ECE 260: Continuous-Time Signals and Systems

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Assignment 5A

6.1 [find Fourier transform by first principles]
(c)

$$\begin{aligned}
x(t) &= 3[u(t) - u(t-2)] \\
X(\omega) &= \int_{-\infty}^{\infty} 3[u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{-\infty}^{\infty} [u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{0}^{2} e^{-j\omega t}dt \\
&= 3[\frac{1}{-j\omega}e^{-j\omega t}]\Big|_0^2 \\
&= \frac{3}{-j\omega}[e^{-j\omega t}]\Big|_0^2 \\
&= \frac{3}{-j\omega}[e^{-j\omega(2)} - e^{-j\omega(0)}] \\
&= \frac{j3}{\omega}[e^{-j2\omega} - 1] \\
&= \frac{j3}{\omega}[e^{-j\omega}][e^{-j\omega} - e^{j\omega}] \\
&= \frac{j3}{\omega}e^{-j\omega}[-2jsin\omega] \text{ eulers relation} \\
&= 6e^{-j\omega}sin\omega
\end{aligned}$$

(d)

$$
\begin{aligned}
x(t) &= e^{-|t|} \\
X(\omega) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{-|t|} e^{-j\omega t} + \int_{0}^{\infty} e^{-|t|} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t} + \int_{0}^{\infty} e^{(-1-j\omega)t} dt \\
&= \frac{1}{1-j\omega}[e^{(1-j\omega)t}]\Big|_{-\infty}^{0} - \frac{1}{1+j\omega}[e^{(-1-j\omega)t}]\Big|_{0}^{\infty} \\
&= \frac{1}{1-j\omega}[e^{(1-j\omega)(0)} - e^{(1-j\omega)(-\infty)}] - \frac{1}{1+j\omega}[e^{(-1-j\omega)(\infty)} - e^{(-1-j\omega)(0)}] \\
&= \frac{1}{1-j\omega}[1-0] - \frac{1}{1+j\omega}[0-1] \\
&= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \\
&= \frac{1+j\omega+1-j\omega}{(1+j\omega)(1-j\omega)} \\
&= \frac{2}{1+\omega^2}
\end{aligned}
$$

6.3 [find Fourier transform]
(c)

$$
\begin{aligned}
x(t) &= cos(t)u(t) \\
x(t) &= v_1(t)v_2(t) \\
X(\omega) &= \frac{1}{2\pi} V_1 * V_2(\omega) \\
V_1(\omega) &= \pi[\delta(\omega-1)+\delta(\omega+1)] \text{ Fourier Transform Pair 6} \\
V_2(\omega) &= \pi\delta(\omega) + \frac{1}{j\omega} \text{ Fourier Transform Pair 2}
\end{aligned}
$$

$$
\begin{aligned}
x(\omega) &= \frac{1}{2\pi} V_1 * V_2(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi[\delta(\lambda-1)+\delta(\lambda+1)][\pi\delta(\omega-\lambda) + \frac{1}{j(\omega-\lambda)}]d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi[\pi\delta(\lambda-1)\delta(\omega-\lambda) + \delta(\lambda-1)\frac{1}{j(\omega-\lambda)} + \pi\delta(\lambda+1)\delta(\lambda+1)\frac{1}{j(\omega-\lambda)}]d\lambda \\
&= \frac{1}{2}[\pi\delta(\omega-1) + \frac{1}{j(\omega-1)} + \pi\delta(\omega+1) + \frac{1}{j(\omega+1}] \\
&= \frac{1}{2}[\pi\delta(\omega-1) + \pi\delta(\omega+1) + \frac{-j(\omega-1)-j(\omega+1)}{\omega^2-1}] \\
&= \frac{1}{2}[\pi\delta(\omega-1) + \pi\delta(\omega+1) + \frac{-j\omega+j-j\omega-j}{\omega^2-1}] \\
&= \frac{\pi}{2}[\delta(\omega-1) + \delta(\omega+1) - \frac{j\omega}{\omega^2-1}]
\end{aligned}
$$

(d)

$$x(t) = 6[u(t) - u(t-3)]$$

Let $v_1(t) = rect(t)$, $v_2(t) = v_1(t - \frac{1}{2})$, and $v_3(t) = v_2(\frac{t}{3})$ so that $x(t) = 6v_3(t)$. Taking the Fourier transform, we have

$$\begin{aligned}
V_1(\omega) &= sinc(\frac{\omega}{2}) \text{ Fourier transform Pair 8} \\
V_2(\omega) &= e^{-j\omega/2} V_1(\omega) \text{ Time-Domain Shifting} \\
V_3(\omega) &= 3V_2(3\omega) \\
X(\omega) &= 6V_3(\omega)
\end{aligned}$$

Combining the Fourier transforms we get

$$\begin{aligned}
X(\omega) &= 6V_3(\omega) \\
&= 18V_2(3\omega) \\
&= 18e^{-3j\omega/2} V_1(3\omega) \\
&= 18e^{-3jw/2} sinc(\frac{3\omega}{2})
\end{aligned}$$

(e)

$$\begin{aligned}
x(t) &= \frac{1}{t} \\
sgn &\overset{CTFT}{\longleftrightarrow} \frac{2}{j\omega}
\end{aligned}$$

By the duality property:

$$\begin{aligned}
\mathcal{F}\{\frac{2}{jt}\}(\omega) &= 2\pi sgn(-\omega) \\
&= -2\pi sgn\omega \quad \text{since } sgn(-\omega) = -sgn(\omega)
\end{aligned}$$

By the linearity property:

$$\begin{aligned}
X(\omega) &= \mathcal{F}\{\frac{1}{t}\}(\omega) \\
&= \frac{j}{2}\mathcal{F}\{\frac{2}{jt}\}(\omega) \\
&= -j\pi sgn\omega
\end{aligned}$$

(f)

$$\begin{aligned}
x(t) &= t\, rect(2t) \\
x(t) &= tv_2(t) \\
v_1(t) &= rect(t) \\
v_2(t) &= v_1(2t) \\
V_1(\omega) &= sinc(\frac{\omega}{2}) \text{ Fourier Transform Pair 9} \\
V_2(\omega) &= \frac{1}{2}V_1(\frac{\omega}{2}) \text{ Time/Frequency-Domain Scaling} \\
X(\omega) &= j\frac{d}{d\omega}V_2(\omega) \text{ Frequency-Domain Differentiation}
\end{aligned}$$

$$
\begin{aligned}
X(\omega) &= j\frac{d}{d\omega}V_2(\omega) \\
&= \frac{j}{2}\frac{d}{d\omega}V_1(\frac{\omega}{2}) \\
&= \frac{j}{2}\frac{d}{d\omega}sinc(\frac{\omega}{4}) \\
&= \frac{j}{2}\Big[\frac{\frac{\omega}{4}(\frac{1}{4}cos(\frac{\omega}{4})) - \frac{1}{4}sin(\frac{\omega}{4})}{\frac{\omega^2}{16}}\Big] \\
&= \frac{j}{2}[\frac{1}{\omega}cos(\frac{\omega}{4}) - \frac{4}{\omega^2}sin(\frac{\omega}{4})] \\
&= \frac{j}{2\omega}cos(\frac{\omega}{4}) - \frac{j2}{\omega^2}sin(\frac{\omega}{4})
\end{aligned}
$$

Thus we have shown that:

$$t\ rect(2t) \underleftrightarrow{CTFT} \frac{j}{2\omega}cos(\frac{\omega}{4}) - \frac{j2}{\omega^2}sin(\frac{\omega}{4})$$

(g)

$$
\begin{aligned}
x(t) &= e^{-j3t}sin(5t-2) \\
x(t) &= e^{-j3t}v_3(t) \\
v_3(t) &= v_2(5t) \\
v_2(t) &= v_1(t-2) \\
v_1(t) &= sint
\end{aligned}
$$

$$
\begin{aligned}
V_1(\omega) &= \frac{\pi}{j}[\delta(\omega-1) - \delta(\omega+1)] \text{ Fourier Transform Pair 7} \\
V_2(\omega) &= e^{-j2\omega}V_1(\omega) \text{ Time-Domain Shifting} \\
V_3(\omega) &= \frac{1}{5}V_2(\frac{\omega}{5}) \text{ Time/Frequency-Domain Scaling} \\
X(\omega) &= V_3(\omega+3) \text{ Frequency-Domain Scaling}
\end{aligned}
$$

$$
\begin{aligned}
X(\omega) &= V_3(\omega+3) \\
&= \frac{1}{5}V_2(\frac{\omega+3}{5}) \\
&= \frac{1}{5}e^{-j2(\omega+3)/5}V_1(\frac{\omega+3}{5}) \\
&= -\frac{\pi}{j5}e^{-j2(\omega+3)/5}\Big[\delta(\frac{\omega-2}{5}) - \delta(\frac{\omega+8}{5})\Big] \\
&= -\frac{\pi}{j5}e^{-j2(\omega+3)/5}[5\delta(\omega-2) - 5\delta(\omega+8)] \\
&= j\pi[e^{-j2(\omega+3)/5}\delta(\omega+8) - e^{-j2(\omega+3)/5}\delta(\omega-2)] \\
&= j\pi([e^{-j2(\omega+3)/5}]\Big|_{\omega=-8}\delta(\omega+8) - [e^{-j2(\omega+3)/5}]\Big|_{\omega=2}\delta(\omega-2)) \\
&= j\pi[e^{j2}\delta(\omega+8) - e^{-j2}\delta(\omega-2)]
\end{aligned}
$$

Thus we have shown that:

$$e^{-j3t}sin(5t-2) \underleftrightarrow{CTFT} j\pi[e^{j2}\delta(\omega+8) - e^{-j2}\delta(\omega-2)]$$

6.4 [find Fourier transform]

(a)

$$y(t) = x(at - b), \text{ where } a \text{ and } b \text{ are constants and } a \neq 0$$

Let $v_1(t) = x(t-b)$ so that $y(t) = v_1(at)$. Taking the Fourier transform, we have

$$V_1(\omega) = e^{-j\omega b}X(\omega), \text{ Time-Domain Shifting}$$
$$Y(\omega) = \frac{1}{|a|}V_1(\frac{\omega}{a}), \text{ Time/Frequency-Domain Scaling}$$

Combining the Fourier transforms we get

$$\begin{aligned} Y(\omega) &= \frac{1}{|a|}V_1(\frac{\omega}{a}) \\ &= \frac{1}{|a|}e^{-j\omega b/a}X(\frac{\omega}{a}) \end{aligned}$$

(b)

$$y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$$

Let $v_1(t) = \int_{-\infty}^{t} x(\tau)d\tau$ so that $y(t) = v_1(2t)$. Taking the Fourier transform, we have

$$\begin{aligned} V_1(\omega) &= \mathcal{F}\Big\{\int_{-\infty}^{t} x(\tau)d\tau\Big\}(\omega) \\ &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) \text{ Time-Domain Integration} \\ Y(\omega) &= \mathcal{F}\{v_1(2t)\}(\omega) \\ &= \frac{1}{2}V_1(\frac{\omega}{2}) \text{ Time/Frequency-Domain Scaling} \end{aligned}$$

Combining the Fourier transforms we get

$$\begin{aligned} Y(\omega) &= \frac{1}{2}V_1(\frac{\omega}{2}) \\ &= \frac{1}{2}\Big[\frac{1}{j(\frac{\omega}{2})}X(\frac{\omega}{2}) + \pi X(0)\delta(\frac{\omega}{2})\Big] \\ &= \frac{1}{2j(\frac{\omega}{2})}X(\frac{\omega}{2}) + \frac{1}{2}(\pi X(0)\delta(\frac{\omega}{2})) \\ &= \frac{1}{j\omega}X(\frac{\omega}{2}) + \frac{\pi}{2}X(0)\delta(\frac{\omega}{2}) \end{aligned}$$

(c)

$$y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$$

Let $v_1(t) = x^2(t)$ so that $y(t) = \int_{-\infty}^{t} v_1(\tau)d\tau$. Taking the Fourier transform, we have

$$V_1(\omega) = \frac{1}{2\pi}X * X(\omega) \text{ Frequency-Domain Convolution}$$
$$Y(\omega) = \frac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega) \text{ Time-Domain Integration}$$

Combining the Fourier transforms we get

$$\begin{aligned}
Y(\omega) &= \frac{1}{j\omega} V_1(\omega) + \pi V_1(0)\delta(\omega) \\
&= \frac{1}{j\omega}\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda\Big] + \pi\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda\Big]\delta(\omega) \\
&= \frac{1}{j2\pi\omega}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda + \frac{1}{2}\delta(\omega)\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda
\end{aligned}$$

(d)

$$y(t) = \mathcal{D}(x * x)(t), \text{ where } \mathcal{D} \text{ denotes the derivative operator}$$

Let $v_1(t) = x * x(t)$ so that $y(t) = \frac{d}{dt} v_1(t)$. Taking the Fourier transform, we have

$$\begin{aligned}
V_1(\omega) &= \mathcal{F}\{x * x\}(\omega) \\
&= X^2(\omega) \text{ Time-Domain Convolution} \\
Y(\omega) &= \mathcal{F}\{\frac{d}{dt} v_1(t)\}(\omega) \\
&= j\omega V_1(\omega) \text{ Time-Domain Differentiation}
\end{aligned}$$

Combining the Fourier transforms we get

$$\begin{aligned}
Y(\omega) &= j\omega V_1(\omega) \\
&= j\omega X^2(\omega)
\end{aligned}$$

(e)

$$y(t) = tx(2t-1)$$

Let $v_2(t) = x(t-1)$ and $v_1(t) = v_2(2t)$ so that $y(t) = tv_1(t)$. Taking the Fourier transform, we have

$$\begin{aligned}
V_2(\omega) &= \mathcal{F}\{x(t-1)\}(\omega) \\
&= e^{-j\omega} X(\omega) \text{ Time-Domain Shifting} \\
V_1(\omega) &= \mathcal{F}\{v_2(2t)\}(\omega) \\
&= \frac{1}{2} V_2(\frac{\omega}{2}) \text{ Time/Frequency-Domain Scaling} \\
Y(\omega) &= \mathcal{F}\{tv_1(t)\}(\omega) \\
&= j\frac{d}{d\omega} V_1(\omega) \text{ Frequency-Domain Differentiation}
\end{aligned}$$

Combining the Fourier transforms we get

$$\begin{aligned}
Y(\omega) &= j\frac{d}{d\omega} V_1(\omega) \\
&= j\frac{d}{d\omega}[\frac{1}{2} V_2(\frac{\omega}{2})] \\
&= j\frac{d}{d\omega}[\frac{1}{2} e^{-j\omega/2} X(\frac{\omega}{2})] \\
&= \frac{j}{2}[\frac{d}{d\omega} e^{-j\omega/2} X(\frac{\omega}{2})]
\end{aligned}$$

(f)

$$y(t) = e^{j2t}x(t-1)$$

Let $v_1(t) = x(t-1)$ so that $y(t) = e^{j2t}v_1(t)$. Taking the Fourier transform, we have

$$\begin{aligned} V_1(\omega) &= \mathcal{F}\{x(t-1)\}(\omega) \\ &= e^{-j\omega}X(\omega) \text{ Time Domain Shifting} \\ Y(\omega) &= \mathcal{F}\{e^{j2t}v_1(t)\}(\omega) \\ &= V_1(\omega-2) \text{ Frequency-Domain Shifting} \end{aligned}$$

Combining the Fourier transforms we get

$$\begin{aligned} Y(\omega) &= V_1(\omega-2) \\ &= e^{-j(\omega-2)}X(\omega-2) \end{aligned}$$

6.5 [find Fourier transform of periodic signal]
(a)

$$T = 4,\ \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$
$$c_k = -\delta(t+1) + \delta(t-1),\ \text{where } c_k \text{ represents the period between } -2 \leq t < 2$$

Taking the Fourier transform, we have

$$\begin{aligned} C_k(\omega) &= \mathcal{F}\{\delta(t-1) - \delta(t+1)\}(\omega) \\ &= \mathcal{F}\{\delta(t-1)\}(\omega) - \mathcal{F}\{\delta(t+1)\}(\omega) \\ &= e^{-j\omega} - e^{j\omega} \\ &= -2jsin\omega \end{aligned}$$

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 C_k(k\omega_0)\delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} -j\pi(sin(\frac{\pi k}{2}))\delta(\omega - \frac{\pi k}{2}) \end{aligned}$$

6.10 [find frequency/magnitude/phase spectrum]
(a)

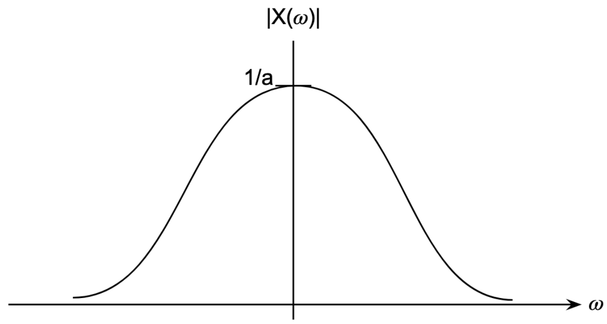
$$x(t) = e^{-at}u(t),\ \text{where } a \text{ is a positive real constant}$$

Taking the Fourier transform, we have

$$\begin{aligned} x(\omega) &= \mathcal{F}\{e^{-at}u(t)\}(\omega) \\ &= \frac{1}{a+f\omega} \text{ Fourier Transform Pair 10 where and } Re\{a\} > 0 \end{aligned}$$

Magnitude:

$$\begin{aligned} |X(\omega)| &= |\frac{1}{a+j\omega}| \\ &= \frac{1}{\sqrt{a^2+\omega^2}} \end{aligned}$$



Phase Spectrum:

$$
\begin{aligned}
argX(\omega) &= arg\Big[\frac{1}{a+j\omega}\Big] \\
&= arg1 - arg(a+j\omega) \\
&= -arg(a+j\omega) \\
&= -tan^{-1}\frac{\omega}{a}
\end{aligned}
$$