3, we want to solve that $\int_0^1 e^{\sin(x)} dx = u'(1) - u'(0)$, that is $\alpha = \int_0^1 e^{\sin(x)} dx$
.
By $u'(0) = 0$, then $u'(x) = \int e^{\sin(x)}\, dx$. Therefore, $u(x) = \int_0^x (\int_0^s e^{\sin(t)}\, dt)\, ds$.

Similar to problem 1,3, A has the form (β), for $i = N$, we have $2u_{N-1} - 2u_N = f_N - 2h\alpha$, that is

$$Au = b \Rightarrow -\frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_N - \frac{2\alpha}{h} \end{pmatrix}$$



> 5. Consider the linear boundary value problem:
>
> $$\epsilon u'' + (1+\epsilon)u' + u = 0, \quad u(0) = 0, \quad u(1) = 1.$$
>
> Solve the problem and check the accuracy of your solutions. Choose $\epsilon = 0.01$.

sol:

Choose $\epsilon = 0.01$, characteristic polynomial, we have $0.01r^2 + (1.01)r + 1 = 0$, and so that

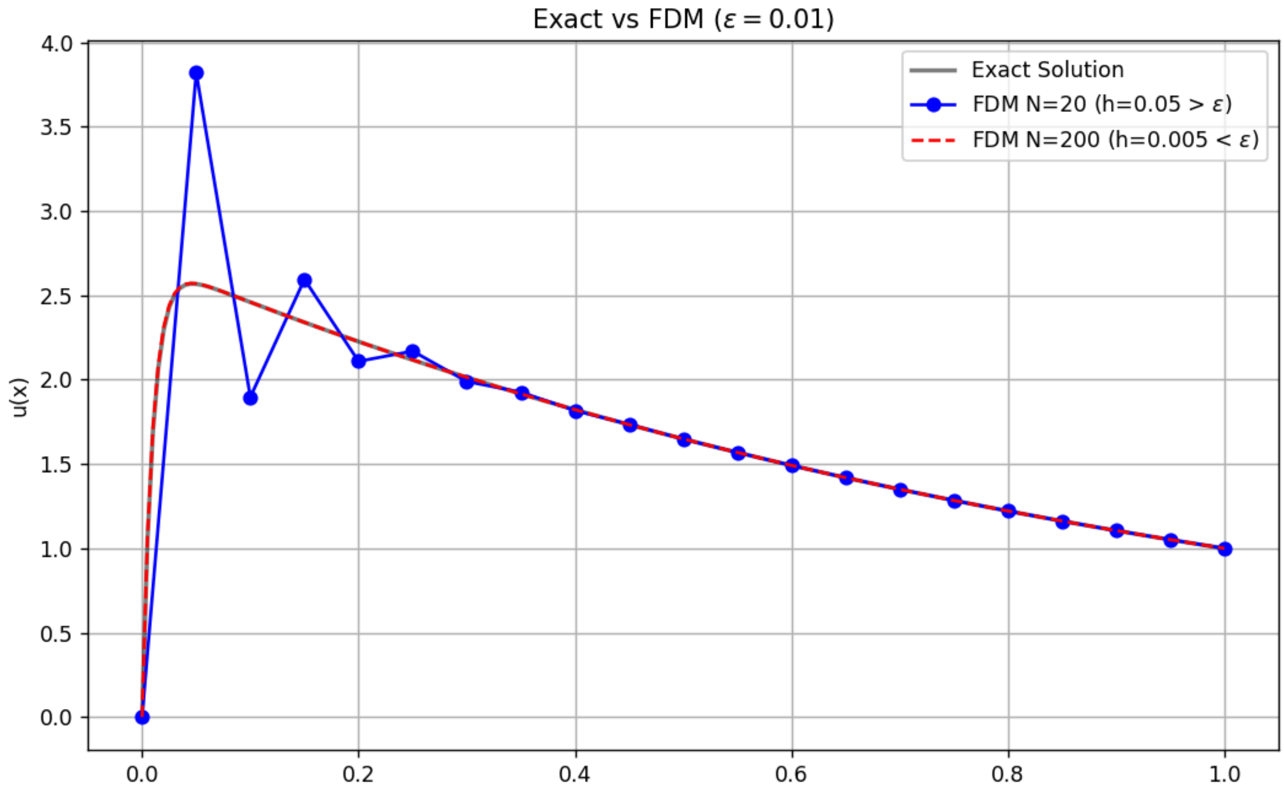
$$\begin{aligned} r &= \frac{-1.01 \pm \sqrt{1.01^2 - 4 \cdot 0.01 \cdot 1}}{2 \cdot 0.01} \\ &= -1, -100 \end{aligned}$$

Hence, $u(t) = C_1 e^{-t} + C_2 e^{-100t}$.

Solve

$$\begin{cases} u(0) = C_1 + C_2 = 0 \\ u(1) = C_1 e^{-1} + C_2 e^{-100} = 1 \end{cases}$$

$C_1 = -C_2 = \frac{1}{e^{-1} - e^{-100}}$. Thus, $u(t) = \frac{e^{-t} - e^{-100t}}{e^{-1} - e^{-100}}$.



Exact vs FDM ($\varepsilon = 0.01$)
Exact Solution
FDM N=20 (h=0.05 > ε)
FDM N=200 (h=0.005 < ε)
u(x)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0