16-899 Assignment 1

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1 Part I - System

1.1

Yes, the system is control-affine. The system is linear in control

1.2

Continuous Time, Time-Invariant, Deterministic, Continuous State

1.3

$$x_e = \begin{bmatrix} P_x^r \\ P_y^r \\ V^r \\ \theta^r \end{bmatrix}$$

$$z = x - x_e \therefore \dot{z} = \dot{x}$$

$$f(x,u) = \begin{bmatrix} x_3 cos x_4 \\ x_3 sin x_4 \\ u_1 \\ u_2 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial z}|_{x=x_e} = \begin{bmatrix} 0 & 0 & cos z_4 & -z_3 sin z_4 \\ 0 & 0 & sin z_4 & z_3 cos z_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & cos\theta^r & -v^r sin\theta^r \\ 0 & 0 & sin\theta^r & v^r cos\theta^r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{z} = Az + Bu$$

$$\dot{x} = Ax + Bu - Ax_e$$

1.4

$$\dot{x} = Ax + Bu - Ax_e$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The system is not controllable because the controllable matrix is not full row rank.

1.5

$$C = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} cos\theta^r & -v^r sin\theta^r & 0 & 0 \\ sin\theta^r & v^r cos\theta^r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In order for the C matrix to be full row-rank

$$\frac{cos\theta^r}{sin\theta^r} \neq \frac{-v^r sin\theta^r}{v^r cos\theta^r}$$

$$\therefore v^r \neq 0 (sufficientandnecessary).$$

2 Part II - Continuous Time Optimal Control

2.1

$$
\begin{aligned}
cost - to - go: J_k(x_t; u_{t:T}) &= \int_{\tau=t}^{T} l_\tau(x_\tau, u_\tau) d\tau + l_T(x_T) \\
&= \int_{\tau=t}^{T} \frac{1}{2}(x_\tau - x_G)^T Q (x_\tau - x_G) + \frac{1}{2} u_\tau^T R u_\tau \quad d\tau + \frac{1}{2}(x_T - x_G)^T S (x_T - x_G) \\
J_0 &= \int_{\tau=0}^{T} \frac{1}{2}(x_\tau - x_G)^T Q (x_\tau - x_G) + \frac{1}{2} u_\tau^T R u_\tau \quad d\tau + \frac{1}{2}(x_T - x_G)^T S (x_T - x_G) \\
valuefunction: V(t, x_t) &= \min_{u_{r:T}} \int_{\tau=t}^{T} \frac{1}{2}(x_\tau - x_G)^T Q (x_\tau - x_G) + \frac{1}{2} u_\tau^T R u_\tau \quad d\tau + \frac{1}{2}(x_T - x_G)^T S (x_T - x_G)
\end{aligned}
$$

2.2

Hamilton-Jacobi-Bellman Eq

$$
0 = \frac{\partial}{\partial t} V(t, x_t) + \min_{u_t} \{ l_t(x_t, u_t) + \frac{\partial}{\partial x} V(t, x_t) \cdot f(x_t, u_t) \}
$$

$$
f(x_t, u_t) = \begin{bmatrix} x_3 cos x_4 \\ x_3 sin x_4 \\ u_1 \\ u_2 \end{bmatrix}
$$

2.3

co-state:

$$
\lambda_t = \frac{\partial}{\partial x} V(t, x_t) = \frac{\partial}{\partial x} \min_{u_{r:T}} \int_{\tau=t}^{T} \frac{1}{2}(x_\tau - x_G)^T Q (x_\tau - x_G) + \frac{1}{2} u_\tau^T R u_\tau \quad d\tau + \frac{1}{2}(x_T - x_G)^T S (x_T - x_G)
$$

Hamilton:

$$
\begin{aligned}
H(t, x_t, y_t, \lambda_t) :&= l_t(x_t, u_t) + \lambda_t^T f(x_t, u_t) \\
&= \frac{1}{2}(x_t - x_G)^T Q (x_t - x_G) + \frac{1}{2} u_t^T R u_t + \lambda_t^T \begin{bmatrix} x_3 cos x_4 \\ x_3 sin x_4 \\ u_1 \\ u_2 \end{bmatrix}
\end{aligned}
$$

2.4

$$
\begin{aligned}
&-\dot{\lambda}_t^* = \frac{\partial}{\partial x} l_t(x_t^*, u_t^*) + [\frac{\partial}{\partial x} f(x_t^*, u_t^*)]^T \lambda_t^* \\
&\frac{\partial}{\partial x} f(x_t^*, u_t^*) = \begin{bmatrix} 0 & 0 & cos\theta & -vsin\theta \\ 0 & 0 & sin\theta & vcos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\frac{\partial}{\partial x} l_t(x_t^*, u_t^*) = Q x_t
\end{aligned}
$$

B.C:

$$
\lambda_t^* = \frac{\partial}{\partial x} l_T(x_T^*) = S x_T
$$

2.5

$$
\begin{aligned}
&u = arg \min_u H \\
&0 = \frac{\partial H}{\partial u} = R u_t + \lambda_t^T \frac{\partial}{\partial u} f(x_t^*, u_t^*)
\end{aligned}
$$

$$
\therefore u_t = -R^{-1} (\frac{\partial}{\partial u} f(x_t^*, u_t^*)^T \lambda_t)
$$

$$
\frac{\partial}{\partial u} f(x_t^*, u_t^*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

2.6

HJB equation:

$$
\begin{aligned}
&-ln(\delta) V(x_t) = \min_{u_t} \{ l(x_t, u_t) + \nabla V(x_t) f(x_t, u_t)) \} \\
&\delta = 1 \\
&\therefore 0 = \min_{u_t} \{ l(x_t, u_t) + \nabla V(x_t) f(x_t, u_t)) \}
\end{aligned}
$$

2.7

$$\begin{aligned}\dot{x}_t &= Ax_t - BR^{-1}B^T\lambda \\ \dot{\lambda}_t &= -Qx_t - A^T\lambda_t \\ A &= \frac{\partial}{\partial x}f(x_t, u_t) \\ B &= \frac{\partial}{\partial u}f(x_t, u_t)\end{aligned}$$

2.8

$$\begin{aligned}u &= arg\min_u H \\ 0 &= \frac{\partial H}{\partial u} = Ru_t + \lambda_t^T\frac{\partial}{\partial u}f(x_t^*, u_t^*)\end{aligned}$$

$$\therefore u_t = -R^{-1}B^T\lambda_t$$

2.9

$$u = -R^{-1}\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\lambda$$

$$\begin{aligned}\dot{x}_t &= Ax_t - BR^{-1}B^T\lambda \\ \dot{\lambda}_t &= -Qx_t - A^T\lambda_t \\ A &= \frac{\partial}{\partial x}f(x_t, u_t) \\ B &= \frac{\partial}{\partial u}f(x_t, u_t) \\ assuming \quad \lambda &= Px \\ PA + A^TP - PBR^{-1}B^TP + Q &= 0 \\ u &= -R^{-1}B^TPx\end{aligned}$$

2.10

The system is unable to the control gain or P matrix when take the linearization at x_G. When linearized at x_G A and B matrix are shown as follow. In this scenarios, there is no converge solution for the Riccati equation so there is no solution. Therefore the system cannot reach the goal location.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When we change the linearization point to [1,1,1,0]. It is able to find a converge solution for the Riccati equation. The state trajectories are shown in 1. Noted that only x_2 reaches the desire location, and this is also expected. When the system is linearized at [1,1,1,0], A and B matrix are as shown. It is now in full row rank which is then controllable. However, we try to use a control obtained from a linearized model on the original non-linear model. It cannot fully control the system to the desire goal states.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

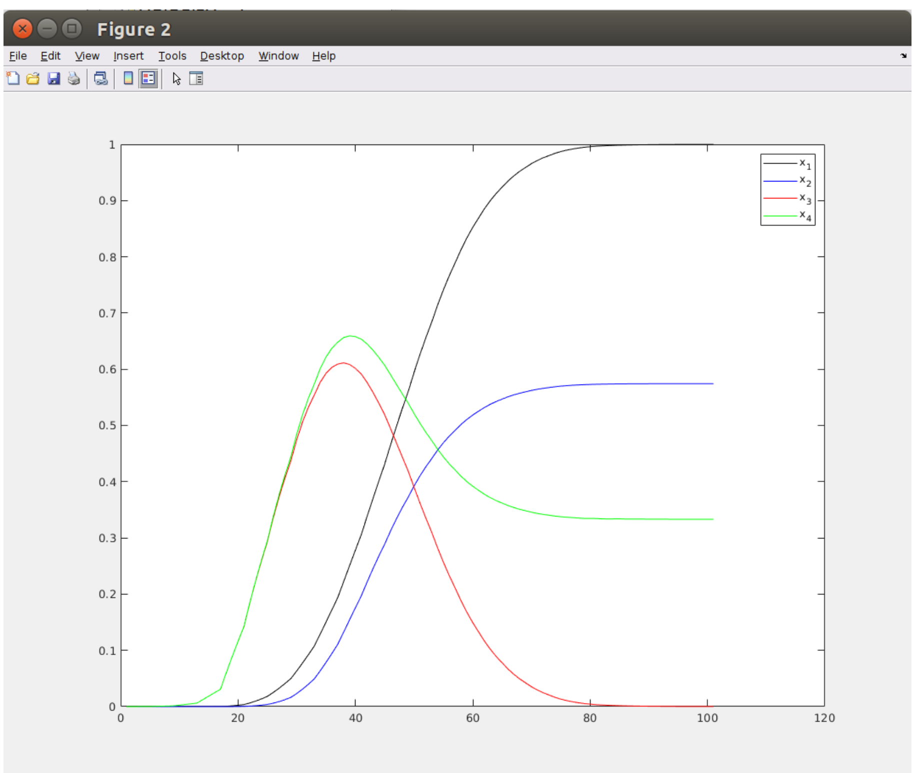


Figure 1: Continuous Time Optimal Control Trajectories

3 Part III - Discrete Time Optimal Control

3.1

$$\dot{x}_t = \frac{x_{t+\Delta t} - x_t}{\Delta t}$$

$$\frac{x_{t+\Delta t} - x_t}{\Delta t} = \begin{bmatrix} V cos\theta \\ V sin\theta \\ u1 \\ u2 \end{bmatrix}$$

$$x_{k+1} = \begin{bmatrix} 0.1V cos\theta \\ 0.1V sin\theta \\ 0.1u1 \\ 0.1u2 \end{bmatrix} + x_k = f(x_k, u_k) = \begin{bmatrix} x_{k1} + 0.1x_{k3}cosx_{k4} \\ x_{k2} + 0.1x_{k3}sinx_{k4} \\ x_{k3} + 0.1uk1 \\ x_{k4} + 0.1uk2 \end{bmatrix}$$

3.2

cost-to-go:

$$\begin{aligned} J_0 &= \Sigma_{k=0}^{N-1} l_k(x_k, u_k) + l_N(x_N) \\ &= \Sigma_{k=0}^{N-1}(\frac{1}{2}(x_k - x_G)^T Q(x_k - x_G) + \frac{1}{2}(x_N - x_G)^T S(x_N - x_G)) + l_N(x_N) \end{aligned}$$

value:

$$V_0 = \min_{u_0:u_{N-1}} J_0(x_0; u_0 : u_{N-1})$$

Bellman equation:

$$V_k(x_k) = \min_{u_k}\{l_k(x_k, u_k) + V_{k+1}(x_{k+1})\}$$

3.3

Hamilton

$$H_k(x_k, u_k, \lambda_{k+1}) = l_k(x_k, u_k) + \lambda_{k+1}^T f(x_k, u_k)$$

co-state

$$\lambda_k = \frac{\partial}{\partial x} V_k(x_k)$$

co-state dynamics

$$\lambda_k = \frac{\partial}{\partial x} l_k(x_k, u_k) + [\frac{\partial}{\partial x} f(x_k, u_k)]^T \lambda_{k_1}$$

B.C

$$\begin{aligned} J_N^*(x_N) &= \frac{1}{2}(x_N - x_G)^T S (x_N - x_G) \\ &= \frac{1}{2} x_N^T S x_N - x_N^T S x_G + \frac{1}{2} x_G^T S x_G \\ \lambda_N &= \frac{\partial}{\partial x} J_N^*(x_N) = S x_N - S x_G \end{aligned}$$

3.4

$$\begin{aligned} u_k &= arg \min_u H_k(x_k, u_k, \lambda_{k+1}) \\ \frac{\partial H_k}{\partial u} = 0 &= \frac{\partial l_k(x_k, u_k)}{\partial u_k} + [\frac{\partial f(x_k, u_k)}{\partial u_k}]^T \lambda_{k+1} \\ 0 &= R u_k + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \lambda_{k+1} \end{aligned}$$

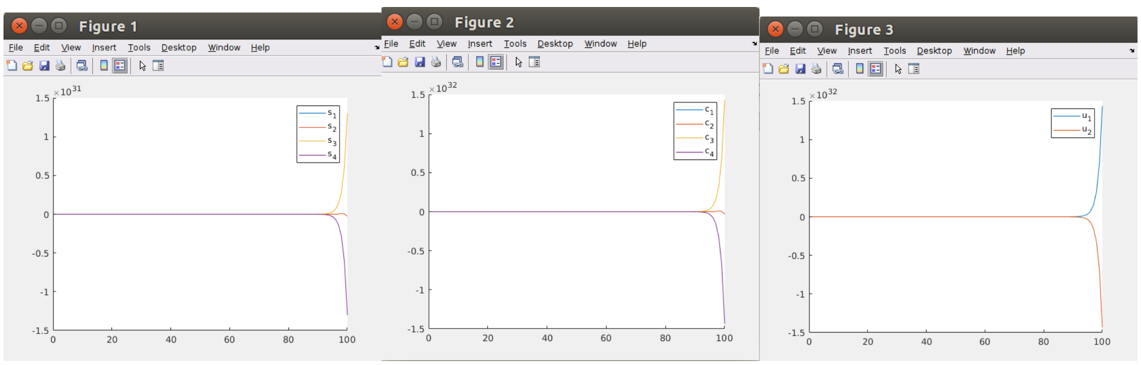
3.5

$$\begin{aligned} \lambda_k &= \frac{\partial}{\partial x} + [\frac{\partial}{\partial x} f(x_k, u_k)]^T \lambda_{k+1} \\ &= Q x_k - Q x_G + [\frac{\partial}{\partial x} f(x_k, u_k)]^T \lambda_{k+1} \end{aligned}$$

$$\frac{\partial}{\partial x} f(x_k, u_k) = \begin{bmatrix} 1 & 0 & 0.1 cos x_4 & -0.1 x_3 sin x_4 \\ 0 & 1 & 0.1 sin x_4 & 0.1 x_3 cos x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \lambda_N &= S x_N - S x_G \\ \lambda_{N-1} &= Q x_k - Q x_G + \begin{bmatrix} 1 & 0 & 0.1 cos x_4 & -0.1 x_3 sin x_4 \\ 0 & 1 & 0.1 sin x_4 & 0.1 x_3 cos x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T (S x_N - S x_G) \end{aligned}$$

$$
\begin{aligned}
x_{N-1} &= [x_1, x_2, x_3, x_4]^T \\
x_G &= [xg1, xg2, xg3, xg4]^T \\
\lambda_{N-1} &= \begin{bmatrix} 11x_1 - 11x_g1 \\ 11x_2 - 11x_g2 \\ 11x_3 - 11x_{g3} + \frac{cos(x_4)(10x_1 - 10x_{g1})}{10} + \frac{sin(x_4)(10x_2 - 10x_{g2})}{10} \\ 11x_4 - 11x_{g4} + \frac{x_3 cos(x_4)(10x_2 - 10x_{g2})}{10} - \frac{x_3 sin(x_4)(10x_1 - 10x_{g1})}{10} \end{bmatrix}
\end{aligned}
$$



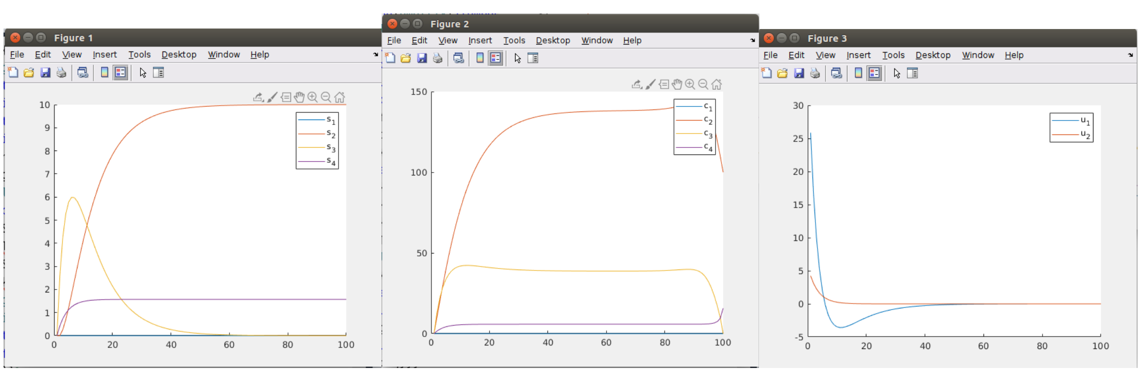
(a) state (b) co-state (c) control

Figure 2: Discrete time model

3.6

For a finite-time LQR linearized discrete-time system the following equations hold:

$$
\begin{aligned}
\lambda_k &= P_k x_k \\
P_k &= Q_k + A_k^T P_{k+1} A_k - A_k^T P_{k+1} B_k [R_k + B_k^T P_{k+1} B_k]^{-1} B_k^T P_{k+1} A_k \\
u_k &= -[R_k + B_k^T P_{k+1} B_k]^{-1} B_k^T P_{k+1} A_k (x_k - x_g)
\end{aligned}
$$

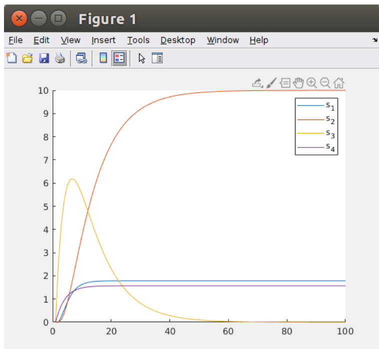


(a) state (b) co-state (c) control

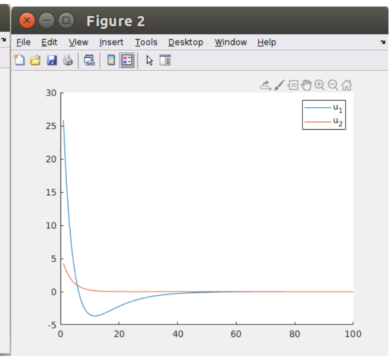
Figure 3: LQR linearized discrete-time system

3.7

The results of applying the result obtained in 3.6 on the original non-linear system are shown as follow:



(a) state



(b) co-state

Figure 4: LQR non-linear discrete-time system

The behavior are similar between 3.7 and 3.6. However in 3.6 x1 is zero all the way and in 3.7 it is able to control to a certain degree. Other states are able to reach the goal positions both in 3.6 and 3.7.

4 Part IV - Linear Quadratic Regulator

4.1

$$\begin{aligned} H_k(x_k, y_k, \lambda_{k+1}) :&= l_k(x_k, u_k) + \lambda_{k+1}^T f(x_k, u_k) \\ &= x_k^T Q x_k + u_k^T R u_k + 2x_k^T N u_k + \lambda_{k+1}^T (A_k x_k + B_k u_k) \\ \lambda_k &= \frac{\partial H}{\partial x} \end{aligned}$$

4.2

$$\begin{aligned} &\frac{\partial H}{\partial u} = 0 = 2N^T x_k + 2Ru_k + B^T \lambda_{k+1} \\ &\lambda_k = 2Qx_k + 2Nu_k + A_k^T \lambda_{k+1} = Px_k \\ &\lambda_{k+1} = Px_{k+1} = P(Ax_k + Bu_k) \end{aligned}$$

$$\begin{aligned} 2N^T x_k + 2Ru_k + B^T \lambda_{k+1} &= 2N^T x_k + 2Ru_k + B^T P(Ax_k + Bu_k) \\ &= 2N^T x_k + B^T PAx_k + 2Ru_k + B^T PBu_k = 0 \\ (2N^T + B^T PA)x_k &= -(2R + B^T PB)u_k \\ u_k &= -(2R + B^T PB)^{-1}(2N^T + B^T PA)x_k \end{aligned}$$

4.3

$$\begin{aligned} \lambda_k = Px_k &= 2Qx_k + 2Nu_k + A_k^T \lambda_{k+1} \\ &= A^T[P(Ax_k + Bu_k)] + 2Nu_k + 2Qx_k \\ &= A^T PAx_k + A^T PBu_k + 2Nu_k + 2Qx_k \\ &= (A^T PA + 2Q)x_k + (A^T PB + 2N)u_k \\ &= (A^T PA + 2Q)x_k + (A^T PB + 2N)[-(2R + B^T PB)^{-1}(2N^T + B^T PA)]x_k \\ &\therefore P = (A^T PA + 2Q) + (A^T PB + 2N)[-(2R + B^T PB)^{-1}(2N^T + B^T PA)] \end{aligned}$$

4.4

The result converged P matrix and K matrix are:

$$P = \begin{bmatrix} 4.6206 & 2.0271 \\ 2.0271 & 5.1117 \end{bmatrix}$$

$$K = [0.4408 1.0183]$$

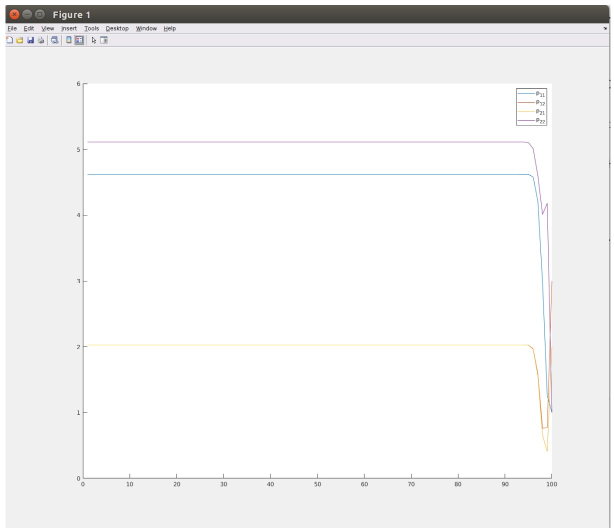


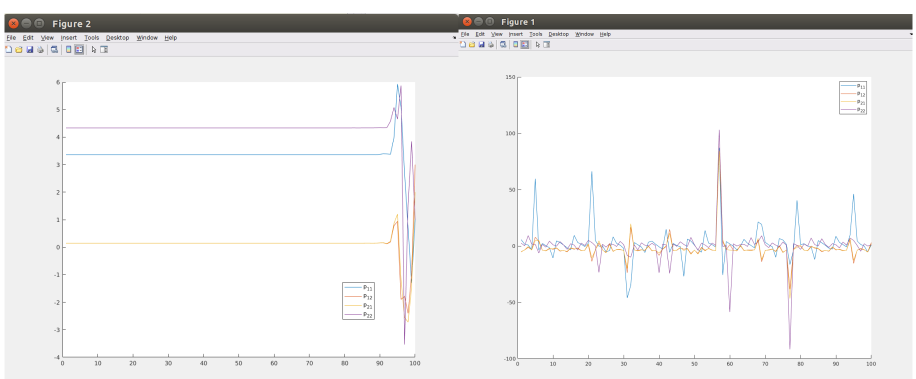
Figure 5: Discrete time LQR Riccati solution P

The results from using dlqr are:

$$P = \begin{bmatrix} 4.6778 & 2.1316 \\ 2.1316 & 5.1437 \end{bmatrix}$$

$$K = [0.4376 1.0235]$$

N will effect the convergence of the result. As show in 6, if N is increased, then the system will take a longer time to converge. And N has to satify that $Q - NR^{-1}N^T >= 0$



(a) N = 1, 0.1

(b) N = 2,1

Figure 6: Result with different