Sequences and Series

Definition of a Sequence

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ where $a(n)$ represents the nth term of the sequence.

Example: $a_n = \frac{1}{n}$ is the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$

Useful Sequences to Remember

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{1}{n!} = 0$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

$\lim_{n\to\infty} \sqrt[n]{a} = 1$ for any $a > 0$

Series Convergence Tests

p-test: $\sum \frac{1}{n^p}$ converges if $p > 1$

Example: $\sum \frac{1}{n^2}$ converges

Solution: $p = 2 > 1$, so it converges

Comparison Test

Example: $\sum \frac{1}{n^2+1}$, compare with $\sum \frac{1}{n^2}$

Solution: $\frac{1}{n^2+1} < \frac{1}{n^2}$, since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^2+1}$ also converges

Squeeze Test

Example: $\sum \sin\left(\frac{1}{n}\right)$

Solution: $0 \le \sin\left(\frac{1}{n}\right) \le \frac{1}{n}$, since $\sum \frac{1}{n}$ diverges, $\sum \sin\left(\frac{1}{n}\right)$ diverges

nth Term Test

Example: $\sum \frac{n}{n+1}$

Solution: $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, so $\sum \frac{n}{n+1}$ diverges

Ratio Test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

Example: $\sum \frac{n!}{2^n}$

Solution: $\frac{a_{n+1}}{a_n} = \frac{(n+1)!/2^{n+1}}{n!/2^n} = \frac{n+1}{2} \to \infty$, so it diverges

Geometric Series

Sum: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$

Example: $\sum_{n=0}^{\infty} \frac{1}{2^n}$

Solution: $\frac{1}{1-\frac{1}{2}} = 2$

Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: Diverges

Limits and Continuity

Important Limits

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to 0} (1+x)^{1/x} = e$

L'Hopital's Rule

If $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Example: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

Solution: $\lim_{x\to 0} \frac{\cos(x)}{1} = 1$

Continuity

f is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$

Example: $f(x) = \begin{cases} 0, & x \le 0 e^{-1/x^2}, \\ x > 0 & \end{cases}$

Solution: Continuous at $x = 0$ since $\lim_{x\to 0} e^{-1/x^2} = 0$

Derivatives

Product Rule, Chain Rule

$(fg)' = f'g + fg'$

$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Example: $\frac{d}{dx}(x^2 e^x) = 2xe^x + x^2 e^x$

Solution: $(x^2 e^x)' = 2xe^x + x^2 e^x$

Example: $\frac{d}{dx} \sin(x^2) = 2x\cos(x^2)$

Solution: $f(g(x)) = \sin(x^2)$, $f'(g(x)) = \cos(x^2)$, $g'(x) = 2x$

Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Example: $\frac{d}{dx}\left(\frac{x^2}{x+1}\right)$

Solution: $\frac{(2x)(x+1)-(x^2)(1)}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$

Derivatives and Antiderivatives of $\ln(x)$, $\frac{1}{x}$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$

$\int \frac{1}{x} dx = \ln|x| + C$

Example: $\frac{d}{dx} \ln(5x) = \frac{1}{x}$

Solution: $\frac{d}{dx} \ln(5x) = \frac{1}{5x} \cdot 5 = \frac{1}{x}$

Derivatives and Antiderivatives of Trig Functions

$\frac{d}{dx} \sin(x) = \cos(x)$

$\frac{d}{dx} \cos(x) = -\sin(x)$

$\frac{d}{dx} \tan(x) = \sec^2(x)$

$\int \sin(x)\, dx = -\cos(x) + C$

$\int \cos(x)\, dx = \sin(x) + C$

$\int \sec^2(x)\, dx = \tan(x) + C$

Example: $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

Solution: $\sec^2(x)$

Power Series and Taylor Series

Radius of Convergence

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Example: $\sum (2n+1)x^n$

Find radius of convergence

Solution: $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{2(n+1)+1}{2n+1}\right| = 2$

$R = \frac{1}{2}$

Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then

$\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

Binomial Series

$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

Example: $(1+x)^2 = 1 + 2x + x^2$

Tangent Plane

$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

Example: $f(x,y) = x^2 + y^2$ at $(1,1,2)$

Solution: $z - 2 = 2(x-1) + 2(y-1)$

Integration

Integration by Parts

$\int u\, dv = uv - \int v\, du$

Example: $\int xe^x\, dx$

Solution: $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$

$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$

Common Integrals

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int e^x\, dx = e^x + C$

$\int \frac{1}{x}\, dx = \ln|x| + C$

$\int \sin(x)\, dx = -\cos(x) + C$

$\int \cos(x)\, dx = \sin(x) + C$

$\int \sec^2(x)\, dx = \tan(x) + C$

Properties of Definite Integrals

$\int_a^b f(x)\, dx = F(b) - F(a)$ where $F'(x) = f(x)$

$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

$\int_a^a f(x)\, dx = 0$

$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

Trigonometric Substitution

For integrals involving $\sqrt{a^2 - x^2}$, use $x = a\sin(\theta)$

For integrals involving $\sqrt{a^2 + x^2}$, use $x = a\tan(\theta)$

For integrals involving $\sqrt{x^2 - a^2}$, use $x = a\sec(\theta)$

Example: $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solution: Use $x = a\sin(\theta)$, then $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos(\theta)\, d\theta}{a\cos(\theta)} = \int d\theta = \theta + C = \sin^{-1}\left(\frac{x}{a}\right) + C$

Integrals Involving $\ln$ Function

$\int \ln(x)\, dx = x\ln(x) - x + C$

Example: $\int x\ln(x)\, dx$

Solution: Integration by parts with $u = \ln(x)$, $dv = x\, dx$, then $du = \frac{1}{x}\, dx$, $v = \frac{x^2}{2}$

$\int x\ln(x)\, dx = \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\, dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$

Partial Derivatives

Linear Approximation

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Example: $f(x,y) = e^x\ln(1+e^{x-y})$ at $(0,0)$

Approximate $f(0.05, 0.1)$

Solution: $L(0.05, 0.1) = 0 + 1(0.05) + 0(-0.1) = 0.05$

Directional Derivatives

$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$

Example: $f(x,y) = e^x\ln(1+e^{x-y})$, direction $-\mathbf{i} - \mathbf{j}$

Calculate $D_{\mathbf{u}} f(0,0)$

Solution: $\nabla f = (1,0)$, $\mathbf{u} = \frac{-\mathbf{i}-\mathbf{j}}{\sqrt{2}}$

$D_{\mathbf{u}} f = (1,0) \cdot \frac{(-1,-1)}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$

Steepest Ascent

Direction: ∇f, Magnitude: $\|\nabla f\|$

Example: $f(x,y) = e^x\ln(1+e^{x-y})$ at $(0,0)$

Find direction and magnitude at $(0,0)$

Solution: Direction: $(1,0)$, Magnitude: 1

Partial Fractions

Decompose $\frac{P(x)}{Q(x)}$ into simpler fractions

Example: $\frac{4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$

Solution: $4 = A(x-2) + Bx$, solve for A and B