4 Transformations and symmetry

In the previous chapter, various graphs of functions and relations were reviewed or introduced. This chapter deals with transformations of these graphs under vertical and horizontal translations, under reflections in the x-axis and y-axis, and under rotations of 180° about the origin (dilations are introduced in Year 12). These procedures allow a wide variety of new graphs to be obtained, and relationships amongst different graphs to be discovered.

Many graphs are unchanged under one or more of these transformations, which means that they are symmetric in some way. This chapter deals with line symmetry under reflection in the y-axis, and point symmetry under rotation of 180° about the origin. These transformations and symmetries are described geometrically and algebraically, and the theme remains the interrelationship between the algebra and the graphs.

The absolute value function is then introduced, together with its own transformations and reflection symmetry. The final section generalises the transformations of this chapter to the far more general idea of composite functions.

As always, computer sketching of curves is very useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and to gain familiarity with the variety of graphs and their interrelationships.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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4A Translations of known graphs

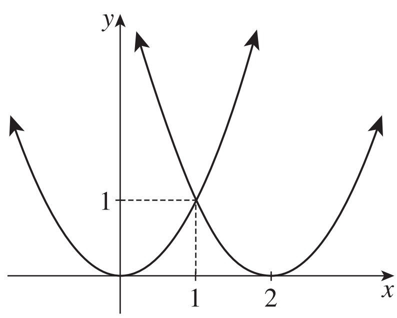
Once a graph has been drawn, it can be *shifted* (or *translated*) vertically or horizontally to produce further graphs. These procedures work generally on all functions and relations, and greatly extend the range of functions and relations whose graphs can be quickly recognised and drawn.

In particular, translations are very helpful when dealing with parabolas and circles, where they are closely related to completing the square.

Shifting right and left

The graphs of $y = x^2$ and $y = (x - 2)^2$ are sketched from their tables of values.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x-2)^2$	16	9	4	1	0	1	4



- The values for $(x - 2)^2$ in the third row are the values of x^2 in second row shifted 2 steps to the right.
- Hence the graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.

1 SHIFTING (OR TRANSLATING) RIGHT AND LEFT

- To shift a graph h units to the *right*, replace x by $x - h$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x - h)$.

Shifting a graph h units to the left means shifting it $-h$ units to the right, so x is replaced by $x - (-h) = x + h$.

Example 1 4A

a Draw up tables of values for $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+1}$.

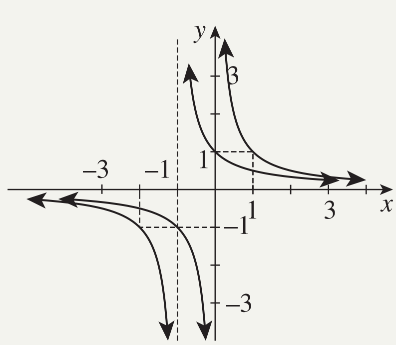
b Sketch the two graphs, and state the asymptotes of each graph.

c What transformation maps $y = \dfrac{1}{x}$ to $y = \dfrac{1}{x+1}$?

SOLUTION

a

x	-3	-2	-1	0	1	2	3
$\frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{x+1}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



b $y = \dfrac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

$y = \dfrac{1}{x+1}$ has asymptotes $x = -1$ and $y = 0$.

c Because x is replaced by $x + 1 = x - (-1)$, it is a shift left of 1 unit.

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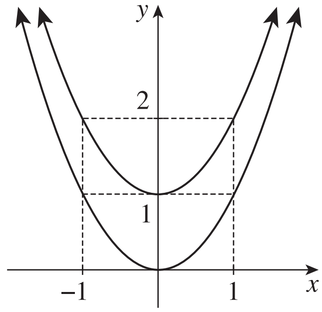
Shifting up and down

The graphs of $y = x^2$ and $y = x^2 + 1$ are sketched on the right from their tables of values.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 1$	10	5	2	1	2	5	10



- The values for $x^2 + 1$ in the third row are each 1 more than the corresponding values of x^2 in second row.
- Hence the graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit.

Rewriting the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule that is completely analogous to that for horizontal shifting.

2 SHIFTING (OR TRANSLATING) UP AND DOWN

- To shift a graph k units *upwards*, replace y by $y - k$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x) + k$.

Shifting a graph k units down means shifting it $-k$ units up, so y is replaced by $y - (-k) = y + k$.

Example 2 — 4A

The graph of $y = 2^x$ is shifted down 2 units.

a Write down the equation of the shifted graph.

b Construct tables of values, and sketch the two graphs.

c State the asymptotes of the two graphs.

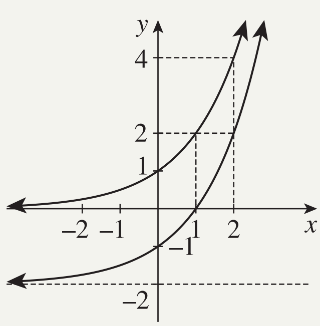
SOLUTION

a Replace y by $y - (-2) = y + 2$, so the new function is

$$y + 2 = 2^x, \qquad \text{that is,} \qquad y = 2^x - 2.$$

b

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$2^x - 2$	$-1\frac{3}{4}$	$-1\frac{1}{2}$	-1	0	2



c $y = 2^x$ has asymptote $y = 0$.

$y = 2^x - 2$ has asymptote $y = -2$.

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Combining horizontal and vertical translations

When a graph is shifted horizontally and vertically, the order in which the translations are applied makes no difference. The following example shows the effect of two translations on a cubic graph.

Example 3 4A

a How is the graph of $y = (x + 2)^3 - 4$ obtained from the graph of $y = x^3$ by a horizontal translation followed by a vertical translation?

b Draw up a table of values for the two functions and the intermediate function.

c Sketch the two curves, together with the intermediate graph.

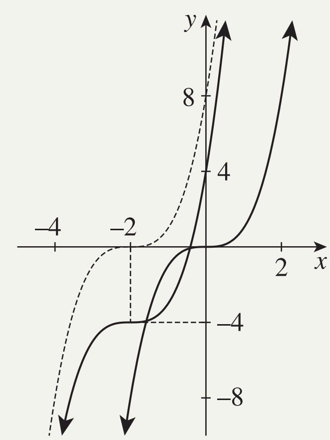
SOLUTION

a Shifting $y = x^3$ left 2 gives $y = (x + 2)^3$.
Shifting $y = (x + 2)^3$ down 4 gives $y + 4 = (x + 2)^3$,
which can be written as $y = (x + 2)^3 - 4$.

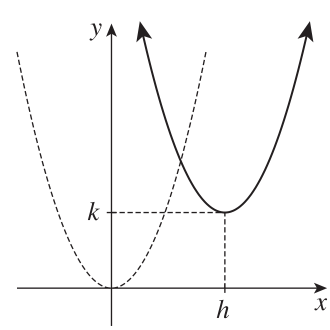
b

x	-4	-3	-2	-1	0	1	2
x^3	-64	-27	-8	-1	0	1	8
$(x + 2)^3$	-8	-1	0	1	8	27	64
$(x + 2)^3 - 4$	-12	-5	-4	-3	4	23	60

c



Translations and the vertex of a parabola



When we complete the square in a quadratic, it has the form

$$y = a(x - h)^2 + k, \qquad \text{that is} \qquad y - k = a(x - h)^2.$$

This is a translation of the quadratic $y = ax^2$. The parabola has been shifted h units right and k units up.

This gives a clear and straightforward motivation for completing the square.

3 THE COMPLETED SQUARE AND THE VERTEX OF A PARABOLA

The completed square form of a quadratic

$$y = a(x - h)^2 + k \qquad \text{or} \qquad y - k = a(x - h)^2$$

displays its graph as the parabola $y = ax^2$ shifted right h units and up k units.

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Example 4 — 4A

In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin, and sketch the two graphs.

a $y = x^2 - 4x + 5$ **b** $y = -2x^2 - 4x$

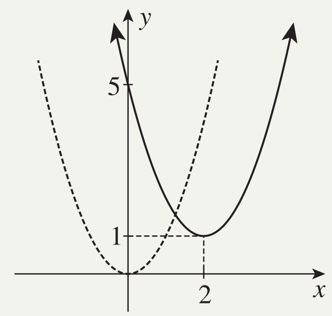
SOLUTION

a $y = x^2 - 4x + 5$

$y = (x^2 - 4x + 4) - 4 + 5$

$y = (x - 2)^2 + 1$ or $y - 1 = (x - 2)^2$

This is $y = x^2$ shifted right 2 and up 1.



b $y = -2x^2 - 4x$

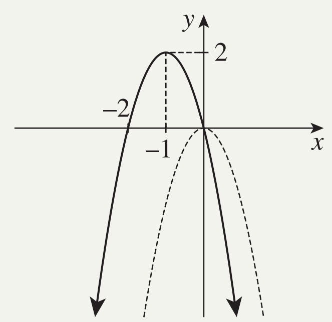
$-\frac{y}{2} = x^2 + 2x$

$-\frac{y}{2} = (x^2 + 2x + 1) - 1$

$-\frac{y}{2} = (x + 1)^2 - 1$

$y = -2(x + 1)^2 + 2$ or $y - 2 = -2(x + 1)^2$

This is $y = -2x^2$ shifted left 1 and up 2.



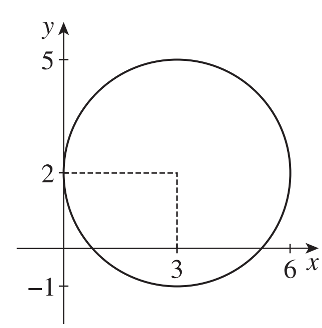
Translations and the centre of a circle

The circle drawn to the right has centre (3, 2) and radius 3. To find its equation, we start with the circle with centre the origin and radius 3,

$$x^2 + y^2 = 9,$$

then translate it 3 to the right and 2 up,

$$(x - 3)^2 + (y - 2)^2 = 9.$$



This formula can also be established directly by Pythagoras' theorem in the form of the distance formula, but as we saw with parabolas, translations make things clearer and more straightforward.

When the squares in the equation of a circle have both been expanded, the centre and radius can be found by completing the squares in x and in y.

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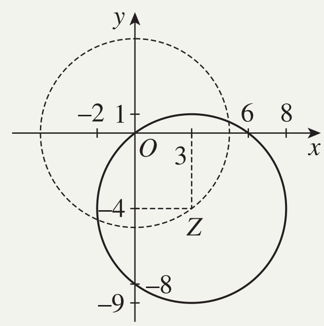
Example 5 4A

a Complete the squares in x and in y of the relation $x^2 + y^2 - 6x + 8y = 0$.
b Identify the circle with centre the origin that can be translated to it, and state the translations.
c Sketch both circles on the same diagram, and explain why each circle passes through the centre of the other circle.

SOLUTION

a Completing the squares in x and in y,

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$
$$(x^2 - 3)^2 + (y + 4)^2 = 25.$$



b It is the circle $x^2 + y^2 = 5^2$ shifted right 3 and down 4, so its centre is $Z(3, -4)$ and its radius is 5.
c Using the distance formula,

$$OZ^2 = 3^2 + 4^2$$
$$OZ = 5,$$

which is the radius of each circle.

Exercise 4A

FOUNDATION

1 **a** Copy and complete the table of values for $y = x^2$ and $y = (x - 1)^2$.

x	-2	-1	0	1	2	3
x^2						
$(x-1)^2$						

b Sketch the two graphs and state the vertex of each.
c What transformation maps $y = x^2$ to $y = (x - 1)^2$?

2 **a** Copy and complete the table of values for $y = \frac{1}{4}x^3$ and $y = \frac{1}{4}x^3 + 2$.

x	-3	-2	-1	0	1	2	3
$\frac{1}{4}x^3$							
$\frac{1}{4}x^3 + 2$							

b Sketch the two graphs and state the y-intercept of each.
c What transformation maps $y = \frac{1}{4}x^3$ to $y = \frac{1}{4}x^3 + 2$?

3 How far and in which direction has the parabola $y = x^2$ been shifted to produce each of these parabolas?
a $y = x^2 + 2$ **b** $y = x^2 - 5$ **c** $y = (x + 4)^2$ **d** $y = (x - 3)^2$

4 How far and in which direction has the hyperbola $y = \dfrac{1}{x}$ been shifted to produce each of these hyperbolas?
a $y = \dfrac{1}{x - 2}$ **b** $y = \dfrac{1}{x + 3}$ **c** $y = \dfrac{1}{x} - 4$ **d** $y = \dfrac{1}{x} + 5$

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5 Sketch each parabola by shifting $y = x^2$ either horizontally or vertically. Mark all intercepts with the axes.

a $y = x^2 + 1$ b $y = x^2 - 1$ c $y = (x - 1)^2$ d $y = (x + 1)^2$

6 Sketch each hyperbola by shifting $y = \frac{1}{x}$ either horizontally or vertically. Mark any intercepts with the axes.

a $y = \frac{1}{x+1}$ b $y = \frac{1}{x-1}$ c $y = \frac{1}{x} + 1$ d $y = \frac{1}{x} - 1$

7 Sketch each circle by shifting $x^2 + y^2 = 1$ either horizontally or vertically. Mark all intercepts with the axes.

a $(x - 1)^2 + y^2 = 1$ b $x^2 + (y - 1)^2 = 1$ c $x^2 + (y + 1)^2 = 1$ d $(x + 1)^2 + y^2 = 1$

DEVELOPMENT

8 Write down the new equation for each function or relation after the given translation has been applied. Then sketch the graph of the new curve.

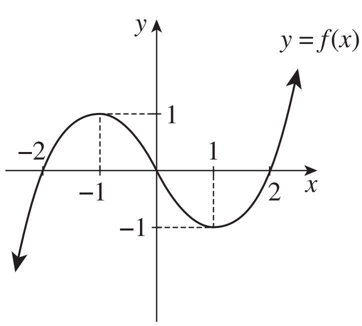
a $y = x^2$: right 1 unit b $y = 2^x$: down 3 units c $y = x^3$: left 1 unit

d $y = \frac{1}{x}$: right 3 units e $x^2 + y^2 = 4$: up 1 unit f $y = x^2 - 4$: left 1 unit

g $xy = 1$: down 1 unit h $y = \sqrt{x}$: up 2 units

9 In each case an unknown function has been drawn. Draw the functions specified below it.

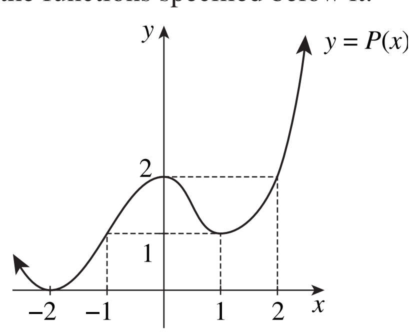
a



i $y = f(x - 2)$

ii $y = f(x + 1)$

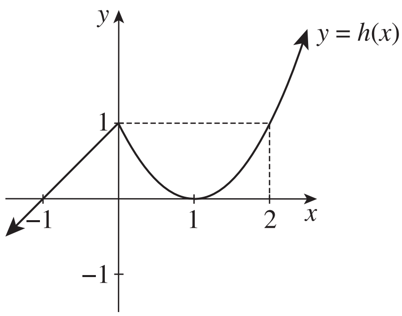
b



i $y = P(x + 2)$

ii $y = P(x + 1)$

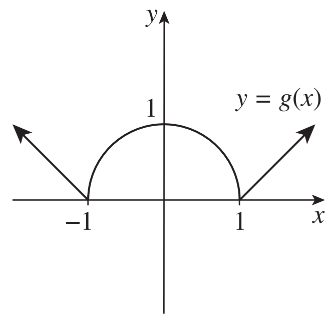
c



i $y - 1 = h(x)$

ii $y = h(x) - 1$

d



i $y - 1 = g(x)$

ii $y = g(x - 1)$

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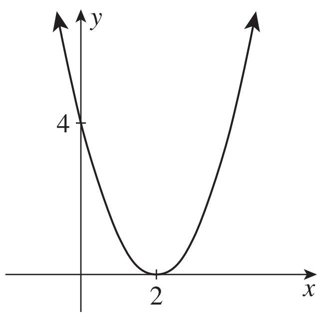
10 In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin. Finally, sketch its graph.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 2x - 2$ **c** $y = -x^2 + 4x + 1$

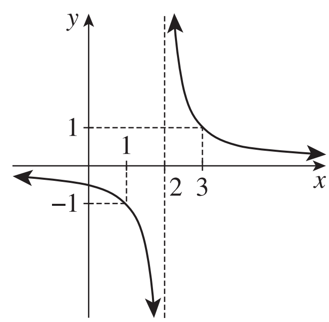
d $y = -x^2 - 4x - 5$ **e** $y = 2x^2 - 4x - 2$ **f** $y = \frac{1}{2}x^2 - x - 2$

11 Describe each graph below as the parabola $y = x^2$ or the hyperbola $xy = 1$ transformed by shifts, and hence write down its equation.

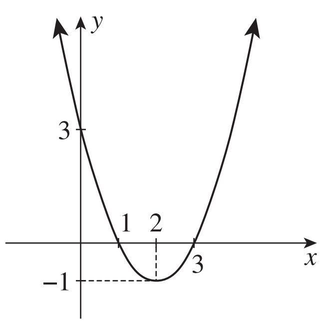
a



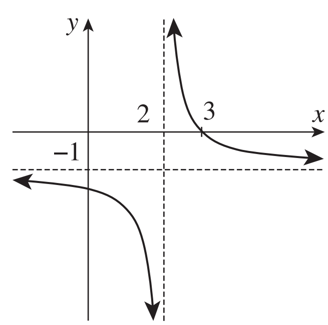
b



c



d



12 Use shifting, and completing the squares where necessary, to determine the centre and radius of each circle.

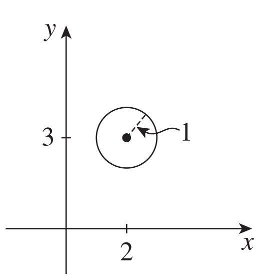
a $(x + 1)^2 + y^2 = 4$ **b** $(x - 1)^2 + (y - 2)^2 = 1$

c $x^2 - 2x + y^2 - 4y - 4 = 0$ **d** $x^2 + 6x + y^2 - 8y = 0$

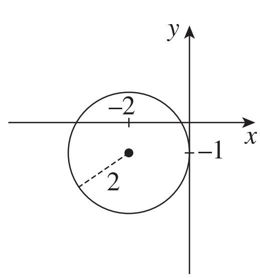
e $x^2 - 10x + y^2 + 8y + 32 = 0$ **f** $x^2 + 14x + 14 + y^2 - 2y = 0$

13 Describe each graph below as the circle $x^2 + y^2 = r^2$ transformed by shifts, and hence write down its equation.

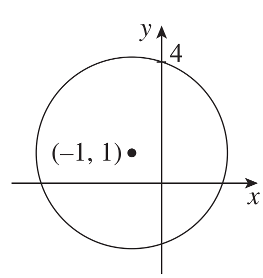
a



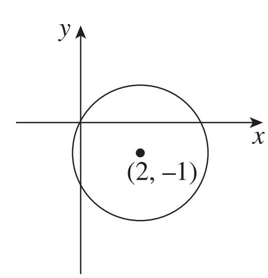
b



c



d



14 a Use a table of values to sketch $y = \frac{1}{2}x^3$. Then use translations to sketch:

i $y = \frac{1}{2}x^3 - 2$ **ii** $y = \frac{1}{2}(x - 2)^3$ **iii** $y = \frac{1}{2}(x + 3)^3 + 1$

b Use a table of values to sketch $y = -2x^3$. Then use translations to sketch:

i $y = 3 - 2x^3$ **ii** $y = -2(x + 3)^3$ **iii** $y = -2(x - 1)^3 - 2$

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CHALLENGE

15 Consider the straight line equation $x + 2y - 4 = 0$.

a The line is translated 2 units left. Find the equation of the new line.

b The original line is translated 1 unit down. Find the equation of this third line.

c Comment on your answers, and draw the lines on the same number plane.

16 Sketch $y = \dfrac{1}{x}$, then use shifting to sketch the following graphs. Find any x-intercepts and y-intercepts, and mark them on your graphs.

a $y = \dfrac{1}{x - 2}$ **b** $y = 1 + \dfrac{1}{x - 2}$ **c** $y = \dfrac{1}{x - 2} - 2$

d $y = \dfrac{1}{x + 1} - 1$ **e** $y = 3 + \dfrac{1}{x + 2}$ **f** $y = \dfrac{1}{x - 3} + 4$

17 In each part, explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each function.

a From $y = 2x$:

i $y = 2x + 4$ **ii** $y = 2x - 4$

b From $y = x^2$:

i $y = x^2 + 9$ **ii** $y = x^2 - 9$ **iii** $y = (x - 3)^2$

c From $y = -x^2$:

i $y = 1 - x^2$ **ii** $y = -(x + 1)^2$ **iii** $y = -(x + 1)^2 + 2$

d From $y = \sqrt{x}$:

i $y = \sqrt{x + 4}$ **ii** $y = \sqrt{x} + 4$ **iii** $y = \sqrt{x + 4} - 2$

e From $y = \dfrac{2}{x}$:

i $y = \dfrac{2}{x} + 1$ **ii** $y = \dfrac{2}{x + 2}$ **iii** $y = \dfrac{2}{x + 2} + 1$

18 a The circle $x^2 + y^2 = r^2$ has centre the origin and radius r. This circle is shifted so that its centre is at $C(h, k)$. Write down its equation.

b The point $P(x, y)$ lies on the circle with centre $C(h, k)$ and radius r. That is, P lies on the shifted circle in part **a**. This time use the distance formula for the radius PC to obtain the equation of the circle.

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4B Reflections in the x-axis and y-axis

Reflecting only in the x-axis and the y-axis may seem an unnecessary restriction, but in fact these two transformations are the key to understanding many significant properties of functions, particularly the symmetry of graphs.

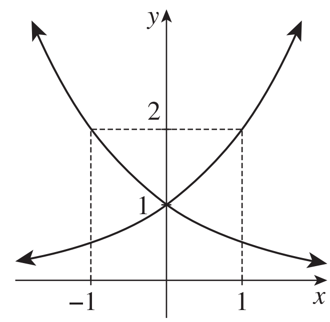
When these two reflections are combined, they produce a rotation of 180° about the origin, which again is the key to the symmetry of many graphs.

Reflection in the y-axis

The graphs of $y = 2^x$ and $y = 2^{-x}$ have been sketched to the right from their tables of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

- The second and third rows are the reverse of each other.
- Hence the graphs are reflections of each other in the y-axis.



4 REFLECTION IN THE Y-AXIS

- To reflect a graph in the y-axis, replace x by $-x$.
- Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

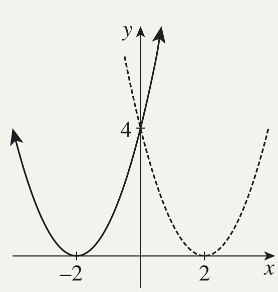
Reflection is mutual — it maps each graph to the other graph.

Example 6 4B

a Sketch the parabola $y = (x - 2)^2$, and on the same set of axes, sketch its reflection in the y-axis.
b Use the rule in the box above to write down the equation of the reflected graph.
c Why can this equation be written as $y = (x + 2)^2$?
d What are the vertices of the two parabolas?
e What other transformation would move the first parabola to the second?

SOLUTION

a



b Replacing x by $-x$ gives the equation
$y = (-x - 2)^2$

c Taking out the factor -1 from the brackets,
$$y = (-x - 2)^2$$
$$y = (-1)^2 \times (x + 2)^2$$
$$y = (x + 2)^2$$

d The vertices are $(2, 0)$ and $(-2, 0)$.

e The second parabola is also the first parabola shifted left 4 units.
This replaces x by $x + 4$, so the new equation is the same as before,
$$y = \big((x + 4) - 2\big)^2, \quad \text{or} \quad y = (x + 2)^2.$$
The reason why there are two possible transformations is that the parabola has line symmetry in its axis of symmetry.

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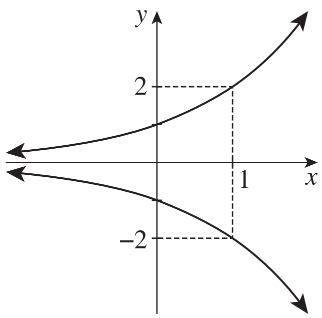
Reflection in the x-axis

The graphs of $y = 2^x$ and $y = -2^x$ have been sketched to the right from the table of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8



- The values in the second and third rows are the opposites of each other.
- Hence the graphs are reflections of each other in the x-axis.

Rewriting the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule that is completely analogous to that for reflection in the y-axis.

5 REFLECTION IN THE X-AXIS

- To reflect a graph in the x-axis, replace y by $-y$.
- Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

Again, reflection is mutual — it maps each graph to the other graph.

Example 7 4B

The graph of $y = (x - 2)^2$ is reflected in the x-axis.

a Write down the equation of the reflected graph.

b Construct tables of values, and sketch the two graphs.

c What are the vertices of the two parabolas?

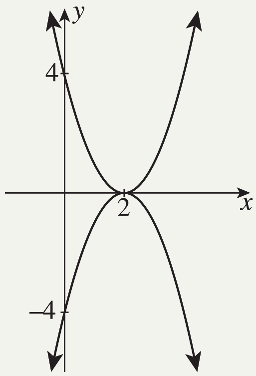
SOLUTION

a Replace y by $-y$, so the new function is

$$-y = (x-2)^2, \qquad \text{that is,} \qquad y = -(x-2)^2.$$

b

x	0	1	2	3	4
$(x-2)^2$	4	1	0	1	4
$-(x-2)^2$	-4	-1	0	-1	-4



c They both have vertex $(2, 0)$.

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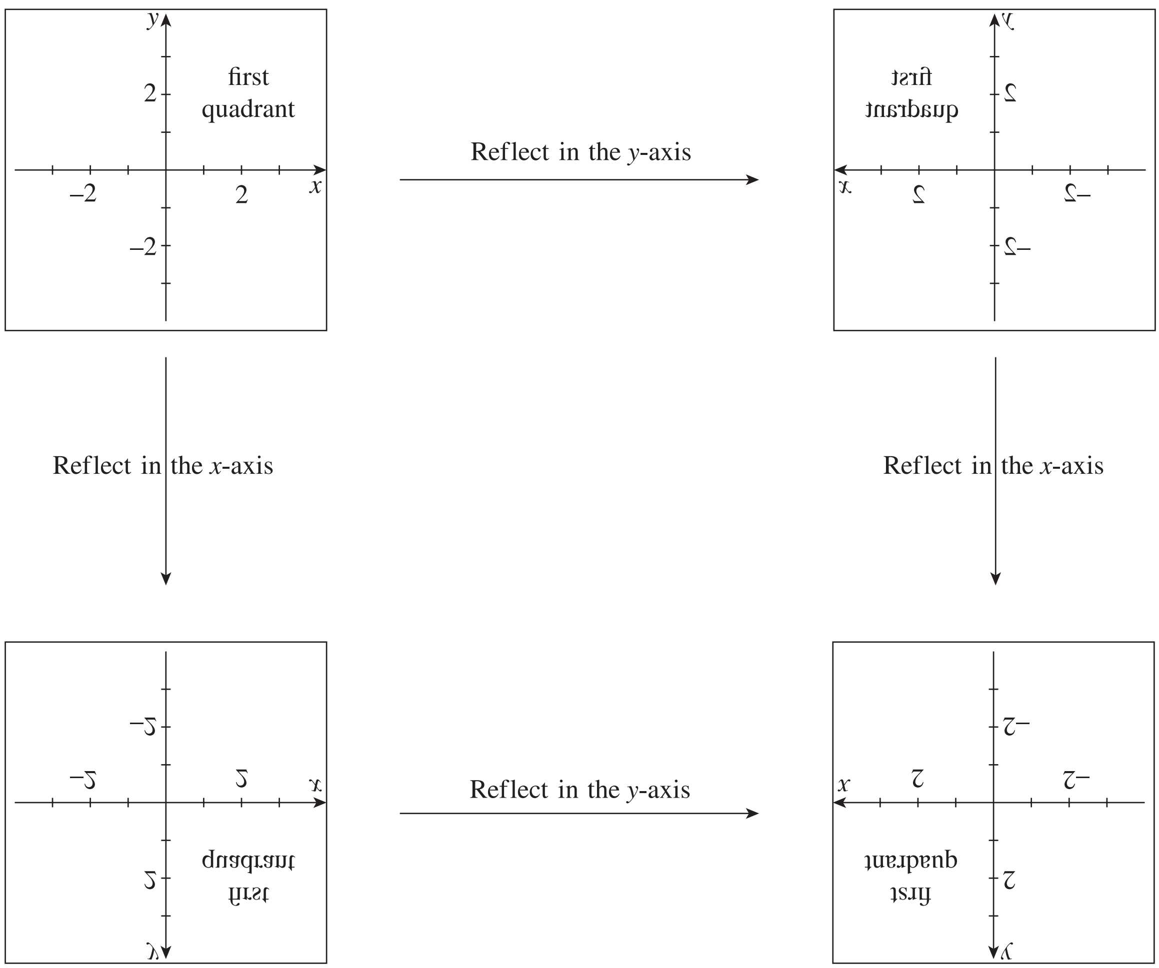
Combining the two reflections — rotating 180° about the origin

- Draw an x-axis and y-axis on a thin, semi-transparent sheet of paper.
- Hold the paper out flat, and regard it as a two-dimensional object.
- Reflect it in the x-axis — do this by holding the sheet steady at the two ends of the x-axis and rotating it 180° so that you are now looking at the back of the sheet. Then reflect it in the y-axis — do this by holding the sheet at the two ends of the y-axis and rotating it 180° so that you are looking at the front of the sheet again. What has happened?
- Reflect it in the y-axis, then in the x-axis. What happens?



This little experiment should convince you of two things:

- Performing successive reflections in the x-axis and in the y-axis results in a rotation of 180° about the origin.
- The order of these two reflections does not matter.

This rotation of 180° about the origin is sometimes called *reflection in the origin* — every point in the plane is moved along a line through the origin to a point the same distance from the origin on the opposite side.

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6 ROTATION OF 180° ABOUT THE ORIGIN

- To rotate a graph 180° about the origin, replace x by $-x$ and y by $-y$.
- Successive reflections in the x-axis and the y-axis are the same as a rotation of 180° about the origin.
- The order in which these two successive reflections are done does not matter.
- Rotation of 180° about the origin is also called *reflection in the origin*, because every point is moved through the origin to a point the same distance from the origin on the opposite side.

Rotation of 180° about the origin is also mutual — it maps each graph to the other graph.

Example 8 4B

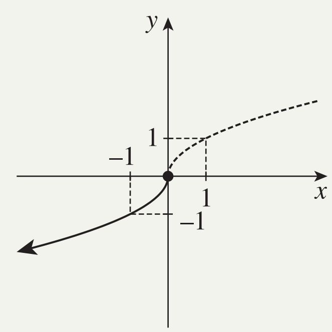
a From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$ using reflections.
b What single transformation maps each graph to the other?

SOLUTION

a The equation $y = -\sqrt{-x}$ can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is obtained from the graph of $y = \sqrt{x}$ by successive reflections in the x-axis and the y-axis, where the reflections may be done in either order.



b This is the same as rotation of 180° about the origin.

Exercise 4B FOUNDATION

1 Consider the parabola $y = x^2 - 2x$.

a Show that when y is replaced by $-y$, the equation becomes $y = 2x - x^2$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = 2x - x^2$.

x	-2	-1	0	1	2	3	4
$x^2 - 2x$							
$2x - x^2$							

c Sketch the two parabolas and state the vertex of each.

d What transformation maps $y = x^2 - 2x$ to $y = 2x - x^2$?

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2 Consider the hyperbola $y = \dfrac{2}{x-2}$.

a Show that when x is replaced by $-x$ the equation becomes $y = -\dfrac{2}{x+2}$.

b Copy and complete the table of values for $y = \dfrac{2}{x-2}$ and $y = -\dfrac{2}{x+2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$\dfrac{2}{x-2}$							*		
$-\dfrac{2}{x+2}$			*						

c Sketch the two hyperbolas and state the vertical asymptote of each.

d What transformation maps $y = \dfrac{2}{x-2}$ to $y = -\dfrac{2}{x+2}$?

3 **a** Sketch the graph of the quadratic function $y = x^2 - 2x - 3$, showing the intercepts and vertex.

b In each case, determine the equation of the result when this parabola is reflected as indicated. Then draw a sketch of the new function.

i In the y-axis **ii** In the x-axis **iii** In both axes

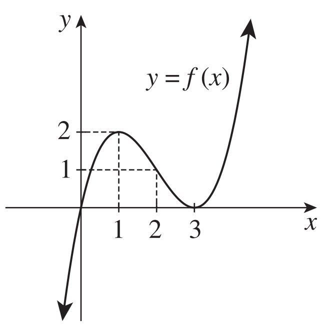
4 **a** Sketch the graph of the exponential function $y = 2^{-x}$, showing the y-intercept and the coordinates at $x = -1$, and clearly indicating the asymptote.

b In each case, find the equation of the curve when this exponential graph is reflected as indicated. Then draw a sketch of the new function.

i In the y-axis **ii** In the x-axis **iii** In both axes

5 In each case, an unknown function has been drawn. Draw the reflections of the function specified below it.

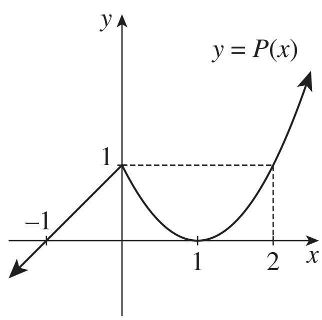
a



i $y = f(-x)$

ii $y = -f(x)$

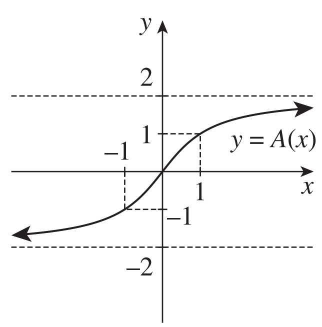
b



i $y = -P(x)$

ii $y = -P(-x)$

c



i $y = A(-x)$

ii $y = -A(-x)$

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DEVELOPMENT

6 Write down the new equation for each function or relation after the given transformation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: reflect in the x-axis

b $y = x^3$: reflect in the y-axis

c $y = 2^x$: rotate by 180°

d $y = 2x - x^2$: rotate by 180°

e $x^2 + y^2 = 9$: reflect in the y-axis

f $y = \dfrac{1}{x}$: reflect in the x-axis

7 Consider the hyperbola $y = \dfrac{1}{x+2} - 1$.

a Sketch this hyperbola.

b In each case, determine the reflection or rotation required to achieve the specified result. Then write down the equation of the new hyperbola and sketch it.

i The vertical asymptote is unchanged, but the horizontal asymptote changes sign.

ii The intercepts with the axes are positive.

8 a Sketch the circles $(x - 3)^2 + y^2 = 4$ and $(x + 3)^2 + y^2 = 4$.

b What transformation maps each circle onto the other?

c Confirm your answer by making an appropriate substitution into the first equation.

9 Consider $x^2 + y^2 = r^2$, the circle with centre the origin and radius r.

a Show that this equation is unchanged when reflected in either the x-axis or the y-axis.

b Explain this result geometrically.

CHALLENGE

10 In each part, explain how the graph of each subsequent equation is a reflection of the first graph or a rotation of 180°, then sketch each one.

a From $y = \frac{1}{2}x + 1$:

i $y = -\frac{1}{2}x + 1$ ii $y = -\frac{1}{2}x - 1$ iii $y = \frac{1}{2}x - 1$

b From $y = 4 - x$:

i $y = x - 4$ ii $y = x + 4$ iii $y = -x - 4$

c From $y = (x - 1)^2$:

i $y = -(x + 1)^2$ ii $y = (x + 1)^2$ iii $y = -(x - 1)^2$

d From $y = \sqrt{x}$:

i $y = -\sqrt{-x}$ ii $y = -\sqrt{x}$ iii $y = \sqrt{-x}$

e From $y = 3^x$:

i $y = -3^x$ ii $y = -3^{-x}$ iii $y = 3^{-x}$

f From $y = 1 + \dfrac{1}{x-1}$:

i $y = 1 - \dfrac{1}{x+1}$ ii $y = -1 + \dfrac{1}{x+1}$ iii $y = -1 + \dfrac{1}{1-x}$

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11 Consider the two parabolas $y = x^2 - 4x + 3$ and $y = x^2 + 4x + 3$.

- **a** Sketch both quadratic functions on the same set of axes.
- **b** What reflection maps each parabola onto the other?
- **c** How can the second parabola be obtained by shifting the first?
- **d** Confirm your answer to part **c** algebraically.
- **e** Investigate which parts of Question **10** could also have been achieved by shifting instead.

12 **a** Let $c(x) = \dfrac{2^x + 2^{-x}}{2}$. Show that $c(-x) = c(x)$, and explain this geometrically.

- **b** Let $t(x) = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$. Show that $-t(-x) = t(x)$, and explain this geometrically.
- **c** [Technology]
  Confirm your observations in parts **a** and **b** by plotting each function using graphing software.

13 Consider the parabola $y = (x - 1)^2$. Sketches or plots done on graphing software may help answer the following questions.

- **a** **i** The parabola is shifted right 1 unit. What is the new equation?
  - **ii** This new parabola is then reflected in the y-axis. Write down the equation of the new function.
- **b** **i** The original parabola is reflected in the y-axis. What is the new equation?
  - **ii** This fourth parabola is then shifted right 1 unit. What is the final equation?
- **c** Parts **a** and **b** both used a reflection in the y-axis and a shift right 1 unit. Did the order of these affect the answer?
- **d** Investigate other combinations of shifts and reflections. In particular, what do you notice if the shift is parallel with the axis of reflection?

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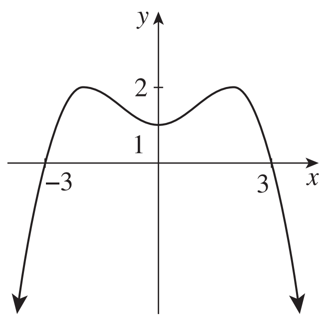
4C Even and odd symmetry

It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

Even functions and line symmetry in the y-axis



A relation or function is called *even* if its graph has *line symmetry in the y-axis*. This means that the graph is unchanged by reflection in the y-axis, as with the graph to the right.

As explained in Section 4B, when the graph of $y = f(x)$ is reflected in the y-axis, the new curve has equation $y = f(-x)$. Hence for a function to be *even*, the graphs of $y = f(x)$ and $y = f(-x)$ must coincide, that is,

$$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

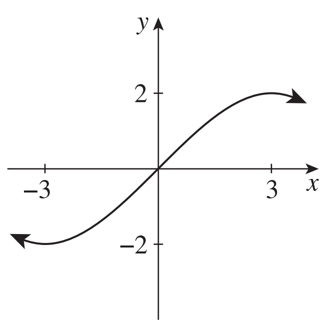
7 EVEN FUNCTIONS

- A relation or function is called *even* if its graph has *line symmetry in the y-axis*.
- Algebraically, a function $f(x)$ is even if

  $$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

  More generally, a relation is even if its equation is unchanged when x is replaced by $-x$.

Odd functions and point symmetry in the origin



A relation or function is called *odd* if its graph has *point symmetry in the origin*. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently, by successive reflections in the x-axis and the y-axis.

When the graph of $y = f(x)$ is reflected in the x-axis and then in the y-axis, the new curve has equation $y = -f(-x)$. Hence for a function to be *odd*, the graph of $-f(-x)$ must coincide with the graph of $f(x)$, that is,

$$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

8 ODD FUNCTIONS AND POINT SYMMETRY IN THE ORIGIN

- A relation or function is called *odd* if its graph has *point symmetry in the origin*.
- Algebraically, a function $f(x)$ is odd if

  $$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

  More generally, a relation is odd if its equation is unchanged when x is replaced by $-x$ and y is replaced by $-y$.
- *Point symmetry in the origin* means that the graph is mapped onto itself by a rotation of 180° about the origin.
- Equivalently, it means that the graph is mapped onto itself by successive reflections in the x-axis and the y-axis. The order of these two reflections does not matter.

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Testing functions algebraically for evenness and oddness

A single test will pick up both these types of symmetry in functions.

9 TESTING FOR EVENNESS AND ODDNESS (OR NEITHER)

- Simplify $f(-x)$ and note whether it is $f(x)$, $-f(x)$ or neither.

Most functions are neither even nor odd.

Example 9 4C

Test each function for evenness or oddness, then sketch it.

a $f(x) = x^4 - 3$ **b** $f(x) = x^3$ **c** $f(x) = x^2 - 2x$

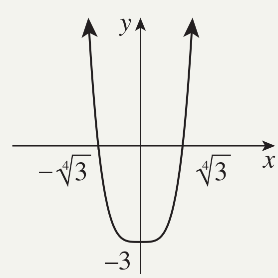
SOLUTION

a Here $f(x) = x^4 - 3$.

Substituting $-x$ for x,

$$\begin{aligned} f(-x) &= (-x)^4 - 3 \\ &= x^4 - 3 \\ &= f(x). \end{aligned}$$

Hence $f(x)$ is an even function.

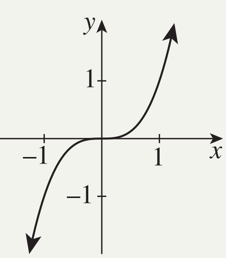


b Here $f(x) = x^3$.

Substituting $-x$ for x,

$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned}$$

Hence $f(x)$ is an odd function.



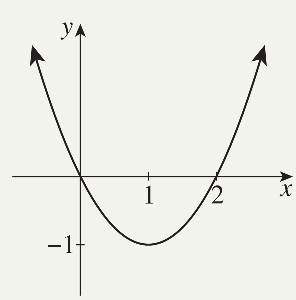
c Here $f(x) = x^2 - 2x$.

Substituting $-x$ for x,

$$\begin{aligned} f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x. \end{aligned}$$

Because $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.

(The parabola does, however, have line symmetry, not in the y-axis, but in its axis of symmetry $x = 1$.)



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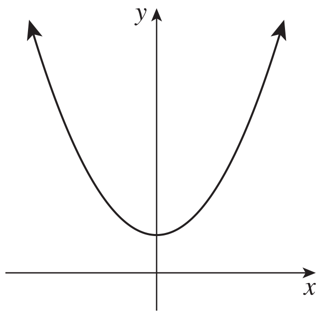
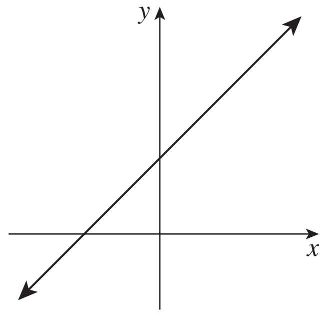
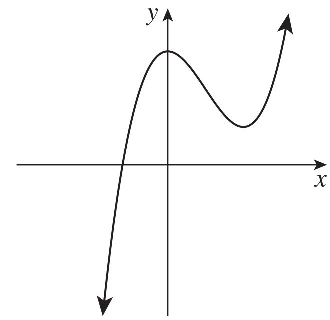
Exercise 4C

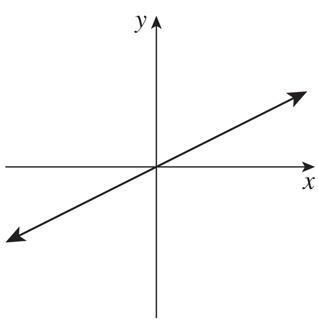
FOUNDATION

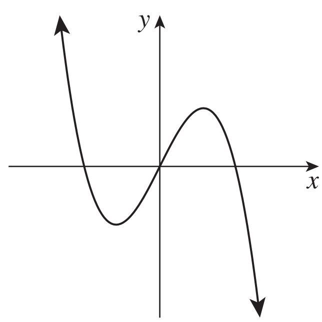
1 Classify each function $y = f(x)$ as even, odd or neither.

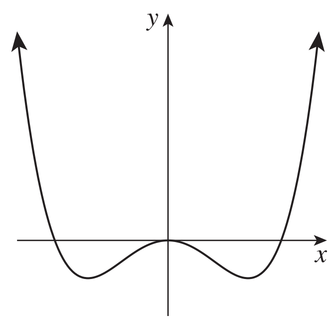
a


b


c


d


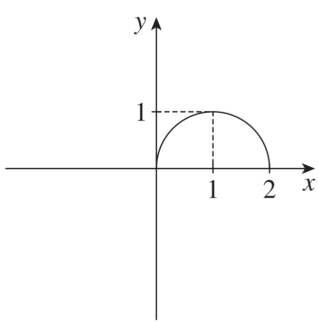
e


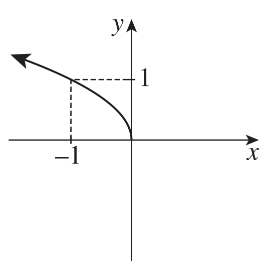
f


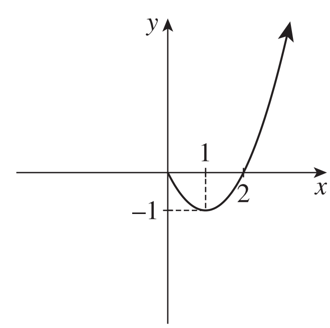
2 In each diagram below, complete the graph so that:

i $f(x)$ is even

ii $f(x)$ is odd.

a


b


c


3 Consider the function $f(x) = x^4 - 2x^2 + 1$.

a Simplify $f(-x)$.

b Hence show that $f(x)$ is an even function.

4 Consider the function $g(x) = x^3 - 3x$.

a Simplify $g(-x)$.

b Hence show that $g(x)$ is an odd function.

5 Consider the function $h(x) = x^3 + 3x^2 - 2$.

a Simplify $h(-x)$.

b Hence show that $h(x)$ is neither even nor odd.

6 Simplify $f(-x)$ for each function, and hence determine whether it is even, odd or neither.

a $f(x) = x^2 - 9$

b $f(x) = x^2 - 6x + 5$

c $f(x) = x^3 - 25x$

d $f(x) = x^4 - 4x^2$

e $f(x) = x^3 + 5x^2$

f $f(x) = x^5 - 16x$

g $f(x) = x^5 - 8x^3 + 16x$

h $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

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7 On the basis of the previous questions, copy and complete these sentences:

a 'A polynomial function is odd if …'.

b 'A polynomial function is even if …'.

DEVELOPMENT

8 Factor each polynomial in parts **a–f** of Question **6** above and write down its zeroes (that is, its x-intercepts). Then use a table of values to sketch its graph. Confirm that the graph exhibits the symmetry established above.

9 [Algebra and Technology]

In Questions **3–8**, the odd and even functions were all polynomials. Other functions can also be classified as odd or even. In each case following, simplify $f(-x)$ and compare it with $f(x)$ and $-f(x)$ to determine whether the function is odd or even. Then confirm your answer by plotting the function on appropriate graphing software.

a $f(x) = \dfrac{2^x + 2^{-x}}{2}$ **b** $f(x) = \dfrac{2^x - 2^{-x}}{2}$ **c** $f(x) = \sqrt[3]{x}$ **d** $f(x) = \left(\sqrt[3]{x}\right)^2$

e $f(x) = \dfrac{x}{x^2 - 4}$ **f** $f(x) = \dfrac{2}{x^2 - 4}$ **g** $f(x) = \sqrt{9 - x^2}$ **h** $f(x) = x\sqrt{9 - x^2}$

10 Determine whether each function is even, odd or neither.

a $f(x) = 2^x$ **b** $f(x) = 2^{-x}$ **c** $f(x) = \sqrt{3 - x^2}$

d $f(x) = \dfrac{1}{x^2 + 1}$ **e** $f(x) = \dfrac{4x}{x^2 + 4}$ **f** $f(x) = 3^x + 3^{-x}$

g $f(x) = 3^x - 3^{-x}$ **h** $f(x) = 3^x + x^3$

CHALLENGE

11 a Explain why the relation $x^2 + (y - 5)^2 = 49$ is even.

b Explain why the relation $x^2 + y^2 = 49$ is both even and odd.

12 a Prove that if $f(x)$ is an odd function defined at $x = 0$, then $y = f(x)$ passes through the origin. Hint: Apply the condition for a function to be odd to $f(0)$.

b If $f(x)$ is an even function defined at $x = 0$, does the graph of $y = f(x)$ have to pass through the origin? Either prove the statement or give a counter-example.

13 a Suppose that $h(x) = f(x) \times g(x)$.

i Show that if f and g are both even or both odd, then $h(x)$ is even.

ii Show that if one of $f(x)$ and $g(x)$ is even and the other odd, then $h(x)$ is odd.

b Suppose that $h(x) = f(x) + g(x)$.

i Show that if $f(x)$ and $g(x)$ are both even, then $h(x)$ is even.

ii Show that if $f(x)$ and $g(x)$ are both odd, then $h(x)$ is odd.

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4D The absolute value function

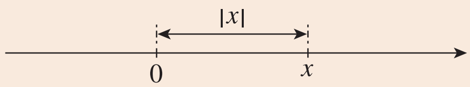
Often it is the size or magnitude of a number that is significant, rather than whether it is positive or negative. *Absolute value* is the mathematical name for this concept.

Absolute value as distance

Distance is the clearest way to define absolute value.

10 ABSOLUTE VALUE AS DISTANCE

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



  For example, $|-5| = 5$ and $|0| = 0$ and $|5| = 5$.
- Distance is always positive or zero, so
  $|x| \geq 0$, for all real numbers x.
- The numbers x and $-x$ are equally distant from the origin, so
  $|-x| = |x|$, for all real numbers x.

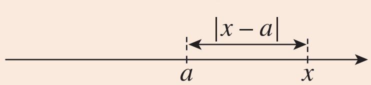
Thus absolute value is a measure of the *size* or *magnitude* of a number. In the examples above, the numbers -5 and $+5$ both have the same magnitude 5, and differ only in their signs.

Distance between numbers

Replacing x by $x - a$ in the previous definition gives a measure of the distance from x to a on the number line.

11 DISTANCE BETWEEN NUMBERS

- The distance from x to a on the number line is $|x - a|$.



  For example, the distance between 5 and -2 is $|5 - (-2)| = 7$.
- If follows that $|x - a| = |a - x|$, for all real numbers x and a.

An expression for absolute value involving cases

If x is a negative number, then the absolute value of x is $-x$, the opposite of x. This gives an alternative definition:

12 ABSOLUTE VALUE INVOLVING CASES

For any real number x, define $|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$

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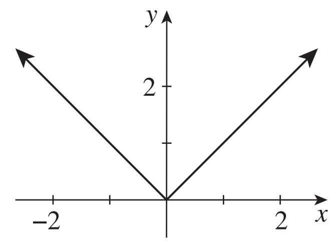
The two cases lead directly to the graph of $y = |x|$.
A table of values confirms the sharp point at the origin where the two branches meet at right angles.



x	-2	-1	0	1	2
$\lvert x\rvert$	2	1	0	1	2

- The domain is the set of all real numbers, and the range is $y \ge 0$.
- The function is even, because the graph has line symmetry in the y-axis.
- The function has a zero at $x = 0$, and is positive for all other values of x.

Graphing functions with absolute value

Transformations can now be applied to the graph of $y = |x|$ to sketch many functions involving absolute value. More complicated functions, however, require the approach involving cases.

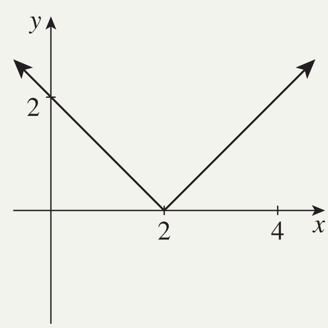
A short table of values is always an excellent safety check.

Example 10 4D

a Sketch $y = |x - 2|$ using shifting.
b Check the graph using a table of values.
c Write down the equations of the two branches.

SOLUTION

a This is $y = |x|$ shifted 2 units to the right.



b

x	0	1	2	3	4
y	2	1	0	1	2

c From the expression using cases, or from the graph:

$$y = \begin{cases} x - 2, & \text{for } x \ge 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$

Example 11 4D

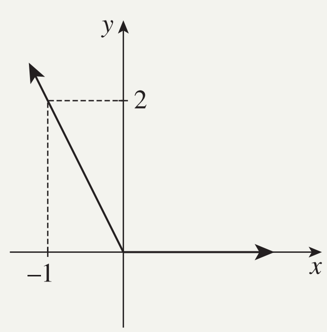
a Use cases to sketch $y = |x| - x$.
b Check using a table of values.

SOLUTION

a Considering separately the cases $x \ge 0$ and $x < 0$,

$$y = \begin{cases} x - x, & \text{for } x \ge 0, \\ -x - x, & \text{for } x < 0, \end{cases}$$

that is, $y = \begin{cases} 0, & \text{for } x \ge 0, \\ -2x, & \text{for } x < 0. \end{cases}$



b Checking using a table of values,

x	-2	-1	0	1	2
y	4	2	0	0	0

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Solving absolute value equations:

Three observations should make everything clear:

- An equation such as $|3x + 6| = -21$ has no solutions, because an absolute value can never be negative.
- An equation such as $|3x + 6| = 0$ is true when $3x + 6 = 0$.
- An equation such as $|3x + 6| = 21$ can be solved by realising that:

$$|3x + 6| = 21 \quad \text{is true when} \quad 3x + 6 = 21 \quad \text{or} \quad 3x + 6 = -21.$$

13 TO SOLVE THE EQUATION $|ax + b| = k$

- If $k < 0$, the equation has no solutions.
- If $k = 0$, the equation has one solution, found by solving
  $ax + b = 0$.
- If $k > 0$, the equation has two solutions, found by solving
  $ax + b = k \qquad \text{or} \qquad ax + b = -k.$

Example 12 4D

Solve each absolute value equation.

a $|3x + 6| = -21$ **b** $|3x + 6| = 0$ **c** $|3x + 6| = 21$

SOLUTION

a $|3x + 6| = -21$ has no solutions, because an absolute value cannot be negative.

b
$$\begin{aligned} |3x + 6| &= 0 \\ 3x + 6 &= 0 \\ \boxed{-6} \qquad 3x &= -6 \\ \boxed{\div 3} \qquad x &= -2 \end{aligned}$$

c
$$\begin{aligned} |3x + 6| &= 21 \\ 3x + 6 &= 21 \text{ or } 3x + 6 = -21 \\ \boxed{-6} \qquad 3x &= 15 \text{ or } 3x = -27 \\ \boxed{\div 3} \qquad x &= 5 \text{ or } x = -9 \end{aligned}$$

Example 13 4D

Solve each absolute value equation.

a $|x - 2| = 3$ **b** $|7 - \frac{1}{4}x| = 3$

SOLUTION

a
$$\begin{aligned} |x - 2| &= 3 \\ x - 2 &= 3 \text{ or } x - 2 = -3 \\ \boxed{+2} \qquad x &= 5 \text{ or } x = -1 \end{aligned}$$

b
$$\begin{aligned} |7 - \tfrac{1}{4}x| &= 3 \\ 7 - \tfrac{1}{4}x &= 3 \text{ or } 7 - \tfrac{1}{4}x = -3 \\ \boxed{-7} \qquad -\tfrac{1}{4}x &= -4 \text{ or } -\tfrac{1}{4}x = -10 \\ \boxed{\times (-4)} \qquad x &= 16 \text{ or } x = 40 \end{aligned}$$

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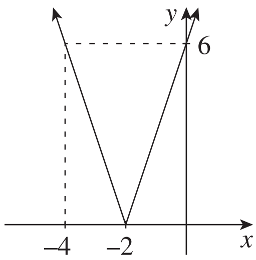
Sketching $y = |ax + b|$:

To sketch a function such as $y = |3x + 6|$, the first step is always to find the x-intercept and y-intercept. As in the example above,

Put $y = 0$, then $|3x + 6| = 0$
$$3x + 6 = 0$$
$$x = -2.$$

Put $x = 0$
Then $y = |0 + 6|$
$$= 6.$$

Plot those two points $(-2, 0)$ and $(0, 6)$.



The graph is symmetric about the vertical line $x = -2$, so the point $(-4, 6)$ also lies on the curve.
Now join the points up in the characteristic V shape.
Alternatively, draw up a small table of values,

x	-4	-3	-2	-1	0
y	6	3	0	3	6

A good final check: The two branches of the curve should have gradients 3 and –3.

Absolute value as the square root of the square

Taking the absolute value of a number means stripping any negative sign from the number. We already have algebraic functions capable of doing this job — we can square the number, then apply the function $\sqrt{\ }$ that says ‘take the positive square root (or zero)’.

14 ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE

- For all real numbers x, $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$.

For example, $|-3|^2 = 9 = (-3)^2$ and $|-3| = \sqrt{9} = \sqrt{(-3)^2}$.

Identities involving absolute value

Here are some standard identities.

15 IDENTITIES INVOLVING ABSOLUTE VALUE

- $|-x| = |x|$, for all x.
- $|x - y| = |y - x|$, for all x and y.
- $|xy| = |x||y|$, for all x and y.
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, for all x, and for all $y \neq 0$.

Substitution of some positive and negative values for x and y should be sufficient to demonstrate these results.

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Exercise 4D

FOUNDATION

1 Evaluate:

a $|3|$ b $|-3|$ c $|4-7|$ d $|7-4|$

e $|14-9-12|$ f $|-7+8|$ g $|3^2-5^2|$ h $|11-16|-8$

2 Solve each absolute value equation, then graph the solution on a number line.

a $|x|=1$ b $|x|=3$ c $|4x|=8$

d $|2x|=10$ e $|2x|=6$ f $|3x|=12$

3 Solve each equation and graph the solution on a number line.

a $|x-4|=1$ b $|x-3|=7$ c $|x-3|=3$ d $|x-7|=2$

e $|x+5|=2$ f $|x+2|=2$ g $|x+1|=6$ h $|x+3|=1$

4 a Copy and complete the tables of values for the functions $y=|x-1|$ and $y=|x|-1$.

x	-2	-1	0	1	2	3
$\lvert x-1\rvert$						

x	-2	-1	0	1	2	3
$\lvert x\rvert-1$						

b Draw the graphs of the two functions on separate number planes, and observe the similarities and differences between them.

c Explain how each graph is obtained by shifting $y=|x|$.

5 Show that each statement is true when $x=-3$.

a $|5x|=5|x|$ b $|-x|=|x|$ c $|x|^2=x^2$

d $|x-7|=|7-x|$ e $x\le|x|$ f $-|x|\le x$

6 Show that each statement is false when $x=-2$.

a $|x|=x$ b $|-x|=x$ c $|x+2|=|x|+2$

d $|x+1|=x+1$ e $|x-1|<|x|-1$ f $|x|^3=x^3$

DEVELOPMENT

7 In each case, use the rules of Box 13 to solve the equation for x.

a $|7x|=35$ b $|2x+1|=3$ c $|2x-1|=11$

d $|7x-3|=-11$ e $|3x+2|=-8$ f $|5x+2|=0$

g $|3x-5|=0$ h $|6x-7|=5$ i $|5x+4|=6$

8 a Consider the equation $|1-2x|=3$.

i Explain why this equation has the same solution as $|2x-1|=3$. (Refer to Box 11.)

ii Hence, or otherwise, solve the equation.

b Likewise, solve these equations.

i $|3-2x|=1$ ii $|1-3x|=2$

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9 **a** Use cases to help sketch the branches of $y = |x|$.

b In each part, identify the shift or shifts of $y = |x|$ and hence sketch the graph. Then write down the equations of the two branches.

i $y = |x - 3|$ **ii** $y = |x + 2|$ **iii** $y = |x| - 2$

iv $y = |x| + 3$ **v** $y = |x - 2| - 1$ **vi** $y = |x + 1| - 1$

10 Sketch each function using cases. Check the graph with a table of values.

a $y = |2x|$ **b** $y = \left|\frac{1}{2}x\right|$

11 Find the x-intercept and y-intercept of each function. Then sketch the graph using symmetry, and confirm with a small table of values.

a $y = |2x - 6|$ **b** $y = |9 - 3x|$ **c** $y = |5x|$

d $y = |4x + 10|$ **e** $y = -|3x + 7|$ **f** $y = -|7x|$

12 [Technology]

Use suitable graphing software to help solve these problems.

a **i** Sketch $y = |x - 4|$ and $y = 1$ on the same set of axes, clearly showing the points of intersection.

ii Hence write down the solution of $|x - 4| = 1$.

b Now use similar graphical methods to solve each of the following.

i $|x + 3| = 1$ **ii** $|2x + 1| = 3$ **iii** $|3x - 3| = -2$ **iv** $|2x - 5| = 0$

CHALLENGE

13 Consider the absolute value function $f(x) = |x|$.

a Use the result $f(x) = \sqrt{x^2}$ given in Box 14 to help prove that the absolute value function is even.

b Why was this result obvious from the graph of $y = |x|$?

14 **a** For what values of x is $y = \dfrac{|x|}{x}$ undefined?

b Use a table of values of x from -3 to 3 to sketch the graph.

c Hence write down the equations of the two branches of $y = \dfrac{|x|}{x}$.

15 Sketch each graph by drawing up a table of values for $-3 \le x \le 3$. Then use cases to determine the equation of each branch of the function.

a $y = |x| + x$ **b** $y = |x| - x$

c $y = 2(x + 1) - |x + 1|$ **d** $y = x^2 - |2x|$

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4E Composite functions

Shifting, reflecting, and taking absolute value, are all examples of a far more general procedure of creating a composite function from two given functions. The example below shows how a translation left 3 and a translation up 3 are obtained using composites. In this section, however, attention is on the algebra rather than on the final graph.

Composition of functions

Suppose that we are given the two functions $f(x) = x + 3$ and $g(x) = x^2$. We can put them together by placing their function machines so that the output of the first function is the input of the second function

$$\begin{array}{ccccccccc} x & & & & f(x) & & & & g\big(f(x)\big) \\ -2 & \longrightarrow & & \longrightarrow & 1 & \longrightarrow & & \longrightarrow & 1 \\ -1 & \longrightarrow & & \longrightarrow & 2 & \longrightarrow & & \longrightarrow & 4 \\ 0 & \longrightarrow & \boxed{\text{Add 3}} & \longrightarrow & 3 & \longrightarrow & \boxed{\text{Square}} & \longrightarrow & 9 \\ 1 & \longrightarrow & & \longrightarrow & 4 & \longrightarrow & & \longrightarrow & 16 \\ 2 & \longrightarrow & & \longrightarrow & 5 & \longrightarrow & & \longrightarrow & 25 \\ x & \longrightarrow & & \longrightarrow & x + 3 & \longrightarrow & & \longrightarrow & (x + 3)^2 \end{array}$$

The middle column is the output of the first function 'Add 3'. This output is then the input of the second function 'Square'. The result is the *composite function*

$$g\big(f(x)\big) = (x + 3)^2 \qquad \text{'Add 3, then square.'}$$

If the functions are composed the other way around, the result is different,

$$f\big(g(x)\big) = x^2 + 3 \qquad \text{'Square, then add 3.'}$$

Notice how in this example, both ways around are examples of translations.

- The composite graph $y = g\big(f(x)\big)$ is $y = g(x)$ shifted left 3.
- The composite graph $y = f\big(g(x)\big)$ is $y = g(x)$ shifted up 3.

Example 14 4E

Find and simplify $k\big(h(x)\big)$ and $h\big(k(x)\big)$ when $h(x) = 2x + 3$ and $k(x) = 1 - 5x$.

SOLUTION

$$\begin{aligned} k\big(h(x)\big) &= k(2x + 3) \\ &= 1 - 5(2x + 3) \\ &= -10x - 14 \end{aligned}$$

$$\begin{aligned} h\big(k(x)\big) &= h(1 - 5x) \\ &= 2(1 - 5x) + 3 \\ &= -10x + 5 \end{aligned}$$

Domain and range of the composite function

In Example 14 above, $h(x)$ and $k(x)$ have domain and range all real numbers, so there are no problems, and the domains and ranges of both composites are all real numbers.

In the original example with the function machines, $f(x)$ and $g(x)$ again both have domain all real numbers, and so do $g\big(f(x)\big)$ and $f\big(g(x)\big)$. But while $f(x)$ has range all real numbers, $g(x)$ has range $y \geq 0$. Thus the range of $g\big(f(x)\big) = (x + 3)^2$ is $y \geq 0$, and the range of $f\big(g(x)\big) = x^2 + 3$ is $y \geq 3$.

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In general, neither the domain nor the range of $g\big(f(x)\big)$ are all real numbers.

- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

Example 15 4E

Find the domain and range of $g\big(f(x)\big)$ if $f(x) = \sqrt{x-4}$ and $g(x) = \dfrac{1}{x}$.

SOLUTION

The domain of $f(x)$ is $x \ge 4$, and the domain of g(x) is $x \ne 0$.
When $x = 4$, $f(4) = 0$, and when $x > 4$, $f(x) > 0$, so $g\big(f(x)\big)$ is not defined at $x = 4$, but is defined for $x > 4$.
Thus the domain of $g\big(f(x)\big)$ is $x > 4$.
When restricted to $x > 4$, the range of $f(x)$ is $y > 0$, and when g(x) is restricted to $x > 0$, its range is $y > 0$ (note the change of variable as output becomes input). Thus the range of $g\big(f(x)\big)$ is $y > 0$.

This approach, however, is rather elaborate. It is almost always enough to find the equation of the composite and look at it as a single function. In this example,

$$g\big(f(x)\big) = \frac{1}{\sqrt{x-4}},$$

from which it is easily seen that the domain is $x > 4$ and the range is $y > 0$.

16 COMPOSITE FUNCTIONS:

- The *composites* of two functions $f(x)$ and $g(x)$ are $g\big(f(x)\big)$ and $f\big(g(x)\big)$.
- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and then $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

It is almost always enough to read the domain and range from the equation of the composite function.

The notations $(g \circ f)(x)$ for the composite $g\big(f(x)\big)$, and $(f \circ g)(x)$ for $f\big(g(x)\big)$, are widely used, but are not required in this course.

The empty function

Let $f(x) = -x^2 - 1$ and $g(x) = \sqrt{x}$. Then

$$g\big(f(x)\big) = \sqrt{-x^2 - 1}.$$

This is a problem, because $\sqrt{-x^2-1}$ is undefined, whatever the value of x. The range of $f(x)$ is all real numbers less than or equal to -1, and g(x) is undefined on all of these because you can't take the square root of a negative. Thus $g\big(f(x)\big)$ has domain the empty set, and its range is therefore also the empty set. It is *the empty function*.

The empty function has domain the empty set, and its range is therefore also the empty set. For those interested in trivialities, the empty function is one-to-one.

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Exercise 4E

FOUNDATION

1 Consider the function $f(x) = x + 2$.

a Find the values of:

i $f\big(f(0)\big)$ ii $f\big(f(3)\big)$ iii $f\big(f(-1)\big)$ iv $f\big(f(-8)\big)$

b Find expressions for:

i $f\big(f(x)\big)$ ii $f\Big(f\big(f(x)\big)\Big)$

c Find the value of x for which $f\big(f(x)\big) = 0$.

2 Consider the function $F(x) = 2x$.

a Find the values of $F\big(F(0)\big)$, $F\big(F(7)\big)$, $F\big(F(-3)\big)$ and $F\big(F(-11)\big)$.

b Find expressions for $F\big(F(x)\big)$ and for $F\Big(F\big(F(x)\big)\Big)$

c Find the value of x for which $F\big(F(x)\big) = 32$.

3 Consider the function $g(x) = 2 - x$.

a Find the values of $g\big(g(0)\big)$, $g\big(g(4)\big)$, $g\big(g(-2)\big)$ and $g\big(g(-9)\big)$.

b Show that $g\big(g(x)\big) = x$.

c Show that $g\Big(g\big(g(x)\big)\Big) = g(x)$.

4 Consider the function $h(x) = 3x - 5$.

a Find the values of $h\big(h(0)\big)$, $h\big(h(5)\big)$, $h\big(h(-1)\big)$ and $h\big(h(-5)\big)$.

b Find expressions for $h\big(h(x)\big)$ and for $h\Big(h\big(h(x)\big)\Big)$.

5 Two linear functions are defined by $f(x) = x + 1$ and $g(x) = 2x - 3$.

a Find the values of $f\big(g(7)\big)$, $g\big(f(7)\big)$, $f\big(f(7)\big)$ and $g\big(g(7)\big)$.

b Find expressions for:

i $f\big(g(x)\big)$ ii $g\big(f(x)\big)$ iii $f\big(f(x)\big)$ iv $g\big(g(x)\big)$

c What transformation maps the graph of $y = g(x)$ to the graph of $y = g\big(f(x)\big)$?

d What transformation maps the graph of $y = g(x)$ to the graph of $y = f\big(g(x)\big)$?

6 Two functions $\ell(x)$ and $q(x)$ are defined by $\ell(x) = x - 3$ and $q(x) = x^2$.

a Find the values of $\ell\big(q(-1)\big)$, $q\big(\ell(-1)\big)$, $\ell\big(\ell(-1)\big)$ and $q\big(q(-1)\big)$.

b Find:

i $\ell\big(q(x)\big)$ ii $q\big(\ell(x)\big)$ iii $\ell\big(\ell(x)\big)$ iv $q\big(q(x)\big)$

c Determine the domains and ranges of:

i $\ell\big(q(x)\big)$ ii $q\big(\ell(x)\big)$

d What transformation maps the graph of $y = q(x)$ to the graph of $y = q\big(\ell(x)\big)$?

e What transformation maps the graph of $y = q(x)$ to the graph of $y = \ell\big(q(x)\big)$?

DEVELOPMENT

7 Suppose that $F(x) = 4x$ and $G(x) = \sqrt{x}$.

a Find the values of $F\big(G(25)\big)$, $G\big(F(25)\big)$, $F\big(F(25)\big)$ and $G\big(G(25)\big)$.

b Find $F\big(G(x)\big)$.

c Find $G\big(F(x)\big)$.

d Hence show that $F\big(G(x)\big) = 2G\big(F(x)\big)$.

e State the domain and range of $F\big(G(x)\big)$.

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8 Two functions f and h are defined by $f(x) = -x$ and $h(x) = \frac{1}{x}$.

a Find the values of $f\big(h\big(-\frac{1}{4}\big)\big)$, $h\big(f\big(-\frac{1}{4}\big)\big)$, $f\big(f\big(-\frac{1}{4}\big)\big)$ and $h\big(h\big(-\frac{1}{4}\big)\big)$.

b Show that for all $x \neq 0$:

i $f\big(h(x)\big) = h\big(f(x)\big)$ **ii** $f\big(f(x)\big) = h\big(h(x)\big)$

c Write down the domain and range of $f\big(h(x)\big)$.

d Describe how the graph of $h(x)$ is transformed to obtain the graph of $h\big(f(x)\big)$.

9 Suppose that $f(x) = -5 - |x|$ and $g(x) = \sqrt{x}$.

a Find $f\big(g(x)\big)$, state its domain and range, and sketch its graph.

b Explain why $g\big(f(x)\big)$ is the empty function.

10 a Show that if $f(x)$ and $g(x)$ are odd functions, then $g\big(f(x)\big)$ is odd.

b Show that if $f(x)$ is an odd function and $g(x)$ is even, then $g\big(f(x)\big)$ is even.

c Show that if $f(x)$ is an even function, then $g\big(f(x)\big)$ is even.

CHALLENGE

11 Find the composite functions $g\big(f(x)\big)$ and $g\big(f(x)\big)$.

a $f(x) = 4$, for all x, and $g(x) = 7$, for all x.

b $f(x) = x$, $g(x)$ any function.

12 Let $f(x) = 2x + 3$ and $g(x) = 5x + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the value of b so that $g\big(f(x)\big) = f\big(g(x)\big)$, for all x.

13 a Let $f(x)$ be any function, and let $g(x) = x - a$, where a is a constant. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

b Let $f(x)$ be any function, and let $g(x) = -x$. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

14 Let $f(x) = 2x + 3$ and $g(x) = ax + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the values of a and b so that $g\big(f(x)\big) = x$, for all x.

c Show that if a and b have these values, then $f\big(g(x)\big) = x$, for all x.

15 Let $f(x) = x^2 + x - 3$ and $g(x) = |x|$.

a Find $f\big(g(0)\big)$, $g\big(f(0)\big)$, $f\big(g(-2)\big)$ and $g\big(f(-2)\big)$.

b Write an expression for $f\big(g(x)\big)$ without any use of absolute value, given that:

i $x \geq 0$ **ii** $x < 0$

16 Let $L(x) = x + 1$ and $Q(x) = x^2 + 2x$.

a State the ranges of $L(x)$ and $Q(x)$.

b Find $L\big(Q(x)\big)$ and determine its range.

c Find $Q\big(L(x)\big)$ and determine its range.

d Find the zeroes of $Q\big(L(x)\big)$.

e Show that $Q\big(L\big(\frac{1}{x+1}\big)\big) = \dfrac{(x+2)(3x+4)}{(x+1)^2}$, provided that $x \neq -1$.

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Chapter 4 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 **a** Copy and complete the table of values for $y = x^2$ and $y = (x - 2)^2$.

x	-2	-1	0	1	2	3	4
x^2							
$(x-2)^2$							

b Sketch the two graphs and state the vertex of each.

c What transformation maps $y = x^2$ to $y = (x - 2)^2$?

2 Consider the parabola $y = x^2 - 2x$.

a Show that when this is reflected in the y-axis the equation becomes $y = x^2 + 2x$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = x^2 + 2x$.

x	-3	-2	-1	0	1	2	3
$x^2 - 2x$							
$x^2 + 2x$							

c Sketch the two parabolas and state the vertex of each.

3 Evaluate:

a $|-7|$ **b** $|4|$ **c** $|3 - 8|$

d $|-2 - (-5)|$ **e** $|-2| - |-5|$ **f** $|13 - 9 - 16|$

4 Solve for x:

a $|x| = 5$ **b** $|3x| = 18$ **c** $|x - 2| = 4$

d $|x + 3| = 2$ **e** $|2x - 3| = 5$ **f** $|3x - 4| = 7$

5 Explain how to shift the graph of $y = x^2$ to obtain each function.

a $y = x^2 + 5$ **b** $y = x^2 - 1$ **c** $y = (x - 3)^2$ **d** $y = (x + 4)^2 + 7$

6 Write down the equation of the monic quadratic with vertex:

a $(1, 0)$ **b** $(0, -2)$ **c** $(-1, 5)$ **d** $(4, -9)$

7 Write down the centre and radius of each circle. Shifting may help locate the centre.

a $x^2 + y^2 = 1$ **b** $(x + 1)^2 + y^2 = 4$

c $(x - 2)^2 + (y + 3)^2 = 5$ **d** $x^2 + (y - 4)^2 = 64$

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8 In each case, find the function obtained by the given reflection or rotation.

a $y = x^3 - 2x + 1$: reflect in the y-axis

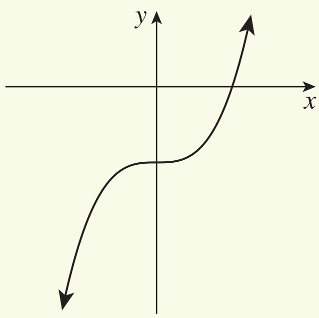
b $y = x^2 - 3x - 4$: reflect in the x-axis

c $y = 2^x - x$: rotate 180° about the origin

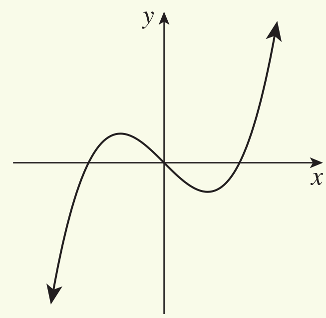
d $y = \sqrt{9 - x^2}$: reflect in the y-axis

9 Classify each function $y = f(x)$ as odd, even or neither.

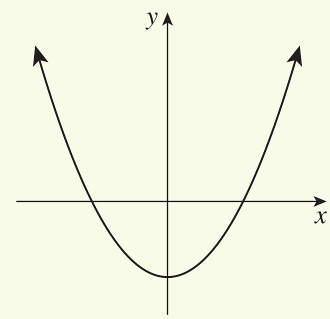
a



b



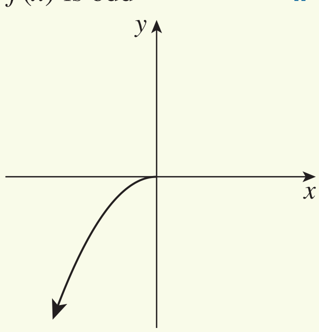
c



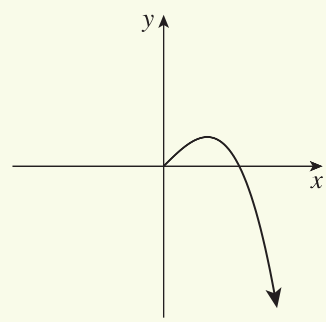
10 In each diagram below, complete the graph so that:

i $f(x)$ is odd **ii** $f(x)$ is even.

a



b



11 Sketch each graph by shifting $y = |x|$, or by using a table of values. Mark all x- and y-intercepts.

a $y = |x| - 2$ **b** $y = |x - 2|$ **c** $y = |x + 2|$ **d** $y = |x| + 2$

12 Sketch each graph by finding the x-intercept and y-intercept and then using symmetry. Perhaps also use a table of values to confirm the graph.

a $y = |3x + 9|$ **b** $y = -|2x - 8|$ **c** $y = |4x + 13|$

13 Solve these absolute value equations:

a $|3x| = 15$ **b** $|x + 4| = 5$ **c** $|x + 4| = -5$ **d** $|5 - x| = 7$

e $|2x + 7| = 9$ **f** $|3x - 8| = 4$ **g** $|7x + 2| = 0$ **h** $|x^2 - 25| = 0$

14 Find $f(-x)$ for each function, and then decide whether the function is odd, even or neither.

a $f(x) = x + 3$ **b** $f(x) = 2x^2 - 5$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = \dfrac{x}{x^2 + 1}$

15 For each parabola, complete the square to find the coordinates of the vertex.

a $y = x^2 - 2x + 5$ **b** $y = x^2 + 4x - 3$ **c** $y = 2x^2 + 8x + 11$ **d** $y = -x^2 + 6x + 1$

16 Use completion of the square to help sketch the graph of each quadratic function. Indicate the vertex and all intercepts with the axes.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 4x + 1$ **c** $y = 2 + 2x - x^2$ **d** $y = x^2 - x - 1$

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17 Complete the squares to find the centre and radius of each circle.

a $x^2 + y^2 - 2y = 3$ **b** $x^2 + 6x + y^2 + 8 = 0$

c $x^2 - 4x + y^2 + 6y - 3 = 0$ **d** $x^2 + y^2 - 8x + 14y = 35$

18 Consider the cubic with equation $y = x^3 - x$.

a Use an appropriate substitution to show that when the graph of this function is shifted right 1 unit the result is $y = x^3 - 3x^2 + 2x$.

b [Technology]
Plot both cubics using graphing software to confirm the outcome.

19 Given that $f(x) = 5x - 2$ and $g(x) = x^2 + 3$, find:

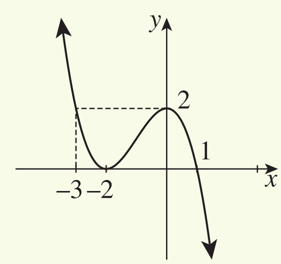
a $f\big(g(0)\big)$ **b** $g\big(f(0)\big)$ **c** $f\big(g(4)\big)$

d $g\big(f(4)\big)$ **e** $f\big(g(a)\big)$ **f** $g\big(f(a)\big)$

20 Find the domain and range of the composite functions $f\big(g(x)\big)$ and $f\big(g(x)\big)$ given that:

a $f(x) = x - 1$ and $g(x) = \sqrt{x}$ **b** $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 1$

21 The graph drawn to the right shows the curve $y = f(x)$. Use this graph to sketch the following.



a $y = f(x + 1)$ **b** $y = f(x) + 1$

c $y = f(x - 1)$ **d** $y = f(x) - 1$

e $y = f(-x)$ **f** $y = -f(x)$

g $y = -f(-x)$

22 [A revision medley of curve sketches]
Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

a $y = 2x$, $y = 2x + 3$, $y = 2x - 2$

b $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$

c $y = x + 3$, $y = 3 - x$, $y = -x - 3$

d $y = (x - 2)^2 - 1$, $y = (x + 2)^2 - 1$, $y = -(x + 2)^2 + 1$

e $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$

f $(x - 1)^2 + y^2 = 1$, $(x + 1)^2 + y^2 = 1$, $x^2 + (y - 1)^2 = 1$

g $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$

h $y = (x + 2)^2$, $y = (x + 2)^2 - 4$, $y = (x + 2)^2 + 1$

i $y = -|x|$, $y = -|x| + 1$, $y = |x - 2|$

j $y = \sqrt{x}$, $y = \sqrt{x} + 1$, $y = \sqrt{x + 1}$

k $y = 2^x$, $y = 2^x - 1$, $y = 2^{x-1}$

l $y = \dfrac{1}{x}$, $y = \dfrac{1}{x - 2}$, $y = \dfrac{1}{x + 1}$

m $y = x^3$, $y = x^3 - 1$, $y = (x - 1)^3$

n $y = x^4$, $y = (x - 1)^4$, $y = x^4 + 1$

o $y = \sqrt{x}$, $y = -\sqrt{x}$, $y = 2 - \sqrt{x}$

p $y = 2^{-x}$, $y = 2^{-x} - 2$, $y = 2 - 2^x$

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