Math7501 Problem Set 1

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1 Question 1

1.1 1.a

$$m_k = 1^k p_1 + 2^k p_2 + 3^k p_3 + 4^k p_4$$

Suppose $m_0 = 1$, $m_1 = 2.3$, $m_2 = 6.6$, $m_3 = 21.6$
For m_0 sub in $k = 0$
$m_0 = 1^0 p_1 + 2^0 p_2 + 3^0 p_3 + 4^0 p_4 = 1$

We can see that we can solve these simultaneous equations using matrix multiplication by converting it to the form $Ax = b$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2.3 \\ 6.6 \\ 21.6 \end{pmatrix}$$

$A \quad . \quad x \quad = \quad b$

1.2 1.b

We can solve for the determinant of A using the "det" function in MATLAB. See below.
MATLAB Code:

```
A = [1 1 1 1; 1 2 3 4; 1 4 9 16; 1 8 27 64]
detA = det(A);
disp(detA);
```

Output:
$detA = 12$

The system does have a unique solution as there are 4 unknowns matching the 4 linear equations. We can solve for p with matrix division.

1.3 1.c

We can use the mldivide function in MATLAB to find the values (p_1, p_2, p_3, p_4), or in the $Ax = b$ formula, the x.

MATLAB Code:

```
A = [1 1 1 1; 1 2 3 4; 1 4 9 16; 1 8 27 64];

m0 = 1;
```

```
m1 = 2.3;
m2 = 6.6;
m3 = 21.6;

B = [m0; m1; m2; m3];

p = mldivide(A,B)
```

Output:

$$p = \begin{pmatrix} 0.33 \\ 0.25 \\ 0.2 \\ 0.2167 \end{pmatrix}$$

Therefore:

$p_1 = 0.33$
$p_2 = 0.25$
$p_3 = 0.2$
$p_4 = 0.2167$

2 Question 2

Let u and v be orthogonal vectors.
Therefore, $u.v = 0$

2.1 2.a

If $x - proj_u x - proj_v x$ is orthogonal to u then the dot product $= 0$.
If $x - proj_u x - proj_v x$ is orthogonal to v then the dot product $= 0$.

Let's test this. Check u.
$(x - proj_u x - proj_v x).u = 0$
$x.u - u.proj_u x - u.proj_v x = 0$

The projection of x onto u multiplied by u is the magnitude of u multiplied by the magnitude of the projection of x onto u. This is because it is projecting in the direction of u.

$u.proj_u x = ||u||.||proj_u x||$ $u.proj_u x = x.u$

The projection of x onto v multiplied by u, is 0, as the vectors are orthogonal.

$u.proj_v x = 0$

We can sub these values into the original equation:

$x.u - u.proj_u x - u.proj_v x = 0$
$x.u - x.u - 0 = 0$
$0 = 0$

This can be repeated for v, with the same result. This shows that for any vector x, $x - proj_u x - proj_v x$ is orthogonal to both u and v

2.2 2.b

$u = \frac{1}{\sqrt{14}}\left(1, 2, 3\right)'$, $v = \frac{1}{\sqrt{27}}\left(-5, 1, 1\right)'$

The cross product of two vectors is a vector orthogonal to both of them. Let w be a unit vector that is orthogonal to both u and v. Therefore the cross product of u and v is w.

MATLAB Code:

```
% Define vectors u and v
u = (1/sqrt(14)) * [1; 2; 3]
```

```
v = (1/sqrt(27)) * [-5; 1; 1]

% Calculate the cross product of u and v
w = cross(u, v)

% Normalize w to obtain a unit vector
w = w / norm(w)
```

Output:

$$w = \begin{pmatrix} -0.0514 \\ -0.823 \\ 0.5658 \end{pmatrix}$$

Vector w is orthogonal to both u and v.

2.3 2.c

Show that any vector $y \in \mathbb{R}^3$ can be expressed as $y = a_1 u + a_2 v + a_3 w$

Given u, v, and w from the 2.b, and by using a random vector y, we can solve for a_1, a_2 and a_3.
We do this by using matrix multiplication, and converting the simultaneous equations into the form $Ax = b$. We can construct our coefficient matrix by using the vectors u, v and w from before, and then use a random y to solve for x, which in this case is our a.

MATLAB Code:

```
% Define the vectors u, v, and w
u = sqrt(1/14) * [1; 2; 3];
v = sqrt(1/27) * [-5; 1; 1];
w = [-0.0514; -0.823; 0.5658];

% Generate a random vector y
y = randn(3, 1);

% Construct the coefficient matrix A
A = [sqrt(1/14), sqrt(1/27), -0.0514;
     2*sqrt(1/14), sqrt(1/27), -0.823;
     3*sqrt(1/14), sqrt(1/27), 0.5658];

% Solve for the coefficients a1, a2, a3
a = A \ y;

% Display the coefficients
disp(['a1 = ', num2str(a(1))]);
disp(['a2 = ', num2str(a(2))]);
disp(['a3 = ', num2str(a(3))]);
```

Output:

For random $y = \begin{pmatrix} 0.3192 \\ 0.3129 \\ -0.8649 \end{pmatrix}$

$a1 = -1.5891$

$a2 = 3.7207$

$a3 = -0.54221$

Given that we are able to solve for a_1, a_2 and a_3 for any random vector y, it can be expressed as $y = a_1 u + a_2 v + a_3 w$

2.4 2.d

Considering we know that y can be expressed as $y = a_1 u + a_2 v + a_3 w$.
We know that there are 3 dimensions in $\mathbb{R}^3$, and there are 3 orthogonal vectors, there is only one vector that could possibly be orthogonal to them all, and that is the 0 vector.
For a vector to be orthogonal to another, it's dot product must equal 0.

E.g. $\begin{pmatrix} -0.0514 \\ -0.823 \\ 0.5658 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

If there was any other y value that was not the 0 vector, then the dot product would not be 0, and therefore not orthogonal.

3 Question 3

$$c_1 = \begin{pmatrix}-1 & 1\end{pmatrix}^T, c_2 = \begin{pmatrix}1 & 1\end{pmatrix}^T, c_3 = \begin{pmatrix}0 & 2\end{pmatrix}^T,$$

and

$$A = \begin{bmatrix}1 & 0.5\\ 0.5 & 2\end{bmatrix}$$

$$D(x;c) = (x-c)^T A(x-c)$$

3.1 3.a

3.1.1 i

For c_1:
$(x-c_1)^T = \begin{pmatrix}x_1+1\\ x_2-1\end{pmatrix}$
$D(x;c_1) = \begin{pmatrix}x_1+1 & x_2-1\end{pmatrix}\begin{bmatrix}1 & 0.5\\ 0.5 & 2\end{bmatrix}\begin{pmatrix}x_1+1\\ x_2-1\end{pmatrix}$
$= \begin{pmatrix}x_1+1+0.5(x_2-1) & 0.5(x_1+1)+2(x_2-1)\end{pmatrix}\begin{pmatrix}x_1+1\\ x_2-1\end{pmatrix}$
$= (x_1+0.5+0.5x_2)(x_1+1)+(0.5x_1+2x_2-1.5)(x_2-1)$
$= x_1^2+0.5x_1+0.5x_2x_1+x_1+0.5+0.5x_2+0.5x_1x_2+2x_2^2-1.5x_2-0.5x_1-2x_2+1.5$
$= x_1^2+x_1+x_1x_2-3x_2+2x_2^2+2$

For c_2:
$(x-c_2)^T = \begin{pmatrix}x_1-1\\ x_2-1\end{pmatrix}$
$D(x;c_2) = \begin{pmatrix}x_1-1 & x_2-1\end{pmatrix}\begin{bmatrix}1 & 0.5\\ 0.5 & 2\end{bmatrix}\begin{pmatrix}x_1-1\\ x_2-1\end{pmatrix}$
$= \begin{pmatrix}x_1-1+0.5(x_2-1) & 0.5(x_1-1)+2(x_2-1)\end{pmatrix}\begin{pmatrix}x_1-1\\ x_2-1\end{pmatrix}$
$= (x_1+0.5x_2-1.5)(x_1-1)+(0.5x_1+2x_2-2.5)(x_2-1)$
$= x_1^2+0.5x_2x_1-1.5x_1-x_1-0.5x_2+1.5+0.5x_1x_2+2x_2^2-2.5x_2-0.5x_1-2x_2+2.5$
$= x_1^2+x_1x_2-3x_1-5x_2+2x_2^2+4$

If $D(x;c_1) = D(x;c_2)$ then $D(x;c_1) - D(x;c_2) = 0$

$0 = (x_1^2+x_1+x_1x_2-3x_2+2x_2^2+2)-(x_1^2+x_1x_2-3x_1-5x_2+2x_2^2+4)$

$$0 = 4x_1+2x_2-2$$

3.1.2 ii

For c_3:
$(x - c_3)^T = \begin{pmatrix} x_1 \\ x_2 - 2 \end{pmatrix}$
$D(x; c_3) = \begin{pmatrix} x_1 & x_2 - 2 \end{pmatrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 - 2 \end{pmatrix}$
$= \begin{pmatrix} x_1 + 0.5(x_2 - 2) & 0.5x_1 + 2(x_2 - 2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - 2 \end{pmatrix}$
$= (x_1 + 0.5x_2 - 1)(x_1) + (0.5x_1 + 2x_2 - 4)(x_2 - 2)$
$= x_1^2 + 0.5x_2x_1 - x_1 + 0.5x_1x_2 + 2x_2^2 - 4x_2 - x_1 - 4x_2 + 8$
$= x_1^2 + x_1x_2 - 2x_1 + 2x_2^2 - 8x_2 + 8$

If $D(x; c_1) = D(x; c_3)$ then $D(x; c_1) - D(x; c_3) = 0$

$0 = (x_1^2 + x_1 + x_1x_2 - 3x_2 + 2x_2^2 + 2) - (x_1^2 + x_1x_2 - 2x_1 + 2x_2^2 - 8x_2 + 8)$

$$0 = 3x_1 + 5x_2 - 6$$

3.2 3.b

Determine the point $x \in \mathbb{R}^2$ for which $D(x; c_1) = D(x; c_2) = D(x; c_3)$

We can solve for this by using the simultaneous equations from $3.b$

$$2 = 4x_1 + 2x_2$$
$$6 = 3x_1 + 5x_2$$

This set of equations can be written using matrix multiplication in the form of $Ax = b$

$$\begin{array}{ccccc} \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\ A & . & x & = & b \end{array}$$

If the matrix A is invertible, we can solve the system of equations for x by left multiplying both side by A^{-1}, to get $x = A^{-1}b$

Check if A is invertible, $|A| \neq 0$:

$$|A| = (4 * 5) - (2 * 3) = 14$$

As the determinant (Δ) of A is 14 (non-zero) we can find the inverse of A

$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{14}\begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix}$

$x = A^{-1}b$

$x = \frac{1}{14}\begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$x = \frac{1}{14}\begin{pmatrix} 10 - 12 \\ -6 + 24 \end{pmatrix}$

$x = \frac{1}{7}\begin{pmatrix} -1 \\ 9 \end{pmatrix}$

Therefore the point $x \in \mathbb{R}^2$ for which $D(x; c_1) = D(x; c_2) = D(x; c_3)$ is $(-0.143, 1.286)$

3.3 3.c

Sketch the regions B_1, B_2, and B_3 on $\mathbb{R}^2$

We can do this in MATLAB, by finding the minimum distance from our centers of every point, and then classifying them into regions $B1$ $B2$ or $B3$ based on which center they are closest to.

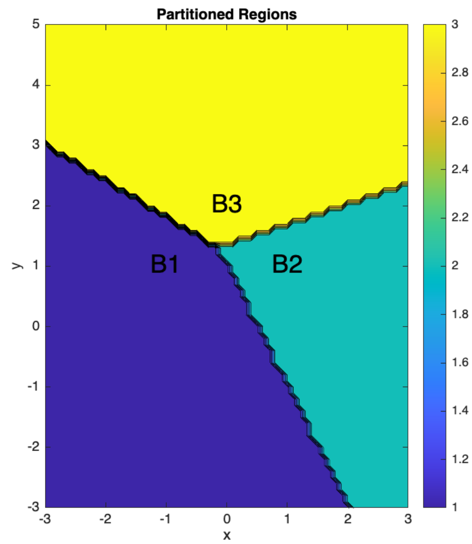

Figure 1: B1, B2 and B3 Plotted

This was made using the code below. MATLAB Code:

```
% Centers from previous question
c1 = [-1; 1];
c2 = [1; 1];
c3 = [0; 2];

A = [1, 0.5; 0.5, 2];

% Partitioning function D(x, c)
D = @(x, c) (x - c)' * A * (x - c);

% Create a grid of points over the region
[x, y] = meshgrid(-3:0.1:3, -3:0.1:5);
points = [x(:)'; y(:)'];
```

```
% Store the region labels
regions = zeros(size(points, 2), 1);

% Iterate through each point and assign it to the
   region with the minimum distance
for i = 1:size(points, 2)
    point = points(:, i);
    distances = [D(point, c1), D(point, c2), D(point,
        c3)];
    [~, min_idx] = min(distances);
    regions(i) = min_idx;
end

% Reshape the region labels to the same size as the
   grid
regions = reshape(regions, size(x));

% Plot the partitioned regions
figure;
contourf(x, y, regions);
title('Partitioned Regions');
xlabel('x');
ylabel('y');
colorbar;
axis equal;

% Label the regions
hold on;
text(c1(1), c1(2), 'B1', 'HorizontalAlignment', '
   center', 'VerticalAlignment', 'middle', 'FontSize',
    18);
text(c2(1), c2(2), 'B2', 'HorizontalAlignment', '
   center', 'VerticalAlignment', 'middle', 'FontSize',
    18);
text(c3(1), c3(2), 'B3', 'HorizontalAlignment', '
   center', 'VerticalAlignment', 'middle', 'FontSize',
    18);
```

4 Question 4

Compute the determinant of this matrix, lets call it A:

$$\begin{bmatrix} 2 & 0.7 & 0 & 0 \\ 0.7 & 2 & 0.7 & 0 \\ 0 & 0.7 & 2 & 0.7 \\ 0 & 0 & 0.7 & 2 \end{bmatrix}$$

$|A| = 2C_{11} + 0.7C_{12}$

$C_{11} = -1^{1+1} \begin{bmatrix} 2 & 0.7 & 0 \\ 0.7 & 2 & 0.7 \\ 0 & 0.7 & 2 \end{bmatrix}$

$|C_{11}| = 2D_{11} + 0.7D_{12}$

$D_{11} = -1^{1+1} \begin{bmatrix} 2 & 0.7 \\ 0.7 & 2 \end{bmatrix}$
$= 3.51$

$D_{12} = -1^{1+2} \begin{bmatrix} 0.7 & 0.7 \\ 0 & 2 \end{bmatrix}$
$= -1.4$

Therefore $|C_{11}| = 2(3.51) + 0.7(-1.4) = 6.04$

$C_{12} = -1^{1+2} \begin{bmatrix} 0.7 & 0.7 & 0 \\ 0 & 2 & 0.7 \\ 0 & 0.7 & 2 \end{bmatrix}$
$-|C_{12}| = 0.7E_{11} + 0.7E12$

$E_{11} = (2 * 2) - (0.7 * 0.7) = 3.51$
$E_{12} = 0$

Therefore, $-|C_{12}| = 0.7(3.51) = 2.457$
$|C_{12}| = -2.457$

$|A| = 2C_{11} + 0.7C_{12}$
$= 2(6.04) + 0.7(-2.457)$

$= 10.3601$ This is the determinant.

5 Question 5

We can use MATLAB to create the Vandermonde Matrix, using the "sym" function.
We can then calculate the determinant of the matrix using the "det" function.

To check if the calculated determinant is equal to $(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_0)(\alpha_1 - \alpha_0)$, we can use the isAlways function.

See MATLAB code below:

```
syms alpha0 alpha1 alpha2

% Define the Vandermonde matrix
V = sym([1 alpha0 alpha0^2; 1 alpha1 alpha1^2; 1
    alpha2 alpha2^2])

% Compute the determinant of the Vandermonde matrix
det_V = det(V)

% Define the expected determinant expression
expected_det = expand((alpha2 - alpha1)*(alpha2 -
    alpha0)*(alpha1 - alpha0))

% Check if the determinant of the Vandermonde matrix
    matches the expected expression
result = isAlways(det_V == expected_det)
```

This is the output:

$$result = logical$$
$$1$$

This indicates that it is true that the determinant of the matrix equals

$$(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_0)(\alpha_1 - \alpha_0)$$