Version 2.67   **The Sturm-Louville-Webster Horn equation**   **March 8, 2016**

Univ. of Illinois   **Prof. Allen**

Title: The reactive radiation impedance of acoustic horns.

Abstract

Here we seek to resolve a subtle paradox about the solutions of the horn, the class of differential equations describing guided waves, with spatial varying coefficients. This equation is cast as a generalization of the Sturm-Louville problem, in the range variable. On one hand, the solution may be written as a superposition of independent out- and in-bound waves. On the other hand, the area of the horn may be computed from the reactive component of the input impedance, thus forming an *inverse problem*. If this knowledge of the area is available at the input, then how can the signal only be traveling outbound? The resolution of this paradox is that the outbound horn solution is really a superposition of forward and backward plane waves, induced by the variation in area at the propagating wave-front, clarifying a basic misunderstanding about the nature of out- and in-bound waves and their relation to the reactive part of the radiation impedance. The observations presented here must generalized to other wave based systems (i.e., Maxwell's Equations), however this is not done here.

Contents

1 Introduction

Horns provide an important generalization of the solution of the 1D wave equation, in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns is vocal tract acoustics, loudspeaker design, cochlear mechanics, any case having wave propagation (Brillouin, 1953). The typical experimental setup is shown in Fig. 1.

The traditional formulation of the horn is widely known and is discussed in many papers and books (Hanna and Slepian, 1924; Morse, 1948; Mawardi, 1949) and in greater detail by (Olson, 1947, p. 101) and (Pierce, 1981, p. 360). Extensive experimental testing for various types of horns (conical, exponential, parabolic) along with a review of horn theory is provided in Goldsmith and Minton (1924).

It is frequently stated that the Webster Horn equation (WHEN) Webster (1919) is an approximation that only applies to low frequencies, because it is assumed that the area function is the cross-sectional area, not the area of the wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981). This serious limitation is discussed in the majority of the discussions Morse (1948); Shaw (1970); Pierce (1981). By a proper derivation based on Gauss' Law, as provided in Appendix A, this restriction may be avoided however, *making the Webster theory of the Horn equation exact.* Here the variables are redefined as the average pressure and the volume velocity, each defined on the wave-front boundary (Hanna and Slepian, 1924).

The horn equation is a generalization of Sturm-Louville theory, for cases where the pressure (i.e., force potential) does not form a "separable" coordinate system.

The study of wave propagation begins at least as early as Huygens (ca. 1678) and following soon after (ca. 1687) with Sir Isaac Newton's calculation of the speed of sound (Pierce, 1981). To obtain a wave, one must include two basic components, the stiffness of air, and its mass. These two equations shall be denoted *Hooke's Law* ($F = kx$) and *Newton's 2nd Law* ($F = ma$), respectively. In vector form these equations are *Euler's equation* (i.e., conservation of momentum density)

$$-\nabla p(\mathbf{x}, t) = \rho_0 \frac{\partial}{\partial t} \boldsymbol{u}(\mathbf{x}, t) \tag{1}$$

and the *continuity equation* (i.e., conservation of mass density)

$$-\nabla \cdot \boldsymbol{u} = \frac{1}{\eta_0 P_0} \frac{\partial}{\partial t} p(\mathbf{x}, t) \tag{2}$$

(Pierce, 1981, page 15).

Webster Horn Equation: These two equations are transformed into the WHEN by integration over the wave-front surface, transforming the equation into the acoustic variables, the *average pressure $\varrho(x, t)$* and the *volume velocity $\nu(x, t)$*

$$\frac{d}{dx} \begin{bmatrix} \mathcal{P}(x, \omega) \\ \mathcal{V}(x, \omega) \end{bmatrix} = - \begin{bmatrix} 0 & \mathscr{Z}(s, x) \\ \mathscr{Y}(s, x) & 0 \end{bmatrix} \begin{bmatrix} \mathcal{P}(x, \omega) \\ \mathcal{V}(x, \omega) \end{bmatrix}, \tag{3}$$

as discussed in Appendix A, Eqs. A.5,A.7. The Fourier-transform pair of the average pressure and volume velocity are denoted as $\varrho(x, t) \leftrightarrow \mathcal{P}(x, \omega)$ and $\nu(x, t) \leftrightarrow \mathcal{V}(x, s)$, respectively.[1] Here we use the complex *Laplace frequency* $s = \sigma + j\omega$ when referring to the *per-unit-length impedance*

$$\mathscr{Z}(s, x) \equiv s \frac{\rho_0}{A(x)} = sM(x) \tag{4}$$

---

[1]**Notation:** Lower case variables (i.e., $\varrho(x, t), \nu(x, t)$) denote time-domain functions while upper case letters (i.e., $\mathcal{P}(x, \omega), \mathcal{V}(x, \omega), Z(x, s), \mathscr{Z}, \mathcal{Z}(s)$) indicate frequency domain Fourier (ω) or Laplace (s) transforms, the latter being *analytic* in $s = \sigma + j\omega$ for $\sigma > 0$. Matrix notation provides notational transparency.
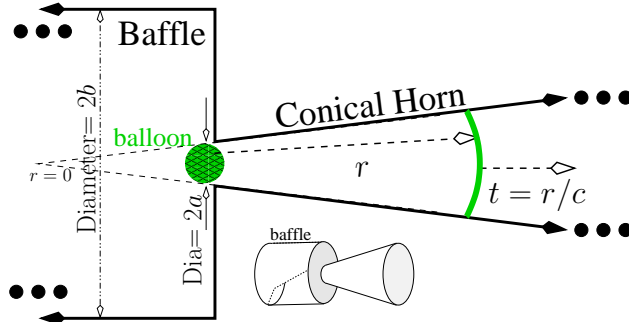
Figure 1: *Experimental setup showing a large pipe on the left terminating the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the balloon is pricked and a pressure pulse is created. The baffle on the left is meant to represent a semi-∞ long tube having a large radius compared to the horn input diameter $2a$, such that the acoustic admittance looking to the left ($A/\rho_0 c$ with $A \to \infty$), is very large compared to the horn's throat admittance (Eq. 12). At time T the outbound pressure pulse $p(r, T) = \delta(t - T/c)/r$ has reached radius T/c. Here $r = x + r_0$ where x is the location of the source at the throat of the horn and r is measured from the vertex.*

and *per-unit-length admittance*

$$\mathcal{Y}(s, x) \equiv s\frac{A(x)}{\eta_0 P_0} = sC(x), \tag{5}$$

to clearly indicate that these functions must be *causal*, and except at their poles, *analytic* in s.[2] Here $M(x) = \rho_0/A(x)$ is the horn's *per-unit-length mass*, $C(x) = A(x)/\eta_0 P_0$ *per-unit-length compliance*, $\eta_0 = c_p/c_v \approx 1.4$ (air), $\kappa(s) \equiv \sqrt{\mathcal{Z}\mathcal{Y}} = s/c$ is the *propagation function*,[3] $c = \sqrt{\eta_0 P_0/\rho_0}$ is the speed of sound and $r_0(x) = \sqrt{\mathcal{Z}/\mathcal{Y}} = \sqrt{\rho_0\eta_0 P_0}/A(x) = \rho_0 c/A(x)$ is denoted the *surge resistance*. Equation 3 is the differential equation in the *conjugate variables* $\mathcal{P}, \mathcal{V}$. The product of conjugate variables defines the *intensity*, and the ratio, an *impedance* (Pierce, 1981, p. 37-41).

We shall discuss two alternative matrix formulations of these equations, the ABCD *transmission matrix*, use for computation, and the *impedance matrix*, used when working with experimental measurements (Pierce, 1981, Chapter 7). For each formulation *reciprocity* and *reversibility* show up as different matrix symmetries, as discussed by (Pierce, 1981, p. 195-203) and further here.

The quasistatic approximation: Having said that, there is an approximation that must play a subtle role, that of quasistatics across the radius. In the typical application of the horn, the wavelength is much greater than the diameter of the horn. When this condition is violated higher order modes can play a significant role. In fact there is a frequency where the radius will be equal to a quarter wavelength. When this happens, the quasistatic approximation fails, and cross-modes will become significant. I have not seen any discussion of this in the context of Eq. 9.

Role of Impedance Since every impedance or admittance is *causal*, it is a function of the Laplace frequency s, and has a Laplace transform (poles and zeros). We shall maintain the distinction that functions of ω are Fourier transforms, not analytic in $j\omega$, while functions of s correspond to Laplace transforms, which *are* necessarily analytic in s, in the *right half plane* (RHP) *region of convergence* (ROC) (i.e., a causal function). This distinction is critical, since we typically describe impedance $Z(s)$ and admittance $Y(s)$ in terms of their poles and zeros, as analytic functions.[4]

---

[2]A function $F(s)$ is said to be analytic in s at point a if it may be represented by a convergent Taylor series in s the neighborhood of $s = a$. That is if $F(s) = \sum_{n=0}^{\infty} f_n(a)(s-a)^n$, where $f_n(a) = \frac{d^n F(s)}{ds^n}\Big|_{s=a}$. Note that $F(s) = 1/s$ is analytic at every point in the $s = \sigma + j\omega$ plane, other than $s = 0$, where it has a pole.

[3]The *propagation function* is commonly called the *wave-number* (Sommerfeld, 1952, p. 152) and the *propagation-constant*. However since $\kappa(s)$ is neither a number, nor constant, we have appropriately rename it. The meaning of a complex $\kappa(s)$ is addressed in (Sommerfeld, 1952, p. 154). The units of $\kappa(s)$ are reciprocal length, namely $\kappa(s) = 1/\lambda(s)$. Since $\kappa(s)$ is complex-analytic, the wavelength $\lambda(s)$ is a complex-analytic function of s. It follows that when $\Re\lambda(j\omega) = 0$, the wave must be evanescent, loss-less and causal.

[4]When an analytic function of complex variable s includes the pole it is called a *Laurent series* in s. For example, the impedance of a capacitor C is $Z_c(s) = 1/sC$, which is analytic in s everywhere other than $s = 0$. The capacitor has a voltage time response given by the integral of the current, i.e., $v(t) = \frac{1}{C}\int^t i(t)dt = \frac{1}{C}u(t) \star i(t)$, where $u(t)$ is the Heaviside step function and $\star$ represents convolution.

In this document we shall consider four different cases of $A(x)$, as summarized in Table 1. Given an area function $A(x)$, each horn has a distinct wave equation, and thus a distinct solution.

Table 1: *Table of horns and their properties discussed in this document. The horn axis is defined by x, the radius at x is $r(x)$, the area is $A(x) \equiv 2\pi r^2(x)$, $F(x)$ is the coefficient on $\mathcal{P}_x$ and $\kappa(s) \equiv s/c$, where c is the speed of sound. A dimensionless range variable may be defined as $x \equiv (\xi - \xi_0)/(L - \xi_0)$, with ξ the linear distance along the horn, where $x = \xi_0$ to L corresponding to $x = 0$ to 1. The horn's primitive solutions are $\mathcal{P}^{\pm}(x, s) \leftrightarrow \varrho^{\pm}(x, t)$. When $\pm$ is indicated, the outbound solution corresponds to the negative sign. The last column is the radiation admittance normalized by $A(x)/\rho c$, for a simplified expression.*

#D	Name	radius	Area/A_0	$F(x)$	$\mathcal{P}^{\pm}(x, s)$	$\varrho^{\pm}(x, t)$	$Y_{rad}^{\pm}/\mathcal{Y}$
1D	plane	1	1	0	$e^{\mp\kappa(s)x}$	$\delta(t \mp x/c)$	1
2D	parabolic	$\sqrt{x/x_0}$	x/x_0	$1/x$	$H_0^{\pm}(-j\kappa(s)x)$	—	$-jxH_1^{\pm}/H_0^{\pm}$
3D	conical	$x + r_0$	$(x + r_0)^2$	$2/(x + r_0)$	Eq. 28	$\delta(t \mp x/c)/x$	$1 \pm c/sx$
	exp	e^{mx}	e^{2mx}	$2m$	$e^{-(m\pm\sqrt{m^2+\kappa^2})x}$	$e^{-mx}E(t)$	—

Goals of the study: A primary focus of this study is to identify and characterize the physical significance of the reactance (imaginary) component of the loss-less horn's radiation admittance $Y_{rad}(s)$ (and impedance $Z_{rad}(s)$). Of course for the case of the uniform horn (i.e., plane-waves), the input (driving-point) admittance $Y_{rad} = \mathcal{Y}$ (impedance) must be real and positive (zero imaginary part).[5] When $A(x)$ is not constant the radiation admittance is always complex (Salmon, 1946a,b; Olson, 1947; Morse, 1948; Leach, 1996). Interestingly, in some special cases, Y_{rad} is purely reactive (the real part is zero), as for the exponential horn below the cut-off frequency (see Fig. 2). We will see that the Horn equation generalizes Strum-Louville theory to situations where due to the choice of $A(x)$, a separation of variables fails. This has interesting and wide-reaching consequences.

As shown by[6] Youla (1964); Sondhi and Gopinath (1971), there is a unique relationship between the reactive part of the input "driving point" admittance Y_{rad} and the horn area function $A(x)$. Determining $A(x)$ from $Y_{rad}(0, s)$ is known as *the inverse problem*.

2 Webster Horn Equation

Equation 3 is equivalent to the traditional second-order Webster horn equation in the pressure. To see this take the partial derivative with respect to x (abbreviated ∂_x) of the Newton pressure equation, giving $\mathcal{P}_{xx} + \mathscr{L}_x\mathcal{V} + \mathscr{L}(s)\mathcal{V}_x = 0$. Next use the Newton and Hooke equations once again, to remove the velocity, obtaining the *Webster wave equation* for the Laplace Transform pair $\varrho(x, t) \leftrightarrow \mathcal{P}(x, \omega)$ for pressure

$$\mathcal{P}_{xx} - (\mathscr{L}_x/\mathscr{L})\,\mathcal{P}_x = \frac{s^2}{c^2}\mathcal{P} \leftrightarrow \frac{1}{c^2}\varrho_{tt}. \tag{6}$$

This is the more traditional form of the Webster equation (Webster, 1919; Morse, 1948; Morse and Feshbach, 1953; Pierce, 1981).

2.1 General solutions of the Horn Equation

In the mathematical literature the general solution of equation of this form are known as *Strum-Liouville* (SL) problems, the solution of which may be obtained by *integration by parts*, following the multiplication by an "integration factor" $\sigma(x)$:

$$\frac{1}{\sigma}\partial_x(\sigma\mathcal{P}_x) \equiv \mathcal{P}_{xx} + \partial_x\ln\sigma(x)\,\mathcal{P}_x(x) \tag{7}$$

---

[5]Here we are only considering loss-less horns, i.e., those that satisfy Eq. 3.
[6]The publication of Youla (1964) seems easier to understand.

By a direct comparison of the above to Eq. 6, we see that $\sigma_x/\sigma \equiv -\mathscr{Z}_x/\mathscr{Z}$. Thus $\partial_x \ln(\sigma)/ \equiv -\partial_x \ln(\mathscr{Z}) = \partial_x \ln(A)$, or

$$\sigma(x) \equiv A(x), \tag{8}$$

that is, the integration factor is physically the area function.

Following this approach we put Eq. 8 into Eq. 7 to obtain the Webster Horn equation (Morse, 1948, p. 269)

$$\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial \varrho}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 \varrho}{\partial t^2}. \tag{9}$$

This "integrated" Sturm-Louville (i.e. , *self-adjoint*) format obscures the natural symmetry seen in the equivalent relation Eq. 3.

Thus we see that for the Webster Horn equation the physical meaning of the integration factor is simply the area function. Physically this makes sense. It explains, for example, why the integration factor in the Sturm-Liouville theory must be strictly positive. While it seems likely that Morse (1948) must have been aware of this connection, to our knowledge he did explicitly state so in writing.

In summary, if one starts with the 3D wave equation, and by the application of Gauss' law, transforms it into a "1 dimensional" (i.e., single range variable) equivalent horn equation, then a Sturm-Louville-like equation results. This may be integrated by the application of the area function as the integration factor.

2.2 Primitive solutions $\varrho^\pm(x,t)$

For each choice of area function used in Eq. 3 (or Eq. 9), there are two causal *primitive solutions* of the homogeneous (i.e., undriven) equation, identified as an *outbound* (right-traveling) and *inbound* (left-traveling) wave, denoted as the Laplace transform pair $\varrho^\pm(x,t) \leftrightarrow \mathcal{P}^\pm(x,s)$, and normalized such that $\mathcal{P}^\pm(x_0,s) = 1$, where x_0 defines the *throat* (e.g., the input) location, as show in Fig. 1.

Propagation function $\kappa(s)$ The primitive solutions of the horn equation always depend on a complex wave *propagation function* $\kappa(s)$, defined as the square root of the *product* of $\mathscr{Z}$ and $\mathscr{Y}$:

$$\kappa(s) \equiv \sqrt{\mathscr{Z}(s,x)\mathscr{Y}(s,x)} = \sqrt{\frac{s\rho_0}{A(x)} \times \frac{sA(x)}{\eta_0 P_0}} = \frac{s}{c}, \tag{10}$$

where the speed of sound is given by $c = \sqrt{\eta_0 P_0/\rho_0}$. Note how the area $A(x)$ cancels in this expression, making the speed of sound constant. While horns are, in general, dispersive, the wave-front speed is always constant.

Examples: The primitive solutions for the four horns are summarized in Table 1. For the plane wave solution in a pipe, the two primitive solutions are $\delta(t \mp x/c) \leftrightarrow P^\pm(x,\kappa) = e^{\mp\kappa(s)x}$. These two directed wave solutions are functions of Laplace frequency s, since they must be causal. They may be viewed as the impulse response of a semi-infinite section of horn, namely the causal solutions of Eq. 3, driven at the input end by an impulse at $t = 0$. It is a convention that these primitive solutions are normalized to 1 at the input ($x = 0$ in this example).

For the spherical geometry the "outbound" wave is $e^{-\kappa r}/r$. It contains a reflected component due the change in the area of the wave front. This gives rise to a reactive mass-component in the radiation impedance.

The exponential horn is of special interest because the radiation impedance is purely reactive below the horn's cutoff frequency. Thus below cutoff the reflection coefficient magnitude is unity (no energy can radiate from an open horn.
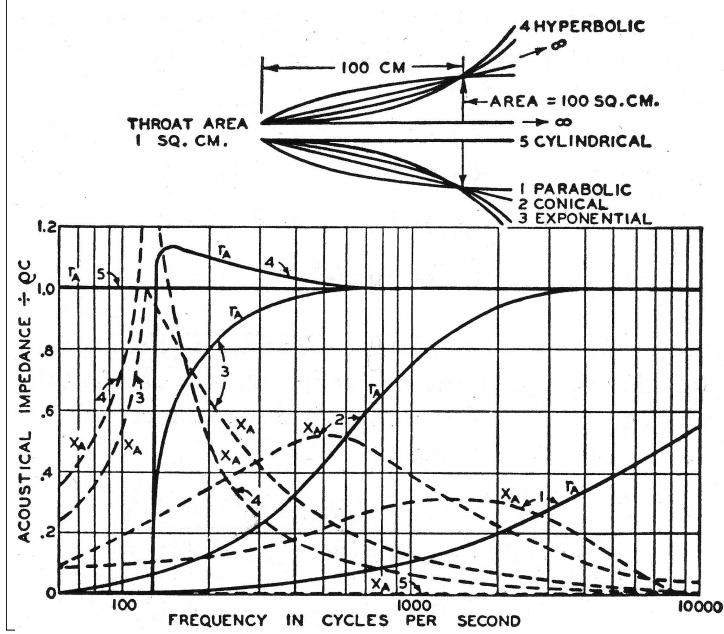
Figure 2: *Throat acoustical resistance r_A and acoustical reactance x_A, frequency characteristics of infinite parabolic, conical, exponential, hyperbolic and cylindrical horns having a throat area of 1 square centimeter. Note how the "critical" frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (1), the cutoff is around 3 kHz. For the conical horn (2), the cut-off is at 0.6 [kHz]. For the exponential horn the critical frequency is around 0.18 [kHz], which is 16 times smaller than for the parabolic horn. For each horn the cross-sectional area of the parabolic, conical exponential and hyperbolic horns is defined as 100 square centimeters at a distance of 100 centimeters from the throat (Olson, 1947, p. 101).*

Characteristic Admittance $\mathcal{Y}$: A second key definition is the wave *characteristic admittance* $\mathcal{Y}(x)$ defined as the square root of the *ratio* of $\mathscr{Z}$ and $\mathscr{Y}$

$$\mathcal{Y}(x) \equiv \sqrt{\frac{\mathscr{Y}(x,s)}{\mathscr{Z}(x,s)}} = \frac{\kappa}{\mathscr{Z}} = \frac{A(x)}{\rho_0 c}, \tag{11}$$

which depends specifically on $A(x)$, but not on frequency s. Based on physical requirements that the admittance must be positive, only the positive square root is allowed. The characteristic impedance is $\mathcal{Z} = 1/\mathcal{Y}$.

Since Eq. 3 is loss less, $\mathcal{Y}(x)$ must be real (and positive). If (when) losses are introduced, $\mathcal{Y}(s,x)$ must be a function of the Laplace frequency s. While such cases are interesting, and obviously more realistic, they are outside of the traditional loss-less Webster horn equation formulation.

Radiation Admittance: The *radiation admittance* at x is defined as the admittance looking into a semi-infinite horn (Fig. 2)

$$Y_{rad}^{\pm}(x,s) \equiv \pm \frac{\mathcal{V}^{\pm}(x,\omega)}{\mathcal{P}^{\pm}(x,\omega)} = \mp \frac{1}{\mathscr{Z}(x,s)} \frac{\mathcal{P}_x^{\pm}}{\mathcal{P}^{\pm}}\bigg|_x = \mp \frac{\mathcal{Y}(x)}{\kappa(s)} \frac{\partial \ln \mathcal{P}^{\pm}}{\partial x}\bigg|_x. \tag{12}$$

In general, $Y_{rad}^{\pm}$ depends on the direction of the velocity, but the real part of the radiation impedance must always be positive. Care must be taken when fixing the signs to obey these conventions. It is helpful to always define the volume velocity $\mathcal{V}^{\pm}(x,s)$ into the port.

Typically the velocity and pressure are functions of frequency ω, not complex frequency s, since they need not be causal functions. However the ratio of the two, defines an admittance, which necessarily is causal, and therefore is necessarily a function of s. Since the primitive solutions must be causal and stable functions, they must be analytic functions of s for $\sigma > 0$.

Surge admittance: Every radiation admittance may be written as $y_{rad}(t) \leftrightarrow Y_{rad}(s)$ which may be further split into a real *surge admittance*[7] $y_0 \delta(t)$ (Campbell, 1922), and a causal *remainder admittance* $y_r(t)$, as[8]

$$y_{rad}(t) = y_0 \delta(t) + y_r(t).$$

---

[7]Since it is real it would best be called a *surge conductance*.
[8]It seem obvious that $y_0 \equiv \mathcal{Y}$?

Alternatively this may also be written as the sum of an impedance surge and remainder components $z_{rad}(t) = z_0\delta(t) + z_r(t)$. These functions are inverses of each other in the convolution sense, namely $y_{rad}(t) \star z_{rad}(t) = \delta(t)$, which follows from $Z_{rad}(s)Y_{rad}(s) = 1$. Any function having a causal inverse is said to be *minimum phase* thus every impedance and admittance must be minimum phase. The remainder characterizes the dispersive component of the impedance, thus when it is zero, the impedance is purely real (e.g., the reactance is zero).

Wave admittance $Y(x, s)$:   The driving-point *wave admittance*, defined as the ratio of the volume velocity $\mathcal{V}(x, s)$ to the average pressure $\mathcal{P}(x, s)$ at any point x along the range axis, may be interpreted as follows: If the horn were split at any point x (i.e., a node is defined), the pressure at the two throats are the same, and the driving point velocity into the node is the sum of the two horn currents, looking in each direction. Due to this definition, the total current is the difference of these two driving point node currents. We define the wave admittance as the sum of the two radiation impedances

$$Y(x, s) \equiv Y_{rad}^+(x, s) + Y_{rad}^-(x, s)$$

since the driving-point pressures are the same while the driving-point currents add.

ABCD Transmission matrix: The transmission matrix is useful for computing the cascade of several system, such as a horn driven by a Thévenin system and loaded by the radiation impedance or a cascade of several horns. The solution of a horn having *finite length* may be expressed in terms of a 2-port ABCD matrix, that relates the pressure and volume velocity at the input and output ports (the two ends) of the horn ($x = 0$ and $x = L$)

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}. \tag{13}$$

Note that $\mathcal{A}(s) \equiv \frac{\mathcal{P}_0}{\mathcal{P}_L}\Big|_{\mathcal{V}_L = 0}$ is not to be confused with the horn area $A(x)$ (note the change of font).

By definition, the output velocity $\mathcal{V}_L$, of an ABCD matrix is *out of* the port, hence the negative sign, since $\mathcal{V}_0, \mathcal{V}_L$ are defined into their respective ports (Orfanidis, 2009). When the system is reversible, $\mathcal{A} = \mathcal{D}$, *reciprocal* when $\Delta_T \equiv \mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} = 1$, and *anti-reciprocal* when $\Delta_T = -1$.

Boundary Conditions: The pressure and velocity at any point x be written in terms of a superposition of the *two homogeneous solutions* $\mathcal{P}^+(x, s)$ and $\mathcal{P}^-(x, s)$ of Eq. 3. In matrix notation this superposition may be written as

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix} = \begin{bmatrix} \mathcal{P}^+(x) & \mathcal{P}^-(x - L) \\ Y_{rad}^+\mathcal{P}^+(x) & -Y_{rad}^-\mathcal{P}^-(x - L) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{14}$$

Coefficients $\alpha(\omega)$ and $\beta(\omega)$, which depend on frequency ω (but not x), are determined by the boundaries conditions at $x = L$. The *radiation admittance* $Y_{rad}^\pm(x, s)$ depends on the range x and Laplace frequency s.

To find the four parameters $[\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s)]$ we must evaluate the inverse of Eq. 14 at $x = L$, substitute this result into 14, and then evaluated the matrix product at $x = 0$. We need a simplified subscript notation for compactness: $\mathcal{P}_0^\pm \equiv \mathcal{P}^\pm(x = 0, s)$, $\mathcal{P}_L^\pm \equiv \mathcal{P}^\pm(x = L, s)$, i.e., $\mathcal{P}_L \equiv \mathcal{P}(x = L)$ and $\mathcal{V}_L \equiv \mathcal{V}(x = L)$. The normalization of the primitive solutions is $\mathcal{P}_0^+ = 1$ and $\mathcal{P}_L^- = 1$. Δ_L is the determinant of matrix Eq. 14 evaluated at $x = L$. Thus

$$-\Delta_L = \mathcal{P}_L^+ \ [Y_{rad}^+(L) + Y_{rad}^-(L)].$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{-1}{\Delta_L} \begin{bmatrix} Y_{rad}^-\mathcal{P}^-(x - L) & \mathcal{P}^-(x - L) \\ Y_{rad}^+\mathcal{P}^+(x) & -\mathcal{P}^+(x) \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ \mathcal{V}_L \end{bmatrix}, \tag{15}$$

which when simplified is

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{-1}{\Delta_L} \begin{bmatrix} Y_{rad}^- & -1 \\ Y_{rad}^+ \mathcal{P}_L^+ & \mathcal{P}_L^+ \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}. \tag{16}$$

This detailed set of computations results in the following:

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \frac{-1}{\Delta_L} \begin{bmatrix} 1 & \mathcal{P}_{-L}^- \\ Y_{rad}^+ & -Y_{rad}^- \mathcal{P}_{-L}^- \end{bmatrix}_0 \begin{bmatrix} Y_{rad}^- & -1 \\ Y_{rad}^+ \mathcal{P}_L^+ & \mathcal{P}_L^+ \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}. \tag{17}$$

The subscript to the right of each matrix indicates it is evaluated at $x = 0$ or $x = L$. Here $\mathcal{P}_{-L}^-$ is $\mathcal{P}^-(x - L)$ at $x = 0$. The sign of $\mathcal{V}_L$ must be negative to satisfy the definition of every ABCD matrix, that the output velocity (i.e., $-\mathcal{V}_L$) is out of the port.

The relationship of β/α has special significance because it specifies the ratio of the reflected wave amplitude $\beta(\omega)$ in terms of the incident wave amplitude $\alpha(\omega)$. This ratio is know as the *reflectance*

$$\gamma_L(t) \leftrightarrow \Gamma_L(s) \equiv \frac{\beta}{\alpha}. \tag{18}$$

It has a critical role in the theory of horns, as we shall see as it is determined by the relative rate of change of the impedance (i.e., area) with range (i.e., $d\ln(\mathscr{Z})/dx$).

Impedance Matrix: For a finite section of horn, the 2×2 impedance matrix (a generalization of Ohm's Law) may expressed in terms of the ABCD matrix elements (Van Valkenburg, 1964) as

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{P}_L \end{bmatrix} = \frac{1}{\mathcal{C}(s)} \begin{bmatrix} \mathcal{A}(s) & \Delta_T \\ 1 & \mathcal{D}(s) \end{bmatrix} \begin{bmatrix} \mathcal{V}_0 \\ \mathcal{V}_L \end{bmatrix}. \tag{19}$$

Note that $\Delta_T = 1$ since the horn must be *reciprocal* (Morse and Feshbach, 1953; Hunt, 1982; Pierce, 1981).

While the Transmission (ABCD) matrix is convenient when modeling, the impedance matrix (and its inverse, the *admittance matrix*) are useful when one makes experimental measurements. For example

$$Y_0|_{\mathcal{V}_L=0} \equiv \frac{\mathcal{C}(s)}{\mathcal{A}(s)} \qquad \text{and} \qquad Y_L|_{\mathcal{V}_0=0} \equiv \frac{\mathcal{C}(s)}{\mathcal{D}(s)}$$

are the *unloaded input admittances* of the horn looking into the two ports (Eq. 12). These admittances are typically easily measured experimentally, given access to the endpoints.

In section 2.3 we work out these relationships for the trivial case of the 1D horn (Goldsmith and Minton, 1924; Olson, 1947).

2.3 Example of the Uniform Horn

All these formulae are well known for the case of the uniform plane-wave horn of constant area $A(x) = A_0$ where Eq. 3 reduces to the classical 1D plane-wave wave equation

$$\mathcal{P}_{xx} + \frac{\omega^2}{c^2}\mathcal{P} = 0.$$

Wave solutions: The solutions is the well-known d'Alembert (plane-wave) solution

$$\varrho(x,t) = \varrho^+(t - x/c) + \varrho^-(t + x/c) \leftrightarrow \alpha(\omega)e^{-\kappa x} + \beta(\omega)e^{-\kappa L}e^{\kappa x},$$

messed up normalization of $\varrho-$

where $a(t) \leftrightarrow \alpha(\omega)$ and $b(t) \leftrightarrow \beta(\omega)$ are Fourier transform pairs representing the forward and backward traveling wave amplitudes. As for the general case given above, it is convenient to define $P_0^+ = 1$ ($\alpha = 1$) and $P_L^- = 1$ ($\beta = e^{-\kappa L}$), thus the normalized primary solutions are

$$\varrho^+(x,t) = \delta(t - x/c) \leftrightarrow e^{-\kappa x}$$

and

$$\varrho^-(x,t) = \delta(t + (x - L)/c) \leftrightarrow e^{\kappa(x-L)}.$$

Radiation Admittance: Since $\partial_x \ln \mathcal{P}^\pm = \mp\kappa = s/c$, from Eq. 12 we find that $Y_{rad}^\pm \equiv \mathcal{Y} = A_0/\rho_0 c$. The signs are chosen to assure that the real admittance $\mathcal{Y} > 0$ for both types of waves, where the velocity $\mathcal{V}^\pm$ is always *into* the port.

1^d ABCD matrix: Repeating the entire argument leading to the ABCD matrix (i.e., Eq. 17)

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix} = \begin{bmatrix} e^{-\kappa x} & e^{\kappa(x-L)} \\ \mathcal{Y}e^{-\kappa x} & -\mathcal{Y}e^{\kappa(x-L)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{20}$$

Solving Eq. 20 for α and β and evaluating at $x = L$ (and fixing all the signs to be consistent with Eq. 13) we find the amplitudes in terms of the pressure and velocity at any point along the line

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{-1}{2\mathcal{Y}e^{-\kappa L}} \begin{bmatrix} -\mathcal{Y}e^{-\kappa(x-L)} & -e^{\kappa(x-L)} \\ -\mathcal{Y}e^{-\kappa x} & e^{-\kappa x} \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{\kappa L} & \mathcal{Z}e^{\kappa L} \\ 1 & -\mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}, \tag{21}$$

where $\mathcal{Z} \equiv 1/\mathcal{Y}$. We have changed the sign of $-\mathcal{V}_L$ so as to be positive looking into the port at $x = L$. Since in this case $\Delta_L = -2\mathcal{Y}e^{-\kappa L}$ is independent of x, evaluating it at $x = L$ has no effect. Substitution of Eq. 21 into Eq. 20 and evaluating the result at $x = 0$ gives the final ABCD matrix

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & e^{-\kappa L} \\ \mathcal{Y} & -\mathcal{Y}e^{-\kappa L} \end{bmatrix} \begin{bmatrix} e^{\kappa L} & \mathcal{Z}e^{\kappa L} \\ 1 & -\mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix} \tag{22}$$

Multiplying these out gives the final transmission matrix as

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \begin{bmatrix} \cosh(\kappa L) & \mathcal{Z}\sinh(\kappa L) \\ \mathcal{Y}\sinh(\kappa L) & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix},$$

with $\kappa = s/c$, $\mathcal{Y} = 1/\mathcal{Z} = A_0/\rho_0 c$ (Pipes, 1958). The two velocities are defined into their respective ports.

Impedance matrix: The impedance matrix (Eq. 19) is therefore

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{P}_L \end{bmatrix} = \frac{\mathcal{Z}}{\sinh(\kappa L)} \begin{bmatrix} \cosh(\kappa L) & 1 \\ 1 & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} \mathcal{V}_0 \\ -\mathcal{V}_L \end{bmatrix}.$$

The input admittance: Given the input admittance of the horn, it is already possible to determine if it is uniform, without further analysis. Namely if the horn is uniform and infinite in length, the input impedance at $x = 0$ is

$$Y_{in}(0, s) \equiv \frac{\mathcal{V}(0, \omega)}{\mathcal{P}(0, \omega)} = \mathcal{Y},$$

since $\alpha = 1$ and $\beta = 0$. That is for an infinite uniform horn, there are no reflections.

When the horn is terminated with a fixed impedance Z_L at $x = L$, one may substitute pressure and velocity measurements into Eq. 21 to find α and β, and given these, one may calculate the *reflectance* at $x = L$ (see Eq. 18)

$$\Gamma_L(s) \equiv \left.\frac{\mathcal{P}^-}{\mathcal{P}^+}\right|_{x=L} = \frac{e^{-\kappa L}\beta}{\alpha} = \frac{\mathcal{P}(L, \omega) - \mathcal{Z}\mathcal{V}(L, \omega)}{\mathcal{P}(L, \omega) + \mathcal{Z}\mathcal{V}(L, \omega)} = \frac{Z_L - \mathcal{Z}}{Z_L + \mathcal{Z}}$$

given accurate measurements of the throat pressure $\mathcal{P}(0, \omega)$, velocity $\mathcal{V}(0, \omega)$, and the characteristic impedance of the input $\mathcal{Z} = \rho_0 c/A(0)$.

9

2.4   2D parabolic Horn

For 2D cylindrical waves the area function is $A(x) = A_0 x$, (horn radius $r \propto \sqrt{x}$) thus the Webster horn equation reduces to the cylindrical wave equation (Appendix A)

$$\mathcal{P}_{xx}(x,\omega) + \frac{1}{x}\mathcal{P}_x(x,\omega) = \kappa^2(s)\mathcal{P}(x,\omega)$$

having Bessel function primitive solutions

$$\mathcal{P}^+(x,\omega) = J_0(\omega x/c) - iY_0(\omega x/c) = H_0^+(-j\kappa x)$$

and

$$\mathcal{P}^-(x,\omega) = J_0(\omega x/c) + iY_0(\omega x/c) = H_0^-(-j\kappa x),$$

where J_0 and Y_0 are the standard Bessel (and Neumann) functions, and $H_0^{\mp}(x)$ are the Hankel function of the first (-) and second (+) kind (respectively), all of order zero (indicated by the subscript) (Salmon, 1946a; Olson, 1947; Morse and Feshbach, 1953).

Radiation Admittance: Given a half-infinite section of parabolic horn, from Eq. 12[9]

$$Y_{rad}^{\pm}(x,s) = \mp\frac{\mathcal{Y}}{\kappa}\frac{\partial}{\partial x}\ln H_0^{\pm}(-j\kappa x) = \mp j\mathcal{Y}\frac{H_1^{\pm}}{H_0^{\pm}}. \tag{23}$$

2^d **ABCD Transmission matrix:**   Based on Eq. 17

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \frac{-1}{\Delta_L}\begin{bmatrix} 1 & \mathcal{P}_{-L}^- \\ Y_{rad}^+ & -Y_{rad}^-\mathcal{P}_{-L}^- \end{bmatrix}_0 \begin{bmatrix} Y_{rad}^- & -1 \\ Y_{rad}^+\mathcal{P}_L^+ & \mathcal{P}_L^+ \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}. \tag{24}$$

Verify the following

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = -\frac{1}{\Delta_L}\begin{bmatrix} 1 & H_0^-(x-L) \\ Y_{rad}^+ & -Y_{rad}^-H_0^-(x-L) \end{bmatrix}_{x=0} \begin{bmatrix} Y_{rad}^- & -1 \\ Y_{rad}^+H_0^-(x) & H_0^+(x) \end{bmatrix}_{x=L} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}, \tag{25}$$

where the subscript on each matrix indicates the value of x at which it is evaluated.

Impedance matrix:

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{P}_L \end{bmatrix} = \frac{1}{\mathcal{C}(s)}\begin{bmatrix} \mathcal{A}(s) & 1 \\ 1 & \mathcal{D}(s) \end{bmatrix}\begin{bmatrix} \mathcal{V}_0 \\ \mathcal{V}_L \end{bmatrix}. \tag{26}$$

2.5   3D Conical Horn

The Webster equation for this area is identical to the spherical wave equation (Appendix A)

$$\mathcal{P}_{rr}(r,\omega) + \frac{2}{r}\mathcal{P}_r(r,\omega) = \kappa^2\mathcal{P}(r,\omega). \tag{27}$$

The area $A(x)$ as a function of the range variable $x = r - r_0$ is

$$A(x) = A_0 r^2 = A_0(x + r_0)^2,$$

---

[9]Note that

$$\frac{\partial \ln H_0^{\pm}(kx)}{\partial x} = -k\frac{H_1^{\pm}(kx)}{H_0^{\pm}(kx)}$$
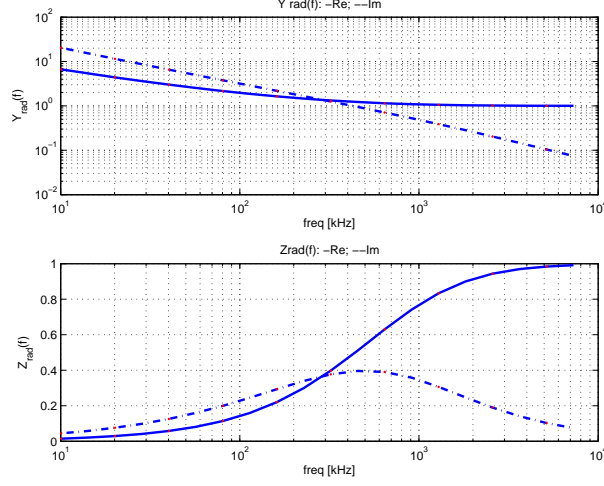
Figure 3: *Real and imaginary parts of the radiation admittance $Y_{rad}/\mathcal{Y}_0$ (upper) and impedance (lower), as a function of frequency, for a infinite section of horn, as computed from Eq. 23. The real and imaginary parts of the impedance are very similar to the results of Olson (1947) shown in Fig. 2. The real and imaginary parts of the admittance are much simpler to characterize: the imaginary part is a mass (i.e., is $\propto 1/s$) while the real part $\rightarrow 1$ above the critical frequency (where the real and imaginary parts are equal) at ≈ 0.3 [kHz].*

where $x = r - r_0$ is measured from $r = r_0$ and A_0 is the area at $x = 0$ (Olson, 1947; Morse, 1948; Benade, 1988).[10]

The primitive d'Alembert solutions of Eq. 27 are the zero order *spherical Hankel functions* (i.e., $h_0^{\pm}(\kappa r)$)

$$\mathcal{P}^+(r,s) \equiv \mathcal{P}_0^+ \frac{e^{-\kappa r}}{r}, \qquad \mathcal{P}^-(r,s) \equiv \mathcal{P}_L^- \frac{e^{\kappa r}}{r},$$

Based on our primitive solution normalization convention (i.e., $\mathcal{P}^+(0,s) = 1$ and $\mathcal{P}^-(L,s) = 1$),

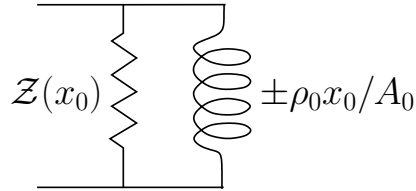$$\mathcal{P}_0^+ \frac{e^{-\kappa r_0}}{r_0} = 1, \qquad \mathcal{P}_L^- \frac{e^{\kappa r_L}}{r_L} = 1.$$

Figure 4: *The equivalent circuit for the radiation impedance $Z_{rad}^+(s,x)$ for a conical is a parallel combination of the radiation resistance $\mathcal{Z}(x) = \rho_0 c/A(x)$ and a mass given by $M(x) = \pm\rho_0 x/A_0$. The inbound primitive wave has a negative mass? The normalized radiation mass is $M/\mathcal{Z} = x/c$ is simply the delay from the origin.*

Thus in terms of the range variable x, the primitive solutions are:

$$\mathcal{P}^+(x,s) \equiv \frac{r_0}{x+r_0} e^{-\kappa x}, \qquad \mathcal{P}^-(x,s) \equiv \frac{r_L}{(x-L)+r_L} e^{\kappa(x-L)}. \qquad (28)$$

---

[10]Olson's definition of $A(x)$ was somewhat unclear.

Radiation Admittance: Given a half-infinite section of conical horn Eq. 12 gives[11]

$$Y_{rad}^\pm(x,s) = \mp \frac{\mathcal{Y}(x)}{\kappa} \left. \frac{\partial \ln P^\pm}{\partial x} \right|_{r_0} = \mathcal{Y}(x)\left(1 \pm \frac{c}{xs}\right) = \leftrightarrow \mathcal{Y}(x)\left(\delta(t) \pm \frac{c}{x}\mathbb{1}(t)\right) \tag{29}$$

where $\delta(t)$ and $\mathbb{1}(t)$ are the Dirac and Heaviside functions, respectively. Thus as shown in Fig. 4 we see that there are two additive admittance terms, a real admittance $\mathcal{Y}$ equal to the characteristic impedance, and a term having a mass equal to $\pm \mathcal{Z}x/c = \pm\rho_0 x/A_0$. For the reverse traveling primitive wave, these conventions result in *negative mass*, an interesting and perhaps surprising result. Since the admittances are summed, the impedances are in parallel.

Figure 2 from Olson (1947, Fig. 5.3, p. 101) provides the real and imaginary impedance for each of the horns discussed here. The difference between Olson and the present work, is in our intent here to explain the physics behind the reactive term, in the time domain.

3^d ABCD Transmission matrix: For clarity the relations are re-derive "from scratch"

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix} = \begin{bmatrix} \frac{r_0}{x+r_0}e^{-\kappa x} & \frac{r_L}{(x-L)+r_L}e^{\kappa(x-L)} \\ Y_{rad}^+(x)\frac{r_0}{x+r_0}e^{-\kappa x} & -Y_{rad}^-(x)\frac{r_L}{(x-L)+r_L}e^{\kappa(x-L)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{30}$$

Taking the inverse and evaluating at the boundary $x = L$ gives the relationship between the weights α, β and the boundary (load) impedance

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_L = \frac{-1}{\Delta_L} \begin{bmatrix} Y_{rad}^-(L) & -1 \\ Y_{rad}^+(L)\frac{r_0}{L+r_0}e^{-\kappa L} & \frac{r_0}{L+r_0}e^{-\kappa L} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix} \tag{31}$$

where

$$-\Delta_L = \left(Y_{rad}^-(L) + Y_{rad}^+(L)\right)\frac{r_0}{L+r_0}e^{-\kappa L} = 2\mathcal{Y}_L\frac{r_0}{L+r_0}e^{-\kappa L}.$$

Substituting Eq. 31 back into Eq. 32

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix} = \frac{-1}{\Delta_L}\begin{bmatrix} \frac{r_0}{x+r_0}e^{-\kappa x} & \frac{r_L}{(x-L)+r_L}e^{\kappa(x-L)} \\ Y_{rad}^+(x)\frac{r_0}{x+r_0}e^{-\kappa x} & -Y_{rad}^-(x)\frac{r_L}{(x-L)+r_L}e^{\kappa(x-L)} \end{bmatrix} \begin{bmatrix} Y_{rad}^-(L) & -1 \\ Y_{rad}^+(L)\frac{r_0}{L+r_0}e^{-\kappa L} & \frac{r_0}{L+r_0}e^{-\kappa L} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix} \tag{32}$$

or at $x = 0$

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = \frac{-1}{\Delta_L}\begin{bmatrix} \frac{r_0}{r_0} & \frac{r_L}{(-L)+r_L}e^{\kappa(-L)} \\ Y_{rad}^+(0) & -Y_{rad}^-(0)\frac{r_L}{(-L)+r_L}e^{\kappa(x-L)} \end{bmatrix} \begin{bmatrix} Y_{rad}^-(L) & -1 \\ Y_{rad}^+(L)\frac{r_0}{L+r_0}e^{-\kappa L} & \frac{r_0}{L+r_0}e^{-\kappa L} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix} \tag{33}$$

Impedance matrix:

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{P}_L \end{bmatrix} = \frac{1}{\mathcal{C}(s)}\begin{bmatrix} \mathcal{A}(s) & 1 \\ 1 & \mathcal{D}(s) \end{bmatrix}\begin{bmatrix} \mathcal{V}_0 \\ \mathcal{V}_L \end{bmatrix}. \tag{34}$$

2.6 Exponential Horn

Starting from the basic transmission line equations with an area function given by $A(x) = A_0 e^{2mx}$, Eq.3 is

$$\frac{\partial^2 \mathcal{P}(x,\omega)}{\partial x^2} + 2m\frac{\partial \mathcal{P}(x,\omega)}{\partial x} = \kappa^2 \mathcal{P}(x,\omega), \tag{35}$$

thus $F(x) \equiv \frac{1}{A(x)}\frac{\partial A(x)}{\partial x} = 2m$ is a constant. Since Eq. 35 is an ordinary constant coefficient differential equation, it has a closed form solution (Olson, 1947; Salmon, 1946a,b; Morse, 1948; Beranek,

---

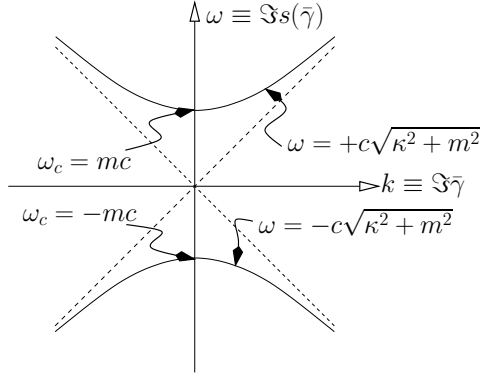[11]I'm not sure about Y_{rad}^-

12

Figure 5: *This shows the dispersion relation between the frequency and the wave number k for the exponential horn, with wave propagation disallowed below the critical frequency $\omega_c = mc$.*

1954; Leach, 1996; Beranek and Mellow, 2012). By the substitution $\mathcal{P}(x,\omega) = P(\kappa(s))e^{-\kappa_\pm(s)x}$, one may solve for the *characteristic roots* $\kappa_\pm(s) = m \pm \sqrt{m^2 + \kappa^2}$. Thus

$$\mathcal{P}^\pm(x) = e^{-mx}e^{\mp\sqrt{m^2+\kappa^2}x} = e^{-mx}e^{\mp j\sqrt{\omega^2 - \omega_c^2}x/c}, \tag{36}$$

which represent the horn's right (+) and left (-) traveling pressure waves. The dispersion diagram corresponding to the horn is shown in Fig. 5. Note that the exponential horn has a dispersion diagram identical to the electron's wave properties in a semiconductor. *Simply by changing the flare function from conical to exponential, the horn impedance switches from a mass in parallel with a resistor, to a horn in cutoff, with conduction and stop band (an evanescent wave band region), as in a semiconductor.*

The outbound time domain solution of the exponential (exp) horn is $e^{-mx}E(t)$, with[12]

$$E(t) = \delta(t - x/c) + \frac{x}{c}\frac{J_1(\sqrt{t^2 - x^2/c^2})}{\sqrt{t^2 - x^2/c^2}}U(t - x/c) \leftrightarrow e^{-\sqrt{m^2+\kappa^2}x}$$

2.6.1 Exp-Horn ABCD Transmission matrix:

From Eq. 18

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{V}_0 \end{bmatrix} = -\frac{1}{\Delta_L}\begin{bmatrix} 1 & \mathcal{P}^- \\ Y_{rad}^+ & -Y_{rad}^-\mathcal{P}^- \end{bmatrix}_0 \begin{bmatrix} Y_{rad}^- & -1 \\ Y_{rad}^+\mathcal{P}^+ & \mathcal{P}^+ \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}. \tag{37}$$

Radiation Admittance: Given a half-infinite section of the exponential horn (Salmon, 1946b; Leach, 1996), from Eq. 12

$$Y_{rad}^\pm(x_0, s) = \frac{\mathcal{Y}}{s}\left(\omega_c \mp \sqrt{\omega_c^2 + s^2}\right). \tag{38}$$

where $\omega_c \equiv mc$ is the horn *cutoff frequency*. At very high frequencies this approaches a real value of $\mathcal{Y}$. Below cutoff it is a purely reactive admittance.

Discussion: Since Eq. 35 contains no viscous or other loss terms, the solution is always loss-less. The propagation function roots $\kappa_\pm(s)$ are imaginary when $\omega > \omega_c$, but change to a purely real value below the *cutoff frequency*, i.e., $\omega < \omega_c$.

At all frequencies the wave propagation is dispersive, meaning that the speed is frequency dependent. Above the cutoff frequency there is normal wave propagation. However the impedance of the wave changes, making the horn an "ideal" acoustic transformer. As shown in Fig. fig:DispersionDiagram, at very high frequencies ($\omega \geq \omega_c = mc$) the wave propagates without dispersion, but still with a decay, due to the exponential change in area.

---

[12]See PowerWavesE paper for a more detailed expression.

When the diameter of the horn becomes greater than half a wavelength, higher order modes start to come into play, and the solution is no longer unique, as the quasistatic approximation totally breaks down. Any imperfection in the area will introduce cross modes, which can then propagate.

In regions where the diameter is less than a half wavelength, higher order modes will not propagate. This is best described by the quasistatic approximation. But the exact solution may be obtained by solving Eq. eq:WHEN locally for the pressure.

Below cutoff requires further analysis, as the wave solution is still causal, but *everywhere in phase*. To see this we may take the inverse Laplace transform of $P^+(x, s)$ to obtain the explicit wave behavior in the time domain. It is very useful to look at this case in both the time and frequency domains, in order to fully appreciate what is happening.

Appendices

A   Derivation of the Webster Horn Equation

In this Appendix we transform the acoustic equations Eq. 1 and 2 into their equivalent integral form. This derivation is similar (but not identical) to that of Hanna and Slepian (1924); Pierce (1981, p. 360).

Conservation of momentum: The first step is an integration the normal component of Eq. 1 over the pressure iso-surface $\mathcal{S}$, defined by $\nabla p = 0$,

$$-\int_{\mathcal{S}} \nabla p(\mathbf{x}, t) \cdot d\mathbf{A} = \rho_0 \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{A}. \tag{A.1}$$

The *average pressure* $\varrho(x, t)$ is defined by dividing by the total area

$$\varrho(x, t) \equiv \frac{1}{A(x)} \int_{\mathcal{S}} p(x, t)\, \hat{\boldsymbol{n}} \cdot d\boldsymbol{A}. \tag{A.2}$$

From the definition of the gradient operator

$$\nabla p = \frac{\partial p}{\partial x} \hat{\boldsymbol{n}}, \tag{A.3}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the iso-pressure surface $\mathcal{S}$. Thus the left side of Eq. 1 reduces to $\partial \varrho(x, t)/\partial x$.

The integral on the right side defines the *volume velocity*

$$\nu(x, t) \equiv \int_{S} \mathbf{u}(x, t) \cdot d\boldsymbol{A}. \tag{A.4}$$

Thus the integral form of Eq. 1 becomes

$$\frac{\partial}{\partial x} \varrho(x, t) = -\frac{\rho_0}{A(x)} \frac{\partial}{\partial t} \nu(x, t). \tag{A.5}$$

Conservation of mass: Integrating Eq. 2 over the volume $\mathcal{V}$ gives

$$-\int_{\mathcal{V}} \nabla \cdot \boldsymbol{u}\, dV = \frac{1}{\eta_0 P_0} \frac{\partial}{\partial t} \int_{\mathcal{V}} p(\mathbf{x}, t) dV. \tag{A.6}$$

Volume $\mathcal{V}$ is defined by two iso-pressure surfaces between x and $x + \delta x$ (Fig. 6, shaded blue). On the right hand side we use our definition for the *average pressure* (i.e., Eq. A.2), integrated over the thickness δx.

Applying Gauss' law to the left hand side[13], and using the definition of ϱ (on the right), in the limit $\delta x \to 0$, gives

$$\frac{\partial \nu}{\partial x} = -\frac{A(x)}{\eta_0 P_0} \frac{\partial \varrho}{\partial t}. \tag{A.7}$$

Finally, writing Eq. A.5 and Eq. A.7 in matrix form, in the frequency domain ($\partial_t \leftrightarrow s$), results in Eq. 3.

---

[13]As shown in Fig. 6, we convert the divergence into the difference between two volume velocities, namely $\nu(x + \delta x) - \nu(x)$, and $\partial \nu/\partial x$ as the limit of this difference over δx, as $\delta x \to 0$
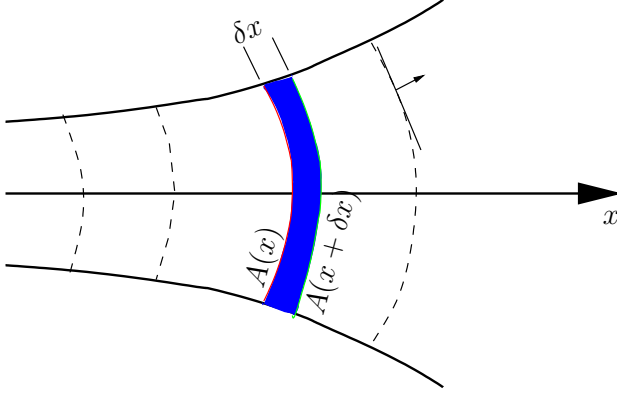
Figure 6: Derivation of horn equation using Gauss's Law: The divergence of the velocity $\nabla \cdot \mathbf{u}$, within δx, shown as the filled shaded region, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities $\delta \nu = \nu(x + \delta x) - \nu(x) = [\mathbf{u}(x + \delta x) \cdot \mathbf{A}(x + \delta x) - \mathbf{u}(x) \cdot \mathbf{A}(x)]$.

B   The inverse problem

For each area function $A(x)$ there is a unique driving point input impedance $z_{rad}(t) \leftrightarrow Z_{rad}(s)$. We define $z_r(t)$ as the *impedance remainder* after normalizing by the surge impedance z_0 and then subtracting the initial impulse (delta function)

$$z_r(t) \equiv z_{rad}(t)/z_0 - \delta(t). \tag{B.8}$$

In the frequency domain $Z_r(s) = Z_{rad}(s)/z_0 - 1$. The remainder is related to the reactive portion of $Z(s)$.

For each remainder there is a corresponding unique $A(x)$. This relation is

$$f(a, \xi_0) + \frac{1}{2c} \int_{-a}^{a} z_r(|\xi_0/c - \xi/c|) f(\xi_0, \xi) d\xi = 1. \tag{B.9}$$

Once this integral equation has been solved for $f(x, x)$, the area function is $A(x) = f^2(x, x)$ (Sondhi and Gopinath, 1971; Sondhi and Resnick, 1983).

C   WKB method

This method is use in approximating the solution to the horn equation under the assumption that the reflections are minimal (no strong change in the area function with the range variable. Briefly, if

$$\frac{d^2}{dx^2} \Phi(x, s) = F(x) \, \Phi(x, s)$$

where the effective *wave number* is

$$F(x) = \frac{2m}{\hbar^2} \left[V(x) - E \right].$$

The the WKB approximation gives the solution

$$\Phi(x, s) \approx C_0 \frac{e^{j \int \sqrt{F(x)} dx}}{\sqrt[4]{F(x)}}.$$

Picking the right branch of the square root is essential in the application of the WKB method, as is unwrapping the phase of each branch of the wave number.

This method is integrating the phase delay of the wave number as a function of the range variable. Thus its complex-analytic properties are essential to obtaining the correct answer (the wave number is not a single valued function of x and s).

D Rydberg series

Fundamental the quantum mechanics is the Rydberg series, which describes the quantized energy levels of atoms

$$\nu_{n,m} = cRZ_n^2 \left(\frac{1}{n^2} - \frac{1}{m^2} \right) \tag{D.10}$$

where $\nu_{n,m}$ are the possible eigen-frequencies, c is the speed of light, $R \approx 1.097 \times 10^7$ is the Rydberg constant, Z_n is the atomic number, along with positive integers n and m ($m > n$) which represent two *principal quantum numbers* that label all possible allowed atomic eigen-states. Integer n indicates the lowest (initial) atomic eigen-state while m is the higher state[14] When $n = 1$ the series is the Lyman series corresponding to Hydrogen ($Z_1 = 1$).

Given observed frequencies $\nu_{n,m}$ it is possible to find the area function that traps the photons into the Rydberg eigen-states.

One way to think of eigen-modes is to make an analogy to a piano string, or an organ pipe. In these much simpler systems, there is an almost constant delay, say τ due to a characteristic length, say $L = \tau c$, such that the eigen-modes are given by integer multiples of a half wavelength $\nu_n = nc/2L$. This assumes the endpoint boundary conditions are pinned displacement (i.e, zero velocity). For an organ pipe, closed at one end and open at the other the corresponding formula is multiples of a quarter wavelength $\nu_n = nc/4L$. In each case $\nu = n/\tau$ where $\tau = 2L/c$ is the round trip delay, thus $\nu = nc/2L$. We suggest looking at the Rydberg series in the say way, but with the very different eigen frequencies (Eq. D.10). Sommerfeld (1949, p. 201) makes a very interesting comment regarding Eq. D.10:

> This equation reduces to a simple mathematical the enigma of the spectral lines, with their finite cummulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

E Laplacian operator in N dimensions

To show that the Webster equation is in agreement with the wave equation in 2 and three dimensions, we need to express the Laplacian, and then determine $F(n)$ of Table 1. In general it may be shown that, in N dimensions (Sommerfeld, 1949, p. 227)[15]

$$\nabla_r^2 P \equiv \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial P}{\partial r} \right)$$

This may be expanded as

$$\frac{\partial^2 P}{\partial r^2} + F(r) \frac{\partial P}{\partial r},$$

where $F(r) = (N-1)/r$ is the same as in Table 1. In terms of the Webster Horn equation, $F(x) \equiv -\partial \ln \mathscr{Z}(x,s)/\partial x$. Thus we see there is a fundamental relation between $\mathscr{Z}$ and the dimensionality of the horn.

For each N, r^{N-1} is proportional to the area function $A(x)$. This generalizes to the Webster Horn equation Eq. 9. For the case of $N = 3$ (the conical Horn) there is a special relationship

$$\nabla_r^2 P \equiv \frac{1}{r^2} \partial_r r^2 \partial_r P = \frac{1}{r} \partial_{rr} r P.$$

---

[14]http://en.wikipedia.org/w/index.php?title=Rydberg_formula&action=edit&section=2.
[15]http://en.wikipedia.org/wiki/Laplacian#Three_dimensions

References

Benade, A. (**may 1988**), "Equivalent circuits for conical waveguides," J. Acoust. Soc. Am. **83**(5), 1764–9.

Beranek, L. L. (**1954**), *Acoustics* (McGraw–Hill Book Company, Inc., 451 pages, New York).

Beranek, L. L. and Mellow, T. J. (**2012**), *Acoustics: Sound Fields and Transducers* (Academic Press - Elsevier Inc., Waltham, MA).

Brillouin, L. (**1953**), *Wave propagation in periodic structures* (Dover, London), updated 1946 edition with corrections and added appendix.

Campbell, G. (**1922**), "Physical Theory of the Electric Wave Filter," Bell System Tech. Jol. **1**(2), 1–32.

Goldsmith, A. N. and Minton, J. (**1924**), "The Performance and Theory of Loud Speaker Horns," Radio Engineers, Proceedings of the Institute of **12**(4), 423–478.

Hanna, C. R. and Slepian, J. (**1924**), "The Function and Design of Horns for Loud Speakers," American Institute of Electrical Engineers, Transactions of the **XLIII**, 393–411.

Hunt, F. V. (**1982**), *Electroacoustics, The analysis of transduction, and its historical background* (The Acoustical Society of America, 260 pages, Woodbury, NY 11797).

Leach, W. (**1996**), "A 2-port analogous circuit and SPICE model for Salmon's family of acoustic horns," J. Acoust. Soc. Am. **99**(3), 1459–1464.

Mawardi, O. K. (**jul 1949**), "Generalized Solutions of Webester's Horn Theory," J. Acoust. Soc. Am. **21**(4), 323–30.

Morse, P. and Feshbach, H. (**1953**), *Methods of theoretical physics. Vol. I. II* (McGraw-Hill, Inc., New York).

Morse, P. M. (**1948**), *Vibration and sound* (McGraw Hill, 468 pages, NYC, NY).

Olson, H. F. (**1947**), *Elements of Acoustical Engineering (2^n Edition)* (D. Van Nostrand Company, 539 pages).

Orfanidis, S. J. (**2009**), *Electromagnetic Waves and Antennas* (web, `http:www.ece.rutgers.edu/~orfanidi/ewa`).

Pierce, A. D. (**1981**), *Acoustics: An Introduction to its Physical Principles and Applications* (McGraw-Hill, 678 pages, New York).

Pipes, L. A. (**1958**), *Applied Mathematics for Engineers and Physicists* (McGraw Hill, NYC, NY).

Salmon, V. (**1946**a), "Generalized plane wave horn theory," J. Acoust. Soc. Am. **17**(3), 199–211.

Salmon, V. (**1946**b), "A new family of horns," J. Acoust. Soc. Am. **17**(3), 212–218.

Shaw, E. (**1970**), "Acoustic Horns with Spatially varying density or elasticity," J. Acoust. Soc. Am. **50**(3), 830–840.

Sommerfeld, A. (**1949**), *Partial differential equations in Physics, Lectures on Theoretical Physics, Vol. I* (Academic Press INC., New York).

Sommerfeld, A. (**1952**), *Electrodynamics, Lectures on Theoretical Physics, Vol. III* (Academic Press INC., New York).

Sondhi, M. and Gopinath, B. (**1971**), "Determination of vocal-tract shape from impulse responses at the lips," J. Acoust. Soc. Am. **49**(6:2), 1867–73.

Sondhi, M. and Resnick, J. (**1983**), "The inverse problem for the vocal tract: Numerical methods, acoustical experiments, and speech synthesis," J. Acoust. Soc. Am. **73**(3), 985–1002.

Van Valkenburg, M. E. (**1964**), *Modern Network Synthesis* (John Weily & Sons, Inc., New York, NY).

Webster, A. G. (**1919**), "Acoustical impedance, and the theory of horns and of the phonograph," Proc. Nat. Acad. Sci. **5**, 275–282.

Youla, D. (**Sep 1964**), "Analysis and Synthesis of arbitrarily terminated lossless nonuniform lines," IEEE Transactions on Circuit Theory **11**(3), 363–372.