A Simplified Approach to Three Mutually Orthogonal Latin Squares of Order 10

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Abstract

This research considers the problem of finding three Mutually Orthogonal Latin Squares of order 10 and propose an approach looking at the frequency distributions in the 5×5 sub-squares I and IV. Based on an extension of Mann theorem, an IP model is formulated to find assignments between 141 frequency distributions, which were found for Latin Squares of order 10. A feasible IP model shows there are enough elements in sub-squares I and IV to form a pair of orthogonal Latin Squares. The solutions of the IP model are used to find triples of mutually assignable distributions of Latin squares of order 10. The results for Latin Squares of order 6 are also present as an example to illustrate how the structural approach works.

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Chapter 1

Introduction

The existence of three mutually orthogonal Latin squares of order 10 has been debated by a number of researchers, across different mathematical disciplines. The main challenge has been the large number of Latin squares of order 10 which means that exhaustively searching through all possibilities is limited by computational capability. This paper follows a simplified approach, looking at the 5×5 sub-square structures of Latin squares of order 10 and thus reducing the number of Latin squares in the search space.

This introductory chapter starts by presenting the basics of Latin squares, orthogonality, and mutually orthogonal Latin squares. Then, it dwells deeper into the theories behind Mutually Orthogonal Latin Squares. Once all the basic ideas are set out, the research questions are introduced. The final section of this chapter goes through the structure of the rest of the paper.

1.1 The Basics of Latin Squares

A Latin square of order m is an $m \times m$ matrix, where the numbers 1 to m occur exactly once in each row and column. Latin squares have applications in many fields including experimental design, coding theory, and algebra (Laywine and Mullen, 1998). Consider an example in experimental design. Let there be 16 test subjects, each belonging to one of 4 age groups and one of 4 health statuses. The experiment is to test 4 types of drug on these subjects such that no 2 subjects in the same age group or health status get the same type of drug. This experiment can be designed into a Latin square of order 4, where the rows account for the health statuses (A, B,

	a	b	b	c
A	1	2	3	4
B	2	3	4	1
C	3	4	1	2
D	4	1	2	3

Table 1.1: A Latin square of order 4 (Laywine and Mullen, 1998)

C, D) and the columns account for the age group (a, b, c, d). Table 1.1 shows one way the types of drug (1, 2, 3, 4) can be assigned to the subjects following the stated requirements. Without loss of generality, we usually keep the first row of the Latin square in the natural order. Any Latin square with both the first row and column in natural order would be called a reduced Latin square.

1.2 Orthogonality

To best illustrate the matter of orthogonality, let us continue with the previous example. Additionally, we also want to test the subjects under 4 different treatments under similar constraints that no 2 subjects in the same age group or health status get the same treatment. An additional arrangement such that no 2 test subjects receive the same combination of drug type and treatment could be made. This problem can be solved by finding two Latin squares of order 4, L_1 and L_2, such that when the cells of L_1 and L_2 are super-imposed, all 16 distinct combinations of 4 drugs and 4 treatments are presented in the combined matrix (Table 1.2). Such L_1 and L_2 are said to be orthogonal.

(a) L_1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) L_2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

(c) (L_1,L_2)

1,1	2,2	3,3	4,4
2,3	1,4	4,1	3,2
3,4	4,3	1,2	2,1
4,2	3,1	2,4	1,3

Table 1.2: (a) L_1 represents the arrangement for 4 types of drug; (b) L_2 represents arrangement for 4 treatment; (c) The arrangement of L_1 and L_2 is one solution to the describe problems

The formal definition is that two Latin squares of order m are orthogonal if their super-imposed matrix contains all m^2 ordered pairs of numbers from 1 to m.

1.3 Mutually orthogonal Latin squares

A set of Latin squares is said to be a set of Mutually Orthogonal Latin Squares (MOLS) if any two distinct squares within the set are orthogonal. For the above example, consider another requirement that these 16 subjects are exposed under 4 conditions such that each test subject receives a distinct combination of drug types, treatments and condition exposure. Hence, we need another Latin square L_3, which must be orthogonal to L_1 and L_2. The set consisting of L_1, L_2 and L_3 is the complete set of three mutually orthogonal Latin squares of order 4.

(a) L_3

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

(b) (L_1,L_2,L_3)

1,1,1	2,2,2	3,3,3	4,4,4
2,3,4	1,4,3	4,1,2	3,2,1
3,4,2	4,3,1	1,2,4	2,1,3
4,2,3	3,1,4	2,4,1	1,3,2

Table 1.3: L_1, L_2, and L_3 give a complete set of MOLS of order 4

This set of three mutually orthogonal Latin squares of order 4 is the largest possible set because it is impossible to find another Latin square that is mutually orthogonal to the 3 existing Latin squares.

Theorem 1.1. *Let $N(m)$ be the maximum number of MOLS of order m. For each $m \geq 2$, $N(m) \leq m - 1$.*

Theorem 1.2. *For all Latin Squares of order p where p is a power of a prime number, $N(p) = p - 1$.*

Euler (1782) conjectured that there is no pair of orthogonal Latin squares of order $2(2k+1)$. In other words, when the order is an odd multiple of 2, there is no pair of Latin squares of such order.

For example, let there be a Latin square of order $2n$, where n is an odd number. When $n = 1$ there is no pair of orthogonal Latin Square of order 2. This is easy to show as there are only 2 Latin squares of order 2. One is Table 1.4 (a) and the other one (Table 1.4 (b)) can be made by permuting the columns or rows of the first one. When super-imposed, it is impossible to get distinct pairs of numbers from 1 to 2, in these 4 cells.

When $n = 3$, there is no orthogonal Latin Square of order 6. This is proven by Tarry (1900) through exhaustive search. When $n = 5$, for a very long time, people

(a)

1	2
2	1

(b)

2	1
1	2

Table 1.4: The only 2 Latin square of order 2

believed that Euler's conjecture holds and there is no pair of orthogonal Latin Squares of order 10. However, in 1959, Bose and Shrikhande disproved this conjecture by showing that there exist mutually orthogonal pairs of order 22. Then, in 1960, Bose, Shrikhande, and Parker gave further results for mutually orthogonal Latin squares of order 10. In fact, for every n which is an odd number larger than 3, there exist pairs of mutually orthogonal Latin squares.

Based on Theorem 1.1, we only know the upper bound for the theoretical maximum number of MOLS of order m when m is not a power of a prime number. The first case of m when m is not a power of a prime number is $m = 6$, for which we have seen that no pair of orthogonal Latin square exists. The next case is when $m = 10$, where pairs of orthogonal Latin squares are found. However, we do not know explicitly if $N(10) = 2$. Hence, this leads to the open question, do three mutually orthogonal Latin squares of order 10 exist?

1.4 The structural approach to Orthogonal Latin Squares

The approach to tackle the problem of finding three mutually orthogonal Latin squares of order 10 in this paper is looking at the structure of the 5×5 sub-squares. Figure 1 shows one possible format of these sub-squares.

Sub-square I can be formed from any combination of 5 rows and 5 columns. Sub-square II would be formed from the same rows as sub-square I and the 5 columns that are left. Sub-square III would be formed from the same columns as sub-square I and the 5 rows that are left. Sub-square IV would be formed from the 5 rows and 5 columns not chosen in sub-square I. Note: Without loss of generality, we shall always represent sub-square I as being formed by the first 5 rows and columns as in Figure 1.1.

Any Latin square of even order can be divided into four sub-squares of equal order as described above. The frequency of appearance of each element from 1 to $2n$ in the

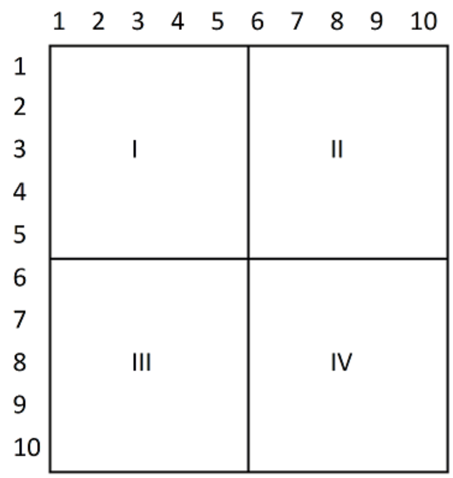

Figure 1.1: Sub-square formation of Latin square of order 10

sub-square I is the same as in sub-square IV. For example, for Latin squares of order 10, if all element 1 appears five times in sub-square I, this means no element 1 can appear in sub-square II or III which are adjacent to sub-square I. Therefore, all of the remaining 5 occurrences of element 1 must appear in sub-square IV. A frequency distribution of $2n$ elements represents the sum of frequency of each element from 1 to $2n$ in sub-squares I and IV.

The frequency distributions are presented as an array of length $2n$ in descending order and denoted by D. Let f_i be the frequency value of element i in distribution D of a $2n \times 2n$ Latin square. We present D by $[f_1\ f_2\ f_3 \dots f_{2n}]$ where $f_1 \geq f_2 \geq f_3 \geq ... \geq f_{2n}$, and $\sum_{i=n}^{2n} f_i = 2n^2$. For example, a frequency distribution of Latin squares of order 10 could be [10 10 10 8 4 4 2 2 0 0].

1.5 The research questions

By investigating the frequency distribution of number from 1 to 10 inside the sub-square matrices of Latin Square of order 10, we aim to answer the following questions:

1. From the frequency distributions, how many possibilities of mutually orthogonal pairs and triplets can be eliminated for Latin squares of order 10?

2. Which structures can form mutually orthogonal pairs?

3. Which structures can form mutually orthogonal triplets?

These questions will be answered using Integer Programming models and Graph Theory.

1.6 The report structure

Chapter 2 of this report will discuss the Methodology which includes the technical notations, the Integer Programming model and application of Graph Theory in answering the above questions. Chapter 3 will present an example of how the methodology is applied to Latin squares of order 6, which are the smaller cases and the result is known. Chapter 4 will report and discuss the results for Latin squares of order 10. Chapter 5 is the summary of the report and suggestion of further research. The full tables of results, documentation for AMPL codes, documentation for Python codes, and details of the Excel spreadsheets are included in the Appendices.

Chapter 2

Methodology

2.1 Introduction

This chapter presents the main idea behind Mann's theorem, and how this is extended to construct an Integer Programming (IP) model to find orthogonal assignments between Latin square structures. From this IP model results, the next goal is to find the structural properties of the orthogonal Latin squares triplets. The approach consists of three phases. In the first phase, all the frequency distributions are found and checked for their upper and lower limits compatibility. The second phase is to use the IP model to find pairs of assignable frequency distributions. Finally, in the third phase, an algorithm based on graph theory is developed to find sets of three frequency distributions which have possibility to form three mutually orthogonal Latin squares of order 10.

2.2 Preliminaries

Definition 2.1 (Isomorphic Latin squares). Let L be a Latin square. If the rows, columns and symbols of L are permuted to produce another Latin square L', then L and L' are isomorphic (Appa et al., 2002).

For example, let L and L' be two Latin squares of order 4 given in figure 2.1; L' can be produced by switching the 2nd and 3rd columns of L. Clearly, L and L' are isomorphic.

1	3	2	4
2	4	3	1
3	1	4	2
4	2	1	3

C2 $\Leftrightarrow$ C3

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Figure 2.1: the Latin square L is shown on the left and L' is shown on the right, columns 2 and 3 are switched

Property 2.1 (Even frequency values in sub-squares I and IV). For a Latin square of order $2n$, $(n \in Z)$, the frequency of numbers from 1 to $2n$ are identical in sub-squares I and IV, and in sub-squares II and III. The frequency distribution in question always has even values and is the representation of sub-squares I and IV combined.

The frequency distributions are important in identifying whether a Latin square can have an orthogonal mate. The idea of using frequency distributions of sub-squares I and IV is an extension of Mann's theorem carried out by Professor Appa (2015, personal communication).

Theorem 2.1 (Mann's Theorem). *If in the Latin square L of size $4k+2$ the sub-square I formed by the first $2k+1$ rows and the first $2k+1$ columns contains fewer than $k+1$ numbers which are different from $1, 2, \ldots, 2k+1$ then L is not a basis square (Mann, 1944).*

We restate Mann in our terminology. A Latin square L of size $4k+2$ cannot have an orthogonal mate if the sub-square I and IV of L has fewer than $2k+2$ numbers which are different from 1 to $2k+1$. This theorem is sufficient in determining whether a frequency distribution is unable to have an orthogonal mate. However, it is not necessary as some frequency distributions of Latin square of order 6 are not ruled out by Mann's theorem but they still have no orthogonal mate.

Property 2.2 (The complement of a distribution). Let D be a distribution of a Latin square of order 10. If D has frequency distribution given by $[f_1, f_2, f_3, ..., f_{10}]$, then the complement of D, denoted by, D', has frequency distribution $[f'_1, f'_2, f'_3, ..., f'_{10}]$ such that $f'_i = 10 - f_{11-i}$ for $i = 1, 2, \ldots, 10$.

For an even order Latin square, when sub-squares I and IV have frequency distribution D, sub-squares II and III will have frequency distribution D' which is the complement.

From definition 2.1, the Latin square remains identical after permutation involving rows, columns and symbols. Hence, the columns of the sub-squares I and II can be completely switched so that sub-square I becomes sub-square II and vice versa. This would effectively switch sub-square III and IV at the same time. Therefore, pairs of complementary distributions are essentially describing the same Latin square.

Definition 2.2 (Self-complement frequency distribution). Let D be a frequency distribution of sub-squares I and IV of an even order Latin square. If the complement D' is the same as D then D is a self-complementary frequency distribution or a self-complement.

2.3 Phase One: General Preparation

2.3.1 Listing frequency distributions

In this phase, we find all frequency distributions for a given n. The systematic way is to partition the structure where only the first n elements are present in sub-squares I and IV of a Latin square of order n. For a 10×10 Latin square, this is obtained by partitioning the frequency distribution where all 10 numbers of the first 5 elements appear in sub-squares I and IV following these rules. (1) They are the combinations of 10 even numbers from 0 to 10; (2) The sum of the frequency equals 50; and (3) The numbers are kept in descending order. For example, from [10 10 10 10 10 0 0 0 0 0], we get [10 10 10 10 8 2 0 0 0 0] by partitioning the last number which is 10 into an 8 and a 2. We continue this until all partitions are exhausted. This leads to 141 distributions listed in Appendix A.1.

2.3.2 Lower and upper limits

We need to define the lower and upper bounds of the frequency distribution as well as the lower and upper limits of frequency values that its orthogonal mate can take.

Definition 2.3 (Lower and upper bounds). We shall call the smallest and largest values of a frequency distribution D the lower and upper bounds and denote this as $[f_1, f_{2n}]$.

Definition 2.4 (Lower and upper limits). The smallest and largest frequency values of a possible orthogonal mate is the lower and upper limits, which will be denoted as $[a, b]$.

No.	1	2	3	4	5	6	7	8	9	10
D	10	10	10	8	4	4	2	2	0	0

Table 2.1: Frequency distribution D

Example 2.1. Let D be the distribution shown in Table 2.1. D is a frequency distribution of sub-squares I and IV of an order 10 Latin square. The smallest and largest values of the frequency distribution D is 0 and 10 respectively so its lower and upper bounds are [0, 10].

Suppose some given 5×5 sub-squares I and IV have distribution D of table 2.1. As all 10 numbers of the first 3 elements present in sub-squares I and IV, the distributions of the orthogonal mates must have all elements occur at least 3 times in sub-squares I and IV. However, as the number of occurrences in sub-square I equals to the number of occurrences in sub-square IV, the frequency values must be an even number (Property 2.1). Hence, the orthogonal mates of the Latin squares of this distribution have to satisfy the lower limit of 4.

Similarly, distribution D has no occurrence of 9 and 10 in sub-squares I and IV. This implies that every element of the orthogonal mates must occur twice outside sub-squares I and IV. Therefore, the upper limit of the frequency of numbers in orthogonal mates of D has to be 8. The lower and upper limits of the orthogonal mates of D is $[4, 8]$.

Lower and upper limits and bounds are important because assignments between two distributions can happen only when their limits are mutually satisfied. In other words, if D is a frequency distribution with lower and upper bounds $[f_1, f_{2n}]$ and the lower and upper limits $[a, b]$, then the orthogonal mate of D must have the lower and upper bounds $[a, b]$ and the lower and upper limits of $[f_1, f_{2n}]$.

2.4 Phase Two: The Integer Programming Model

The Integer Programming model is based on the work in Appa et al. (2002) where it was used to find pairs of mutually orthogonal Latin squares of order m. In this report, we use a simpler IP model to rule out cases by verifying which pairs of frequency distributions cannot form orthogonal Latin squares. It does this by looking at the assignments of k values of L_1 to l values of L_2 in sub-squares I and IV in the following way. Given the frequency distribution in sub-squares I and IV of L_1 and L_2, is it possible to assign k values to l values in such a way that each combination (k, l)

appears in exactly one cell of the super-imposed matrix of L_1 and L_2. Bear in mind that we do not know exactly where this cell is and if it could block either L_1 or L_2 from being a Latin square. Hence, the IP model being feasible does not guarantee an orthogonal pair; but if it is infeasible, it rules out L_1 and L_2 with given distributions forming an orthogonal pair. The IP model to check assignability between pairs of distributions can be formulated as follows;

2.4.1 Variables

Let x_{kl} represent the possibility that element k of L_1 is assigned to element l of L_2.

$$x_{kl} = \begin{cases} 1 & \text{if } k \text{ is assigned to } l \\ 0 & \text{otherwise} \end{cases}$$

Where k is an element in L_1 and l is an element in L_2, such that $k, l \in 1, \ldots, 10$.

2.4.2 Parameters

Let a_k represent the frequency of element k in the distribution of L_1 and b_l represent the frequency of element l in the distribution of L_2.

2.4.3 Constraints

Two distributions are said to be assignable if there exists a solution to the Integer Programming model such that: Each element k of L_1 must be assigned to a_k elements of L_2; similarly, for L_2, each element l of L_2 must be assigned to b_l elements of L_1. These will be two sets of constraints of the IP model.

2.4.4 Objective

We would like to maximise the sum of x_{kl} which counts the number of times element k of L_1 and element l of L_2 appear together in the super-imposed cells in sub-squares I and IV of L_1 and L_2. In fact, if the distributions of L_1 and L_2 are assignable, this matrix must have the sum of all entries to be 50. Any other solutions would imply two frequency distributions are not assignable.

2.4.5 The IP model

$$
\begin{array}{rrcll}
\textit{maximise} & \sum_{k,l} & x_{kl} & & \\
\text{s.t.} & \sum_{l} & x_{kl} & \leq \ a_k & \forall k \in \{1, ..., 10\} \\
 & \sum_{k} & x_{kl} & \leq \ b_l & \forall l \in \{1, ..., 10\} \\
 & & x_{kl} & \in \ 0, 1 &
\end{array}
\tag{2.1}
$$

This is an 0-1 IP model consisting of 20 constraints and 50 binary variables.

Example 2.2. Let the distribution of L_1 be [10 10 10 8 4 4 2 2 0 0] and one of L_2 be [8 8 6 4 4 4 4 4 4 4]. Figure 2.2 shows an example where an assignment between the distribution of L_1 and L_2 is possible. The sums of rows equal the distribution of L_2; and the sums of columns equal the distribution of L_1. The total of all variables is 50 which indicates that these 2 frequency distributions are assignable.

L1

		1	2	3	4	5	6	7	8	9	10	sum
	1	1	1	1	1	1	1	1	1	0	0	8
	2	1	1	1	1	1	1	1	1	0	0	8
	3	1	1	1	1	1	1	0	0	0	0	6
	4	1	1	1	0	1	0	0	0	0	0	4
	5	1	1	1	0	0	1	0	0	0	0	4
L2	6	1	1	1	1	0	0	0	0	0	0	4
	7	1	1	1	1	0	0	0	0	0	0	4
	8	1	1	1	1	0	0	0	0	0	0	4
	9	1	1	1	1	0	0	0	0	0	0	4
	10	1	1	1	1	0	0	0	0	0	0	4
	sum	10	10	10	8	4	4	2	2	0	0	50

Figure 2.2: Assignment of distribution of L_1 and L_2

The solution to this IP model is not unique. In figure 2.2, if the assignment between $(4, 5)$ is swapped for $(4, 6)$, i.e. $x_{45} = 0$, $x_{46} = 1$, $x_{55} = 1$, $x_{56} = 0$. Then the solution is different but still feasible. However, this report will not concern itself with multiple solutions as long as a feasible solution exists.

		L'_1										
		1	2	3	4	5	6	7	8	9	10	sum
L'_2	1	0	0	0	0	0	0	0	0	1	1	2
	2	0	0	0	0	0	0	0	0	1	1	2
	3	0	0	0	0	0	0	1	1	1	1	4
	4	0	0	0	1	0	1	1	1	1	1	6
	5	0	0	0	1	1	0	1	1	1	1	6
	6	0	0	0	0	1	1	1	1	1	1	6
	7	0	0	0	0	1	1	1	1	1	1	6
	8	0	0	0	0	1	1	1	1	1	1	6
	9	0	0	0	0	1	1	1	1	1	1	6
	10	0	0	0	0	1	1	1	1	1	1	6
	sum	0	0	0	2	6	6	8	8	10	10	50

Figure 2.3: Assignments of the complements L'_1 and L'_2, rearranging the frequency values in descending order would give the solution in figure 2.2

Property 2.3 (Assignments of the complements). As any Latin square of order 2n can be divided into 4 sub-squares and the frequency distribution of sub-squares I and IV is directly the complement of the frequency distribution of sub-squares II and III, assignability of a pair of frequency distributions would imply assignability of their complements.

Example 2.3. Let the complements of L_1 and L_2 be L'_1 and L'_2 respectively. L_1 is [10 10 10 8 4 4 2 2 0 0] so L'_1 will be [10 10 8 8 6 6 2 0 0 0]. L_2 is [8 8 6 4 4 4 4 4 4 4] so L'_2 will be [6 6 6 6 6 6 6 4 2 2]. From the assignments shown in Figure 2.2, if we invert the solution of the decision variables x_{kl}, i.e. changing 0 to 1 and 1 to 0, the results become the solution for assigning L'_1 and L'_2 with frequencies in reverse order (Figure 2.3). This holds true for all assignable pairs of L_1 and L_2. We would need to consider this property when counting the actual number of pairs and triples in the result.

A pre-requisite for assignment is that two distributions must have their lower and upper limits mutually satisfied. By checking the limits of each frequency distribution against the set of all distributions, we can obtain the list of frequency distributions where the limits are mutually satisfied. However, solving the IP model 2.1 for each of these pairs would be time consuming. Hence, the IP model is modified to verify assignments between one distribution D against the set of all the distributions which have their limits mutually satisfied with distribution D.

2.4.6 Set

For the frequency distribution of Latin square L_1, let S be the set of distributions L_2 that satisfy L_1's limits.

2.4.7 The modified IP model

Variables

Let x_{kl} represent the super-imposed cells (k, l) where $k \in L_1$, $l \in p$ such that $p \in S$. In other words, p is a distribution of L_2, satisfying lower and upper limits of L_1. Let y_p be 1 if L_1 can be assigned to p and 0 otherwise; where p is a distribution of L_2 in set S.

Parameters

Let a_k represent the frequency of element k in the distribution of L_1 and b_{lp} represent the frequency of element l in the distribution p; where p is a distribution of L_2 in set S.

Constraints

1. Element k of L_1 must be assigned to at most b_{lp} elements l of p, where p is the distribution of L_2 in set S.

2. Element l of p must be assigned to at most a_k elements k of L_1, where p is the distribution of L_2 in set S.

3. Assignment is admissible if the sum of all entries equals 50.

Objective

We want to maximise the number of distributions of L_2 that L_1 can have assignment with.

The model

$$\begin{array}{rrcll} maximise & \sum_p \; y_p & & & \\ \text{s.t.} & \sum_l \; x_{klp} & \leq & a_k & \forall k \in \{1, ..., 10\}; \quad \forall p \in S \\ & \sum_k \; x_{klp} & \leq & b_{lp} & \forall l \in \{1, ..., 10\}; \quad \forall p \in S \\ & 50y_p & \leq & \sum_{k,l} x_{klp} & \forall p \in S \\ & x_{klp} & \in & 0, 1 & \\ & y_p & \in & \{0, 1\} & \end{array} \tag{2.2}$$

This is an 0-1 IP model consisting of $20t$ constraints and $50t$ binary variables x_{klp} and t binary variables of y_p; where t is the size of set S. This reduces the number of IP models to solve from 2273 to 141 in the case of Latin square of order 10.

Example 2.4. Let the frequency distribution of L_1 be $D = [10\ 10\ 10\ 10\ 4\ 2\ 2\ 2\ 0\ 0]$. In terms of lower and upper limits, this frequency distribution is compatible with (a), (b), and (c) where

$$(a) = [8\ 8\ 6\ 4\ 4\ 4\ 4\ 4\ 4\ 4],$$
$$(b) = [8\ 6\ 6\ 6\ 4\ 4\ 4\ 4\ 4\ 4],$$
$$(c) = [6\ 6\ 6\ 6\ 6\ 4\ 4\ 4\ 4\ 4].$$

The assignments of L_1 to each of these are as shown in figure 2.4

Here, we have y_1 represent the assignment between the first pair of distributions, D and (a). As the sum is less than 50, $y_1 = 0$. y_2 and y_3 represent the assignments between the two later pairs, and they are both equal to 1. Therefore, L_1 can only be assigned to 2 out of 3 distributions.

Lemma 2.1. If there are k mutually orthogonal Latin square of order $2n$, at least $k - 1$ of them must have a full house. (Appa, 2015, personal communication)

Full house is when none of the elements in a distribution has frequency 0. The full house property provides a strategy to speed up the process by reducing the number of iterations in solving the IP model. Since at least one distribution of each assignable

	1	2	3	4	5	6	7	8	9	10	sum
1	1	1	1	1	1	1	1	1	0	0	8
2	1	1	1	1	1	1	1	1	0	0	8
3	1	1	1	1	1	0	0	0	0	0	5
4	1	1	1	1	0	0	0	0	0	0	4
5	1	1	1	1	0	0	0	0	0	0	4
6	1	1	1	1	0	0	0	0	0	0	4
7	1	1	1	1	0	0	0	0	0	0	4
8	1	1	1	1	0	0	0	0	0	0	4
9	1	1	1	1	0	0	0	0	0	0	4
10	1	1	1	1	0	0	0	0	0	0	4
sum	10	10	10	10	3	2	2	2	0	0	**49**

	1	2	3	4	5	6	7	8	9	10	sum
1	1	1	1	1	1	1	1	1	0	0	**8**
2	1	1	1	1	1	0	0	1	0	0	**6**
3	1	1	1	1	1	1	0	0	0	0	**6**
4	1	1	1	1	1	0	1	0	0	0	**6**
5	1	1	1	1	0	0	0	0	0	0	**4**
6	1	1	1	1	0	0	0	0	0	0	**4**
7	1	1	1	1	0	0	0	0	0	0	**4**
8	1	1	1	1	0	0	0	0	0	0	**4**
9	1	1	1	1	0	0	0	0	0	0	**4**
10	1	1	1	1	0	0	0	0	0	0	**4**
sum	10	10	10	10	3	2	2	2	0	0	**50**

	1	2	3	4	5	6	7	8	9	10	sum
1	1	1	1	1	1	1	0	0	0	0	**6**
2	1	1	1	1	1	0	1	0	0	0	**6**
3	1	1	1	1	1	1	0	0	0	0	**6**
4	1	1	1	1	0	0	1	1	0	0	**6**
5	1	1	1	1	1	0	0	1	0	0	**6**
6	1	1	1	1	0	0	0	0	0	0	**4**
7	1	1	1	1	0	0	0	0	0	0	**4**
8	1	1	1	1	0	0	0	0	0	0	**4**
9	1	1	1	1	0	0	0	0	0	0	**4**
10	1	1	1	1	0	0	0	0	0	0	**4**
sum	10	10	10	10	3	2	2	2	0	0	**50**

Figure 2.4: Assignments example using the modified IP model

pair must be a full house, the IP model only needs to be solved with respect to the full houses. Out of 141 distributions, there are only 43 full houses, and hence, using this IP model, theoretically, we only have to solve the model 43 times for each of the full houses.

2.5 Phase Three: Triples of frequency distributions that can be mutually assigned

From the pairs of mutually assignable Latin squares distributions of order 10, we represent the system of mutually assignable LS distributions as the graph $G = (N, E)$ where nodes are the distinct frequency distributions of Latin squares of order 10 and the edges represent if a pair of frequency distributions are assignable.

Note: self-loops are allowed for graph G.

The task of finding three mutually assignable frequency distributions is equivalent to finding 3 nodes that are connected by 3 arcs on this graph G. For example, we have a graph in figure 2.5

We have the following pseudo-code

```
initiate visited nodes = empty set
for each node i in graph G:
   initiate temporary visited nodes = empty set;
   for each node j connected to i:
      if j is in visited nodes:
             move to the next node;
      else:
             for each node k connected to j:
                    if k is in visited nodes
                    or temporary visited nodes:
                       move to the next node;
                    else:
                       if k is connected to i:
                          (i, j, k) can be mutually assigned;
                    else:
                           move to the next node;
      add j to temporary visited nodes;
   add i to visited nodes;
```

The first set of visited nodes prevents multiple occurrence of the same triple by marking the nodes as visited. Moreover, the input is present as directed graph so the set of temporary visited nodes prevents the triples to be counted twice.

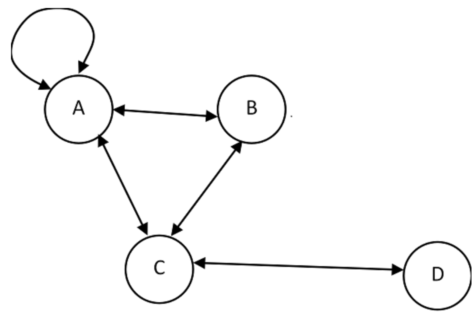

Figure 2.5: Example for graph representation of frequency distributions assignment

Example 2.5. In figure 2.5, starting from node $i = A$. A is linked to A, B and C. At the moment, the set of visited nodes is empty so we can go to $j = A$. Again, A is linked to A, B and C and the set of temporary visited nodes is also empty so we can consider $k = A$. This results in the triple (A, A, A). To continue, we add $j = A$ to the set of temporary visited nodes and consider the next node. We could follow this method to get all the triples which should be (A, A, A), (A, A, B), (A, A, C), and (A, B, C).

Following from the Lemma 2.1 on full houses, each mutually assignable triple must have at least 2 full house frequency distributions. In fact, instead of iterating through every node of graph G, iterating through the set of full house gives the same results. However, there is no significant improvement in the computational effort when iterating through all the nodes as compare to only the full house nodes for Latin squares of order 10. The documentation of Python codes is available in Appendix D.

Chapter 3

The Test Case: Latin Square of Order Six

The most well-known question related to orthogonal Latin squares is possibly the 36 officers' problem, first introduced by Euler (Laywine and Mullen, 1998). The problem considers placing 36 officers of 6 ranks and 6 regiments on a 6×6 square formation so that (1) no two ranks or regiments are repeated in any row or column, and (2) all of 36 combinations of ranks and regiments are present.

According to (Euler, 1782), such Latin square does not exist and it leads to his conjecture that there is no mutually orthogonal Latin square of order $2n$ where n is an odd number. In 1900, through exhaustive search, Tarry proved that no pair of mutually orthogonal Latin squares of order 6 exists. It would be worthwhile to look at the distributions of the 3×3 sub-squares of Latin square of order 6 as an example to illustrate how the structural approach works. Latin square of order 6 is the smallest case of m where m is not a power of a prime number. The number of frequency distributions in this case is small and simple which can be considered as a pilot study for the larger case where $m = 10$ in the next section.

3.1 Phase One: General Preparation

3.1.1 Frequency Distribution

First of all, there are 8 distributions in the 3×3 sub-squares I and IV – Table 3.1. The first column of the table shows the name of each distribution which will be used as

	1	2	3	4	5	6	L	U	Mann	Comp
D1	6	6	6				4	2	Excl	D1
D2	6	6	4	2			2	4	Excl	D2
D3	6	6	2	2	2		2	4		D4
D4	6	4	4	4			2	4		D3
D5	6	4	4	2	2		2	4		D5
D6	6	4	2	2	2	2	2	6		D7
D7	4	4	4	4	2		0	4		D6
D8	4	4	4	2	2	2	0	6		D8

Table 3.1: Details of 8 frequency distributions of sub-squares I and IV of 6×6 Latin squares

reference in later discussion. The next 6 columns are the frequency values of elements 1 to 6 in each frequency distribution. L and U denote the lower and upper limits. Those frequency distributions which are excluded under Mann's theorem are noted as 'Excl' in the Mann column. Finally, the Comp column shows the complement of each frequency distribution.

In order to obtain these frequency distributions, we start from [6 6 6 0 0 0] which is when only three elements appear in sub-squares I and IV. Comparing to the general method described in the Methodology chapter, we get D_2 by partitioning 6 numbers of element 3 into 4 of 3's and 2 of 4's. Similarly, the rest of the distributions are obtained by further splitting of the previous distribution in a systematic way.

3.1.2 Mann rejectable distribution

By Mann's theorem, for a Latin square of order 6, if sub-squares I and IV has fewer than 4 numbers which are different from 1 to 3, this Latin square could not have an orthogonal mate of the same order. Hence, D_1 and D_2 are excluded.

3.1.3 Lower and Upper limits

At this point, we have 6 frequency distributions and the highest number of assignable pairs is ${}^6C_2 + 6 = 21$, consisting of ${}^6C_2$ ways to choose a distinct pair from 6 distributions and 6 combinations of D_i with D_i. This number can be reduced through checking which two distributions mutually satisfy each other's limits. The limits of D_3, D_4, D_5, D_6, D_7 and D_8 satisfy D_8's. D_6 satisfies its own limits and can be assigned to itself. D_7 also satisfies its own limits and can be assigned to itself.

Therefore, we have 8 pairs where the limits are mutually satisfied. These are (D_3, D_8), (D_4, D_8), (D_5, D_8), (D_6, D_8), (D_7, D_8), (D_8, D_8), (D_6, D_6), and (D_7, D_7).

3.1.4 The Complements

Out of the six distributions, two of them, D_5 and D_8, are the complements of themselves; D_3-D_4 and D_6-D_7 are pairs of distinct complements. The assignment between a pair of (D_3, D_8) and (D_4, D_8) are the same because D_3 and D_4 are complements. Hence, here we pick only the pair (D_3, D_8) as the assignment between D_4 and D_8 would be the inverse of the assignment between D_3 and D_8. Similarly, the assignment between the pairs (D_6, D_6) and (D_7, D_7) would be the inverse of each other as D_6 and D_7 are complements. This leaves 5 pairs which are (D_3, D_8), (D_5, D_8), (D_6, D_8), (D_8, D_8), and (D_6, D_6).

3.2 Phase Two: Applying IP model

The IP model 2.1 is applied on individual pairs where the distributions mutually satisfy each other's limits. The assignments for each of the five pairs of frequency distributions are displayed in figure 3.1. All of them are assignable.

3.3 Phase Three: Triples of mutually assignable frequency distribution

From the pairs found, we can employ our algorithm to find triples of mutually assignable distributions. The solutions obtained above is transformed into a graph representation. Let $G = (N, V)$ be an undirected graph where self-loops are allowed. We get $N = \{D_3, D_5, D_6, D_8\}$, $V = \{(D_3, D_8), (D_5, D_8), (D_6, D_6), (D_6, D_8), (D_8, D_8)\}$. The question here is to find a group of 3 nodes that are connected by 3 edges.

We can either represent this as in figure 3.3 or a list of nodes with their mutually assignable partners as follows;

D3 and D8

	1	2	3	4	5	6	
1	1	1	1	0	1	0	**4**
2	1	1	0	1	1	0	**4**
3	1	1	1	1	0	0	**4**
4	1	1	0	0	0	0	**2**
5	1	1	0	0	0	0	**2**
6	1	1	0	0	0	0	**2**
	6	**6**	**2**	**2**	**2**	**0**	**18**

D5 and D8

	1	2	3	4	5	6	
1	1	1	1	0	1	0	**4**
2	1	1	1	1	0	0	**4**
3	1	1	1	0	1	0	**4**
4	1	0	1	0	0	0	**2**
5	1	1	0	0	0	0	**2**
6	1	0	0	1	0	0	**2**
	6	**4**	**4**	**2**	**2**	**0**	**18**

D6 and D8

	1	2	3	4	5	6	
1	1	1	1	1	0	0	**4**
2	1	1	0	1	1	0	**4**
3	1	1	1	0	1	0	**4**
4	1	0	0	0	0	1	**2**
5	1	1	0	0	0	0	**2**
6	1	0	0	0	0	1	**2**
	6	**4**	**2**	**2**	**2**	**2**	**18**

D8 and D8

	1	2	3	4	5	6	
1	1	1	1	1	0	0	**4**
2	1	1	1	0	1	0	**4**
3	1	1	1	1	0	0	**4**
4	0	1	0	0	0	1	**2**
5	1	0	1	0	0	0	**2**
6	0	0	0	0	1	1	**2**
	4	**4**	**4**	**2**	**2**	**2**	**18**

D6 and D6

	1	2	3	4	5	6	
1	1	1	1	1	1	1	**6**
2	1	1	0	1	1	0	**4**
3	1	0	1	0	0	0	**2**
4	1	0	0	0	0	1	**2**
5	1	1	0	0	0	0	**2**
6	1	1	0	0	0	0	**2**
	6	**4**	**2**	**2**	**2**	**2**	**18**

Figure 3.1: Assignments of frequency distributions in sub-squares I and IV of 6 × 6 Latin squares

$$
\begin{aligned}
D_3 &= [D_8] \\
D_5 &= [D_8] \\
D_6 &= [D_6, D_8] \\
D_8 &= [D_3, D_5, D_6, D_8]
\end{aligned}
$$

Example 3.1. From node D_3, look inside the list of nodes that D_3 can be assigned to. We find node D_8. Iterate the list of nodes that D_8 can be assigned to: D_3 is the first one on the list. In order for (D_3, D_8, D_3) to form a triangle, there must be an arc from D_3, going back to D_3. However, D_3 can only be assigned to D_8, so we have to move to the next node on the list. The only node that have both D_3 and D_8 inside its list is D_8 and so the triple we found here is (D_3, D_8, D_8). Add D_3 into a list of visited nodes.

L_1 Dist	L_2 Dist	L_3 Dist
D_3	D_8	D_8
D_5	D_8	D_8
D_6	D_6	D_6
D_6	D_6	D_8
D_6	D_8	D_8
D_8	D_8	D_8

Table 3.2: The list of assignable triples found for Latin square of order 6

3.4 Further discussion

It seems to hold true at least for $m = 6$ that when 2 distributions have their limits mutually satisfied, there will be an assignment between them. However, there might be counterexamples for Latin squares of higher order.

The assignments between distributions of the 3×3 sub-squares of order 6 Latin squares also show the greatest limitation of this approach. There is no mutually orthogonal Latin squares of order 6 yet assignments are possible between these frequency distributions. Hence, there can be assignment between two distributions but this does not necessary mean there will be two orthogonal Latin squares of such form. From the formation of the IP model 2.1, we can only determine whether there are enough elements in the sub-squares distributions for all 36 distinct pairs of numbers from 1 to 6 to exist. When using this model, we are not putting any constraint to specify that both L_1 and L_2 must be Latin squares.

3.4.1 An alternative proof for the infeasibility of $n = 6$

Furthermore, this also emphasises that if both D_8 or D_6 could be proven to have no orthogonal mate, it would show another proof of non-existence of mutually orthogonal Latin squares of order 6. This proof will be different from the existing proof by Tarry (1900) as only the Latin squares that have this frequency distributions in sub-squares I and IV are considered. This reduces the number of Latin squares to look at as opposed to exhaustively searching through all possibilities.

3.4.2 Hidden Mann structures

From definition 2.1, a Latin square would remain identical after permuting rows, columns and symbols. Hence, there can be structures ruled out by Mann's theorem (i.e. D_1 and D_2) hidden as 3×3 sub-squares I and IV.

To tackle this problem, we need to iterate over all possible combinations of rows and columns for sub-square I and IV in a Latin square. As the structure of sub-square I and IV are identical, for simplification, only sub-square I is considered.

Example 3.2. Let us consider the following 6×6 Latin square

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

C3 $\leftrightarrow$ C4

1	2	4	3	5	6
2	3	5	1	6	4
3	1	6	2	4	5
4	5	1	6	2	3
5	6	2	4	3	1
6	4	3	5	1	2

Figure 3.2: Example of 2 isomorphic Latin squares of order 6

The one on the right is formed by switching column 3 and column 4 from the original Latin square (left). From definition 2.1, these 2 Latin squares are essentially the same one. Looking at the structure of the 3×3 sub-square, the left Latin square contains distribution rejected by Mann's theorem so it could not have an orthogonal mate. However, the structure of the right Latin square's sub-squares is the one where assignment with another distribution is possible. Therefore, we need a mechanism to find all the combinations for the 3×3 sub-square I and determine whether any combination produces a distribution rejected by Mann's theorem.

There are ${}^6C_3$ ways to choose 3 columns and ${}^6C_3$ ways to choose 3 rows from a 6×6 Latin square. However, choosing the columns and rows for the 3×3 sub-square I would fix the columns and rows of sub-squares II, III, and IV. Any distribution found in sub-square I would reflect sub-square IV and the distribution of sub-square II and III are the complements of sub-square I. Hence, we only need to iterate over $({}^6C_3 \times {}^6C_3) \div 4 = 100$ combinations for sub-square I.

We need a list of 10 combinations for rows and columns such that if 3 rows (or columns) are chosen, the 3 other rows (or columns) are not included. For example, if columns 1,2,4 are chosen, the list does not have to include columns 3,5,6 as this would describe the same Latin square. In order to do this, let us choose to fix 1 row (or column) out of the three. For example, if the first row and column are fixed.

We have the list of 10 combinations $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,5\}$, $\{1,4,6\}$, $\{1,5,6\}$.

These 10 combinations would form 10 sets of rows and 10 sets of columns. Each set of rows would be combined with set of columns. For example, rows $\{1,2,3\}$ and columns $\{1,2,3\}$, rows $\{1,2,3\}$ and columns $\{1,2,4\}$, etc.

After finding all permutations of rows and columns in the Latin square, the next step is to count the frequency of numbers from 1 to 6 in each sub-square. Although any set of symbols can be used in the Latin square without having any impact on orthogonality, as the symbols can also be permuted, the frequency of numbers from 1 to 6 in each sub-square is not necessarily in descending order. This problem would be solved by looking at the sum of three highest values in the frequency distribution. For the 6×6 Latin squares, the structures rejected by Mann's theorem would have the sum equal to 8 or 9. Therefore, when these values are found, the Latin square in question cannot have an orthogonal mate.

We have obtained the results for 12 main classes of 6×6 Latin squares mentioned in the book Latin Squares and Their Applications (Denes and Keedwell, 1974). This method has been able to prove that Mann rejected structures are present in at least 7 of the 12 classes and directly show that these Latin squares do not have orthogonal mate. This result is available in the Appendix F. Another observation is that when one representative of a class is shown to have a Hidden Mann structure, the rest of the class also contains a Hidden Mann structure. Hence any Latin square belongs to that class would not have an orthogonal mate. This should be true as the representatives of each main class are isomorphic.

Chapter 4

Main results

This chapter extends Mann's theorem by reducing the number of assignable frequency distributions of Latin squares of order 10. By using Mann's theorem, we are able to eliminate 6 distributions for $n = 10$ as they cannot be paired with any other frequency distribution, i.e. they cannot have orthogonal mates. The IP model is then applied to find solutions for the assignments. Finally, we attempted different methods to estimate the number of triples of assignable distributions.

4.1 Phase One: General Preparation

4.1.1 Frequency distributions

As mentioned in the Methodology Chapter, for Latin square of order 10, using the method described in part 2.3.1, we found 141 frequency distributions of the 5×5 sub-squares I and IV.

The full list of all the frequency distributions is included in the APPENDIX. Table 4.1 shows some examples of the frequency distributions and the attributes associated with them.

Here are some explanations related to table 4.1. The number in the first column is the identity of that distribution. In this paper, we shall refer to the specific distribution using this identity number, i.e. D_i, where $i = \{1, ..., 141\}$. The next 10 columns indicate the frequency of each element in the distribution. The values are even numbers between 0 and 10, where 0's are presented as blank cells. The

D. No	1	2	3	4	5	6	7	8	9	10	L	U	Mann	Compl	Self-comp	Fullhous
1	10	10	10	10	10						6	4	Excl	1	Yes	
2	10	10	10	10	8	2					4	6	Excl	2	Yes	
3	10	10	10	10	6	4					4	6	Excl	3	Yes	
4	10	10	10	10	6	2	2				4	6	Excl	8		
5	10	10	10	10	4	4	2				4	6		10		
6	10	10	10	10	4	2	2	2			4	8		27		
7	10	10	10	10	2	2	2	2	2		4	8		58		
8	10	10	10	8	8	4					4	6	Excl	4		
9	10	10	10	8	8	2	2				4	6	Excl	9	Yes	
10	10	10	10	8	6	6					4	6		5		
11	10	10	10	8	6	4	2				4	6		11	Yes	
12	10	10	10	8	6	2	2	2			4	8		28		
13	10	10	10	8	4	4	4				4	6		17		
14	10	10	10	8	4	4	2	2			4	8		30		
15	10	10	10	8	4	2	2	2	2		4	8		59		
16	10	10	10	8	2	2	2	2	2	2	4	10		99		Yes

Table 4.1: Frequency distributions example of 10×10 Latin squares

frequency distributions in this table are the standard ones where the values are arranged in descending order, i.e. $[f_1 \geq f_2 \geq f_3 \geq \ldots \geq f_10]$. The columns L and U respectively show the lower and upper limits of the orthogonal mates' frequency distribution. The column which is headed 'Mann' indicates if the distribution can be ruled out by Mann's theorem. There are 6 of them and they are denoted by 'excl' for excluded. 'Compl' column shows the complement of each distribution. If the distribution is the complement of itself, it would be indicated by 'Yes' in the next column ('Self-comp'). The last column states whether the distribution is full house. Full house is when all elements of a frequency distribution are present in sub-squares I and IV, i.e. $f_i > 0 \forall i = \{1, ..., 10\}$.

4.1.2 Mann rejectable structures

Applying Mann's theorem to Latin squares of order 10, an orthogonal mate does not exist for distributions with less than 6 elements different from the elements 1 to 5. Hence, there are 6 frequency distribution excluded from 141 distributions mentioned in the previous section. These are D_1, D_2, D_3, D_4, D_8 and D_9. This reduces the assignable distributions to 135.

4.1.3 Lower and Upper limits

Without considering the lower and upper limits, there are ${}^{135}C_2 + 135 = 9180$ possible combinations of 2 distributions from the 135 distributions found previously, consisting of ${}^{135}C_2$ ways of choosing two distinct ones and 135 combinations of D_i with D_i. However, when the lower and upper limits of two distributions are not mutually satisfied, the IP model would be infeasible. In other words, compatible assignments are not possible as there is an excess or shortage of number of elements in sub-squares I and IV of both L_1 and L_2. Effectively, there are 2273 pairs of distributions where the lower and upper limits are mutually satisfied. This is less than a quarter of the number of problems to solve if we run the IP model exhaustively on all possible pairwise combinations of 135 distributions.

Dist No	Compl	# LU satisfied	# Assgn	List Assignable	List not Assignable
5	10	1	1	141	none
10	5	1	1	141	none
6	27	3	2	136, 141	129
27	6	3	2	140, 141	139
7	58	3	3	129, 136, 141	none
58	7	3	3	139, 140, 141	none
11	11	1	1	141	none
12	28	3	2	136, 141	129
28	12	3	2	140, 141	139
13	17	1	1	141	none
17	13	1	1	141	none
14	30	3	3	129, 136, 141	none
30	14	3	3	139, 140, 141	none
15	59	3	3	129, 136, 141	none
59	15	3	3	139, 140, 141	none
16	99	5	5	90, 98, 126, 136, 141	none
99	16	5	5	137, 138, 139, 140, 141	none

Table 4.2: Assignments of distributions in sub-squares I and IV (examples)

4.2 Phase Two: Applying the IP model

4.2.1 Mutually Assignable Distributions

Table 4.2 shows some examples of distributions with their assignable partners. Each pair of complements is shown next to each other to illustrate property 2.3. The 'Dist' No column shows the identity i of distribution D_i. The 'Compl' column shows the identity of the complement of D_i. '#LU satisfied' shows the number of distributions that D_i has its limits mutually satisfied. '# Assgn' shows the number of distributions that D_i can be assigned to, which is the result of the IP model. 'List Assignable' shows the list of distributions that D_i can be assigned to. 'List not Assignable' shows the list of distributions that D_i cannot be assigned to out of all distributions that satisfy D_i's limits.

Example 4.1. Take the pair of D_{12} and D_{28}, which is also a pair of distinct complements. Both of them have the limits compatible with 3 other frequency distributions. However, for D_{12}, the set of lower and upper limits compatible distributions consists of $\{D_{129}, D_{136}, D_{141}\}$; whereas for D_{28}, the set is $\{D_{139}, D_{140}, D_{141}\}$. The distributions of one set are the complements of the distributions in the other set. Where assignment is possible between a pair of distributions, assignment is possible between the complements of that pair. For example, D_{12} and D_{136} can be assigned. As D_{28} is the complement of D_{12} and D_{140} is the complement of D_{136}, assignment is possible between D_{28} and D_{136}. Furthermore, as D_{129} cannot be assigned to D_{12}, the result is D_{139}, which is the complement of D_{129}, cannot be assigned to D_{28}.

As mentioned in the Methodology chapter, the full-house property provides a strategy to speed up the process by reducing the number of iterations in solving the IP model. Since at least one distribution of each assignable pair must be full-house, the IP models only need to be solved with respect to those that have full-house. Out of 135 distributions, there are only 43 which are full-house, and hence, using the modified IP model in 2.2, theoretically, we only have to solve the model 43 times for each of those that are full-house. Although in practice, all 135 models are solved for validation and to facilitate the running of the algorithm to find the triples of mutually assignable distributions in the next section. Solving the models returns 1851 pairs of assignable distributions.

4.2.2 Eliminating similar assignments of the complements

However, the full-house property also creates a challenge in eliminating the complements. Of the 43 distributions that are full house, 31 are self-complements and 12 have distinct complements. Although example 4.1 shows some cases such as D_6 or D_{10} where a distribution is assigned to only one distribution in a distinct complement pairs, we cannot simply ignore one distribution in a pair of distinct complements because some distributions can be assigned to both of them. For example, let F and F' be distributions that are full house and distinct complements of one another. Let A be a distribution that can be assigned to both F and F'; A' is the complement of A. We have 2 cases: (1) A is assigned to F, A' is assigned to F'; or (2) A is assigned to F', A' is assigned to F. The assignments are different and eliminating either F or F' before solving the IP model would cause missing out important results.

In the dataset of 1851 pairs that we found, except for when assignments happen between two distributions that are the complements of each other, all other pairs are counted twice. In total, there are 55 such pairs which includes 49 pairs of self-complements and 6 pairs of distinct and full house complements. Therefore, the total number of distributions after eliminating those that are essentially describing the same assignments is 953.

4.2.3 Assignable pairs where both distributions are full house

According to lemma 2.1, for 3 distributions to be mutually assignable, there must be at least 2 that are full house. From 1851 pairs that we found in 4.2.1, there are 716 pairs where both distributions are full house. The reason for using 1851 instead of 953 pairs is because a third distribution could be assigned to both the pair and the complement pair. Therefore, only these 716 pairs can form triples of mutually assignable distributions. For example, let (A, B) and (A', B') be two pairs where A' is the complement of A and B' is the complement of B. Let C be a distribution that can be assigned to A, A', B, B'. The process of eliminating the complement pairs would results in the pair (A', B') being ignored. Hence, the only set of three distributions found would be (A, B, C). However, as C can still be assigned to (A', B'), if the pair (A', B') is eliminated, the triple (A', B', C) would be neglected.

4.3 Phase Three: Triples of mutually assignable distributions

The algorithm to find triples of mutually assignable distributions is implemented in Python. The result shows 19066 mutually assignable triples of frequency distribution for n=10 before the complements of triples are deleted. In order to eliminate the triples that essentially describe the same assignments of three distributions, there are two solutions:

1. From the 19066 triples of mutually assignable distributions found, for each triple (A, B, C), find and eliminate the triple (A', B', C') which consists of the complements of A, B, and C.

2. From Lemma 2.1, for a set of 3 distributions that can be assigned, at least 2 of them must be full house. In 4.2.3, there are 716 assignable pairs where both distributions are full house. Let us also call a pair with this property a full house pair. Therefore, we can grow triples by finding the third distribution that can be assigned to both distributions of a full house pair. However, there are still some triples of complements that essentially describe the same assignments. We would have to find and eliminate these triples using similar method described in the first solution.

4.3.1 Eliminating the complement triples

Both solutions described in 4.3 give 9624 as the number of triples after eliminating those that are essentially describing the same assignments. Although there is no significant improvement of using one method against the other in terms of the time taken for the problem to be solved, applying both has given the advantage of validating the results. Especially, a critical error in the original data would have been missed if only one method was used. This mistake was due to error in recording the assignment results when solving the IP models.

4.3.2 Full house pairs appearing in the highest number of triples

The number of recurrence of each full house pair would give a good strategy in looking for triples of mutually orthogonal Latin squares. Let the full house pair consist of two distributions (F_1, F_2), where F_1 is the distribution in sub-squares I and IV of Latin square L_1 and F_2 is the distribution in sub-squares I and IV of Latin square L_2. We could look for the third Latin square L_3 orthogonal to both L_1 and L_2 starting from the pairs of (F_1, F_2) which have the highest number of occurrence. The list of 10 full house pairs with the highest number of recurrence in the assignable triples is shown in table 4.3.

F_1	F_2	No. of recurrence
136	136	88
136	141	88
129	136	86
129	141	86
129	129	86
141	141	76
128	128	75
128	135	75
128	129	75
128	136	75

Table 4.3: top 10 full house pairs with highest number of recurrence in triples

4.3.3 Assignable triples with 3 full house distributions

From the result of 19066 mutually assignable triples, we could explore further to find how many triples that have all three distributions being full house. The set of triples with all distributions being full house would be the foundation to find if there exists any quadruple of mutually orthogonal Latin squares of order 10. There are 7544 triples with three full house distributions.

Chapter 5

Summary and Conclusion

Through an exploration into the structural approach using the 5×5 sub-squares, the research has been able to answer questions about certain properties that constitute a set of three mutually orthogonal Latin squares of order 10. In this chapter, we will start by summarising the approach and the findings. Next, we conclude by discussing the implication of this research including the limitations and further research opportunities.

5.1 Summary of the approach

Instead of finding specific pairs of mutually orthogonal Latin squares of order 10, the approach simplifies the problem by looking at the structure of the 5×5 sub-squares I and IV. The frequency of occurrence of numbers 1 to 10 in these 5×5 sub-squares forms properties that can be used to find the structure of its orthogonal mates. The method is based on an extension of Mann's theorem proposed by Professor Appa. The methodology is divided into three phases.

Phase one consists of general preparation to find the frequency distributions of Latin squares of order 10 and their basic attributes. The frequency distribution is defined by the frequency of each element in the sub-square I and IV of the Latin square. The attributes include the lower and upper limits, the complement, and the full house property. The lower and upper limits indicates the bounds for the lowest and highest values for the frequency distributions of the orthogonal mates. As the complement of one distribution describes the same Latin square structure, it is necessary to account for the complements in order to reduce the number of pairs of distributions, which have mutual assignments. If all elements appear in sub-square I and IV, the frequency distribution is said to be full house. The full house property suggests a strategy to speed up the process when solving the IP model and find triples of distributions that can be mutually assigned.

Phase two of the approach aims to solve the IP model by enumerating all the possible pairs of frequency distributions to see which ones can be mutually assigned. While Mann's theorem looks at the frequencies of only one distribution and determine whether the conditions are satisfied for an orthogonal mate to exist, the IP model takes a pair of distributions and checks if a pair of Latin squares of these distributions can be orthogonal mates. Solving this model results in a full set of assignable structures of orthogonal pairs. However, this is a time consuming process as there are theoretically 9180 pairs to check. This number is reduced by lower and upper limits and by modifying the original IP model, bundling several distributions to solve in one integer program. Furthermore, as all pairs of mutually assignable distributions must have at least one distribution being full house, the models can be solved with respect to only the set of distributions which are full house.

Finally, phase three reports the number of frequency distribution triples that can be mutually assigned. The original method to find the triples is to represent the result of the IP model as a graph with the frequency distributions being the nodes and assignments between pairs of distributions being the edges. This method works well when testing for mutually orthogonal Latin squares of order 6 but complication arises when the number of pairs becomes very high for Latin squares of order 10. Hence, we implement another solution by growing the triples from the set of full house pairs. This method has helped to verify the results and identify errors in the original dataset.

5.2 Summary of the findings

We applied the methods described above to both Latin squares of order 6 and 10. For Latin squares of order 6, the aim was to test the approach on a smaller case with known results. This pilot study shows there are 8 standard distributions for the frequencies of numbers 1 to 6 in the 3×3 sub-squares I and IV. Out of these, two can be ruled out by Mann's theorem as the condition for orthogonality is not satisfied, leaving 21 pairs to be considered. By considering the lower and upper limits, as well as eliminating the complements, we reduced this to assignments between 5 pairs of frequency distributions. This result also highlights one weakness of the IP model which is that assignments between a pair of distributions does not guarantee the existence of a pair of mutually orthogonal Latin squares with such distributions. Furthermore, we also found 8 triples of frequency distributions for Latin squares of order 6, from orthogonal assignment point of view.

For Latin squares of order 10, there are 141 distributions of numbers 1 to 10 in the 5x5 sub-squares I and IV. Mann's theorem rules out 6 distributions which do not have enough elements for an orthogonal mate to exist. Out of the remaining 135 distributions, we found that 17 are self-complements and 59 are paired, distinct complements. Furthermore, there are 43 distributions that are full house. As some pairs of complements can both be assigned to one distribution and to themselves, deleting one member of the pair would cause missing results. Hence, the IP model is solved for all 43 distributions and further reduction by removing assignments by the complements found 953 pairs.

Due to complication in reducing the number of complementary triples that represents the same mutual assignments, another method is employed in addition to counting the system of nodes connected by 3 edges. For triples of distributions to be mutually assigned, at least two of them would be full house. Therefore, we have attempted to grow a set of triples, starting from a pair of 2 full house distributions and find the third distribution that can be assigned to both distributions in the initial pair. There are 645 pairs of distributions where both of them are full house. The set grown from this confirms that there are 9624 orthogonally assignable triples.

Furthermore, we also obtain the results for the number of recurrence of the full house pairs (Table 4.3). This give a good strategy to search for the third Latin square in the mutually orthogonal triples as we can start from the mutually orthogonal pairs that have high frequency of recurrence.

The final result that we have acquired is the number of full house triples, which is 7544. Once the triples of orthogonal Latin squares of order 10 are found, we could employ this number to answer the next question, do the set of four mutually orthogonal distributions of order 10 exist?

5.3 Implication of the research

The research aims to reduce the number of Latin squares of order 10 to verify orthogonality by limiting to the distributions for the 5×5 sub-squares I and IV that can be mutually assigned. However, this approach prompts several limitations that need to be addressed in further research.

5.3.1 Limitations of the IP model

The weakness of this approach is that assignments between a pair of distributions does not guarantee the existence of a pair of mutually orthogonal Latin squares with such distributions. Our IP model has only been able to assign elements of the first distribution to the elements of the second distribution. However, of course, when constructing an actual pair of Latin squares, this condition of assignments would become the constraint and potentially block the creation of Latin squares.

One approach is to write a programming model that imposes the Latin squares constraints as well as the distributions for L_1 and L_2. Either Integer Programming, Constraint Programming or a combination of both can be considered. Attempt has been made in the past in Appa et al. (2006) where a combination of Integer Programming and Constraint Programming was used but the approach was to look at orthogonality condition between element k of L_1 and l of L_2 for all 4 sub-squares I, II, III, and IV. To replace this condition with the requirement of assignability between 2 distributions in sub-squares I and IV of L_1 and L_2 would be a new direction.

5.3.2 Structural analysis for Latin squares of order 10

Since the results for both $n = 6$ and $n = 10$ are established, we have thus far confirmed that Mann's theorem is sufficient in both cases to determine if one distribution is not assignable. The evidence is that if one distribution is not ruled out by Mann, it will have assignment with at least one other distribution of the same order. At the end of Chapter 3, we have demonstrated how an analysis of hidden Mann can be implemented and reported the results for 12 classes of 6×6 Latin squares listed in Denes and Keedwell (1974). The same analysis can be modified for pairs of mutually orthogonal Latin squares of order 10 and thus would provide the structures that aid the construction of three mutually orthogonal Latin squares.

For Latin squares of order 6, we have used the list of main class representatives obtained from Denes and Keedwell (1974). The result confirms that in 12 classes, 7 of them are proven unable to have an orthogonal mate. However, the main class representatives of Latin squares of higher orders are not specified. Other resources such as the ANU website (McKay, 2015) provides the list of reduced Latin squares of up to order 7 and the list of main class representatives for up to order 8. However, even for Latin squares of order 8, the number of main class representatives is 283657. This number would grow exponentially large as the size of the Latin squares increases.

Although it would be quite impossible to run the Hidden Mann analysis on all individual Latin squares of order 10 to check which ones cannot have an orthogonal mate, this method can be applied by picking a certain frequency distribution; finding all the possibilities for filling the Latin square where the sub-squares I and IV follow this frequency distribution; and checking the existence of Mann rejectable structures in these Latin squares. If Mann rejectable structures exist in the Latin square in consideration, we could conclude that it does not have an orthogonal mate. Otherwise, a further technique concerning the transversal would be applied to determine the orthogonality of the Latin square. We predict that running the Hidden Mann analysis would decrease the number of Latin squares for each frequency distributions and reduce the effort to apply the further technique.

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Appendix A

The full list of frequency distributions for Latin squares of order 10

Dist No	1	2	3	4	5	6	7	8	9	10	LB	UB	Mann	Compl	Self-comp	Fullhouse
1	10	10	10	10	10						6	4	Excl	1	Yes	
2	10	10	10	10	8	2					4	6	Excl	2	Yes	
3	10	10	10	10	6	4					4	6	Excl	3	Yes	
4	10	10	10	10	6	2	2				4	6	Excl	8		
5	10	10	10	10	4	4	2				4	6		10		
6	10	10	10	10	4	2	2	2			4	8		27		
7	10	10	10	10	2	2	2	2	2		4	8		58		
8	10	10	10	8	8	4					4	6	Excl	4		
9	10	10	10	8	8	2	2				4	6	Excl	9	Yes	
10	10	10	10	8	6	6					4	6		5		
11	10	10	10	8	6	4	2				4	6		11	Yes	
12	10	10	10	8	6	2	2	2			4	8		28		
13	10	10	10	8	4	4	4				4	6		17		
14	10	10	10	8	4	4	2	2			4	8		30		
15	10	10	10	8	4	2	2	2	2		4	8		59		
16	10	10	10	8	2	2	2	2	2	2	4	10		99		Yes
17	10	10	10	6	6	6	2				4	6		13		
18	10	10	10	6	6	4	4				4	6		18	Yes	
19	10	10	10	6	6	4	2	2			4	8		31		
20	10	10	10	6	6	2	2	2	2		4	8		60		
21	10	10	10	6	4	4	4	2			4	8		37		
22	10	10	10	6	4	4	2	2	2		4	8		63		
23	10	10	10	6	4	2	2	2	2	2	4	10		100		Yes
24	10	10	10	4	4	4	4	4			4	8		47		
25	10	10	10	4	4	4	4	2	2		4	8		71		
26	10	10	10	4	4	4	2	2	2	2	4	10		105		Yes
27	10	10	8	8	8	6					2	6		6		
28	10	10	8	8	8	4	2				2	6		12		
29	10	10	8	8	8	2	2	2			2	8		29	Yes	
30	10	10	8	8	6	6	2				2	6		14		
31	10	10	8	8	6	4	4				2	6		19		
32	10	10	8	8	6	4	2	2			2	8		32	Yes	
33	10	10	8	8	6	2	2	2	2		2	8		61		
34	10	10	8	8	4	4	4	2			2	8		38		
35	10	10	8	8	4	4	2	2	2		2	8		64		
36	10	10	8	8	4	2	2	2	2	2	2	10		101		Yes
37	10	10	8	6	6	6	4				2	6		21		
38	10	10	8	6	6	6	2	2			2	8		34		
39	10	10	8	6	6	4	4	2			2	8		39	Yes	
40	10	10	8	6	6	4	2	2	2		2	8		65		

Table A.1: Frequency distributions of 10×10 Latin squares

Dist No	1	2	3	4	5	6	7	8	9	10	LB	UB	Mann	Compl	Self-comp	Fullhouse
41	10	10	8	6	6	2	2	2	2	2	2	10		102		Yes
42	10	10	8	6	4	4	4	4			2	8		48		
43	10	10	8	6	4	4	4	2	2		2	8		72		
44	10	10	8	6	4	4	2	2	2	2	2	10		106		Yes
45	10	10	8	4	4	4	4	4	2		2	8		81		
46	10	10	8	4	4	4	4	2	2	2	2	10		113		Yes
47	10	10	6	6	6	6	6				2	6		24		
48	10	10	6	6	6	6	4	2			2	8		42		
49	10	10	6	6	6	6	2	2	2		2	8		68		
50	10	10	6	6	6	4	4	4			2	8		50	Yes	
51	10	10	6	6	6	4	4	2	2		2	8		74		
52	10	10	6	6	6	4	2	2	2	2	2	10		108		Yes
53	10	10	6	6	4	4	4	4	2		2	8		82		
54	10	10	6	6	4	4	4	2	2	2	2	10		114		Yes
55	10	10	6	4	4	4	4	4	4		2	8		91		
56	10	10	6	4	4	4	4	4	2	2	2	10		122		Yes
57	10	10	4	4	4	4	4	4	4	2	2	10		130		Yes
58	10	8	8	8	8	8					2	6		7		
59	10	8	8	8	8	6	2				2	6		15		
60	10	8	8	8	8	4	4				2	6		20		
61	10	8	8	8	8	4	2	2			2	8		33		
62	10	8	8	8	8	2	2	2	2		2	8		62	Yes	
63	10	8	8	8	6	6	4				2	6		22		
64	10	8	8	8	6	6	2	2			2	8		35		
65	10	8	8	8	6	4	4	2			2	8		40		
66	10	8	8	8	6	4	2	2	2		2	8		66	Yes	
67	10	8	8	8	6	2	2	2	2	2	2	10		103		Yes
68	10	8	8	8	4	4	4	4			2	8		49		
69	10	8	8	8	4	4	4	2	2		2	8		73		
70	10	8	8	8	4	4	2	2	2	2	2	10		107		Yes
71	10	8	8	6	6	6	6				2	6		25		
72	10	8	8	6	6	6	4	2			2	8		43		
73	10	8	8	6	6	6	2	2	2		2	8		69		
74	10	8	8	6	6	4	4	4			2	8		51		
75	10	8	8	6	6	4	4	2	2		2	8		75	Yes	
76	10	8	8	6	6	4	2	2	2	2	2	10		109		Yes
77	10	8	8	6	4	4	4	4	2		2	8		83		
78	10	8	8	6	4	4	4	2	2	2	2	10		115		Yes
79	10	8	8	4	4	4	4	4	4		2	8		92		
80	10	8	8	4	4	4	4	4	2	2	2	10		123		Yes
81	10	8	6	6	6	6	6	2			2	8		45		
82	10	8	6	6	6	6	4	4			2	8		53		
83	10	8	6	6	6	6	4	2	2		2	8		77		

Table A.2: Frequency distributions of 10×10 Latin squares

Dist No	1	2	3	4	5	6	7	8	9	10	LB	UB	Mann	Compl	Self-comp	Fullhouse
84	10	8	6	6	6	6	2	2	2	2	2	10		111		Yes
85	10	8	6	6	6	4	4	4	2		2	8		85	Yes	
86	10	8	6	6	6	4	4	2	2	2	2	10		117		Yes
87	10	8	6	6	4	4	4	4	4		2	8		93		
88	10	8	6	6	4	4	4	4	2	2	2	10		124		Yes
89	10	8	6	4	4	4	4	4	4	2	2	10		131		Yes
90	10	8	4	4	4	4	4	4	4	4	2	10		137		Yes
91	10	6	6	6	6	6	6	4			2	8		55		
92	10	6	6	6	6	6	6	2	2		2	8		79		
93	10	6	6	6	6	6	4	4	2		2	8		87		
94	10	6	6	6	6	6	4	2	2	2	2	10		119		Yes
95	10	6	6	6	6	4	4	4	4		2	8		95	Yes	
96	10	6	6	6	6	4	4	4	2	2	2	10		126		Yes
97	10	6	6	6	4	4	4	4	4	2	2	10		133		Yes
98	10	6	6	4	4	4	4	4	4	4	2	10		138		Yes
99	8	8	8	8	8	8	2				0	6		16		
100	8	8	8	8	8	6	4				0	6		23		
101	8	8	8	8	8	6	2	2			0	8		36		
102	8	8	8	8	8	4	4	2			0	8		41		
103	8	8	8	8	8	4	2	2	2		0	8		67		
104	8	8	8	8	8	2	2	2	2	2	0	10		104	Yes	Yes
105	8	8	8	8	6	6	6				0	6		26		
106	8	8	8	8	6	6	4	2			0	8		44		
107	8	8	8	8	6	6	2	2	2		0	8		70		
108	8	8	8	8	6	4	4	4			0	8		52		
109	8	8	8	8	6	4	4	2	2		0	8		76		
110	8	8	8	8	6	4	2	2	2	2	0	10		110	Yes	Yes
111	8	8	8	8	4	4	4	4	2		0	8		84		
112	8	8	8	8	4	4	4	2	2	2	0	10		116		Yes
113	8	8	8	6	6	6	6	2			0	8		46		
114	8	8	8	6	6	6	4	4			0	8		54		
115	8	8	8	6	6	6	4	2	2		0	8		78		
116	8	8	8	6	6	6	2	2	2	2	0	10		112		Yes
117	8	8	8	6	6	4	4	4	2		0	8		86		
118	8	8	8	6	6	4	4	2	2	2	0	10		118	Yes	Yes
119	8	8	8	6	4	4	4	4	4		0	8		94		
120	8	8	8	6	4	4	4	4	2	2	0	10		125		Yes
121	8	8	8	4	4	4	4	4	4	2	0	10		132		Yes
122	8	8	6	6	6	6	6	4			0	8		56		
123	8	8	6	6	6	6	6	2	2		0	8		80		
124	8	8	6	6	6	6	4	4	2		0	8		88		
125	8	8	6	6	6	6	4	2	2	2	0	10		120		Yes
126	8	8	6	6	6	4	4	4	4		0	8		96		

Table A.3: Frequency distributions of 10×10 Latin squares

Dist No	1	2	3	4	5	6	7	8	9	10	LB	UB	Mann	Compl	Self-comp	Fullhouse
127	8	8	6	6	6	4	4	4	2	2	0	10		127	Yes	Yes
128	8	8	6	6	4	4	4	4	4	2	0	10		134		Yes
129	8	8	6	4	4	4	4	4	4	4	0	10		139		Yes
130	8	6	6	6	6	6	6	6			0	8		57		
131	8	6	6	6	6	6	6	4	2		0	8		89		
132	8	6	6	6	6	6	6	2	2	2	0	10		121		Yes
133	8	6	6	6	6	6	4	4	4		0	8		97		
134	8	6	6	6	6	6	4	4	2	2	0	10		128		Yes
135	8	6	6	6	6	4	4	4	4	2	0	10		135	Yes	Yes
136	8	6	6	6	4	4	4	4	4	4	0	10		140		Yes
137	6	6	6	6	6	6	6	6	2		0	8		90		
138	6	6	6	6	6	6	6	4	4		0	8		98		
139	6	6	6	6	6	6	6	4	2	2	0	10		129		Yes
140	6	6	6	6	6	6	4	4	4	2	0	10		136		Yes
141	6	6	6	6	6	4	4	4	4	4	0	10		141	Yes	Yes

Table A.4: Frequency distributions of 10×10 Latin squares

Appendix B

IP model results

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
1	6	4	1					
2	4	6	2					
3	4	6	3					
4	4	6	8					
5	4	6	10		1	1	141	none
6	4	8	27		3	2	136, 141	no 129
7	4	8	58		3	3	129, 136, 141	none
8	4	6	4					
9	4	6	9					
10	4	6	5		1	1	141	none
11	4	6	11		1	1	141	none
12	4	8	28		3	2	136, 141	no 129
13	4	6	17		1	1	141	none
14	4	8	30		3	3	129, 136, 141	none
15	4	8	59		3	3	129, 136, 141	none
16	4	10	99	Y	5	5	90, 98, 129, 136, 141	none
17	4	6	13		1	1	141	none
18	4	6	18		1	1	141	none
19	4	8	31		3	3	129, 136, 141	none
20	4	8	60		3	3	129, 136, 141	none
21	4	8	37		3	3	129, 136, 141	none
22	4	8	63		3	3	129, 136, 141	none
23	4	10	100	Y	5	5	90, 98, 129, 136, 141	none
24	4	8	47		3	3	129, 136, 141	none
25	4	8	71		3	3	129, 136, 141	none
26	4	10	105	Y	5	5	90, 98, 129, 136, 141	none
27	2	6	6		3	2	140, 141	139
28	2	6	12		3	2	140, 141	139
29	2	8	29		18	3	136, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 139
30	2	6	14		3	3	139, 140, 141	none
31	2	6	19		3	3	139, 140, 141	none
32	2	8	32		18	8	128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 127, 132
33	2	8	61		18	8	128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 127, 132
34	2	8	38		18	9	127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 132
35	2	8	64		18	11	120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 125, 132

Table B.1: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
36	2	10	101	Y	40	18	57, 88, 89, 90, 96, 97, 98, 120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 54, 56, 67, 70, 76, 78, 80, 84, 86, 94, 104, 110, 112, 116, 118, 125, 132
37	2	6	21		3	3	139, 140, 141	none
38	2	8	34		18	9	127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 132
39	2	8	39		18	9	127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 132
40	2	8	65		18	11	120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 125, 132
41	2	10	102	Y	40	18	57, 88, 89, 90, 96, 97, 98, 120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 54, 56, 67, 70, 76, 78, 80, 84, 86, 94, 104, 110, 112, 116, 118, 125, 132
42	2	8	48		18	11	120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 125, 132
43	2	8	72		18	11	120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 125, 132
44	2	10	106	Y	40	20	56, 57, 80, 88, 89, 90, 96, 97, 98, 120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 54, 67, 70, 76, 78, 84, 86, 94, 104, 110, 112, 116, 118, 125, 132
45	2	8	81		18	11	120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 125, 132
46	2	10	113	Y	40	20	56, 57, 80, 88, 89, 90, 96, 97, 98, 120, 121, 127, 128, 129, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 54, 67, 70, 76, 78, 84, 86, 94, 104, 110, 112, 116, 118, 125, 132
47	2	6	24		3	3	139, 140, 141	none
48	2	8	42		18	11	125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121
49	2	8	68		18	15	116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112
50	2	8	50		18	16	112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110
51	2	8	74		18	16	112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110
52	2	10	108	Y	40	30	54, 56, 57, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 67, 70, 76, 104, 110
53	2	8	82		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104

Table B.2: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
54	2	10	114	Y	40	34	52, 54, 56, 57, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 67, 104
55	2	8	91		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
56	2	10	122	Y	40	36	44, 46, 52, 54, 56, 57, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 67, 104
57	2	10	130	Y	40	40	36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
58	2	6	7		3	3	139, 140, 141	none
59	2	6	15		3	3	139, 140, 141	none
60	2	6	20		3	3	139, 140, 141	none
61	2	8	33		18	8	128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 127, 132
62	2	8	62		18	8	128, 129, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121, 125, 127, 132
63	2	6	22		3	3	139, 140, 141	none
64	2	8	35		18	11	125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121
65	2	8	40		18	11	125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121
66	2	8	66		18	13	120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118
67	2	10	103	Y	40	21	57, 88, 89, 90, 94, 96, 97, 98, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 54, 56, 67, 70, 76, 78, 80, 84, 86, 104, 110, 112, 116, 118
68	2	8	49		18	15	112, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 116
69	2	8	73		18	15	112, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 116

Table B.3: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
70	2	10	107	Y	40	28	54, 56, 57, 78, 80, 86, 88, 89, 90, 94, 96, 97, 98, 112, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 67, 70, 76, 84, 104, 110, 116
71	2	6	25		3	3	139, 140, 141	none
72	2	8	43		18	11	125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121
73	2	8	69		18	15	116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112
74	2	8	51		18	16	112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110
75	2	8	75		18	16	112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110
76	2	10	109	Y	40	30	54, 56, 57, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 52, 67, 70, 76, 104, 110
77	2	8	83		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
78	2	10	115	Y	40	34	52, 54, 56, 57, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 67, 104
79	2	8	92		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
80	2	10	123	Y	40	36	44, 46, 52, 54, 56, 57, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 67, 104
81	2	8	45		18	11	125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104, 110, 112, 116, 118, 120, 121
82	2	8	53		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
83	2	8	77		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104

Table B.4: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
84	2	10	111	Y	40	33	52, 54, 56, 57, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 67, 70, 104
85	2	8	85		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
86	2	10	117	Y	40	34	52, 54, 56, 57, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46, 67, 104
87	2	8	93		18	18	104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
88	2	10	124	Y	40	40	36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
89	2	10	131	Y	40	40	36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
90	2	10	137	Y	43	43	16, 23, 26, 36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
91	2	8	55		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
92	2	8	79		18	17	110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	104
93	2	8	87		18	18	104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
94	2	10	119	Y	40	36	52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	36, 41, 44, 46

Table B.5: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
95	2	8	95		18	18	104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
96	2	10	126	Y	40	40	36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
97	2	10	133	Y	40	40	36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
98	2	10	138	Y	43	43	16, 23, 26, 36, 41, 44, 46, 52, 54, 56, 57, 67, 70, 76, 78, 80, 84, 86, 88, 89, 90, 94, 96, 97, 98, 104, 110, 112, 116, 118, 120, 121, 125, 127, 128, 129, 132, 134, 135, 136, 139, 140, 141	none
99	0	6	16		5	5	137, 138, 139, 140, 141	none
100	0	6	23		5	5	137, 138, 139, 140, 141	none
101	0	8	36		40	18	124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123
102	0	8	41		40	18	124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123
103	0	8	67		40	21	119, 120, 121, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 122, 123
104	0	10	104	Y	101	32	57,87,88,89,90,93,94,95,96,97,98,119,120,121,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,91,92,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,122,123
105	0	6	26		5	5	137, 138, 139, 140, 141	none
106	0	8	44		40	20	122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121

Table B.6: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
107	0	8	70		40	28	114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113
108	0	8	52		40	30	111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 113
109	0	8	76		40	30	111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 113
110	0	10	110	Y	101	56	53,54,55,56,57,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,61,62,64,65,66,67,68,69,70,72,73,74,75,76,81,101,102,103,104,106,107,108,109,110
111	0	8	84		40	33	108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 113
112	0	10	116	Y	101	68	50,51,52,53,54,55,56,57,68,69,70,74,75,76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,108,109,110,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,61,62,64,65,66,67,72,73,81,101,102,103,104,106,107
113	0	8	46		40	20	122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121
114	0	8	54		40	34	107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 113
115	0	8	78		40	34	107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 113

Table B.7: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
116	0	10	112	Y	101	68	49,50,51,52,53,54,55,56,57,73,74,75,76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,107,108,109,110,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,61,62,64,65,66,67,68,69,70,72,81,101,102,103,104,106,113
117	0	8	86		40	34	107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104, 106, 113
118	0	10	118	Y	101	71	49,50,51,52,53,54,55,56,57,68,69,70,73,74,75,76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,107,108,109,110,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,61,62,64,65,66,67,81,101,102,103,104,106,113
119	0	8	94		40	36	103, 104, 107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 106, 113
120	0	10	125	Y	101	84	35,36,40,41,42,43,44,45,46,49,50,51,52,53,54,55,56,57,66,67,68,69,70,73,74,75,76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,103,104,107,108,109,110,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,38,39,48,61,62,64,65,72,81,101,102,106,113
121	0	10	132	Y	101	84	35,36,40,41,42,43,44,45,46,49,50,51,52,53,54,55,56,57,66,67,68,69,70,73,74,75,76,77,78,79,80,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,103,104,107,108,109,110,111,112,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,38,39,48,61,62,64,65,72,81,101,102,106,113

Table B.8: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
122	0	8	56		40	36	106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104
123	0	8	80		40	36	106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	101, 102, 103, 104
124	0	8	88		40	40	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
125	0	10	120	Y	101	84	48,49,50,51,52,53,54,55,56,57,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,61,62
126	0	8	96		40	40	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
127	0	10	127	Y	101	96	34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,61,62

Table B.9: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
128	0	10	134	Y	101	100	32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29
129	0	10	139	Y	115	112	7,14,15,16,19,20,21,22,23,24,25,26,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	6,12,29
130	0	8	57		40	40	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
131	0	8	89		40	40	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
132	0	10	121	Y	101	84	48,49,50,51,52,53,54,55,56,57,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,61,62

Table B.10: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
133	0	8	97		40	40	101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
134	0	10	128	Y	101	100	32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29
135	0	10	135	Y	101	100	32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	29
136	0	10	140	Y	115	115	6,7,12,14,15,16,19,20,21,22,23,24,25,26,29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	none
137	0	8	90		43	43	99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none

Table B.11: Assignments between frequency distributions of 10×10 Latin squares

Dist No	LB	UB	Compl	Fullhouse	# LU satisfied	# Assgn	List Assignable	List not Assignable
138	0	8	98		43	43	99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141	none
139	0	10	129	Y	115	112	30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	27,28,29
140	0	10	136	Y	115	115	6,7,12,14,15,16,19,20,21,22,23,24,25,26,29,32,33,34,35,36,38,39,40,41,42,43,44,45,46,48,49,50,51,52,53,54,55,56,57,61,62,64,65,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,101,102,103,104,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141	none
141	0	10	141	Y	135	135	All	none

Table B.12: Assignments between frequency distributions of 10×10 Latin squares

Appendix C

Notes to IP models implemented in AMPLDev

Phase Two are conducted using using AMPLDev which is a software which is capable of solving large Integer Programming problems. The programming language running on this program is AMPL.

C.1 Sets

N = n, where the size of the Latin square is determined by $2n$.

DIM = the set of elements 1 to $2n$ in the Latin square of size $2n$.

ALL = the set of all frequency distributions under examination.

L2 = the subset of ALL, which is the list of all frequency distributions of Latin square L2 that satisfy the LU limits with a frequency distributions of Latin square L1.

L1 = the subset of ALL; the set contains only one member, which is the predefined frequency distribution of Latin square L1. This is for the convenience of changing the model to run the solver for each of the 76 frequency distributions.

C.2 Parameters

$freq\_matrix\{ALL, DIM\}$ = the parameter contains all of 76 frequency distributions. This is a two-dimensional parameter where each row is a frequency distribution and the columns show the corresponding frequencies.

C.3 Variables

$pairs\{DIM, DIM, L2\} = (x_{klp})$ The binary variable to check whether element k of the frequency distribution of L1 can be assigned to element l of any of the frequency distributions of L2, which have their lower and upper limits satisfied with L1.

$assignment\{L2\} = (y_p)$ The binary variable to indicate if frequency distribution p of L2 can be assigned to the predefined frequency distribution of L1.

C.4 Model codes

```
############### MOLS: Assignment Modified ##############

param N;

######## SETS #########

# order of the Latin squares
set DIM = 1..2*N;
# list of all structures under examination
set ALL;
# list of possible structures for Latin square L2,
could be changed for each L1
set L2 within ALL;
# the distribution to find assignments
set L1 within ALL;


######## PARAMETERS ##########
```

```
# all frequency matrix distribution
param freq_matrix{ALL, DIM};

######## VARIABLES #########

# 1 if there is assignment between i and j elements of L1 and L2;
  0 otherwise
var pairs{DIM, DIM, L2} binary;
# 1 if L1 can be assigned to L2; 0 otherwise
var ass{L2} binary;

######## OBJECTIVE #########

maximize LSAssignment: sum{k in L2} ass[k];

######## CONSTRAINTS #########

subject to

#(1) element i of L1 must be assigned to b_j elements of L2
numberAssignments {i in DIM, k in L2, p in L1}:
sum{j in DIM} pairs[i,j,k] <= freq_matrix[p,i];

#(2) every element j of any L2 must be assigned to a_i elements of L1
evenTime {j in DIM, k in L2}:
sum{i in DIM} pairs[i,j,k] <= freq_matrix[k,j];

#(3) assignment is admissible if sum of pairing equals 50
admissibleAssignment {k in L2}:
50*ass[k] = sum{i in DIM, j in DIM} pairs[i,j,k];
```

C.5 Data file sample

```
######## Mutually Orthogonal Latin Squares ########

##### Set parameter ######
param N = 5;
```

```
##### Sets #######

set ALL :=        5        6        7   ...   141;

### for L1 = D5
set L2 :=         141; # the set of limits mutually satisfied of L1

set L1 :=         5;

##### Parameters #####

param freq_matrix:
1      2      3      4      5      ...    10 :=
5      10     10     10     10     ...    0
6      10     10     10     10     ...    0
7      10     10     10     10     ...    0
11     10     10     10     8      ...    0
12     10     10     10     8      ...    0
...
141    6      6      6      6      ...    4;
```

Note: The data file is changed with respect to the frequency distribution.

Appendix D

Notes to Python files

Phase Three is conducted in Python. The code files are available as attachments with the submission of this report. Furthermore, copies of these files are also available on a public online repository at https://github.com/lbhtran/mols.git.

The listing of the Python files are as follows;

- mols10.py: data file of 141 frequency distributions, set of full house distributions, and set of complements.
- molsfunc.py: function file, which contains functions to count pairs/triples of frequency distributions (see below).
- mols.py: execution file, which reports the solutions as shown in the report.

Some important functions in Python are as follows;

1. Functions for assignable pairs of frequency distributions
    1.1. Count pairs: generate_pairs(nodes)
    1.2. Eliminate complement pairs: remove_comp_pairs(pairs)
2. Functions for assignable triples of frequency distributions
    2.1. Count triples: generate_triples(nodes)
    2.2. Eliminate complement triples: remove_comp_triples(triples)

2.3. Count the frequencies of full house pairs in triples: count_fh_freq(fh_grow, fh_pairs)

3. Functions for both pairs and triples

   3.1. Count number of pairs/triples with at least n full house distributions: count_fh(tuples, n)

   3.2. Grow triples from pairs of 2 full house distributions: fh_grow_triples(fh_pairs)

Notes:

nodes : a dictionary where the key is the distribution ID and the value is the list of assignable distributions.

pairs : list of 2-tuples where each 2-tuple indicates a pair of assignable distributions.

triples : list of 3-tuples where each 3-tuple indicates a triple of assignable distributions.

fh_pairs : list of all assignable full house pairs, obtained from list of assignable pairs.

fh_grow : list of triples grown from assignable full house pairs.

tuples : any list of n-tuple, this list is either pairs/triples in this case.

n : the number of full house distributions in an n-tuple.

Appendix E

Notes to Excel spreadsheets

The processes which Excel spreadsheets were used are: recording results, searching for complements, searching for sets of limits satisfied distributions, solving the basic IP model, and performing hidden Mann analysis for Latin squares of order 6. These files are available as attachments with the submission of this report.

1. 10x10 Lower and Upper Limits - part 1.xlsm: contains VBA function to search for sets of limits satisfied distributions.
2. 10x10 Lower and Upper Limits - part 2.xlsx: results for sets of limits satisfied distributions.
3. 141 MOLS 10 Distributions Final.xlsx: contains 2 tables mentioned in Appendix A and B.
4. IP Solver for Higher order LS.xlsm: contains Macro to solve the basic IP model.
5. sub-square combination 6x6-extra.xlsm: contains VBA function to perform hidden Mann analysis for Latin squares of order 6.

Appendix F

More on Latin squares of order 6

Here are the results for checking Mann's rejectable structures for Latin Square of order 6 from the book 'Latin Squares and their applications' by (Denes and Keedwell, 1974). Latin square is the Latin square ID as indicated in the book. The hidden columns and rows are the columns and rows that contains Mann's reject-able structures. There could be more but only one of them are recorded. The sum is the sum of top three frequency values of the hidden Mann's structures, which is always 8 or 9.

No hidden Mann's can be found for class 6, 7, 8, 10 and 11.

*All LS in class 9 has distribution [6 6 6 0 0 0] in sub-square I.

Table F.1: Results for Hidden Mann sub-squares in Latin squares of order 6

Latin square	**Hidden columns**	**Hidden rows**	**Sum**
1.1.1	1-3-5	1-3-5	(did not record)
2.1.1	1-5-6	1-5-6	(did not record)
3.1.1	1-2-3	1-2-3	(did not record)
4.1.1	1-3-5	1-3-5	(did not record)
4.1.2	1-5-6	1-5-6	8
5.1.1	1-3-5	1-3-6	8
5.1.2	1-5-6	1-5-6	8
5.1.3	1-4-5	1-4-5	8
5.1.4	1-4-5	1-4-6	8
5.1.5	1-4-6	1-4-5	8
9.1.1*	1-2-3	1-2-3	9
12.1.1	1-3-4	1-3-4	8
12.1.2	1-5-6	1-5-6	8
12.1.3	1-4-5	1-4-5	8
12.1.4	1-3-6	1-3-6	8
12.1.5	1-4-5	1-4-5	8
12.1.6	1-2-6	1-2-6	8
12.1.7	1-3-5	1-3-6	8
12.1.8	1-3-6	1-3-5	8
12.1.9	1-2-3	1-2-4	8
12.1.10	1-2-4	1-2-3	8
12.2.1	1-5-6	1-5-6	8
12.2.2	1-3-4	1-3-4	8
12.2.3	1-4-5	1-5-6	8
12.2.4	1-3-6	1-3-6	8
12.2.5	1-4-5	1-5-6	8
12.2.6	1-2-6	1-2-6	8
12.2.7	1-3-5	1-3-5	8
12.2.8	1-3-5	1-3-6	8
12.2.9	1-2-3	1-2-3	8
12.2.10	1-2-3	1-2-4	8