Math7501 Problem Set 3

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May 2024

1 Question 1

1.1 1.a

$$\lim_{x\to 1/2} \frac{arcsin(x)}{sin(x)}$$

Substitute x = 1/2 into equation

$$= \frac{arcsin(\frac{1}{2})}{sin(\frac{1}{2})}$$

$$= \frac{\frac{\pi}{6}}{\frac{1}{2}}$$

$$= \frac{\pi}{3}$$

The limit is $\frac{\pi}{3}$

1.2 1.b

$$\lim_{x\to 0} \frac{arcsin(x)}{sin(x)}$$

We can use L'Hôpital's Rule as we know that sin(0) = 0

$$\lim_{x\to 0} \frac{arcsin(x)}{sin(x)} = \lim_{x\to 0} \frac{\frac{d}{dx}arcsin(x)}{\frac{d}{dx}sin(x)}$$

$$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{cos(x)}$$

$$= \frac{\frac{1}{\sqrt{1-0^2}}}{cos(0)}$$

$$= 1$$

The limit is 1

1.3 1.c

$$\lim_{x\to\infty} x(\frac{\pi}{2} - arctan(x))$$

can rearrange and then use L'Hôpital's Rule

$$\lim_{x\to\infty} \frac{(\frac{\pi}{2} - arctan(x))}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}(\frac{\pi}{2} - arctan(x))}{\frac{d}{dx}\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$$

The limit is 1

2 Question 2

$f : [0, \infty) \to \mathbb{R}$
If the derivative is positive, then the function is increasing.

$$f(x) = \frac{x^4}{2} + \sin(x^2) - x^2\cos(x^2),$$

Differentiate each component:

1st component:

$$\frac{d}{dx}\left(\frac{x^4}{2}\right) = 2x^3,$$

2nd component, using the chain rule:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2),$$

3rd component, using product and chain rules:

$$\frac{d}{dx}\left(-x^2\cos(x^2)\right) = -2x\cos(x^2) - x^2(-2x\sin(x^2)),$$
$$= -2x\cos(x^2) + 2x^3\sin(x^2),$$

Combine the derivatives:

$$\frac{d}{dx}\left(\frac{x^4}{2} + \sin(x^2) - x^2\cos(x^2)\right)$$
$$= 2x^3 + 2x\cos(x^2) - 2x\cos(x^2) + 2x^3\sin(x^2),$$
$$= 2x^3(1 + \sin(x^2)),$$

Since $2x^3$ is non-negative for all $x \geq 0$ and $\sin(x^2)$ oscillates between -1 and 1, $1 + \sin(x^2)$ oscillates between 0 and 2.
Thus, $2x^3(1 + \sin(x^2)) \geq 0$ over the interval $[0, \infty)$.
Therefore, the function $f(x) = \frac{x^4}{2} + \sin(x^2) - x^2\cos(x^2)$ is increasing.

3 Question 3

$$\mathbb{P}(\bar{X} > a) \leq e^{-at}\left(1 - \left(\frac{t}{100}\right)^2\right)^{-100}$$

$$f(t) = e^{-at}\left(1 - \left(\frac{t}{100}\right)^2\right)^{-100}$$

To find global min, must differentiate and find when f'(t) = 0

Start with Product Rule:

$$\text{Let } u = e^{-at} \text{ and } v = \left(1 - \left(\frac{t}{100}\right)^2\right)^{-100},$$

$$\frac{d}{dt}(uv) = u'v + uv',$$

$$u' = -ae^{-at},$$

To calculate v', we need to use chain rule:

$$\text{Let } h(t) = 1 - \left(\frac{t}{100}\right)^2, \text{ and } g(t) = h(t)^{-100},$$

$$v' = g'(h(t)) \cdot h'(t),$$

$$h'(t) = -\frac{2t}{100^2},$$

$$v' = -100 \cdot h(t)^{-101} \cdot \left(-\frac{2t}{100^2}\right),$$

$$v' = \frac{200t}{100^2} \cdot h(t)^{-101},$$

$$v' = \frac{t}{50} \cdot h(t)^{-101},$$

Combine all components back into the product rule:

$$f'(t) = -ae^{-at} \cdot h(t)^{-100} + e^{-at} \cdot \left(\frac{t}{50}\right) \cdot h(t)^{-101},$$

Set f'(t) = 0 for critical points:

$$0 = -ae^{-at} \cdot h(t)^{-100} + e^{-at} \cdot \left(\frac{t}{50}\right) \cdot h(t)^{-101},$$

Divide by $e^{-at}$to simplify:

$$a \cdot h(t)^{-100} = \left(\frac{t}{50}\right) \cdot h(t)^{-101},$$

$$a = \frac{\left(\frac{t}{50}\right) \cdot h(t)^{-1}}{1},$$

$$a = \frac{t}{50} \cdot \frac{1}{h(t)},$$

Simplify to find t :

$$a(1 - \left(\frac{t}{100}\right)^2) = \frac{t}{50},$$

$$a - a\left(\frac{t^2}{100^2}\right) = \frac{t}{50},$$

$$\frac{t}{50} + a\left(\frac{t^2}{100^2}\right) - a = 0,$$

Multiply by 10000 to clear the denominator:

$$200t + at^2 - 10000a = 0,$$

Use the quadratic formula to solve for t:

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

$$= \frac{-2000 \pm \sqrt{2000^2 + 4 \cdot a \cdot 10000a}}{2a},$$

$$= \frac{-2000 \pm \sqrt{4000000 + 40000a^2}}{2a},$$

$$= \frac{-2000 \pm \sqrt{40000(1 + a^2)}}{2a},$$

$$= \frac{-1000 \pm 1000\sqrt{1 + a^2}}{a}$$

Given $t \geq 0$ for all a ¿ 0

$$t = \frac{-1000 + 1000\sqrt{1 + a^2}}{a}$$

For a given a = 2:

$$t = \frac{-1000 + 1000\sqrt{1 + 2^2}}{2} = 618.034$$

4 Question 4

$$f(x) = \frac{1}{x^2 + 3x + 2}$$

Rearrange using partial fractions

$$f(x) = \frac{1}{(x+2)(x+1)}$$
$$= \frac{A}{x+2} + \frac{B}{x+1}$$
$$= \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$
$$= \frac{(A+B)x + (A+2B)}{(x+2)(x+1)}$$

So (A + B) = 0 and (A + 2B) = 1

Therefore, A = -1 and B = 1

$$f(x) = \frac{-1}{x+2} + \frac{1}{x+1}$$

Can use Taylor expansion of each component and then join them. At x = 0

$$\text{Given } \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \text{ for } |u| < 1$$

First component:

$$\frac{-1}{x+2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^n}{2^n}\right),$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{2^{n+1}},$$

Second component:

$$\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n,$$

Join components:

$$f(x) = \frac{-1}{x+2} + \frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{2^{n+1}} + (-1)^n x^n,$$
$$= \sum_{n=0}^{\infty}\left((-1)^{n+1}\frac{x^n}{2^{n+1}} + (-1)^n x^n\right),$$
$$= \sum_{n=0}^{\infty}\left((-1)^n\left(\frac{-1}{2^{n+1}} + 1\right)x^n\right),$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(1 - \frac{1}{2^{n+1}}\right)x^n,$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{2^{n+1}-1}{2^{n+1}}\right)x^n$$

To determine the radius of convergence, we can use the ratio test.

$$a_n = (-1)^n \left(\frac{2^{n+1} - 1}{2^{n+1}} \right) x^n$$

$$\lim_{n \to \infty} \left| \frac{a_n + 1}{a_n} \right|$$

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1} \left(\frac{2^{n+2} - 1}{2^{n+2}} \right) x^{n+1}}{(-1)^n \left(\frac{2^{n+1} - 1}{2^{n+1}} \right) x^n} \right|$$

$$= \lim_{n \to \infty} |x| \left| \left(\frac{2^{n+2} - 1}{2^{n+2}} \right) \left(\frac{2^{n+1}}{2^{n+1} - 1} \right) \right|$$

As n increases towards ∞, the ratio terms both simplify to 1

$$= |x|(1 \times 1)$$

Hence the series converges for $|x| < 1$, therefore the radius of convergence is 1.

5 Question 5

$$f(x;\theta) = c_\theta x \exp\left(-\frac{x^2}{2\theta}\right)$$

5.1 5.a

$$\int_0^\infty f(x,\theta)dx = 1,$$

$$\int_0^\infty c_\theta x \exp\left(-\frac{x^2}{2\theta}\right) dx = 1,$$

$$\text{Let } u = \frac{x^2}{2\theta},$$

$$du = \frac{x}{\theta}dx,$$

$$dx = \frac{\theta}{x}du,$$

Substitute u and du into equation,

$$\int_0^\infty c_\theta x e^{-u}\frac{\theta}{x}du = 1,$$

$$\int_0^\infty c_\theta \theta e^{-u}du = 1,$$

$$c_\theta\theta\int_0^\infty e^{-u}du = 1,$$

$$c_\theta\theta[-e^{-u}]_0^\infty = 1,$$

$$c_\theta\theta(-e^{-\infty} + e^{-0}) = 1,$$

$$c_\theta\theta(0+1) = 1,$$

$$c_\theta = \frac{1}{\theta},$$

Therefore, in order for $\int_0^\infty f(x,\theta)dx = 1$, $c_\theta = \frac{1}{\theta}$

5.2 5.b.i

$$\int_0^\infty x^2 f(x,\theta)dx$$

$$\int_0^\infty x^2 c_\theta x \exp\left(-\frac{x^2}{2\theta}\right) dx$$

Subbing in $c_\theta = \dfrac{1}{\theta}$:

$$= \int_0^\infty x^3 \frac{1}{\theta} \exp\left(-\frac{x^2}{2\theta}\right) dx$$

$$= \frac{1}{\theta} \int_0^\infty x^3 \exp\left(-\frac{x^2}{2\theta}\right) dx$$

Using $u = \dfrac{x^2}{2\theta}$ and $du = \dfrac{x}{\theta}dx,$ so $dx = \dfrac{\theta}{x}du$:

$$= \frac{1}{\theta} \int_0^\infty x^2 \theta e^{-u} du$$

Given $u = \dfrac{x^2}{2\theta},$ substitute $x^2 = 2\theta u$ into the integral:

$$= \frac{1}{\theta} \int_0^\infty 2\theta^2 u e^{-u} du$$

$$= 2\theta \int_0^\infty u e^{-u} du$$

Using answer from previous question we can sub it in below:

$$= 2\theta \cdot 1 = 2\theta$$

5.3 5.b.ii

$$\int_0^\infty x^4 f(x,\theta)dx$$
$$\int_0^\infty x^4 c_\theta x \exp\left(-\frac{x^2}{2\theta}\right)dx$$
$$\text{Subbing in } c_\theta = \frac{1}{\theta}:$$
$$= \int_0^\infty x^5 \frac{1}{\theta} \exp\left(-\frac{x^2}{2\theta}\right)dx$$
$$\text{Using } u = \frac{x^2}{2\theta} \text{ and } du = \frac{x}{\theta}dx, \text{ so } dx = \frac{\theta}{x}du:$$
$$= \frac{1}{\theta}\int_0^\infty x^4 \theta e^{-u} du$$
$$x^2 = 2\theta u, \text{ therefore } x^4 = (2\theta u)^2$$
$$= \frac{1}{\theta}\int_0^\infty (2\theta u)^2 \theta e^{-u} du$$
$$= \int_0^\infty 4\theta^2 u^2 e^{-u} du$$
$$= 4\theta^2 \int_0^\infty u^2 e^{-u} du$$

Can then integrate by parts, by setting $v = u^2, dv = 2udu$ and $w = e^{-u}, dw = -e^{-u}du$

$$= 4\theta^2[-u^2e^{-u}]_0^\infty + 2\int_0^\infty ue^{-u}du$$
$$\text{Using } \int_0^\infty ue^{-u}du = 1 \text{ from previous question, and knowing } e^{-\infty} = 0$$
$$= 4\theta^2(0+2)$$
$$= 8\theta^2$$

5.4 5.c

$$\text{5.c.i } \int_0^\infty x f(x,\theta)dx$$
$$\text{and}$$
$$\text{5.c.ii } \int_0^\infty x^3 f(x,\theta)dx$$

MATLAB code below:

```
clearvars

% Define symbolic variables
syms theta x

% Assume theta is positive
assume(theta > 0)

c_theta = 1/theta

% Define the function f(x)
f(x) = c_theta * x * exp(-x^2 / (2 * theta))

% Define g(x) for 5.c i
g(x) = x * f(x)

% Integrate from 0 to infinity
int(g(x), x, 0, inf)

% Define h(x) for 5.c ii
h(x) = x^3 * f(x)

% Integrate from 0 to infinity
int(h(x), x, 0, inf)
```

Using the above MATLAB code, it was determined that

$$\int_0^\infty x f(x,\theta)dx = \frac{\sqrt{2}\sqrt{\theta}\sqrt{\pi}}{2}$$
$$\int_0^\infty x^3 f(x,\theta)dx = \frac{3\sqrt{2}\;\theta^{\frac{3}{2}}\sqrt{\pi}}{2}$$

6 Question 6

$$\int_{-\infty}^\infty f(x,\theta)dx = 1$$
$$\int_{-\infty}^\infty x f(x,\theta)dx = 0$$
$$\int_{-\infty}^\infty x^2 f(x,\theta)dx = 1$$

6.1 6.a

$$\int_c^\infty f(x,\theta)\,dx \le \int_{-\infty}^\infty \frac{(x+t)^2}{(c+t)^2} f(x)\,dx$$

On the interval $[c,\infty),$ we have $x \ge c,$ therefore:

$$1 \le \frac{(x+t)^2}{(c+t)^2}$$

Using the comparison properties of integrals, we can compare:

$$\int_c^\infty f(x,\theta)\,dx \le \int_c^\infty \frac{(x+t)^2}{(c+t)^2} f(x)\,dx$$

Given that x is increased by a positive number t and squared, extending the domain of the integral to $(-\infty,\infty),$ the integral on the right-hand side can only increase:

$$\int_c^\infty f(x,\theta)\,dx \le \int_{-\infty}^\infty \frac{(x+t)^2}{(c+t)^2} f(x)\,dx$$

6.2 6.b

$$\int_c^\infty f(x,\theta)\,dx \le \int_{-\infty}^\infty \frac{(x+t)^2}{(c+t)^2} f(x)\,dx$$

Expand and solve the integral on the right-hand side:

$$\text{RHS} = \frac{1}{(c+t)^2}\int_{-\infty}^\infty (x^2+2xt+t^2) f(x)\,dx$$

Expanding the terms inside the integral:

$$\text{RHS} = \frac{1}{(c+t)^2}\left(\int_{-\infty}^\infty x^2 f(x)\,dx + 2t\int_{-\infty}^\infty x f(x)\,dx + t^2\int_{-\infty}^\infty f(x)\,dx\right)$$

Substitute known values from the properties of $f(x)$:

$$\text{RHS} = \frac{1}{(c+t)^2}(1+0+t^2)$$

$$\text{RHS} = \frac{1+t^2}{(c+t)^2}$$

MATLAB Code:

```
clearvars

% Define symbolic variables
syms c t

% Assume c>0
assume(c>0)
```

```
% Define function
f(t) = (1 + t^2)/(c + t)^2

% Differentiate function
diff_t = diff(f(t))

% Solve differentiation
solve(diff_t)

% Check 2nd derivative for if min or max
second_diff_t = diff(diff_t)
solve(second_diff_t)
```

Output: f'(t) = 1/c
f"(t) = $(3 + c^2)/2c$, this means that the critical point is a minimum.
There is a global minimum at t = 1/c.
For $c = 1, t = 1$, therefore the global min is at t = 1, for when c = 1