ECE 260: Continuous-Time Signals and Systems
Assignment 4

5.1 [find Fourier series]

(a)

$$
\begin{aligned}
x(t) &= 1 + cos(\pi t) + sin^2(\pi t)\\
&= 1 + \frac{1}{2}(e^{j\pi t} + e^{-j\pi t}) + [\frac{1}{2j}(e^{j\pi t} - e^{-j\pi t})]^2\\
&= 1 + \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} - \frac{1}{4}((e^{j\pi t})^2 - 2(e^{j\pi t})(e^{-j\pi t}) + (e^{-j\pi t})^2)\\
&= 1 + \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} - \frac{1}{4}(e^{j2\pi t} - 2 + e^{-j2\pi t})\\
&= -\frac{1}{4}e^{-j2\pi t} + \frac{1}{2}e^{-j\pi t} + \frac{3}{2} + \frac{1}{2}e^{j\pi t} - \frac{1}{4}e^{j2\pi t}
\end{aligned}
$$

where $\omega_0 = \pi$

$$
c_k = \begin{cases} \frac{3}{2} & k = 0\\ \frac{1}{2} & k = \pm 1\\ -\frac{1}{4} & k = \pm 2\\ 0 & otherwise \end{cases}
$$

(c)

$$
\begin{aligned}
T &= \frac{1}{2}\\
\omega_0 &= \frac{2\pi}{1/2} = 4\pi
\end{aligned}
$$

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\\
&= 2\int_0^{1/2} |sin(2\pi t)|e^{-jk4\pi t}dt
\end{aligned}
$$

Since $\int e^{ax} sin(bx)dx = \frac{e^{ax}[asin(bx) - bcos(bx)]}{a^2+b^2} + C$ where a and b are arbitrary complex and nonzero real constants, respectively.

$$
\begin{aligned}
c_k &= 2[\frac{e^{-j4\pi kt}[-j4\pi k sin(2\pi t) - 2\pi cos(2\pi t)]}{(-j4\pi k)^2 + (2\pi)^2}]\Big|_0^{1/2} \\
&= \frac{2(2\pi)}{-16\pi^2k^2 + 4\pi^2}[e^{-j4\pi kt}[-j2k sin(2\pi) - cos(2\pi t)]]\Big|_0^{1/2} \\
&= \frac{1}{\pi(1-4k^2)}[e^{-j4\pi k/2}[-j2k sin(2\pi/2) - cos(2\pi t/2)] + cos(0)] \\
&= \frac{2}{\pi(1-4k^2)}
\end{aligned}
$$

where $\omega_0 = 4\pi$

$$c_k = \frac{2}{\pi(1-4k^2)}$$

5.2 [find Fourier series]
(a)

$$
\begin{aligned}
&T = 4 \text{ is the fundamental period, so } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2} \\
c_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{4}\int_{-2}^{2}[\delta(t-1) - \frac{1}{2}\delta(t+1)]e^{-j\pi kt/2}dt \\
&= \frac{1}{4}\Big[\int_{-2}^{2}\delta(t-1)e^{-j\pi kt/2}dt - \frac{1}{2}\int_{-2}^{2}\delta(t+1)e^{-j\pi kt/2}dt\Big] \\
&= \frac{1}{4}e^{-j\pi k/2} - \frac{1}{8}e^{j\pi k/2} \text{ sifting property} \\
&= \frac{1}{4}(-j)^k - \frac{1}{8}j^k
\end{aligned}
$$

(c)

$$T = 5 \text{ is the fundamental period, so } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{5}$$

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{5}\int_{-5/2}^{5/2} x(t)e^{-j2\pi kt/5}dt \\
&= \frac{1}{5}\Big[\int_{-2}^{-1} e^{-j2\pi kt/5}dt + \int_{-1}^{1} 2e^{-j2\pi kt/5}dt + \int_{1}^{2} e^{-j2\pi kt/5}dt\Big] \\
&= \frac{1}{5}\Big[\frac{1}{-j2\pi k/5}e^{-jk2\pi t/5}\Big|_{-2}^{2} + \frac{2}{-j2\pi k/5}e^{-jk2\pi t/5}\Big|_{-1}^{1} + \frac{1}{-j2\pi k/5}e^{-jk2\pi t/5}\Big|_{1}^{2}\Big] \\
&= \frac{1}{-j2\pi k}\Big[e^{-jk2\pi t/5}\Big|_{-2}^{2} + 2e^{-jk2\pi t/5}\Big|_{-1}^{1} + e^{-jk2\pi t/5}\Big|_{1}^{2}\Big] \\
&= \frac{1}{-j2\pi k}[e^{-jk2\pi(-1)/5} - e^{-jk2\pi(-2)/5} + 2e^{-jk2\pi(1)/5} - 2e^{-jk2\pi(-1)/5} + e^{-jk2\pi(2)/5} - e^{-jk2\pi(1)/5}] \\
&= \frac{1}{-j2\pi k}[e^{jk2\pi/5} - e^{2jk2\pi/5} + 2e^{-jk2\pi/5} - 2e^{jk2\pi/5} + e^{-2jk2\pi/5} - e^{-jk2\pi/5}] \\
&= \frac{1}{-j2\pi k}[e^{-j4\pi k/5} - e^{j4\pi k/5} + e^{-j2\pi k/5} - e^{j2\pi k/5}] \\
&= \frac{1}{-j2\pi k}[-2jsin(4\pi k/5) - 2jsin(2\pi k/5)] \\
&= \frac{1}{\pi k}[sin(4\pi k/5) - sin(2\pi k/5)] \\
&= \frac{sin(4\pi k/5)}{\pi k} + \frac{sin(2\pi k/5)}{\pi k} \\
&= \frac{4}{5}sinc(4\pi k/5) + \frac{2}{5}sinc(2\pi k/5) \text{ for } k = 0
\end{aligned}$$

For $k = 0$

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T x(t) \\
&= \frac{1}{5}\int_{-5/2}^{5/2} x(t) \\
&= \frac{1}{5}\Big[\int_{-2}^{-1} dt + \int_{-1}^{1} 2dt + \int_{1}^{2} dt\Big] \\
&= \frac{1}{5}[-1 - (-2) + 2(1 - (-1)) + 2 - 1] \\
&= \frac{1}{5}(6) \\
&= \frac{6}{5}
\end{aligned}$$

$\therefore$ we get:

$$c_k = \begin{cases} \frac{6}{5} & k = 0 \\ \frac{4}{5}sinc(4\pi k/5) + \frac{2}{5}sinc(2\pi k/5) & \text{otherwise} \end{cases}$$

5.6 [odd harmonic proof]
(b)

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
-x(t = \frac{T}{2}) &= -\sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 (t-T/2)} \\
&= -\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jk\omega_0 T/2} \\
&= -\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-j\pi k} \\
&= -\sum_{k=-\infty}^{\infty} (-1)^k c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} (-1)^{k+1} c_k e^{jk\omega_0 t} \\
c_k &= (-1)^{k+1} \\
c_k &= \begin{cases} -c_k & even \\ c_k & odd \end{cases}
\end{aligned}
$$

$\therefore$ x is odd harmonic iff $x(t) = -x(t - \frac{T}{2})$ for all t

5.8 [find/plot frequency spectrum]

$$T = 2 \text{ is the fundamental period, so } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$
\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{2} \int_0^2 x(t) e^{-j\pi k t} dt \\
&= \frac{1}{2} \int_0^1 e^{-j\pi k t} dt \\
&= \frac{1}{2} \Big[\frac{1}{-j\pi k} e^{-j\pi k t} \Big] \Big|_0^1 \\
&= \frac{1}{2} \Big[\frac{1}{-j\pi k} e^{-j\pi k (1)} - \frac{1}{-j\pi k} e^{-j\pi k (0)} \Big] \\
&= \frac{1}{j2\pi k} [1 - e^{-j\pi k}] \\
&= \frac{1}{j2\pi k} [1 - (-1)^k] \\
c_k &= \begin{cases} -\frac{j}{\pi k} & k \text{ odd} \\ 0 & k \text{ even, } k \neq 0 \end{cases}
\end{aligned}
$$

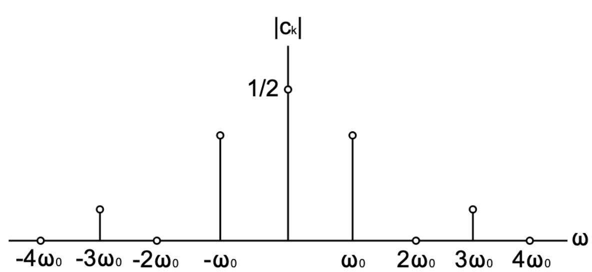
if $k=0$

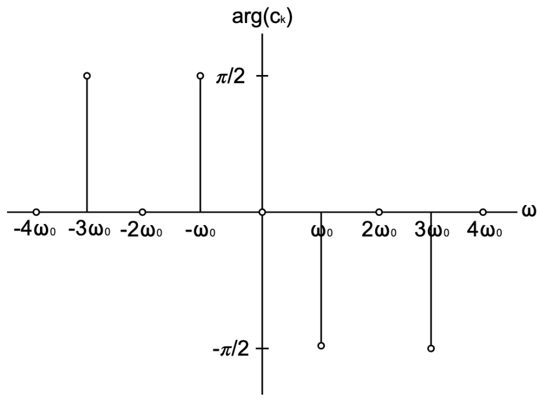
$$\begin{aligned}
c_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{1}{2}\int_0^2 x(t) \\
&= \frac{1}{2}\int_0^1 dt \\
&= \frac{1}{2}[t]\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

$$c_k = \begin{cases} \frac{1}{2} & k=0 \\ -\frac{j}{\pi k} & k \text{ odd} \\ 0 & k \text{ even, } k \neq 0 \end{cases}$$

First Fourier series coefficients:

$$\begin{aligned}
&k=0,\ |c_k| = \frac{1}{2},\ arg(c_k)=0 \\
&k=1,\ |c_k| = \frac{1}{\pi},\ arg(c_k)=-\frac{\pi}{2} \\
&k=2,\ |c_k| = 0,\ arg(c_k)=0 \\
&k=3,\ |c_k| = \frac{1}{3\pi},\ arg(c_k)=-\frac{\pi}{2} \\
&k=4,\ |c_k| = 0,\ arg(c_k)=0 \\
&k=5,\ |c_k| = \frac{1}{5\pi},\ arg(c_k)=-\frac{\pi}{2}
\end{aligned}$$





5.9 [filtering]

$$\begin{aligned} x(t) &= 1 + 2cos(2t) + 2cos(4t) + \frac{1}{2}cos(6t) \\ &= 1 + 2[\frac{1}{2}(e^{j2t} + e^{-j2t})] + 2[\frac{1}{2}(e^{j4t} + e^{-j4t})] + \frac{1}{2}[\frac{1}{2}(e^{j6t} + e^{-j6t})] \\ &= 1 + e^{j2t} + e^{-j2t} + e^{j4t} + e^{-j4t} + e^{j6t} + e^{-j6t} \end{aligned}$$

$$c_k = \begin{cases} 1 & k = 0 \\ 1 & k \pm 1 \\ 1 & k \pm 2 \\ \frac{1}{4} & k \pm \\ 0 & otherwise \end{cases}$$

Since the system is LTI:

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\begin{aligned} b_0 &= a_0 H([0][2]]) = 0 \\ b_1 &= a_1 H([1][2]]) = 0 \\ b_{-1} &= a_{-1} H([-1][2]]) = 0 \\ b_2 &= a_2 H([2][2]]) = 0 \\ b_{-2} &= a_{-2} H([-2][2]]) = 0 \\ b_3 &= a_3 H([3][2]]) = \frac{1}{4} \\ b_{-3} &= a_{-3} H([-3][2]]) = \frac{1}{4} \end{aligned}$$

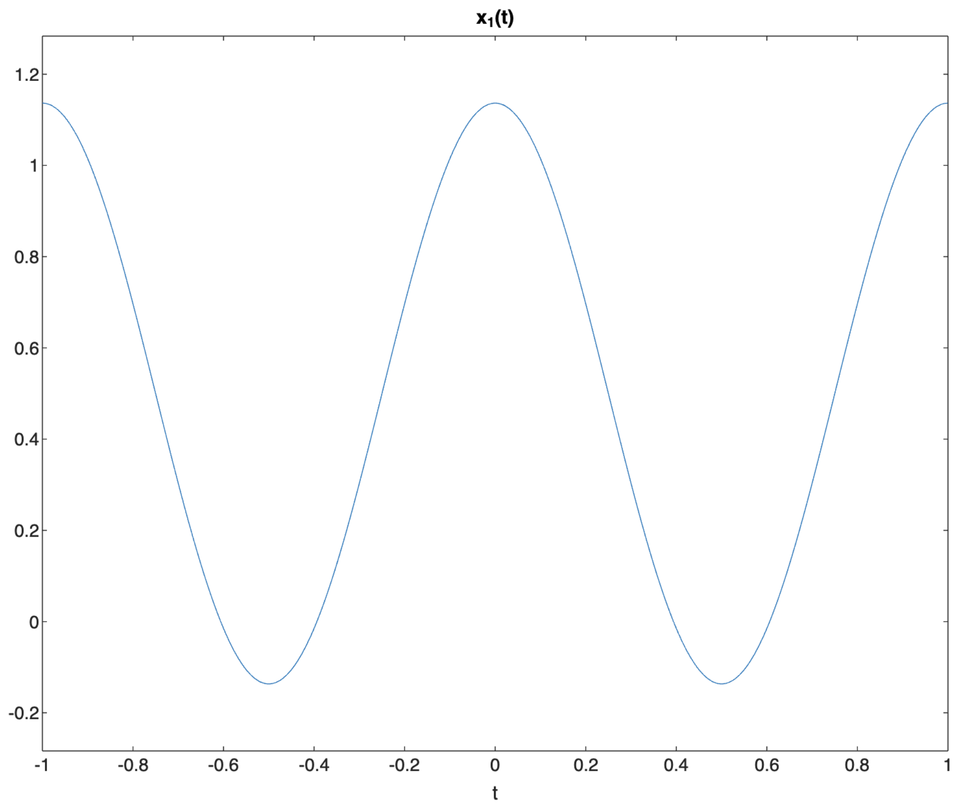
$$c_k = \begin{cases} \frac{1}{4} & k = \pm 3 \\ 0 & otherwise \end{cases}$$

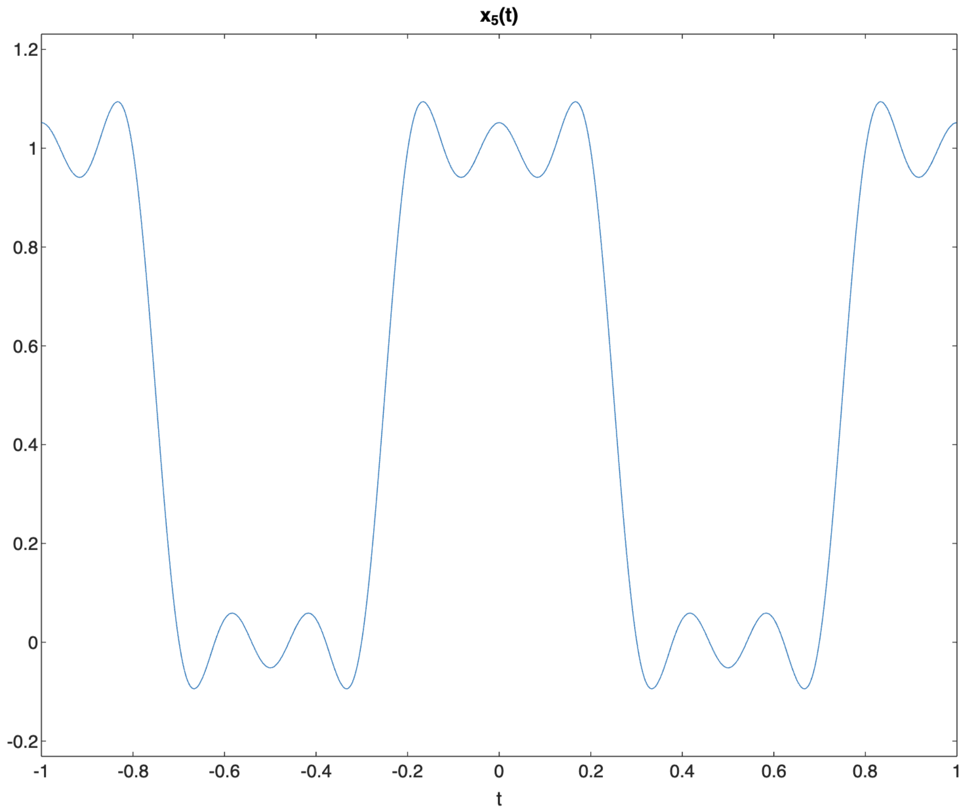
∴ the output for y is:

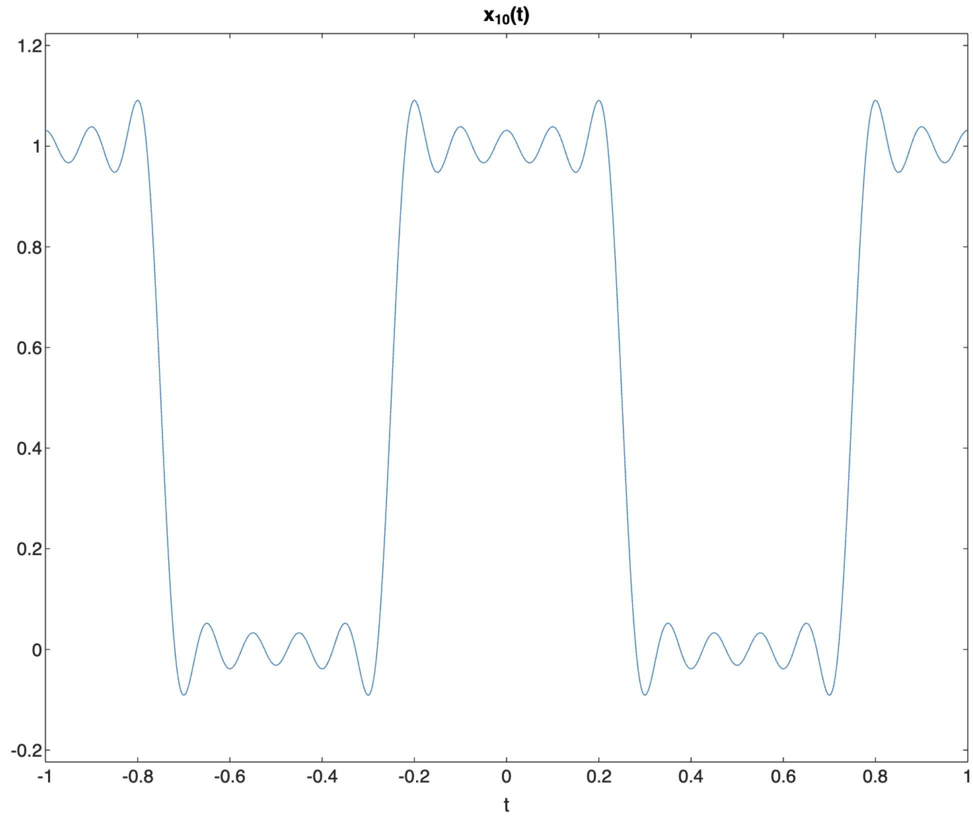
$$
\begin{aligned}
y(t) &= \frac{1}{4}e^{-j6t} + \frac{1}{4}e^{j6t} \\
&= \frac{1}{4}[e^{-j6t} + e^{j6t}] \\
&= \frac{1}{4}[2cos(6t)] \\
&= \frac{1}{2}cos(6t)
\end{aligned}
$$

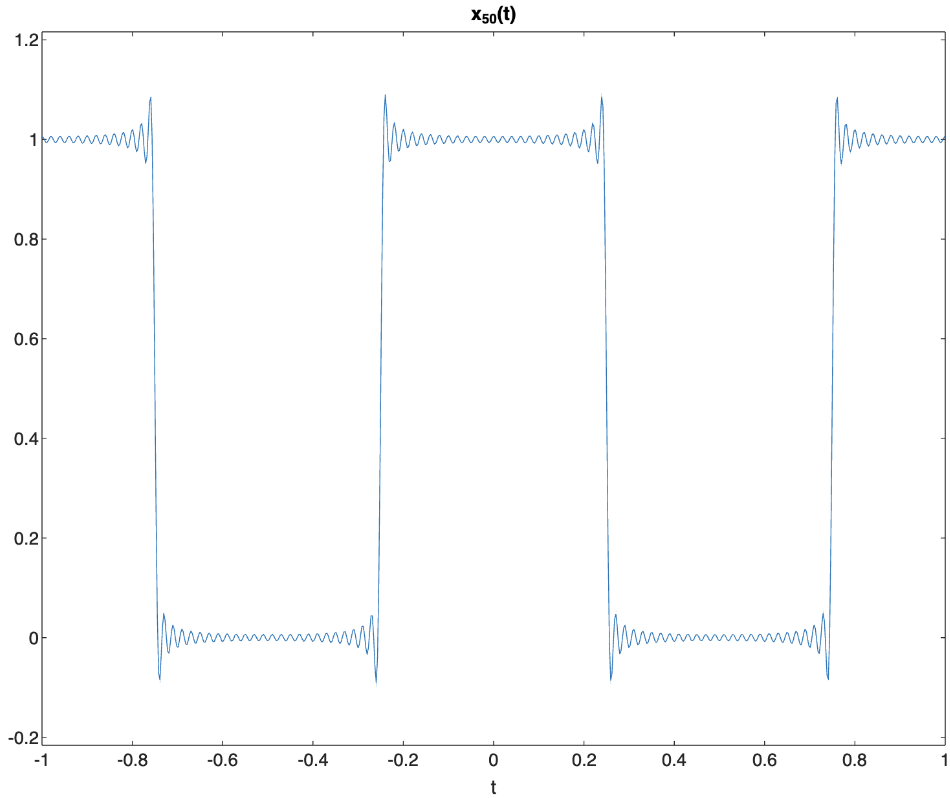
5.101 [Fourier series convergence]
(a)

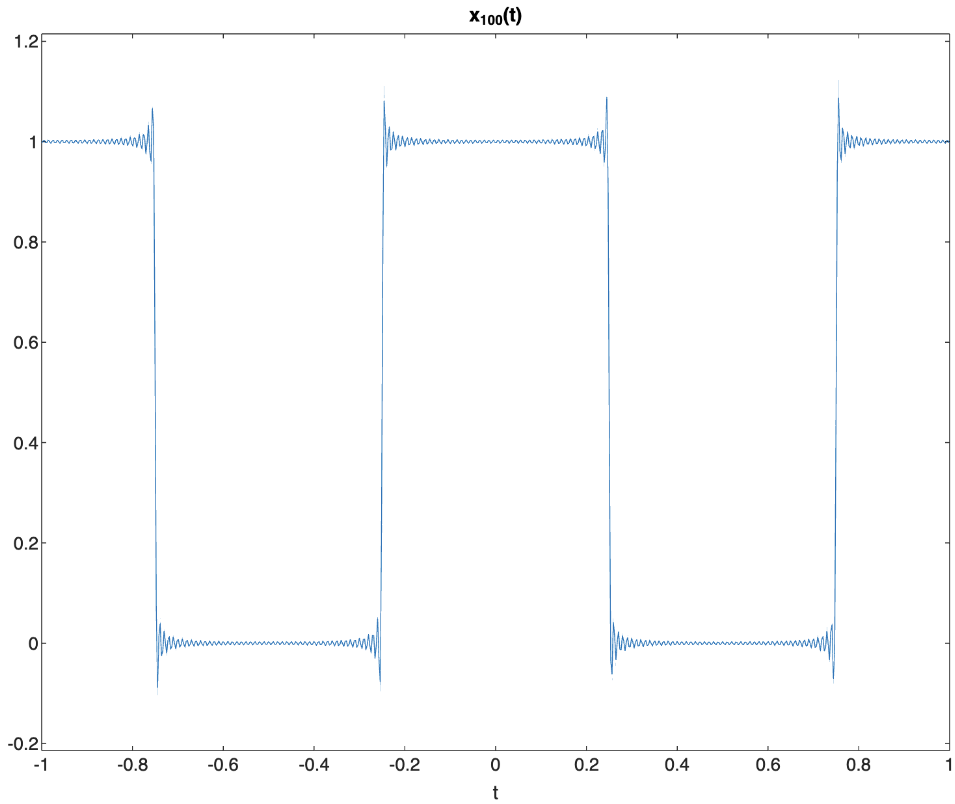
```
syms t;
delta = @(t) 1 - abs(-heaviside(-t) + heaviside(t));
mysinc = @(t) (sin(t) + delta(t)) / (t + delta(t));
A = 0.5;
syms k;
for nVal = [1 5 10 50 100]
    % op = -n
    f = symsum(0.5 * mysinc(pi / 2 * k)
        * exp(j * k * (2 * pi) * t), k, -n, n);
    ezplot(f, [-1 1]);
    title(sprintf('x_{%d}(t)', nVal));
    pause (2);
end
```


x1(t)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
t


x5(t)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
t


x10(t)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
t


$x_{50}(t)$
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
t



(b)
The function does not converge uniformly everywhere. The function converges at a lower rate at the points of discontinuity.

(c)
At the point of discontinuity of $x(t)$ at $t = \frac{1}{4}$ the function appears to converge at a value of $\frac{1}{2}$.