Gaussian Weighted Least Squares

Introduction

Gaussian Weighted Least Squares (GWLS) is a variation of the least squares method, where each observation is assigned a weight based on its variance. Unlike ordinary least squares (OLS), which assumes constant variance, GWLS is used when observations have differing variances. This approach improves model accuracy by accounting for non-uniform variance and is widely used in smoothing, regression, and curve fitting.

Multivariate Gaussian Distribution: The multivariate Gaussian distribution generalises the normal distribution to multiple dimensions. It is defined with a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{V}$. The probability density function is given by:

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{V}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{V}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

The covariance matrix $\boldsymbol{V}$ encodes the correlations between the variables.

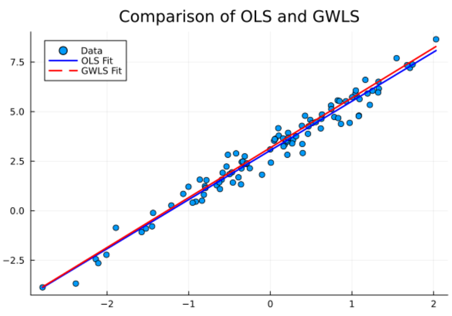

Figure 1: Comparison of GWLS to OLS

Bivariate Distribution: The bivariate distribution is a special case of the multivariate Gaussian distribution where $k = 2$, representing two jointly distributed random variables, X and Y. These variables are characterized by their means μ_X, μ_Y, variances σ_X^2, σ_Y^2, and covariance σ_{XY}, which describes the degree of their linear relationship. The covariance matrix for the bivariate distribution is:

$$\mathbf{V} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$$

When X and Y are correlated, knowledge of one variable provides partial information about the other due to their shared covariance. If the covariance is zero, the variables are independent, and the joint distribution simplifies to the product of two independent normal distributions.

Gaussian Computations: Key computations with Gaussian distributions include calculating probabilities and deriving the maximum likelihood estimates for parameters. The mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V}$ are estimated as:

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i, \quad \hat{\boldsymbol{V}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

The covariance matrix $\boldsymbol{V}$ captures the spread and correlation of the variables.

Random Variable Generation via Cholesky Factorization: Cholesky decomposition is a method used to decompose a positive-definite matrix $\mathbf{V}$ into the product of a lower triangular matrix and its transpose $\mathbf{V} = LL^T$ where L is a lower triangular matrix with real and positive diagonal entries.

To generate a vector $\mathbf{x}$ of correlated Gaussian random variables, we first generate a vector $\mathbf{z}$ of independent standard normal random variables. The correlated random variables are then computed as:

$$\mathbf{x} = \boldsymbol{\mu} + L\mathbf{z}$$