Writing Assignment 2 in LaTeX

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Question 1 Let $a_0, a_1, ...$ be the sequence recursively defined by

$$a_0 = 4, a_1 = 7, \text{and } a_n = a_{n-1} + 6a_{n-2} \text{ for } n \geq 2.$$

prove that, the smallest positive integer n_0 such that $n! > n^3$ for all $n \geq 0$.

Basis: By definition, $a_0 = 4 = (-2)^0 + 3^1$, $a_1 = 7 = (-2)^1 + 3^2$, and $a_2 = 31 = (-2)^2 + 3^3$ Therefore, the statement that $a_n = (-2)^n + 3^{n+1}$ is true when n = 0, 1 or 2.

Induction Hypothesis: Suppose there is an integer $k \geq 2$ such that $a_n = (-2)^n + 3^{n+1}$ for n = 0, 1,...,k.

Induction Step: We want to show that the statement is true when n=k+1, that is that $a_{k+1} = (-2)^{k+1} + 3^{k+2}$. Look at a_{k+1}. Since $k \geq 2$ we know that $k + 1 \geq 3$ and so.

$$\begin{aligned}
a_{k+1} &= a_k + 6a_{k+1} \\
&= (-2)^k + 3^{k+1} + 6((-2)^{k-1} + 3^k) \text{ (since } a_k = (-2^k) + 3^{k+1} \text{ by the induction Hypothesis)} \\
&= (-2)^k + 3^{k+1} + 6(-2)^{k-1} + 6(3)^k \\
&= (-2)^k + (1 + 6(-2)^{-1} + 3^k(3 + 6(1)) \\
&= (-2)^k(-2) + 3^k(9) \\
&= (-2)^{k+1} + 3^k 3^2 \\
&= (-3)^{k+1} + 3^{k+2}
\end{aligned}$$

Conclusion: By strong Principles of Mathematical Induction, $a_n = (-2)^n + 3^{n+1}$ for all $n \geq 0$. $\square$

Question 2 Find, with proof, the smallest positive integer n_0 such that $n! > n^3$ for all $n \geq n_0$.

From the equation $n! \geq n^3$ for all $n \geq n_0$, states that there must be some positive integer n_0 such that any integer n that is greater than or equal to n_0 there will be n! which will always be greater than n^3. We can first start proving this by finding values of n! and n^3 for the first few positive integers at increments of one.

$$
\begin{array}{rr}
n! & n^3 \\
1! = 1 & 1^3 = 1 \\
2! = 2 & 2^3 = 8 \\
3! = 6 & 3^3 = 27 \\
4! = 24 & 4^3 = 64 \\
5! = 120 & 5^3 = 125 \\
6! = 720 & 6^3 = 216
\end{array}
$$

It can be observed that 6 is the first positive integer at which $n! > n^3$. We must now prove that for all $n > 6$ the statement is true.

Basis: When $n_0 = 6$ we have $n! = 6! = 720$ and $n^3 = 6^3 = 216$. Hence the statement to be proved is true when $n_0 = 6$. Let the statement P(n) be defined as $P(n) : n! > n^3$ for $n \geq 6$.

induction Hypothesis: Suppose there exists and integer $P(k) : k! \geq k^3$.

Induction Step: We need to prove that P(k+1) is true, we have

$$
\begin{aligned}
&(k+1)! = (k+1)k! > (k+1)k^3 \text{ (Since } k! > k^3) \\
&(k+1)! > (k+1)k^3 \\
&\text{now } k^3 > (k+1)^2, k \geq 3 \\
&(k+1)! > (k+1) \cdot (k+1)^2 \\
&(k+1)! > (k+1)^3
\end{aligned}
$$

Thus, P(k+1) is true, whenever P(k) is true

Conclusion: Therefore, by Principle of Mathematical Induction $n! \geq n^3$ for all $n \geq 6$ and so the smallest positive integer n_0 is 6. □

Question 3 let $t_0, t_1, ...$ be the sequence recursively defined by $t_0 = 5$, and $t_n = t_{n-1} + 2n + 5$ for $n \geq 1$. Compute t_1, t_2, t_3 and $t_n = t_{n-1} + 2n + 5$ for $n \geq 1$. Compute t_1, t_2, t_3 and t_4 (leave your answers as sums), and then use your work to conjecture a formula for $t_n, n \geq 0$. Then, prove by induction that your conjectured formula holds for all $n \geq 0$.

We first use computation without simplification to look for a pattern that we can conjecture a formula.

$$
\begin{aligned}
&t_0 = 5 \\
&t_1 = t_0 + 2 \cdot 1 + 5 = 5 + 2 \cdot 1 + 5 \\
&t_2 = t_1 + 2 \cdot 1 + 5 = 5 + 2 \cdot 1 + 5 + 2 \cdot 2 + 5 \\
&t_3 = t_2 + 2 \cdot 1 + 5 = 5 + 2 \cdot 1 + 5 + 2 \cdot 2 + 5 + 2 \cdot 3 + 5 \\
&t_4 = t_3 + 2 \cdot 1 + 5 = 5 + 2 \cdot 1 + 5 + 2 \cdot 2 + 5 + 2 \cdot 3 + 5 + 2 \cdot 4 + 5 \\
&\quad = 5 + 2 + 5 + 4 + 5 + 6 + 5 + 8 + 5
\end{aligned}
$$

At this point is seems reasonable to conjecture that

$$
\begin{aligned}
t_n &= 5(n+1) + (2+4+6+8+...+2n) \\
t_n &= 5(n+1) + 2(1+2+3+6+...+n)
\end{aligned}
$$

for all $n \geq 0$. We know that the bracketed expression is a known sum, so our conjecture can be written as $t_n = 5(n+1) + 2n(n+1)/2 = 5(n+1) + n(n+1) = 5n + 5 + n^2 + n = n^2 + 6n + 5$ for all $n \geq 0$.

We can now prove the conjecture by induction. The statement to prove is $n^2 + 6n + 5$ for all $n \geq 0$.

Basis: When $n = 0$ we have $a_n = a_0 = 5$ and $n^2 + 6n + 5 = 0^2 + 6(0) + 5 = 5$, Thus the statement is true when $n = 0$.

Induction Hypothesis: Suppose there is an integer $n \geq 0$ such that $t_k = k^2 + 6k + 5$.

Induction Step: We want to show that $t_{k+1} = (k+1)(k+1)+6(k+1)+5 = k^2+2k+1+6k+6+5 = k^2+8k+12$. Look at t_{k+1}. Since $k+1 \geq 1$, we can use the recursion to write

$$
\begin{aligned}
t_{k+1} &= t_k + 2(k+1) + 5 \\
&= k^2 + 6k + 5 + 2(k+1) + 5 \text{ (since } t_k = k^2 + 6k + 5\text{by the Induction Hypothesis)} \\
&= k^2 + 6k + 5 + 2k + 2 + 5 \\
&= k^2 + 8k + 12
\end{aligned}
$$

as wanted.

Conclusion: Therefore, by Principle of Mathematical Induction $t_n = n^2 + 6n + 5$ for all $n \geq 0$. $\square$