1 [30pt] Model Predictive Control

Consider a double integrator

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k \tag{1}$$

where T is the sample time. We want the system to track a reference trajectory $\mathbf{r} = \{r_0, r_1, \ldots, r_{2N}\}$. The run-time cost is $l_k(x_k, u_k) = \frac{1}{2}(x_k - r_k)^T Q (x_k - r_k) + \frac{1}{2} u_k^T R u_k$ where $Q \in \mathbb{R}^{2\times 2}$ and $R \in \mathbb{R}$ are both positive definite. There is no discount in the system. Let the planning horizon for each MPC step be N and the terminal cost be $l_N(x_N, u_N) = \frac{1}{2}(x_N - r_N)^T S (x_N - r_N)$ where $S \in \mathbb{R}^{2\times 2}$ is positive definite. For all k, there is a constraint on the state $x_k \in [-\bar{x}, \bar{x}]$ for $\bar{x} > 0$ and a constraint on the control $u_k \in [-\bar{u}, \bar{u}]$ for $\bar{u} > 0$.

1.1 [5pt] Write down the MPC problem as a quadratic programming problem in the standard form.

1.2 [5pt] For $T = 0.1s$, $N = 10$, $\mathbf{r} \equiv 0$, $\bar{x} = [5, 5]^T$ and $\bar{u} = 1$. Find the feasible state such that there is a solution for the QP problem in 1.1. Mark the area on a 2D plane.

1.3 [10pt] For $T = 0.1s$, $\mathbf{r} \equiv 0$, $\bar{x} = [5, 5]^T$ and $\bar{u} = 1$. Find the biggest set in the state space that is persistently feasible. Mark the area on a 2D plane. [Hint: Maximal control invariant set. Use the code `www.mpc.berkeley.edu/mpc-course-material/bookexamples/Cinf.m`]

1.4 [10pt] For $x_0 = [2, 0]^T$, $T = 0.1s$, $Q = I$, $R = 0.1$, $S = 10I$, $N = 10$, $\bar{x} = [5, 5]^T$ and $\bar{u} = 1$. Consider the reference trajectory: $r_k = \frac{N-k}{N} x_0$ for $k \leq N$ and $r_k = 0$ for $k \geq N$. Obtain the optimal MPC trajectory. Show the predicted and executed trajectories on a same figure.

2 [20pt] Iterative Learning Control

Suppose there is some unknown disturbance applied on the system

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} (u_k + w_k) \tag{2}$$

Let's use a feedforward controller to minimize the tracking error in the previous problem 1.4.

2.1 [2pt] Randomly generate a sequence of w_k and simulate the performance of the MPC controller you developed in 1.4. Write down the sequence and plot the simulation result including the state trajectory and the control trajectory. (Record this sequence of w_k, it will be used in the following problems.)

2.2 [2pt] Denote the optimal MPC trajectory from 1.4 as x_k^*. Denote the executed trajectory in 2.1 as $x_k^{(1)}$. Compute the error $E^{(1)} := [(x_1^* - x_1^{(1)})^T, (x_2^* - x_2^{(1)})^T, \ldots, (x_N^* - x_N^{(1)})^T]^T$. Plot the error $E^{(1)}$.

2.3 [10pt] Suppose w_k is the same for every trial. According to the error in 2.2, design the learning gain $L_k \in \mathbb{R}^{1\times 2N}$ for the new control law

$$u_k = u_k^{MPC} + L_k E^{(1)} \tag{3}$$

to reduce the error $E^{(2)} := [(x_1^* - x_1^{(2)})^T, (x_2^* - x_2^{(2)})^T, \dots, (x_N^* - x_N^{(2)})^T]^T$, where $x_k^{(2)}$ is the executed trajectory with the new control law, u_k^{MPC} is the feedback control signal generated by MPC. For simplicity, ignore the control constraint in the new control law. Plot $E^{(1)}$ and $E^{(2)}$ on the same graph to show a comparison.

2.4 [6pt] Design the feedforward learning gain such that the error converges to 0 after several iterations. Simulate and plot the resulting errors in different iterations to show convergence. [Hint: The lifted system is not controllable, but stabalizable.]

3 [25pt] Linear Quadratic Gaussian

Now suppose we cannot directly access the state value. We are facing a stochastic system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} (u_k + w_k) && (4) \\ y_k &= x_k + v_k && (5) \end{aligned}$$

where $w_k \sim \mathcal{N}(0, W)$ and $v_k \sim \mathcal{N}(0, V)$ are Gaussian white i.i.d. The noises are all independent from each other. The run-time cost is the same as in problem 1. $x_0 = [2, 0]^T$. $W = 0.0001I$, $\Sigma_0 = 0.001I$ and $V = 0.00001I$.

3.1 [2pt] For $T = 0.1s$, $Q = I$, $R = 0.1$, write down the steady state Riccati equation for infinite horizon LQR with the deterministic system (1).

3.2 [2pt] Solve the Riccati equation for P.

3.3 [2pt] What is the optimal control law $u_k = g(x_k)$ for the deterministic system (1)?

3.4 [2pt] Simulate the deterministic control law $u_k = g(x_k)$ on the stochastic system and plot the state and control trajectories.

3.5 [5pt] Derive a Kalman Filter for the stochastic system. 1) What is the dynamic update? 2) What is the measurement update?

3.6 [5pt] Write down the steady state Riccati equation for the Kalman Filter and solve for the steady state covariance and the steady state Kalman Filter gain.

3.7 [7pt] Apply the following control law on the stochastic system:

$$\begin{aligned} u_k &= g(\hat{x}_k) && (6) \\ \hat{x}_k &= \mathrm{KF}(\hat{x}_{k-1}, u_{k-1}, y_k) && (7) \end{aligned}$$

where g is the deterministic control law we obtained in 3.4 and KF is the Kalman filter we obtained in 3.6. Simulate the resulting trajectory and compare with the trajectory obtained in 3.4 by plotting them on a same figure. Discuss the difference between two trajectories.

4 [25pt] Parameter Adaptation

The discretized double integer model is actually not very accurate, especially when T is big. We need to run online identification of the system model to improve accuracy. Suppose we want to identify a, b, c in the following model:

$$x_{k+1} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} b \\ c \end{bmatrix} (u_k + w_k) \tag{8}$$

$$y_k = x_k + v_k \tag{9}$$

4.1 [5pt] Augment the state space to include the unknown parameters. Write down the resulting dynamics.

Now let us assume the ground truth values are $a = 0.12, b = 0.008, c = 0.11$. But those are unknown to the controller. The initial estimation made by the controller is $\hat{a}_0 = 0.1, \hat{b}_0 = 0.005, \hat{c}_0 = 0.1$. The initial state is $x_0 = [2, 0]^T$. The noise models follow from problem 3.

4.2 [10pt] Apply the following control law on the uncertain stochastic system:

$$u_k = g(\hat{x}_k, \hat{a}_k, \hat{b}_k, \hat{c}_k) \tag{10}$$

$$\hat{x}_k, \hat{a}_k, \hat{b}_k, \hat{c}_k = \text{EKF}(\hat{x}_{k-1}, \hat{a}_{k-1}, \hat{b}_{k-1}, \hat{c}_{k-1}, u_{k-1}, y_k) \tag{11}$$

where g is the deterministic LQR control law with respect to the estimated dynamics; EKF is the Extended Kalman Filter on the augmented system for simultaneous state and parameter estimation. Write down all update equations. Simulate the system for 1000 steps and plot the trajectories of u_k, $\hat{x}_k$, $\hat{a}_k$, $\hat{b}_k$, $\hat{c}_k$, x_k, and y_k.

4.3 [10pt] Apply the following control law on the uncertain stochastic system:

$$u_k = g(\hat{x}_k, \hat{a}_k, \hat{b}_k, \hat{c}_k) \tag{12}$$

$$\hat{x}_k, \hat{a}_k, \hat{b}_k, \hat{c}_k = \text{UKF}(\hat{x}_{k-1}, \hat{a}_{k-1}, \hat{b}_{k-1}, \hat{c}_{k-1}, u_{k-1}, y_k) \tag{13}$$

where g is the deterministic LQR control law with respect to the estimated dynamics; UKF is the Unscented Kalman Filter on the augmented system for simultaneous state and parameter estimation. Write down all update equations. Simulate the system for 1000 steps and plot the trajectories of u_k, $\hat{x}_k$, $\hat{a}_k$, $\hat{b}_k$, $\hat{c}_k$, x_k, and y_k.