Econ 212a: Business Cycles
Lecture 9
Normative Analysis of the New Keynesian Model

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Last two classes

- In lecture 7 we derived the NK model in nonlinear form
  - In lecture 8 we used the linearized equations for positive analysis:
  - 'How does the economy behave in response to shocks'?
  - Considered productivity, monetary policy, government spending, and reduced-form demand shocks

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    - ’How does the economy behave in response to shocks’?
    - Considered productivity, monetary policy, government spending, and reduced-form demand shocks
- We now want to study *optimal policy* in the model
    - ’What should the central bank do?’
    - The answer will be especially interesting for cost-push shocks

Two simplifications

- To simplify the problem, we will:

1. Ignore money (ie the MIU term in welfare)
    - May be a good approximation for low interest rates/low inflation
    - This is known as the 'cashless limit' (justified by eg technological progress)
    - Without this approximation, the MIU term provides a force favoring lower i_t to increase m_t (Friedman rule, cf problem set 3)

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    - Without this approximation, the MIU term provides a force favoring lower i_t to increase m_t (Friedman rule, cf problem set 3)
2. Ignore the 'ZLB' constraint $i_t \geq 0$
    - With MIU this is an endogenous constraint
    - Need to check ex-post the constraint does not bind
    - Will add back at the end

Optimal policy problem: commitment v. discretion

Stating the problem with commitment

- The planner takes Δ_{-1} as given and solves:

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{u\left(C_t\right) - v\left(N_t\right)\right\}\right]$$

by choice of F_0, H_0, $\{\Pi_t, C_t, Y_t, N_t\}$, and possibly τ^w, subject to the constraints:

$$C_t = Y_t = \frac{A_t}{\Delta_t} N_t^{1-\alpha}$$

$$\omega_t = \frac{\epsilon}{\epsilon - 1} \frac{\left(1 + \tau^w\right)}{\left(1 - \alpha\right)} \psi \frac{1}{A_t} \left(\frac{Y_t \Delta_t}{A_t}\right)^{\frac{\alpha+\phi}{1-\alpha}} \psi C_t^{\sigma}$$

$$\frac{1 - \theta \Pi_t^{\epsilon-1}}{1 - \theta} = \left(\frac{F_t}{H_t}\right)^{\frac{1-\epsilon}{1+\frac{\epsilon\alpha}{1-\alpha}}}$$

$$F_t = u'\left(C_t\right) C_t \omega_t + \theta \beta \mathbb{E}_t\left[\Pi_{t+1}^{\epsilon + \frac{\epsilon\alpha}{1-\alpha}} F_{t+1}\right]$$

$$H_t = u'\left(C_t\right) C_t + \theta \beta \mathbb{E}_t\left[\Pi_{t+1}^{\epsilon-1} H_{t+1}\right]$$

$$\triangle_t^{\frac{1}{1-\alpha}} = \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} \triangle_{t-1}^{\frac{1}{1-\alpha}} + (1 - \theta) \left(\frac{1 - \theta \Pi_t^{\epsilon-1}}{1 - \theta}\right)^{\frac{\epsilon}{(1-\alpha)(\epsilon-1)}}$$

Remarks

- Goals of optimal policy:
  1. Static objective: get output 'right' given Δ_t
     - consider welfare in the model with fully sticky prices ($\Delta_t = 1$):

$$\begin{array}{ll} \max & u(C_t) - v(N_t) \\ \text{s.t.} & C_t = \frac{A_t}{\Delta_t} N_t^{1-\alpha} \end{array} \quad \Rightarrow \quad \frac{v'(C_t)}{u'(N_t)} = (1-\alpha)\frac{A_t}{\Delta_t} N_t^{-\alpha}$$

     - equivalently, set allocation at its statically efficient level $Y_t^e = C_t^e$

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        - equivalently, set allocation at its statically efficient level $Y_t^e = C_t^e$
    2. Dynamic objective: get inflation 'right', ie make future Δ_t small
        - by setting $\Pi_t = 1$, law of motion of Δ gives us $\Delta_t \to 1$
- The planner wants to satisfy both objectives at once
    - Cares both about Π and Y relative to its efficient level Y^e
    - If can choose τ^w, optimal to set $\tau^w = -\frac{1}{\epsilon}$ to remove the monopoly distortion
    - Then flexible-price = efficient, so planner cares about output *gap*

Remarks

- How do we solve this? Can use dynamic programming:

$$
\begin{aligned}
V\left(F, H, \Delta_{-}, s\right)=\max_{C, N, F(s'), H(s')} \quad & u(C)-v(N)+\beta \mathbb{E}\left[V\left(F\left(s'\right), H\left(s'\right), \Lambda, s'\right) | s\right] \\
\text{s.t.} \quad & C=\frac{A(s)}{\Delta} N^{1-\alpha} \\
& F=u'(C) C \omega+\theta \beta \mathbb{E}\left[\Pi\left(\frac{F\left(s'\right)}{H\left(s'\right)}\right)^{\epsilon+\frac{\epsilon \alpha}{1-\alpha}} F\left(s'\right) | s\right] \\
& H=u'(C) C+\theta \beta \mathbb{E}\left[\Pi\left(\frac{F\left(s'\right)}{H\left(s'\right)}\right)^{\epsilon-1} H\left(s'\right) | s\right] \\
& \Delta=f\left(\Delta_{-}, \Pi\left(\frac{F}{H}\right)\right)
\end{aligned}
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\text{s.t.} \quad & C = \frac{A(s)}{\Delta} N^{1-\alpha} \\
& F = u'(C)\, C\omega + \theta\beta\mathbb{E}\left[\Pi\left(\frac{F(s')}{H(s')}\right)^{\epsilon + \frac{\epsilon\alpha}{1-\alpha}} F\left(s'\right) | s\right] \\
& H = u'(C)\, C + \theta\beta\mathbb{E}\left[\Pi\left(\frac{F(s')}{H(s')}\right)^{\epsilon - 1} H\left(s'\right) | s\right] \\
& \Delta = f\left(\Delta_-, \Pi\left(\frac{F}{H}\right)\right)
\end{aligned}
$$

- Not very tractable in practice, so we follow the literature and linearize
- Solution is policy for C, N, $F(s')$ and $H(s')$ as functions of (F, H, Δ_-, s)

Remarks on the solution

- We just discussed what is known as the **commitment solution**
  - The variables F_0 and H_0 (hence Π_0) are chosen at $t = 0$
  - Then they become state variables (promises by the planner)
  - The solution is *history dependent*: the planner may decide to promise inflation or an output gap in the future, to increase its objective today

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  - When it does not, we say the solution is *time-inconsistent*
- Alternative is '**discretionary solution**':
  - Planner picks F, H (so Π) and C each period
  - It can no longer control expectations
  - Theme of the literature: commitment improves over discretion
  - We discuss two examples below

Remarks

- Once the planner's problem is solved we can compute prices using

$$u'\left(C_t\right) = \beta \mathbb{E}\left[u'\left(C_{t+1}\right)\frac{1+i_t}{\Pi_{t+1}}\right]$$

$$q\left(s^t, s_{t+1}\right) = \beta\pi\left(s^{t+1}|s^t\right)\frac{u'\left(C\left(s^{t+1}\right)\right)}{u'\left(C\left(s^t\right)\right)}\frac{P\left(s^t\right)}{P\left(s^{t+1}\right)}$$

$$\frac{W_t}{P_t} = \psi C_t^{\sigma} N_t^{\phi}$$

- This is called the 'primal approach' to optimal policy problems:
    - First figure out quantities
    - Then use FOCs to determine the prices that sustain the allocation
    - Other example: Ramsey optimal taxation (see Patrick's classes)

Second-order objective function

- When $\tau^w = -\frac{1}{\epsilon}$, we can compute a second-order approximation to welfare

$$\min \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{(y_t - y_t^e)^2 + \lambda \pi_t^2\right\}\right] \tag{1}$$

where $y_t^e \equiv \log Y_t^e$ is the log efficient output level (see appendix go)

  - It turns out that $\lambda = \frac{\epsilon}{\kappa}$ is optimal weight on inflation vs output

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- Optimal policy maximizes (1) subject to

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t\left[\pi_{t+1}\right] \tag{NKPC}$$

- Once solution is computed, back out nominal interest rate using

$$x_t = \mathbb{E}_t\left[x_{t+1}\right] - \sigma^{-1}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) \tag{DIS}$$

so

$$i_t = r_t^n + \mathbb{E}_t\left[\pi_{t+1}\right] + \sigma\left(\mathbb{E}_t\left[x_{t+1}\right] - x_t\right)$$

The divine coincidence

Efficient steady-state without cost-push shocks

- Since $\tau^w = -\frac{1}{\epsilon}$, have $y_t^e = y_t^n$, and $x_t = y_t - y_t^n$, so (1) is simply

$$\min \quad \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{x_t^2 + \lambda \pi_t^2\right\}\right]$$

$$\text{s.t.} \quad \pi_t = \kappa x_t + \beta \mathbb{E}_t \left[\pi_{t+1}\right]$$

- Optimal solution is clearly $x_t = \pi_t = 0$ at all t
  - This is also true in nonlinear solution if initially $\Delta_{-1} = 1$
  - The optimal policy is time consistent, so commitment = discretion
  - Along this solution, the nominal interest rate is $i_t = r_t^n$

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  - The optimal policy is time consistent, so commitment = discretion
  - Along this solution, the nominal interest rate is $i_t = r_t^n$
- This is the well-known **divine coincidence result**
  - Zero inflation avoids relative price distortions ($\Delta_t = 1$ always)
  - Can get the first best level of output at all times
  - Markups are at firms' desired level, so resetters keep $P_t^* = P_{t-1}$

More on divine coincidence

- Fairly general principle that works in other price-setting models too:
  - If flexible price equilibrium is also first-best
  - and price ridigities is the only constraint that creates a friction
  - then we make sure this constraint does not bind!

  $\Rightarrow$ inflation targeting is optimal, goes hand-in-hand with output stabilization
- We already know from Lecture 8 how to implement this uniquely
  - Most direct: use a Taylor rule

$$i_t = r_t^n + \phi_\pi \pi_t + \phi_x x_t$$

  with Taylor principle satisfied. Requires knowledge of r_t^n
  - Approximately: use a simple Taylor rule

$$i_t = \rho + \phi_\pi \pi_t + \phi_x x_t$$

  with sufficiently large ϕ_π and/or ϕ_x, will get close to $x_t \to 0$

Costs of inflation in reality vs the mode

- Why don't people like inflation in practice?
  - 'Shoe leather' costs (holding less cash, menu costs, costs of searching)
  - Redistributes nominal debt from debtors to creditors (if unexpected)
  - Creates uncertainty/volatility, in addition to high average level
  - Creates confusion between relative price changes and price level changes

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    - captures some of these stories, nicely interpretable in terms of welfare
    - but maybe more of a parable for something else
- Outside of this model: why could $\pi > 0$ be optimal?
    - Risk of hitting the zero lower bound
    - Downward nominal wage rigidities (’greasing the wheels’)

Limits to the divine coincidence

- In several practical cases, the divine coincidence breaks:
  - Not true, for example, with two sticky-price sectors, and sectoral shocks that imply efficient relative price movements
  - Not true if there are wage rigidities in addition to price rigidities
  - Not true without the right labor subsidy (ie $\tau^w \neq -\frac{1}{\epsilon}$). That creates 'inflationary bias' for the central bank: temptation to inflate and create a boom to lower markups
  - Not true when there are 'cost-push' shocks that increase π for given x
- We consider this latter case, which also illustrates the difference between commitment and discretion

Imperfect stabilization: cost-push shocks

Cost-push shocks

- Suppose now that we have time-varying τ_t^w or ϵ_t
  - Creates changes in desired markups by firms conditional on demand
- Then (1) is

$$
\begin{aligned}
\min \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{x_t^2 + \lambda \pi_t^2\right\}\right] \\
\text{s.t.} \quad & \pi_t = \kappa x_t + \beta \mathbb{E}_t\left[\pi_{t+1}\right] + u_t
\end{aligned}
$$

- u_t is called 'cost-push shock'

Discretion solution with cost-push shocks

- Cost-push shocks create a tradeoff betwen π and x stabilization
  - also, divergence between commitment and discretion solution
- Consider first discretion solution:

$$\begin{aligned} \min \quad & x_t^2 + \lambda \pi_t^2 + \mathbb{E}\left[\sum_{s=t+1}^{\infty} \beta^{s-t} \left\{x_s^2 + \lambda \pi_s^2\right\}\right] \\ \text{s.t.} \quad & \pi_t = \kappa x_t + \beta \mathbb{E}_t\left[\pi_{t+1}\right] + u_t \qquad [2\mu_t] \end{aligned}$$

where expectations are taken as given, for each t.

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  where expectations are taken as given, for each t. FOCs:

$$\begin{aligned} x_t &= \mu_t\kappa \\ \lambda\pi_t &= -\mu_t \end{aligned} \quad \Rightarrow x_t = -\kappa\lambda\pi_t \quad \Rightarrow \pi_t = \frac{1}{1+\kappa^2\lambda}\left(\beta\mathbb{E}_t\left[\pi_{t+1}\right] + u_t\right) \qquad (2)$$

  which can be solved forward given a stochastic process for u_t
- In general, $u_t > 0$ implies $\pi_t > 0$ and $x_t < 0$

Commitment solution with cost-push shocks

- Consider now commitment solution

$$\min \quad \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{\widehat{x_t}^2 + \lambda \pi_t^2\right\}\right]$$

$$\text{s.t.} \quad \pi_t = \kappa \widehat{x_t} + \beta \mathbb{E}_t\left[\pi_{t+1}\right] + u_t \qquad \left[2\beta^t \gamma_t\right]$$

- Planner takes as given the fact that it can influence expectations

$$\mathcal{L} = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{\widehat{x_t}^2 + \lambda \pi_t^2 + 2\gamma_t\left(\pi_t - \kappa \widehat{x_t} - \beta \pi_{t+1}\right)\right\}\right]$$

so FOCs are

$$\widehat{x_t} = \gamma_t \kappa$$

$$\lambda \pi_0 = -\gamma_0$$

$$\lambda \pi_t = -\gamma_t + \gamma_{t-1} \qquad t > 0$$

- FOCs imply:

$$x_t = x_{t-1} - \kappa\lambda\pi_t = -\kappa\lambda\left(p_t - p_{-1}\right) \equiv -\kappa\lambda\widehat{p}_t$$

  where $\widehat{p}_t = p_t - p_{-1}$ is price increase since $t = -1$.

- Plug back into NKPC

$$\widehat{p}_t - \widehat{p_{t-1}} = -\kappa^2\lambda\widehat{p}_t + \beta\mathbb{E}_t\left[\widehat{p_{t+1}} - \widehat{p}_t\right] + u_t$$

Solution, continued

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  in sequence space:

$$\mathbf{F}\left(\mathbf{L}^2 - \left(\beta + 1 + \kappa^2\lambda\right)\mathbf{L} + \beta\mathbf{I}\right)\widehat{\mathbf{p}} = -\mathbf{u} \tag{3}$$

- $P(X) \equiv X^2 - (\beta + 1 + \kappa^2\lambda)L + \beta$ has $P(0) > 0$ and $P(1) < 0$

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- $P(X) \equiv X^2 - (\beta + 1 + \kappa^2\lambda)L + \beta$ has $P(0) > 0$ and $P(1) < 0$
- Call $\frac{1}{\delta}$ the unique root > 1, other root is $\beta\delta \in (0, 1)$
    - then, $P(X) = (X - \beta\delta)\left(X - \frac{1}{\delta}\right)$

Solution, continued

- Rewrite (3) as

$$\mathbf{F}(\mathbf{L}-\beta\delta\mathbf{I})\left(\mathbf{L}-\frac{1}{\delta}\mathbf{I}\right)\widehat{\mathbf{p}} = (\mathbf{I}-\beta\delta\mathbf{F})\left(-\frac{1}{\delta}\right)(\mathbf{I}-\delta\mathbf{L})\widehat{\mathbf{p}}$$
$$= -\mathbf{u}$$

Hence the solution is $(\mathbf{I}-\delta\mathbf{L})\widehat{\mathbf{p}} = \delta(\mathbf{I}-\beta\delta\mathbf{F})^{-1}\mathbf{u}$, or

$$\widehat{p}_t - \delta\widehat{p_{t-1}} = \delta\mathbb{E}_t\left[\sum_{k=0}^{\infty}(\beta\delta)^k u_{t+k}\right] \tag{4}$$

- Contrast (2) and (4). Here, the price level reverts to p_{-1} in the long run.

- Assume $u_t = \epsilon_t$ is iid mean 0:

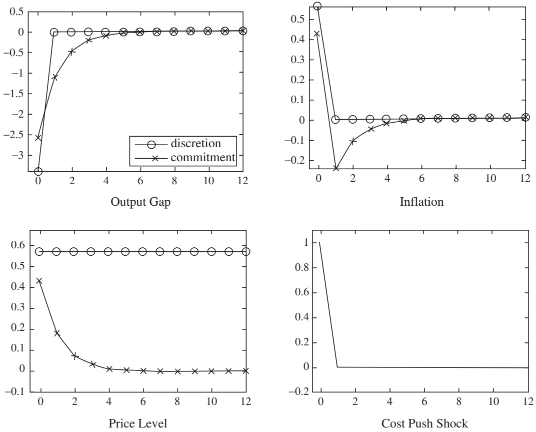

Figure 5.1 Optimal Responses to a Transitory Cost Push Shock

- Assume $u_t = \rho u_{t-1} + \epsilon_t$ is AR process with $\rho = 0.5$

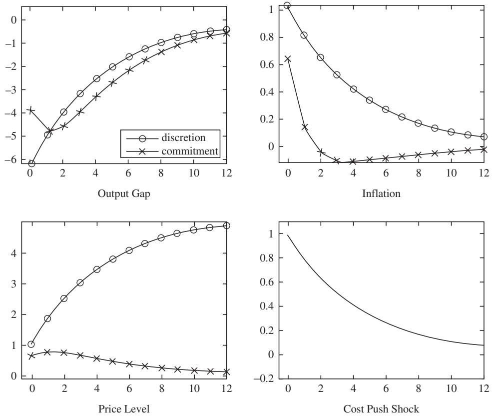

Figure 5.2 Optimal Responses to a Persistent Cost Push Shock

Lesson from optimal policy

- Faced with cost-push shock: tradeoff between stabilizing π and x
  - Under discretion, offset shock by tightening, creating current recession
  - Under commitment, smaller recession but (small) future recession
  - This lower $\mathbb{E}_0[\pi_1]$, ameliorates the date-0 tradeoff for the central bank
  - Lesson: there is a benefit of *forward guidance* under commitment

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    - Lesson: there is a benefit of *forward guidance* under commitment
- Another example in which this theme shows up is at the ZLB

The Zero Lower Bound

A liquidity trap scenario

- Let's go back to divine coincidence case, but assume $r_t^n < 0$
  - Why? Could be financial shock, deleveraging, structural factors such as population aging, inequality, etc
- Our divine coincidence solution implies $i_t = r_t^n < 0$
  - Not allowed! So need to add it back to the problem
  - Situation called a 'liquidity trap': Japan in 1980s, U.S. 2009-2020

- New problem: given $r_t^n = -\underline{r} < 0$ for $t \in [0, T)$, then $r_t^n = \bar{r} > 0$

$$\begin{aligned} \min \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{x_t^2 + \lambda \pi_t^2\right\}\right] \\ \text{s.t.} \quad & \pi_t = \kappa x_t + \beta \mathbb{E}_t\left[\pi_{t+1}\right] \\ & x_t = \mathbb{E}_t\left[x_{t+1}\right] - \sigma^{-1}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) \\ & i_t \geq 0 \end{aligned}$$

- Papers studying this problem:
  - Eggertsson and Woodford (2003) (numerical simulations)
  - Werning (2012) (closed-form solutions in continuous-time)

Solution under discretion

- Assume no uncertainty for simplicity
- Under **discretion**:
    - For $t \geq T$, $r_t^n = \bar{r} > 0$ again, immediately set $i_t = r_t^n = \bar{r}$
    - Implies $x_T = 0$
    - Solving backwards:

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  - Solving backwards:

$$
\begin{aligned}
x_{T-1} &= -\sigma^{-1}\left(0 + \underline{r}\right) < 0 \\
\pi_{T-1} &= \kappa x_{T-1} < 0 \\
x_{T-2} &= x_{T-1} - \sigma^{-1}\left(0 - \pi_{T-1} + \underline{r}\right) < 0 \\
\pi_{T-2} &= \kappa x_{T-1} + \beta \pi_{T-1} < 0 \\
&\cdots
\end{aligned}
$$

    Bad outcome! $x_t < 0$, $\pi_t < 0$ thoughout
  - (At ZLB, deflation increases real rate, making things worse)

Solution under commitment

- **Commitment solution**:
  - Promise to keep interest rates lower for longer
    - $i_t = 0$ for some periods after $t = T$
  - Loose monetary policy in future, so creates future output boom
  - Through logic of Euler equation, this ameliorates recession today
    - $x_T > 0 \Rightarrow x_{T-1} = x_T - \sigma^{-1}(0 + \underline{r})$ higher than under discretion, etc
- Idea: Raise future incomes, people feel richer and spend more today

- This argument appears to have convinced the Fed...

  [There has been] a remarkable transformation of Minneapolis Fed President Narayana Kocherlakota from a policy hawk against the central bank's easy-money policies, to policy dove, who is strongly supportive. [...] There's an economist who played a role in shaping Mr. Kocherlakota's views, and you probably haven't heard of this one. He is a 38-year-old Argentine professor at the Massachusetts Institute of Technology named Ivan Werning. (WSJ 09/28/12)

Conclusion

- The RBC and NK models provide an intellectually coherent framework for thinking about many of today's key macroeconomic issues. Among others:
  - What creates booms and busts? Are these fluctuations efficient?
  - What creates inflation?
  - How does fiscal policy work? How does monetary policy work?
- They also provide important principles that guide policy design:
  - What are appropriate objectives of monetary policy?
  - How should policy respond to shocks?
  - What is forward guidance, what are the benefits from policy commitment?
- We just got started with fascinating questions...
  - More on that next year in the 2nd year sequence. See you there!

Appendix slides

Appendix: derivation of welfare approximation

Linearizing welfare 1/6

- Deriving approximation to welfare requires linearizing to higher order
- Recall $z_t \equiv \log\left(\frac{Z_t}{Z}\right)$ and hat variable is $\widehat{z_t} \equiv dz_t = d\log\left(\frac{Z_t}{Z}\right)$
- Then we have to second order:

$$\frac{dZ_t}{Z} \simeq \widehat{z_t} + \frac{1}{2}\widehat{z_t}^2 \qquad \left(\frac{dZ_t}{Z}\right)^2 \simeq \widehat{z_t}^2$$

- Second order Taylor approximation to any function $F(X, Y)$ is

$$\begin{aligned}\frac{dF_t}{F} &= \frac{F_X X}{F}\widehat{x_t} + \frac{F_Y Y}{F}\widehat{y_t} \\ &+ \frac{1}{2}\left(\frac{F_X X}{F} + \frac{F_{XX}X^2}{F}\right)\widehat{x_t}^2 + \frac{1}{2}\left(\frac{F_Y Y}{F} + \frac{F_{YY}Y^2}{F}\right)\widehat{y_t}^2 + \frac{F_{XY}XY}{F}\widehat{x_t}\widehat{y_t}\end{aligned} \tag{5}$$

- Write period welfare as

$$U_t = u(C_t) - v(N_t)$$

- Using (5), to second order

$$\begin{aligned}dU_t &= u'(C)\,C\widehat{c_t} - v'(N)\,N\widehat{n_t} \\ &+ \frac{1}{2}\left(u'(C)\,C + u''(C)\,C^2\right)\widehat{c_t}^2 - \frac{1}{2}\left(v'(N)\,N + v''(N)\,N^2\right)\widehat{n_t}^2\end{aligned}$$

Linearizing welfare 2/6

- Use $u''(C)C^2 = -\sigma u'(C)C$ and $v''(N)N^2 = \phi v'(N)N$ to get

$$dU_t = u'(C)C\left(\widehat{c}_t + \frac{1-\sigma}{2}\widehat{c}_t^{\,2}\right) - v'(N)N\left(\widehat{n}_t + \frac{1+\phi}{2}\widehat{n}_t^{\,2}\right)$$

- At the zero inflation steady state

$$\frac{v'(N)}{u'(C)} = \frac{W}{P} = \frac{1-\frac{1}{\epsilon}}{1+\tau^w}A(1-\alpha)N^{-\alpha} = (1-\tau)A(1-\alpha)N^{-\alpha}$$

where τ is the steady state labor wedge. Using $C = Y = AN^{1-\alpha}$ we find

$$v'(N)N = (1-\tau)(1-\alpha)Cu'(C)$$

so

$$\frac{dU_t}{u'(C)C} = \widehat{c}_t + \frac{1-\sigma}{2}\widehat{c}_t^{\,2} - (1-\tau)(1-\alpha)\left(\widehat{n}_t + \frac{1+\phi}{2}\widehat{n}_t^{\,2}\right)$$

- Since $Y_t = C_t = \frac{A_t N_t^{1-\alpha}}{\Delta_t}$ we have

$$\widehat{c}_t = \widehat{a}_t + (1-\alpha)\,\widehat{n}_t - \widehat{\triangle}_t$$

resulting in

$$\frac{dU_t}{u'(C)\,C} = \tau\widehat{c}_t - (1-\tau)\,\widehat{\triangle}_t + \frac{1-\sigma}{2}\widehat{c}_t^{\,2} - (1-\tau)\,\frac{1+\phi}{2}\frac{1}{1-\alpha}\left(\widehat{c}_t^{\,2} - 2\widehat{c}_t\widehat{a}_t\right)$$
$$+\underbrace{(1-\tau)\,\widehat{a}_t - (1-\tau)\,\frac{1+\phi}{2}\frac{1}{1-\alpha}\widehat{a}_t^{\,2}}_{\text{t.i.p}}$$

- "t.i.p" mean terms independent of policy (here, productivity)
- Note there is a **first order term** if $\tau \neq 0$
    - Reflects the incentive of the central bank to manipulate activity $\widehat{c}_t$ to undo the static distortion from monopoly
    - From now on, we set $\tau^w = -\frac{1}{\epsilon}$ so $\tau = 0$
    - See Benigno Woodford for how to treat the general case with $\tau \neq 0$

- We are left with

$$\frac{dU_t}{u'(C)\,C} = -\widehat{\triangle}_t - \frac{1}{2}\left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right)\widehat{c}_t^{\,2} + \frac{1}{2}\frac{1+\phi}{1-\alpha}\left(2\widehat{c}_t\widehat{a}_t\right) + \mathrm{t.i.p}$$

- Replace $\widehat{c}_t\widehat{a}_t$ by $\widehat{c}_t\widehat{c}_t^e$, with the natural = efficient level of consumption satisfying

$$\left(\frac{1+\phi}{1-\alpha}\right)\widehat{a}_t = \left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right)\widehat{c}_t^e$$

- Find

$$\frac{dU_t}{u'(C)\,C} = -\widehat{\triangle}_t - \frac{1}{2}\left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right)\left(\widehat{c}_t - \widehat{c}_t^{\,e}\right)^2 + \mathrm{t.i.p} \qquad (6)$$

- Two terms in (6):
    1. Price dispersion, reflecting accumulated inflation
    2. Deviation of output from first best

- Use (5) to linearize the evolution of price dispersion to second order and find

$$\widehat{\Delta_t} = \theta\widehat{\Delta_{t-1}} + \frac{\epsilon}{2}\left(1 + \frac{\epsilon\alpha}{1-\alpha}\right)\frac{\theta}{1-\theta}\pi_t^2$$

- Cumulating,

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t\widehat{\Delta_t} &= \frac{\theta}{1-\theta}\sum_{t=0}^{\infty}\beta^t\sum_{s=0}^{t}\theta^{t-s}\pi_s^2 \\ &= \frac{\epsilon}{2}\left(1 + \frac{\epsilon\alpha}{1-\alpha}\right)\frac{\theta}{(1-\theta)(1-\theta\beta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2\end{aligned}$$

- Hence, the present discounted value of welfare along any realization of shocks is

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t\frac{dU_t}{u'(C)C} = &-\frac{\epsilon}{2}\left(1 + \frac{\epsilon\alpha}{1-\alpha}\right)\frac{\theta}{(1-\theta)(1-\theta\beta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2 \\ &-\frac{1}{2}\left(\sigma + \frac{\alpha+\phi}{1-\alpha}\right)\sum_{t=0}^{\infty}\beta^t\left(\widehat{c}_t - \widehat{c}_t^e\right)^2 + \mathrm{t.i.p}\end{aligned}$$

Linearizing welfare 6/6

- Recall from Lecture 7 that the Phillips curve slope is

$$\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta} \frac{1}{1+\frac{\epsilon\alpha}{1-\alpha}} \left(\frac{\alpha+\phi}{1-\alpha}+\sigma\right)$$

- Hence we also have simply

$$\begin{aligned} dW \equiv & \frac{1}{\left(\sigma+\frac{\alpha+\phi}{1-\alpha}\right)} \sum_{t=0}^{\infty} \beta^t \frac{dU_t}{u'(C)\,C} \\ = & -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left\{ \left(\widehat{c}_t - \widehat{c}_t^{\,e}\right)^2 + \frac{\epsilon}{\kappa}\pi_t^2 \right\} \end{aligned}$$

- Exante, we also have

$$\mathbb{E}[dW] = -\frac{1}{2}\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left\{ \left(\widehat{c}_t - \widehat{c}_t^{\,e}\right)^2 + \frac{\epsilon}{\kappa}\pi_t^2 \right\}\right]$$

- QED back