1 [18pt] System

Consider a unicycle model which has four states and two control inputs. The state $x = [p_x, p_y, v, \theta]^T \in \mathbb{R}^4$ contains the 2D position (p_x, p_y), the speed v, and the heading θ of the vehicle. The control input $u = [\dot{v}, \dot{\theta}]^T \in \mathbb{R}^2$ contains the acceleration $\dot{v}$ and the turning rate $\dot{\theta}$. The dynamic equation is

$$\dot{x} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{1}$$

1.1 [1pt] Is this system control-affine?

1.2 [5pt] Which description is most appropriate? Choose one from each row

Linear	Nonlinear
Continuous-Time	Discrete-Time
Time-invariant	Time-varying
Deterministic	Stochastic
Continuous-State	Discrete-State

1.3 [4pt] Linearize the system at $[p_x^r, p_y^r, v^r, \theta^r]^T$. Write down the resulting dynamics.

1.4 [4pt] Compute the linearized system dynamics at $[1, 1, 0, 0]^T$. Is the linearized system at this point controllable? Justify your answer.

1.5 [4pt] Under which condition on the parameters $p_x^r, p_y^r, v^r, \theta^r$ is the linearized system controllable? Find sufficient and necessary conditions.

2 [28pt] Continuous Time Optimal Control

For the system in (1), we would like to find an optimal trajectory from an intial state $x_0 = [p_{x,0}, p_{y,0}, 0, 0]^T$ to a target state $x_G = [p_{x,G}, p_{y,G}, 0, 0]^T$. The run-time cost is $l(x_t, u_t) = \frac{1}{2}(x_t - x_G)^T Q (x_t - x_G) + \frac{1}{2} u_t^T R u_t$ where $Q \in \mathbb{R}^{4\times4}$ and $R \in \mathbb{R}^{2\times2}$ are both positive definite. There is no discount in the system.

First consider a finite-time optimal control problem. Let the horizon be from 0 to T and the terminal cost be $l_T(x_t, u_t) = \frac{1}{2}(x_t - x_G)^T S (x_t - x_G)$ where $S \in \mathbb{R}^{4\times4}$ is positive definite.

2.1 [2pt] Define 1) the cost-to-go function; 2) the value function.

2.2 [2pt] Write down the Hamilton-Jacobi-Bellman Equation for the system.

2.3 [2pt] Define Hamiltonian and the co-state for the system.

2.4 [2pt] Write down the dynamics of the co-state that satisfy the maximum principle and its boundary condition.

2.5 [2pt] Derive the optimal control law (as a function of the state and the co-state).

Let's now make the problem infinite horizon.

2.6 [2pt] Write down the corresponding HJB equation.

2.7 [2pt] What condition does the optimal co-state need to satisfy?

2.8 [2pt] Derive the optimal control law (as a function of the state and the co-state).

2.9 [4pt] Apply infinite-time LQR on the linearized system in problem 1.3. What is the resulting optimal control law?

2.10 [8pt] Simulate the response of the control law in problem 2.9 on the original nonlinear system using parameters: Q = I, R = $0.1I$, $x_0 = [0,0,0,0]^\top$, $x_G = [1,1,0,0]^\top$, $T = 10$s. Take the linearization at x_G. Discuss your findings. Did the system reach the desired goal location? Why or why not? What may be different if we change the reference point to the linearization to $[1,1,1,0]^\top$? [Hint: You need to use ODE solvers. Useful MATLAB functions *lqr(A,B,Q,R)*, *ode45*]

3 [36pt] Discrete Time Optimal Control

Solving the continuous time optimal control problem requires solving ODEs. Let's now consider its discrete counterpart. All other conditions remain the same. We are still looking for an optimal trajectory from an initial state $x_0 = [p_{x,0}, p_{y,0}, 0, 0]^T$ to a target state $x_G = [p_{x,G}, p_{y,G}, 0, 0]^T$. The run-time cost is $l(x_t, u_t) = \frac{1}{2}(x_t - x_G)^T Q (x_t - x_G) + \frac{1}{2} u_t^T R u_t$ where $Q \in \mathbb{R}^{4\times 4}$ and $R \in \mathbb{R}^{2\times 2}$ are both positive definite. There is no discount in the system. The horizon is N and the terminal cost is $l_N(x_t, u_t) = \frac{1}{2}(x_t - x_G)^T S (x_t - x_G)$ where $S \in \mathbb{R}^{4\times 4}$ is positive definite.

3.1 [2pt] Write down the discrete-time problem of (1). Assume sample time $\Delta t = 0.1s$. [Hint: $\dot{x}_t \approx \frac{x_{t+\Delta t} - x_t}{\Delta t}$.]

3.2 [4pt] Define 1) the cost-to-go function; 2) the value function. And write down the Bellman Equation for the discrete time system.

3.3 [4pt] Define Hamiltonian and the co-state for the discrete time system. And write down the dynamics of the co-state that satisfy the maximum principle and its boundary condition.

3.4 [2pt] Derive the optimal control law (as a function of the state and the co-state).

3.5 [10pt] For $x_0 = [0,0,0,0]^T$, $x_G = [10,10,0,\pi/2]^T$, $Q = I$, $R = 0.1I$, $S = 10I$, and $N = 100$. Approximately solve for the optimal state, co-state, and control trajectories. During the backward computation, you can approximate the co-state as a second-order function of the state. [Hint: You can use symbolic solvers such as MATLAB's Symbolic Toolbox and *SymPy* in Python]

3.6 [10pt] Apply finite-time LQR on the linearized discrete-time system (linearized around x_G). What is the resulting optimal trajectories for the state, co-state, and control? Compare the result with the result in 3.5. Discuss your findings.

3.7 [4pt] Apply the optimal LQR control law obtained in 3.6 on the nonlinear discrete-time system and compare its performance with the solutions in 3.5 and 3.6. Discuss your findings.

4 [18pt] Linear Quadratic Regulator

Consider a discrete time system $x_{k+1} = Ax_k + Bu_k$. The objective function of the system is

$$J = \sum_{0}^{\infty} x_k^\top Q x_k + 2x_k^\top N u_k + u_k^\top R u_k. \tag{2}$$

4.1 [2pt] Write down the Hamiltonian of the problem.

4.2 [4pt] Solve for the optimal control law (Note the control gain should be static since this is an infinite horizon problem).

4.3 [4pt] Derive the Riccati equation for the problem.

4.4 [8pt] Compute the control gain and the P matrix in the Riccati equation using the following parameters:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, Q = I, N = \begin{bmatrix} 0.1 \\ 0.01 \end{bmatrix}, R = 1. \tag{3}$$

Discuss the effect of N in this problem. Discuss if there is any constraint that N needs to satisfy. [Hint: Compare your solution with the mathlab function dlqr.]