Math7501 Problem Set 2

Alex White: 43218307

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1 Question 1

$B_1 = A_1$
$B_n = A_n - \bigcup_{k=1}^{n-1} B_k \qquad \text{for } n \geq 2$

Proof by mathematical induction:

Basis Step: $n = 2$
As $B_1 = A_1$,

$B_2 = A_2 - \bigcup_{k=1}^{1} B_1$
$B_2 = A_2 - A_1$ Holds for basis step, as B_2 does not contain any elements from B_1, and is therefore mutually disjoint.

Inductive Step: Suppose the logic holds for $n = m$, then for $n = m + 1$:

$B_{m+1} = A_{m+1} - \bigcup_{k=1}^{m} B_k$

B_{m+1} is defined as A_{m+1} minus the elements of the union of B_1 to B_m
Therefore for any i, where $1 \leq i \leq m$, B_{m+1} and B_i have no common elements, and are therefore mutually disjoint.

2 Question 2

2.1 2.i

p	q	$\sim p$	$\sim q$	$p \wedge q$	$(p \wedge q) \vee (\sim p)$	$q \vee (\sim p)$
True	True	False	False	True	True	True
True	False	False	True	False	False	False
False	True	True	False	False	True	True
False	False	True	True	False	True	True

2.2 2.ii

$$\begin{aligned} &(p \wedge q) \vee (\sim p) \\ &\equiv ((\sim p) \vee p) \wedge ((\sim p) \vee q) \\ &\equiv t \wedge ((\sim p) \vee q) \\ &\equiv (\sim p) \vee q \end{aligned}$$

3 Question 3

3.1 3.a

To show that f is a one-to-one function, we can prove by contradiction:
Suppose $f(x_1) = f(x_2)$:

$\frac{x_1+\sqrt{x_1^2+4}}{2} = \frac{x_2+\sqrt{x_2^2+4}}{2}$
$x_1 + \sqrt{x_1^2+4} = x_2 + \sqrt{x_2^2+4}$ Divide by 2
$\sqrt{x_1^2+4} = x_2 - x_1 + \sqrt{x_2^2+4}$ Subtract x_1
$x_1^2 + 4 = (x_2 - x_1)^2 + 2(x_2 - x_1)\sqrt{x_2^2+4} + x_2^2 + 4$ Square both sides
$x_1^2 = x_2^2 - 2x_1x_2 + x_1^2 + 2(x_2 - x_1)\sqrt{x_2^2+4} + x_2^2$ Simplify
$0 = x_2^2 - x_1x_2 + (x_2 - x_1)\sqrt{x_2^2+4}$ Simplify and divide by 2

Therefore, either $x_2^2 - x_1x_2 = 0$ or $(x_2 - x_1) = 0$
Since both terms result in $x_1 = x_2$, we arrive at a contradiction. Hence $\frac{x+\sqrt{x^2+4}}{2}$is a one-to-one function.

3.2 3.b

Since $f : \mathbb{R}$, then there exists an x such that
$y = \frac{x+\sqrt{x^2+4}}{2}$
$2y = x + \sqrt{x^2+4}$ Multiply by 2
$2y - x = \sqrt{x^2+4}$ Subtract x both sides
$(2y - x)^2 = x^2 + 4$ Square both sides
$4y^2 - 4yx + x^2 = x^2 + 4$ Expand left side
$4y^2 - 4yx - 4 = 0$ Simplify
$y^2 - yx - 1 = 0$ Simplify
$yx = y^2 - 1$ Simplify and rearrange
$x = \frac{y^2-1}{y}$ Simplify and rearrange
Therefore the inverse of f, $f^{-1}(x) = \frac{x^2-1}{x}$

3.3 3.c

We need to determine the limit of $f(x) = \frac{x+\sqrt{x^2+4}}{2}$ as x approaches $-\infty$.

Given the inequality $1 < \sqrt{1+y} < 1 + \frac{1}{2}y$ for all $y > 0$, we can apply this to the term $\sqrt{1 + \frac{4}{x^2}}$ where $\frac{4}{x^2} > 0$ as $x \to -\infty$.

Noting that $\sqrt{x^2+4} = |x|\sqrt{1 + \frac{4}{x^2}}$ and $|x| = -x$ for $x < 0$:

$$\sqrt{1 + \frac{4}{x^2}} > 1 \quad \text{and} \quad \sqrt{1 + \frac{4}{x^2}} < 1 + \frac{2}{x^2}.$$

Multiplying these inequalities by $|x| = -x$ we have:

$$-x < \sqrt{x^2+4} < -x + \frac{2}{x}.$$

Substitute these bounds into the expression for $f(x)$:

$$\frac{x-x}{2} < \frac{x+\sqrt{x^2+4}}{2} < \frac{x-x+\frac{2}{x}}{2}.$$

Simplifying further:

$$0 < f(x) < \frac{1}{x} \quad \text{as } x \to -\infty.$$

By the Squeeze Theorem, since $\lim_{x\to-\infty} \frac{1}{x} = 0$:

$$\lim_{x\to-\infty} f(x) = 0.$$

3.4 3.d

To show that $f(x) = \frac{x+\sqrt{x^2+4}}{2}$ is continuous on $\mathbb{R}$, we can check each term within $f(x)$.

1. The function $g(x) = x$ is linear and hence continuous everywhere on $\mathbb{R}$.
2. The function $h(x) = x^2 + 4$ is a polynomial, which is continuous everywhere on $\mathbb{R}$.
3. The square root function $\sqrt{x}$ is continuous for all $x \geq 0$. Since $h(x)$ is positive, $\sqrt{h(x)} = \sqrt{x^2+4}$ is continuous.
4. The sum and multiplication of continuous functions is also continuous.

4 Question 4

4.1 4.a

$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)(2n+1)!}$

Can check for convergence using the ratio test

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$
$\lim_{n\to\infty} |\frac{(-1)^{n+1}2^{n+1}}{(2(n+1)+1)(2(n+1)+1)!} \cdot \frac{(2n+1)(2n+1)!}{(-1)^n 2^n}|$
$\lim_{n\to\infty} |\frac{-2(2n+1)(2n+1)!}{(2n+3)(2n+3)!}|$
$\lim_{n\to\infty} |\frac{-2}{(2n+3)(2n+2)}| = 0$

From the ratio test, as $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 < 1$, the sequence converges absolutely.

4.2 4.b

$\sum_{n=0}^{\infty}(-1)^n\frac{n}{n+1}$

Using the n-th term test, if $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=0}^{\infty} a_n$ diverges.
$\lim_{n\to\infty}(-1)^n$ limit does not exist: diverges
$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$ limit = 1: diverges

4.3 4.c

$\sum_{n=1}^{\infty}(-1)^{n+1}(e^{1/n}-1)$

$e^x < 1+2x$ for all x $\in (0,1]$ and $1+x < e^x$ for all $x > 0$

$e^{1/n} < 1+2(\frac{1}{n})$
$e^{1/n} - 1 < 2(\frac{1}{n})$

$\sum_{n=1}^{\infty}(-1)^{n+1}(e^{1/n}-1) < \sum_{n=1}^{\infty}(-1)^{n+1}(\frac{2}{n})$

As $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{2}{n})$ converges to 0, we can use the comparison to test to show that since $\sum_{n=1}^{\infty}(-1)^{n+1}(e^{1/n}-1)$ is bounded by it, it also converges.
However, it doesn't converge absolutely as if we take the absolute value of the equation, we get the harmonic series, which diverges. This means that our equation conditionally converges.

5 Question 5

5.1 5.a

$\lim_{x\to 1}\frac{x^2+6x+5}{3x^2+6x-45}$
$= \frac{1+6+5}{3+6-45}$
$= \frac{12}{-36}$
$= \frac{-1}{3}$

Therefore $\lim_{x\to 1}\frac{x^2+6x+5}{3x^2+6x-45} = -1/3$

5.2 5.b

$\lim_{x\to 5}\frac{x^2+6x+5}{3x^2+6x-45}$
$= \frac{5^2+6*5+5}{3*5^2+6*5-45}$
$= \frac{60}{60}$
$= 1$

Therefore $\lim_{x\to 5} \frac{x^2+6x+5}{3x^2+6x-45} = 1$

5.3 5.c

$\lim_{x\to\infty} \frac{x^2+6x+5}{3x^2+6x-45}$
$= \lim_{x\to\infty} \frac{(x^2/x^2)+(6x/x^2)+(5/x^2)}{(3x^2/x^2)+(6x/x^2)-(45/x^2)}$ Divide by largest power denominator
$= \lim_{x\to\infty} \frac{1+(6/x)+(5/x^2)}{3+(6/x)-(45/x^2)}$
$= \frac{1}{3}$

Therefore, $\lim_{x\to\infty} \frac{x^2+6x+5}{3x^2+6x-45} = 1/3$

5.4 5.d

$\lim_{x\to 3} \frac{x^2+6x+5}{3x^2+6x-45}$
$= \frac{3^2+6*3+5}{3*3^2+6*3-45}$
$= \frac{32}{0}$

Therefore, $\lim_{x\to 3} \frac{x^2+6x+5}{3x^2+6x-45}$ does not exist.

6 Question 6

6.1 6.a

$f(x) = sin(x)/x, \qquad x < 0$
$f(x) = a - bx^2, \qquad 0 \leq x \leq \pi$
$f(x) = sin(x)/x, \qquad x > \pi$

$f(x)$ must be continuous at $x = 0$ and $x = \pi$.
$f(x)$ is continuous if $\lim_{x\to a+} f(x) = f(a), \lim_{x\to a-} f(x) = f(a)$

For $x = 0$
$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$
$\lim_{x\to 0-} \frac{sin(x)}{x} = \lim_{x\to 0+} f(0) = a$
$\lim_{x\to 0+} a - bx^2 = a$
To find a, we can use L'Hopital's Rule on $\lim_{x\to 0} \frac{sin(x)}{x}$
$\lim_{x\to 0} \frac{sin(x)}{x} = \lim_{x\to 0} \frac{cos(x)}{1} = 1$
Therefore, a = 1

For $x = \pi$
$\lim_{x\to \pi-} f(x) = \lim_{x\to \pi+} f(x)$
$\lim_{x\to \pi-} a - bx^2 = \lim_{x\to \pi+} f(\pi) = \frac{sin(\pi)}{\pi}$
$\lim_{x\to \pi+} f(\pi) = \frac{sin(\pi)}{\pi} = 0$
$\lim_{x\to \pi-} a - bx^2 = 0$

Since we know a = 1

$1 - b\pi^2 = 0$
$b = 1/\pi^2$

Therefore, to make f(x) continuous, the values of a and b are: $a = 1, b = \frac{1}{\pi^2}$

7 Question 7

7.1 7.a

$\{a_{n}{}_{n=0}^{\infty}\}$

$a_{n+1} = a_n \exp(r(1 - a_n/K))$

Assuming the limit exists; $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n \exp(r(1 - a_n/K))$
Given a_{n+1} follows the same sequence as a_n, they will have the same limit.
Therefore we can substitute in $\lim_{n\to\infty} a_n = l$

$l = l * \exp(r(1 - l/K)) \qquad$ Given recursive formula

$1 = \exp(r(1 - l/K))$	divide by l
$0 = r(1 - \frac{l}{K})$	take natural log of both sides
$0 = r - \frac{rl}{K}$	expand braces
$r = \frac{rl}{K}$	rearrange
$l = K$	

Therefore, if the limit $\lim_{n\to\infty} a_n = l$ exists, it must be K

7.2 7.b

$\sum_{n=0}^{\infty}(a_n - K)$, is a recursive relationship:

$$a_{n+1} = a_n \exp(r(1 - \frac{a_n}{K})).$$

Let $x_n = a_n - K$ to simplify the recursion:

$$x_{n+1} = (K + x_n)\exp(-r\frac{x_n}{K}) - K.$$

Using the known identity:

$$\exp(-r\frac{x_n}{K}) \approx 1 - r\frac{x_n}{K}.$$

$$x_{n+1} \approx x_n - r\frac{x_n^2}{K}.$$

Comparing with a geometric series:

$$|x_{n+1}| \leq |x_n|(1 - \frac{r}{2}).$$

Thus,

$$\sum_{n=0}^{\infty}|x_n| \leq |x_0|\sum_{n=0}^{\infty}(1 - \frac{r}{2})^n.$$

The RHS is a geometric series which converges when $0 < 1 - \frac{r}{2} < 1$, or $0 < r < 2$.

Therefore, assuming $r < 2$, the series $\sum_{n=0}^{\infty}(a_n - K)$ converges.