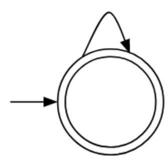
CSc 320: Foundations of Computer Science (Summer 2022)

Alex Holland

Assignment 5

Question 1
We want to show that $L = \{\langle A \rangle | A$ is a DFA over $\sum^*$ and $L(A) = \sum^*\}$ is decidable by giving a high-level description of a decider M with $L(M) = L$. We can design a TM for the given DFA that run finitely and halts.

L accepts all strings including the empty string, therefore the TM must accept all strings and reject none.



$Y =$ "On input $\langle A \rangle$ where A is a DFA:

1. Mark the start state of A
2. Repeat until no new states are marked
   - Mark any state that has an incoming transition from a marked state
3. If any accept state is marked, accept, otherwise, if unmarked, reject"

Since in the constructed DFA there is only 1 state which is also an accept state, all strings are accepted. Only an empty set will be rejected. By step 2 guarantees the DFA will halt because the TM will eventually run out of states to mark.

$EQ_{DFA} = \{\langle A, B \rangle |$A,B are DFA's and $L(A) = L(B)$

All word in $L(A)$ are in $L(B)$ and there is no word that isn't
$L(A) \cup L(B) \neq \emptyset$
$\overline{L(A)} \cup \overline{L(B)} = \emptyset$ if and only if $L(A) = L(B)$
We now need to create a DFA that recognizes exactly $L(C) = \overline{L(A)} \cup \overline{L(B)} = \emptyset$. Since $L(C)$ is regular we can construct such DFA that recognizes $L(C)$.

$Z =$ "On input $\langle A, B \rangle$ where $A, B = DFA$

1. Construct C as show previously
2. Simulate TM Y on input $\langle C \rangle$ where Y is the TM that decides EQ_{DFA}
3. If Y accepts, accept. Reject if Y rejects.

Thus, $L = \{\langle A \rangle | A$ is a DFA over $\sum^*$ and $L(A) = \sum^*\}$ is decidable.

Question 2
a)
The language L_{min} for Minimal DFA Recognition recognizes a DFA M such that there does not exist any other DFA M' that has less states than M for the language L_{min}.

b)
We to show that the problem is decidable, in other words, we want to describe a Turing machine that recognizes L_{min}. The proof from lecture for DFA state minimization states that if there is a pair $\{p, q\}$ such that $\{\delta(p, a), \delta(q, a)\}$ for some $a \in \sum$, such that $\{p, q\}$ can be removed, since $p \sim q$.

We want to create a TM that can run both finitely and recognizes L_{min} which would then prove that L_{min} is decidable.

$M =$ "On input $\langle L_{min} \rangle$, where L_{min} is a DFA:

1. Remove states that are unreachable
2. Collapse states for unmarked pairs $\{p, q\}$ if $\{\delta(p, a), \delta(q, a)\}$ is marked for some $a \in \sum$.
3. If the DFA has been minimized or been has not changed after minimization, accept. Otherwise, reject."

$N =$ "On input $\langle L_{min} \rangle$ where $L_{min} =$ DFA:

1. Simulate the TM M on input $\langle L_{min} \rangle$ where M is the TM that decides the minimization of the DFA.
2. If M accepts, accept. Else reject if M rejects."

Thus, the problem is decidable.

Question 3
We want to show that L_β is undecidable, where
$L_\beta = \{\langle M \rangle |$ at some point during its computation on empty input ϵ, M writes the symbol β on its tape $\}$.
We can do this by using a reduction from A_{TM} to L_β.

Note that $A_{TM} = \{\langle M, w \rangle | M$ is a TM and M accepts $w\}$

Assume L_B is decidable and that there exists some decider R that decides it.

Let S be a decider.

$S =$ "On input $\langle M, w \rangle$"

1. Construct a TM M' as follows:
   (a) $M' =$ "On input string x
      i. If $x \neq \epsilon$, reject
      ii. If $x = \epsilon$, run M on x. Write β to the tape if M accepts, and accept, otherwise, don't write β and reject.
2. Run R on input $\langle M' \rangle$
   (a) If R accepts, accept

(b) If R reject, reject

Since S is a decider for A_{TM}; R only accepts if M' writes a β to the tape iff M accepts the input string x. Likewise, If R rejects, then it means that M' does not write β to the tape if M rejects the input string x. Since, a decider has been created for A_{TM}, this shows a contradiction, hence, it is proved that L_β is undecidable.

Question 4

a)

The language L_{long} for Long Enough Cycle accepts undirected simple graphs that contain a simple cycle with at least k vertices, where none of the vertices except the start and end vertex can be repeated. The cycle is not simple if it has k vertices which can be further broken down into more cycles. We can define the language L_{long} as follows:

$L_{long} = \{\langle G, k\rangle\} |$ G is a graph such that it contains a simple cycle with at least k vertices $\}$

b)

We want to show that Long Enough Cycle is in NP. To prove this, a polynomial verifier that satisfies the definition of a verifier for L_{long} can be given.

$\langle G, K\rangle$ is a subset of the Graph G and is a simple cycle with a at least a minimum of k vertices. Let c be the simple cycle used in the following:

$V =$ "On input $\langle\langle G, K\rangle, c\rangle$

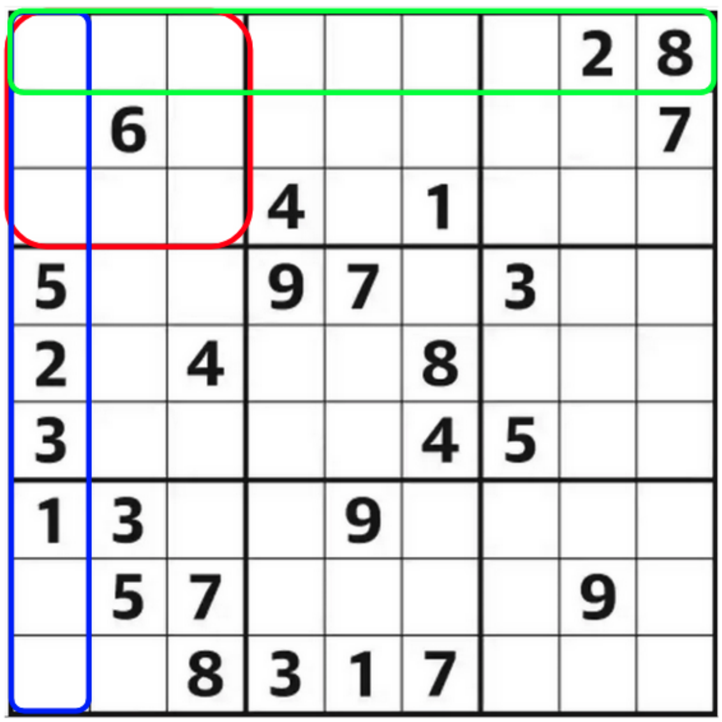
1. Test if c contains edges from one vertex to the next
2. Test if c contains repeated vertices
3. Test if a simple cycle in c contains $\geq k$ vertices
   (a) Traverse through each vertex, incrementing a counter for each visited vertex, once the end vertex has been reached. Then compare the vertice count in the cycle to k
4. Test if a cycle is completed in c
   (a) Traverse through each vertex, mark each vertex until the final vertex is reached, check that no other unmarked vertexes can be reached in c.
5. If all tests pass, accept. Otherwise, reject."

Since traversing through each vertex in the certificate takes linear time for each test, we can say that V_{long} is a polynomial time verifier. Thus, by showing a polynomial verfier for V_{long} with its corresponding certificate, Long Enough Cycles is in NP.

Question 5
We want to prove Sudoku is in NP, where empty cells in the Sudoku grid be completed such that each row, each column, and each block contains each integer from $\{1, 2, ..., n^2\}$. In this proof a block, column, and row are represented as follows:

The following is an example of a Sudoku grid with prefilled numbers and indication of of it rows, columns and blocks.

red = *block*
blue = *column*
green = *row*



To prove Sudoku is in NP, we need to show that Sudoku is both NP-hard and NP-complete.

The input size of Sudoku is a finite grid of $n^2 \times n^2$ squares where each of it's column, row, and $n \times n$ block contains $1 \rightarrow n^2$ integers once, they each have fixed a number of solutions depending on the number of squares filled in prior to solving the Sudoku. As such, any $n^2 \times n^2$ grid can be solved in constant time.

To prove Sudoku is in NP, a polynomial time verifier, V_{sudoku}, can be constructed to check if a solution is valid by verifying that each row, column, and $n \times n$ block has distinct $1 \rightarrow n^2$ integers. The solution is rjected if duplicates are identified in any of corresponding rows, columns, or blocks.

A certificate for V_{sudoku} can be described where each row, column, and $n \times n$ block contain unique integers $1 \rightarrow n^2$.

Thus with the use of a polynomial verifier V_{sudoku} and certificate, Sudoku is in NP.