SCHOOL OF MATHEMATICS AND PHYSICS

MATH7502
Assignment 2
Semester 2 2024

Submit your answers - along with this sheet to blackboard. The due date is 2pm 18/10/2024.

You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

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Marker's use only

Each question marked out of 3.

- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

Q1a:	Q1b:	Q2a:	Q2b:	Q2c:	Q2d:	
Q3b:	Q3c:	Q4a:	Q4b:	Q5a:	Q5b:	Total (ou

Question 1.

(a) Let A be an $n \times n$ matrix. Prove that the determinant of A is the product of its eigenvalues.

Solution: We know that

$$\det(A - \lambda I) = 0$$

The characteristic polynomial of the matrix A is given by:

$$p(\lambda) = \det(A - \lambda I)$$

Using the Fundamental Theorem of Algebra, this polynomial can be factored as:

$$p(\lambda) = (-1)^n(\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda)$$

With $\lambda = 0$, $p(0) = \det(A)$, thus:

$$p(0) = (-1)^n \lambda_1 \lambda_2 \dots \lambda_n$$

Therefore:

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

Therefore, the determinant of A is the product of its eigenvalues

(b) Let A be an $n \times n$ real symmetric matrix. Show that the trace of A is the sum of its eigenvalues.

Solution: The trace of a matrix is defined as the sum of its diagonal elements.
From theorem 69, we know that if S is an $n \times n$ symmetric matrix, then it is diagonalisable by an orthogonal matrix Q:

$$S = Q \Lambda Q^T$$

where Λ is a diagonal matrix of eigenvalues.

Thus, for the matrix S, we have:

$$\mathrm{Tr}(S) = \sum_{i=1}^{n} S_{ii}$$

Since Λ contains the eigenvalues of S on its diagonal, we know that:

$$\mathrm{Tr}(S) = \mathrm{Tr}(\Lambda)$$

Therefore:

$$\mathrm{Tr}(S) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Thus, it's shown that the trace of S is the sum of its eigenvalues.

Question 2. Let $W \subseteq V \subseteq \mathbb{R}^n$ be a subspace of a vector space V.

a) Show that $W^\perp$, the orthogonal complement of W, is a subspace of V.

Solution: $W^\perp$ is a subspace of V if the three conditions for a subspace hold:

1. Additive Identity: The zero vector is orthogonal to all vectors in V, including those in W. Thus, $\mathbf{0} \in W^\perp$.

2. Closure under addition: Let $v_1, v_2 \in W^\perp$. Then for every $w \in W$:

$$v_1 \cdot w = 0 \quad \text{and} \quad v_2 \cdot w = 0.$$

Hence,

$$(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w = 0 + 0 = 0.$$

Therefore, $v_1 + v_2 \in W^\perp$.

3. Closure under scalar multiplication: Let $v \in W^\perp$ and $\alpha \in \mathbb{R}$. Then for all $w \in W$:

$$(\alpha v) \cdot w = \alpha(v \cdot w) = \alpha \cdot 0 = 0.$$

Hence, $\alpha v \in W^\perp$.

Therefore, $W^\perp$ is a subspace of V.

b) Prove that $W \cap W^\perp = \{0\}$.

Solution: Let $v \in W \cap W^\perp$
Therefore, $v \in W$ and $v \in W^\perp$.
As $v \in W^\perp$, therefore it must be orthogonal to every vector in W, including itself (v):

$$v \cdot v = 0.$$

But $v \cdot v = 0$ only works if v is is the zero vector: $v = \mathbf{0}$. Therefore, $W \cap W^\perp = \{0\}$.

c) Show that any vector $v \in V$ can be written as $v = w + u$, where $w \in W$ and $u \in W^\perp$. Let $\{q_1, q_2, \ldots, q_m\}$ be an orthonormal basis for the subspace W.

Solution: Given the vector: $v - \sum_{i=1}^m (v \cdot q_i) q_i$, which we can call u, we can rewrite v in the form $v = w + u$:

$$v = \sum_{i=1}^m (v \cdot q_i) q_i + \left(v - \sum_{i=1}^m (v \cdot q_i) q_i \right).$$

Where $w = \sum_{i=1}^m (v \cdot q_i) q_i$.
As we can see, $u = v - w$, therefore:

$$u \cdot q_i = \left(v - \sum_{j=1}^m (v \cdot q_j) q_j \right) \cdot q_i = v \cdot q_i - (v \cdot q_i) = 0.$$

Therefore, $u \in W^\perp$. Hence, we have shown that any vector $v \in V$ can be written as the sum of a vector $w \in W$ and a vector $u \in W^\perp$.

d) Prove that $(W^\perp)^\perp = W$.

Solution: Let $u \in W^\perp$, so $u \cdot w = 0$ for all $w \in W$.
Let $v \in (W^\perp)^\perp$, which means $v \cdot u = 0$ for all $u \in W^\perp$. Need to show that $v \in W$.

Since $v \in (W^\perp)^\perp$, it is orthogonal to all vectors in $W^\perp$. By the orthogonal decomposition theorem, $V = W \oplus W^\perp$, and thus $v \in W$.

Therefore, $(W^\perp)^\perp \subseteq W$ and also $W \subseteq (W^\perp)^\perp$. Therefore:

$$W = (W^\perp)^\perp.$$

e) Show that $\mathrm{NS}(A) = (\mathrm{R}(A^T))^\perp$

Solution: The row space of A^T, $\mathrm{R}(A^T)$, is the subspace spanned by the rows of A^T:

$$\mathrm{R}(A^T) = \mathrm{span}(r_1, r_2, \ldots, r_m),$$

where $r_1, r_2, \ldots, r_m$ are the rows of A^T.

The null space of A, denoted $\mathrm{NS}(A)$, is the set of vectors $x \in \mathbb{R}^n$ such that:

$$Ax = 0.$$

If $x \in \mathrm{NS}(A)$, then $Ax = 0$. This implies:

$$r_1 \cdot x = 0,\ r_2 \cdot x = 0, \ldots, r_m \cdot x = 0.$$

Since the dot product of x with each row of A^T is zero, x is orthogonal to every row of A^T.
Therefore, $x \in (\mathrm{R}(A^T))^\perp$.

Since every vector in the null space of A is orthogonal to the row space of A^T:

$$\mathrm{NS}(A) \subseteq (\mathrm{R}(A^T))^\perp.$$

By the rank-nullity theorem, the dimensions of these spaces are the same, so we conclude:

$$\mathrm{NS}(A) = (\mathrm{R}(A^T))^\perp.$$

Question 3. Let A be a (not necessarily square or full rank) matrix.

a) Prove that $R(A^T) = R(A^T A)$.

Solution: Need to show that $\mathrm{R}(A^T A) \subseteq \mathrm{R}(A^T)$.

Let $v \in \mathrm{R}(A^T A)$. By definition, there exists some vector $x \in \mathbb{R}^n$ such that:

$$v = A^T Ax.$$

Since $v = A^T(Ax)$, it is a linear combination of the columns of A^T. But the span of the columns of A^T is precisely the row space of A^T. Therefore, $v \in \mathrm{R}(A^T)$, which shows that:

$$\mathrm{R}(A^T A) \subseteq \mathrm{R}(A^T).$$

Also need to show that $\mathrm{R}(A^T) \subseteq \mathrm{R}(A^T A)$.

Let $v \in \mathrm{R}(A^T)$, so there exists a vector $x \in \mathbb{R}^n$ such that:

$$v = A^T x.$$

Now, multiply both sides of the equation by A on the left:

$$Av = A(A^T x) = AA^T x.$$

Since $AA^T x$ is a linear combination of the rows of $A^T A$, it follows that $v \in \mathrm{R}(A^T A)$.

Therefore, we have shown that $\mathrm{R}(A^T) = \mathrm{R}(A^T A)$.

b) Explain why the normal equations $A^T A\hat{x} = A^T b$ always have a solution.

Solution: From part (a), $\mathrm{R}(A^T) = \mathrm{R}(A^T A)$.

We know from q2 that $A^T A$ is symmetric, and a symmetric matrix S is positive semi-definite if for any vector $x \in \mathbb{R}^n$:

$$x^T Sx \geq 0.$$

Checking this for $A^T A$:

$$x^T(A^T A)x = (Ax)^T(Ax) = \|Ax\|^2.$$

Since $\|Ax\|^2 \geq 0$ for all $x \in \mathbb{R}^n$, we know $A^T A$ is positive semi-definite.

From q2, we also know that:

$$\mathrm{NS}(A^T A) = \mathrm{NS}(A).$$

This means that any vector b in the column space of A will lead to a valid least-squares solution for $\hat{x}$.

Thus there will always be a solution to $A^T A\hat{x} = A^T b$ as we know that $A^T A$ is positive semi-definite and symmetric.

c) Show that if $\mathrm{NS}(A) = \{0\}$, the solution to the normal equations is unique.

Solution: If $\mathrm{NS}(A) = \{0\}$, then A is full rank, and the matrix $A^T A$ is invertible. Therefore there is a unique solution to the equation:

$$A^T A\hat{x} = A^T b$$

We can find this by multiplying each side on the left by the inverse of $A^T A$.

$$A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b.$$

Since $A^T A$ is invertible, the solution is unique.

Question 4.

a) Let $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$, $\dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix}$, and let A be a 3×3 matrix. Suppose that $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, with corresponding real eigenvectors x_1, x_2, x_3. Show that the general solution to the coupled differential equation $\dot{y} = Ay$ is

$$y(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} + c_3 x_3 e^{\lambda_3 t},$$

where c_1, c_2, c_3 are constants.

Solution: A can be diagonalised as:

$$A = X \Lambda X^{-1}$$

where $X = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ is the matrix of eigenvectors, and Λ is the diagonal matrix of eigenvalues.

$\dot{y} = Ay$ can then be written as:

$$\dot{y} = X \Lambda X^{-1} y.$$

If we let $z(t) = X^{-1} y(t)$, then we can then decouple the system:

$$\dot{z} = \Lambda z \implies \dot{z}_1 = \lambda_1 z_1$$

Integrating both sides, we get:

$$z_1(t) = c_1 e^{\lambda_1 t}, \quad z_2(t) = c_2 e^{\lambda_2 t}, \quad z_3(t) = c_3 e^{\lambda_3 t}.$$

Rearranging $z(t) = X^{-1} y(t)$, we get:

$$y(t) = X z(t) = X \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{pmatrix}.$$

Expanding this gives:

$$y(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} + c_3 x_3 e^{\lambda_3 t},$$

which is the solution.

b) Find the general solution to the system of coupled differential equations:

$$\dot{y}_1 = y_1, \quad \dot{y}_2 = y_1 + 2y_2, \quad \dot{y}_3 = \alpha y_1 - y_3.$$

Solution:
Using Julia to answer this question:

```
using SymPy

# Define time variable and parameter
t = symbols("t")
a = symbols("a")

# Define the functions y1(t), y2(t), y3(t)
y1 = SymFunction("y1")(t)
y2 = SymFunction("y2")(t)
y3 = SymFunction("y3")(t)

# Solve for y1 first
eq1 = Eq(diff(y1, t), y1)  # dy1/dt = y1
sol1 = dsolve(eq1)  # Solve for y1
println(sol1)

# Substitute the solution for y1 into the equation for y2
y1_sol = rhs(sol1)
eq2 = Eq(diff(y2, t), y1_sol + 2*y2)  # dy2/dt = y1 + 2y2
sol2 = dsolve(eq2)  # Solve for y2
println(sol2)

# Substitute the solution for y1 into the equation for y3
eq3 = Eq(diff(y3, t), a*y1_sol - y3)  # dy3/dt = a*y1 - y3
sol3 = dsolve(eq3)  # Solve for y3
println(sol3)
```

The output is:

```
Eq(y1(t), C1*exp(t))
Eq(y2(t), (-C1 + C2*exp(t))*exp(t))
Eq(y3(t), C1*a*exp(t)/2 + C2*exp(-t))
```

This output can be written more legibly as:

$$\begin{gathered} y_1(t) = c_1 e^t \\ y_2(t) = (-c_1 + c_2 e^t)e^t = -c_1 e^t + c_2 e^{2t} = c_2 e^{2t} - c_1 e^t \\ y_3(t) = \frac{c_1 \alpha}{2} e^t + c_2 e^{-t} \end{gathered}$$

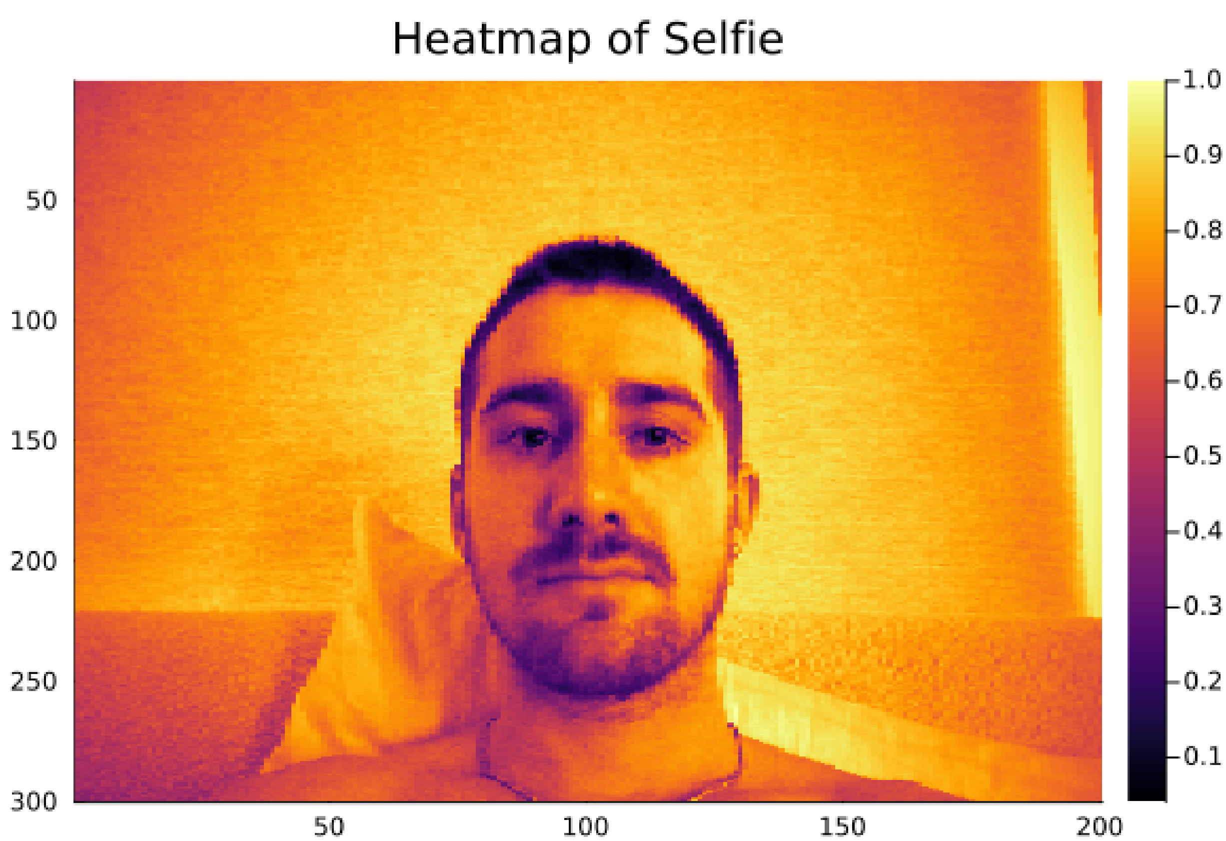
Question 5. Take a "Selfie" of yourself and transform it into a 300 × 200 monochrome matrix with elements in the range [0, 1].

a) Plot your selfie using `heatmap()`.

Solution: Using Julia:

```
using Images, LinearAlgebra, Plots

# Load the image
img = load("black_and_white_selfie.jpg")

# Convert the image to grayscale
gray_img = red.(img) * 1.0 + green.(img) * 1.0 + blue.(img) * 1.0

# Resize the grayscale image to 300x200
resized_img = imresize(gray_img, (300, 200))

# Convert the resized image to a matrix and normalize to [0, 1]
A = float64.(resized_img)
A = A / maximum(A)  # Normalize to the range [0, 1]

# Check the size of the resized matrix (should now be 300x200)
println(size(A))

# Plot the resized grayscale image as a heatmap
heatmap(A, yflip=true, title="Heatmap of Selfie")
```

The result is a grayscale heatmap of the selfie:

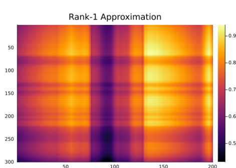


b) Present low-rank, SVD-based approximations of your selfie, including ranks: 1, 5, 10, 15, 20, 40, 80, 160, and 200 (full rank).

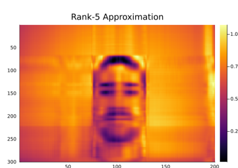
Solution: Using Julia and continuing on from the previous code (same variables are initiated):

```
# Perform SVD on the resized grayscale image matrix
U, E, V = svd(A)

# Low-rank approximation function
function low_rank_approximation(U, E, V, r)
    U_r = U[:, 1:r]  # Take the first r columns of U
    E_r = Diagonal(E[1:r])  # Take the first r singular values
    V_r = V[:, 1:r]  # Take the first r columns of V
    return U_r * E_r * V_r'  # Compute the low-rank approximation
end

# Show heatmaps for different ranks
for r in [1, 5, 10, 15, 20, 40, 80, 160, 200]
    A_approx = low_rank_approximation(U, E, V, r)
    heatmap(A_approx, yflip=true, title="Rank-$(r) Approximation")
    savefig("rank_$(r)_approximation.png")  # Save the low-rank heatma
end
```

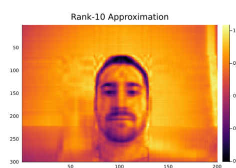
Images:



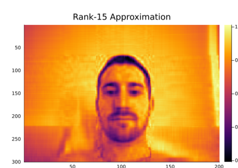
(a) Rank 1



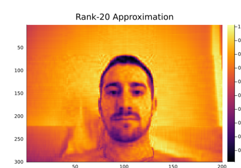
(b) Rank 5



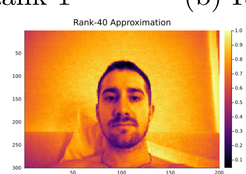
(c) Rank 10



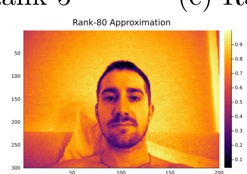
(d) Rank 15



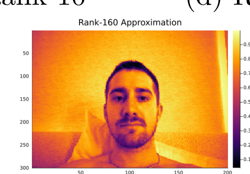
(e) Rank 20



(f) Rank 40



(g) Rank 80



(h) Rank 160

(i) Rank 200