Finding the intersection of a ray with a plane

Suppose we have a plane $H = [a,b,c,d]$, defined as the plane where for any point x,y,z:

$$ax + by + cz + d = 0$$

To find the value of t where ray $V+t*W$ intersects plane H, we can plug $V+t*W$ into H:

$$a * (Vx+t*Wx) + b * (Vy+t*Wy) + c * (Vz+t*Wz) + d = 0$$

We can regroup terms to write this equation using dot products:

$$(H \bullet [Vx,Vy,Vz,1]) + t * (H \bullet [Wx,Wy,Wz,0]) = 0$$

which gives the value of t as a ratio of two dot products:

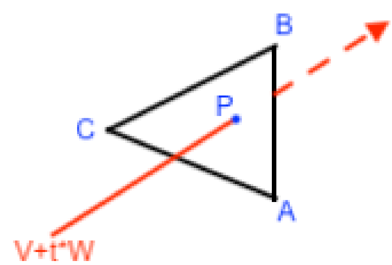
$$t = \frac{-H \bullet [Vx,Vy,Vz,1]}{H \bullet [Wx,Wy,Wz,0]}$$

Finding the intersection of a ray with a triangle

Goal: Find distance t where ray $V+t*W$ intersects triangle A,B,C.
If the ray misses the triangle, return -1.

We can achieve this goal in three steps:

1) Find the plane H that contains the triangle
2) Compute the point P where the ray intersects plane H
3) Figure out whether P lies within the triangle



1) Finding the plane that contains the triangle

The plane of the triangle with vertices A,B,C is found as follows:

1) Compute vector N normal to the plane, giving the first three plane coordinates a,b,c
2) Compute the fourth coordinate d so as to satisfy the plane equation $ax+by+cz+d = 0$

We can compute vector N by choosing any two edges of the triangle, say (A,B) and (B,C), and doing a cross product between them:

$$N = B-A \times C-B$$

Then we can use triangle vertex A to compute d = -N●A, thereby satisfying the plane equation:

$$Nx*a + Ny*b + Nz*c + d \quad = \quad N \bullet A - N \bullet A \quad = \quad 0$$

The plane containing the triangle is therefore given by:

$$H = [\ Nx,\ Ny,\ Nz,\ -N \bullet A\ ]$$

2) Computing the intersection of the ray with plane H

Ray V + t*W will intersect plane H at some value t. To find t, we can just use the method described at the top of this document.

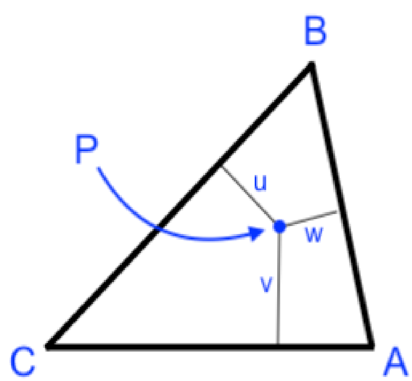
We can then plug t into the ray equation to find intersection point P:

$$P = V + t * W$$

3) Figuring out whether the intersection point lies inside the triangle

Finally we need to figure out whether point P is inside or outside of triangle A,B,C.

We can do this by computing the *Barycentric coordinates* of P within the triangle. Barycentric coordinates provide, for each vertex A of a triangle, a parameter u whose value varies linearly from 0 at the far edge (B,C) to 1 at vertex A.



For all three vertices A,B,C, we need to compute their respective Barycentric coordinates u,v,w. Barycentric coordinates are useful to us because a point is inside a triangle if and only if all three of its Barycentric coordinates are non-negative.

To compute the Barycentric coordinate u for vertex A, we need to compute the direction vector pA in the triangle that is perpendicular to edge (B,C).

We already know the vector N normal to the triangle plane, so we can just take a cross product:

$$pA = N \times C-B$$

We can find all three vectors perpendicular to the three triangle edges in the same way:

$$pA = N \times C-B$$
$$pB = N \times A-B$$
$$pC = N \times B-A$$

Once we have found perpendicular vector pA, we can find the corresponding Barycentric coordinate u by computing the ratio:

$$u = \frac{P\text{-}B \bullet pA}{A\text{-}B \bullet pA}$$

To make the above ratio easier to understand, notice that it satisfies two important conditions:

When P is at point A, the value of u is 1.

When P is at point B (or anywhere else along edge (B,C)), the value u is 0.

In this way, we compute Barycentric coordinates u,v,w for each of the triangle vertices A,B,C.

If none of u,v,w are negative, we return t. Otherwise we return -1.

The complete algorithm is as follows

Given ray V+tW and triangle with vertices A,B,C, return distance t along the ray to the triangle. If the ray misses the triangle, return -1.

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N = B-A × C-B

t = rayIntersectPlane(V, W, [Nx, Ny, Nz, -N ● A])

P = V + t * W

pA = N × C-B
pB = N × A-C
pC = N × B-A

u = (P-B ● pA)  /  (A-B ● pA)
v = (P-C ● pB)  /  (B-C ● pB)
w = (P-A ● pC)  /  (C-A ● pC)

if u >= 0 and v >= 0 and w >= 0
   return t
else
   return -1
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