ECE 260: Continuous-Time Signals and Systems

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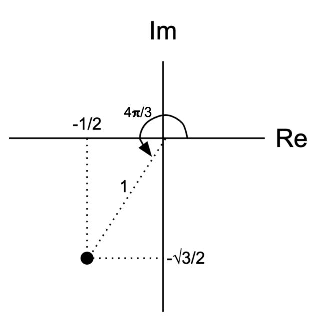
Assignment 1

A.1 (c) [**convert to cartesian form**]

$$z = 2e^{j7\pi/6}$$
$$z = x + jy$$
$$x = 2cos\frac{7\pi}{6}, y = 2sin\frac{7\pi}{6}$$
$$z = -\sqrt{3} - j$$

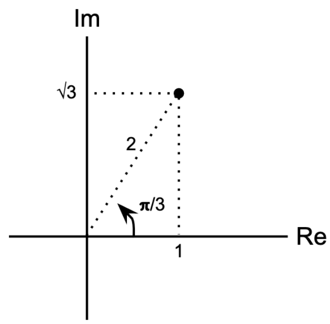
A.2 (b) [**convert to polar form, principal argument**]

$$\frac{-1}{2} - j\frac{\sqrt{3}}{2}$$
$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(-\frac{1}{2})^2(\frac{\sqrt{3}}{2})^2} = 1$$
$$argz = arctan(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}) + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$
$$z = 1e^{j4\pi/3}$$



A.2 (d)

$$
\begin{aligned}
1 + j\sqrt{3} \\
|z| = \sqrt{x^2 + y^2} \\
|z| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2 \\
argz = arctan(\frac{\sqrt{3}}{1}) = \frac{\pi}{3} \\
z = 2e^{j\pi/3}
\end{aligned}
$$



A.3 (a)[**complex arithmetic**]

$$
\begin{aligned}
&2(\frac{\sqrt{3}}{2} - j\frac{1}{2}) + j(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}) \text{ in Cartesian form} \\
&= \sqrt{3} - j + j(\frac{1}{\sqrt{2}}cos(\frac{-3\pi}{4}) + j\frac{1}{\sqrt{2}}sin(\frac{-3\pi}{4})) \\
&= \sqrt{3} - j + j((\frac{1}{\sqrt{2}})(\frac{-\sqrt{2}}{2}) + j(\frac{1}{\sqrt{2}})(\frac{-\sqrt{2}}{2})) \\
&= \sqrt{3} - j + j(-\frac{1}{2} - \frac{1}{2}j) \\
&= \sqrt{3} - j - \frac{1}{2}j + \frac{1}{2} \\
&= \frac{2\sqrt{3} + 1}{2} - j\frac{3}{2}
\end{aligned}
$$

(b)

$$(\frac{\sqrt{3}}{2} - j\frac{1}{2}) + j(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}) \text{ in polar form}$$

$$|z| = \sqrt{x^2 + y2}$$
$$|z| = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2} = 1$$
$$argz = arctan(\frac{-1/2}{\sqrt{3}/2}) = -\frac{\pi}{6}$$

$$\begin{aligned}(\frac{\sqrt{3}}{2} - j\frac{1}{2}) + j(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}) &= (1e^{j(-\pi/6)})(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}) \\ &= \frac{1}{\sqrt{2}}e^{j(\frac{-\pi}{6}+\frac{-3\pi}{4})} \\ &= \frac{1}{\sqrt{2}}e^{j(-11\pi/12)}\end{aligned}$$

(f)

$$(1 + j)^{10} \text{ in cartesian form}$$

$$|z| = \sqrt{x^2 + y2}$$
$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$argz = arctan(\frac{1}{1}) = \frac{\pi}{4}$$

$$\begin{aligned}(1 + j)^{10} &= [\sqrt{2}cos(\frac{\pi}{4}) + \sqrt{2}jsin(\frac{\pi}{4})]^{10} \\ &= (\sqrt{2})^{10}[cos(\frac{\pi}{4}(10) + jsin(\frac{\pi}{4}(10)))] \\ &= 32[0 + j] \\ &= 32j\end{aligned}$$

(g)

$$= \frac{1+j}{1-j} \text{ in polar form}$$
$$|z| = \sqrt{x^2 + y2}$$
$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$argz = arctan(\frac{1}{1}) = \frac{\pi}{4}$$

$$argz = arctan(\frac{-1}{1}) = \frac{-\pi}{4}$$

$$\frac{1+j}{1-j} = \frac{\sqrt{2}e^{j(\pi/4)}}{\sqrt{2}e^{j(-\pi/4)}}$$
$$= e^{j(\frac{\pi}{4} - (-\frac{\pi}{4}))}$$
$$= e^{j\frac{\pi}{2}}$$

A.4 (b) **[properties of complex numbers]**

$$arg(\frac{z_1}{z_2}) = arg_{z_1} - arg_{z_2}, \text{ for } z \neq 0$$
$$\text{let } z_1 = r_1 e^{j\theta_1}$$
$$\text{let } z_2 = r_2 e^{j\theta_2}$$
$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}$$
$$= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$
$$arg(\frac{z_1}{z_2}) = \theta_1 - \theta_2$$
$$= argz_1 - argz_2$$

(e)

$$\begin{aligned}
(z_1z_2)^* &= z_1^*z_2^* \\
\text{let } z_1 &= x_1 + jy_1 \\
\text{let } z_2 &= x_2 + jy_2
\end{aligned}$$

$$\begin{aligned}
(z_1z_2) &= (x_1 + jy_1)(x_2 + jy_2) \\
&= (x_1x_2 + jx_1y_2 + jx_2y_1 - y_1y_2) \\
&= (x_1x_2 = y_1y_2) + j(x_1y_2 + x_2y_1) \\
(z_1z_2)^* &= (x_1x_2 - y_1y_2) - j(x_1y_2 + x_2y_1)
\end{aligned}$$

$$\begin{aligned}
z_1^*z_2^* &= (x_1 - jy_1)(x_2 - jy_2) \\
&= (x_1x_2 - jx_1y_2 - jx_2y_1 - y_1y_2) \\
&= (x_1x_2 - y_1y_2) - j(x_1y_2 + x_2y_1)
\end{aligned}$$

$$\therefore (z_1z_2)^* = z_1^*z_2^*$$

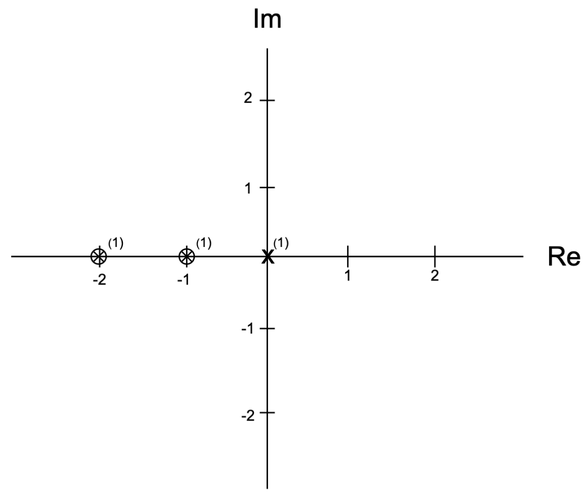
A.5 (b) [**Euler's relation**]

$$\begin{aligned}
sin\theta &= \frac{1}{-2j}[e^{j\theta} - e^{-j\theta}] \\
&= \frac{1}{2j}[cos\theta + jsin\theta - (cos\theta - jsin\theta)] \\
&= \frac{1}{2j}(2jsin\theta) \\
&= sin\theta
\end{aligned}$$

A.6 (b) [**poles/zeros**]

$$\begin{aligned}
r(z) &= z + 3 + 2z^{-1} \\
&= z + 3 + \frac{2}{z} \\
r(-1) &= -1 + 3 + \frac{2}{-1} \\
&= 0 \\
r(-2) &= -2 + 3 + \frac{2}{-2} \\
&= 0
\end{aligned}$$

first order zeros at -1 and -2
first order pole at 0



(c)

$$F(z) = \frac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$$

factor the numerator polynomials:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= -1 \pm 2j \\ &= \{-1+2j, -1-2i\} \\ &= (z+1+2j)(z+1-2j) \end{aligned}$$

$$z^2+1 = (z+j)(z-j)$$

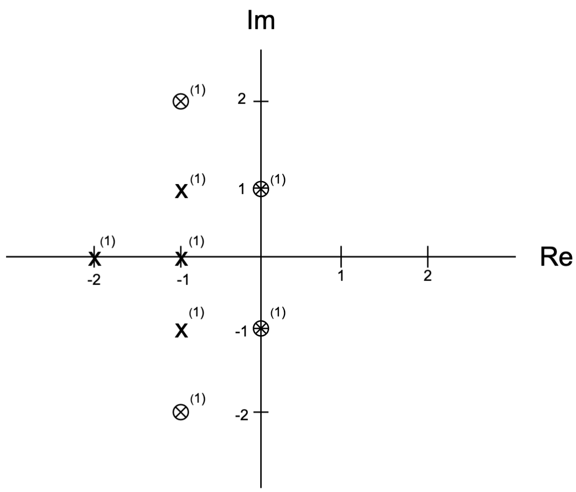
factor the numerator polynomials:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2-4(1)(2)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2j}{2} \\ &= -1 \pm j \\ &= \{-1+j, -1-j\} \\ &= (z+1+j)(z+1-j) \end{aligned}$$

$$z^2+3z+2 = (z+2)(z+1)$$

$$f(z) = \frac{(z+1+2j)(z+1-2j)(z+j)(z-j)}{(z+1+j)(z+1-j)(z+2)(z+1)}$$

First order zeros at $-1-2j, -1+2j, -j, j$
First order poles at $-1-j, -1+j, -2, -1$



A.7 (c) [**continuity, differentiability, analyticity**]

$$\frac{z}{z^4-16}$$
$$= \frac{z}{(z^4+4)(z+2)(z-2)}$$

i) $f(z)$ is continuous everywhere except at $z = \pm 2, \pm 2j$. This is because the denominator goes to zero at these points.

ii)

$$\begin{aligned} f'(z) &= \frac{\frac{d}{dz}(z)\cdot(z^4-16) - z\cdot\frac{d}{dz}[z^4-16]}{(z^4-16)^2} \\ &= \frac{z^4-16-z(4z^3)}{(z^4-16)^2} \\ &= \frac{-3x^4-16}{(z^4-16)^2} \end{aligned}$$

$Ff(z)$ is differentiable everywhere except at $z = \pm 2, \pm 2j$. This is because the denominator goes to zero at these points.

iii) $f(z)$ is analytic everywhere except at $z = \pm 2, \pm 2j$. This is because the denominator goes to zero at these points.

(d)

$$
\begin{aligned}
f'(z) &= z + 2 + z^{-1} \\
&= z + 2 + \frac{1}{2}
\end{aligned}
$$

i) $f(z)$ is continuous everywhere except at $z = 0$. This is because the denominator goes to zero at that point.

ii)

$$
\begin{aligned}
f'(z) &= \frac{d}{dz}(z) + \frac{d}{dz}(2) + \frac{d}{dz}(z^{-1}) \\
&= 1 + (-\frac{1}{z^2}) \\
&= 1 - \frac{1}{z^2}
\end{aligned}
$$

$f(z)$ is differentiable everhwere except at $z = 0$. This is because the denominator goes to zero at that point.

iii) $f(z)$ is analytic everywhere except at $z = 0$. This is because the denominator goes to zero at that point.

A.9 (c) **[magnitude/argument]**

$$
f(\omega) = \frac{2e^{j11\omega}}{(3 + j5\omega)^7}
$$

$$
\begin{aligned}
\theta &= arctan(\frac{5\omega}{3}) \\
|x| &= \sqrt{3^2 + 5\omega^2} \\
&= \sqrt{9 + 25\omega^2}
\end{aligned}
$$

$$
\begin{aligned}
|f(\omega)| &= |\frac{2e^{j11\omega}}{(3 + j5\omega)^7}| \\
&= \frac{|2e^{j11\omega}|}{|3 + j5\omega|^7} \\
&= \frac{2}{(\sqrt{3^2 + (5\omega)^2})^7} \\
arg\, f(\omega) &= arg(\frac{2e^{j11\omega}}{(3 + j5\omega)^7}) \\
&= arg(2e^{j11\omega}) - arg(3 + j5\omega)^7 \\
arg\, f(\omega) &= 11\omega - 7arctan(\frac{5\omega}{3})
\end{aligned}
$$

(f)

$$f(\omega) = \frac{j\omega - 1}{j\omega + 1}$$

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + \omega^2} \\ &= \sqrt{1 + \omega^2} \end{aligned}$$

$$\theta = \pi + arctan(\frac{\omega}{-1}) = \pi + arctan(-\omega)$$

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(1)^2 + \omega^2} \\ &= \sqrt{1 + \omega^2} \end{aligned}$$

$$\theta = arctan(\frac{\omega}{1}) = arctan(\omega)$$

$$\begin{aligned} f(\omega) &= \frac{\sqrt{1+\omega^2}e^{j(\pi + arctan(-\omega)}}{\sqrt{1+\omega^2}e^{j(arctan(\omega)}} \\ &= 1ej^{[\pi + arctan(-\omega) - arctan(\omega)]} \end{aligned}$$

$$|f(\omega)| = 1; arg f(\omega) = \pi + arctan(-\omega) - arctan(\omega)$$