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MTHE 212 — Linear Algebra

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Based on lectures by A. Shaltut – Queen's University

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These notes are my own interpretations of the course material and they are not endorsed by the lecturers.

Feel free to reach out if you point out any errors.

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2.1 Topic 1: Direct Sums & Dimension Formulas

Focus: Equivalence of $U \cap W = \{0\}$ and direct sums; applying $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

Question 1.1 (Concrete Application):

Consider the vector space $\mathcal{P}_3(\mathbb{R})$ of polynomials with degree at most 3. Let U and W be the following subspaces:

$$U = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \mid p(0) = 0 \text{ and } p'(0) = 0\}$$

$$W = \{ax + b \mid a, b \in \mathbb{R}\}$$

(a) Find the dimension of U and the dimension of W.

(b) Determine the dimension of $U \cap W$.

(c) Use the dimension formula to find $\dim(U + W)$. Is the sum $U + W$ a direct sum? Explain why or why not using the appropriate equivalence.

Question 1.2 (Abstract/Theoretical):

Let V be a finite-dimensional vector space with $\dim V = 10$. Let U and W be subspaces of V such that $\dim U = 6$ and $\dim W = 5$.

(a) Prove that $U \cap W \neq \{0\}$.

(b) What are the possible values for $\dim(U \cap W)$?

(c) Prove that there does not exist a subspace Z such that $U \oplus Z = V$ and $W \oplus Z = V$ simultaneously (Hint: Use dimensions).

2.2 Topic 2: Definitions & Basic Proofs

Focus: Basis, Linear Independence, Span, Injectivity, Surjectivity, Isomorphisms.

Question 2.1 (Linear Independence & Injectivity):

Let V and W be vector spaces and let $T \in \mathcal{L}(V, W)$ be an injective linear map.

Prove that if $v_1, \ldots, v_n$ is a linearly independent list of vectors in V, then the list $Tv_1, \ldots, Tv_n$ is linearly independent in W.

(Note: This is a classic "follow the definitions" proof that Axler emphasizes).

Question 2.2 (Surjectivity & Spanning):

Let V and W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$.

Prove that T is surjective if and only if T maps a spanning list of V to a spanning list of W.

(Specifically: Show that if span$(v_1, \ldots, v_n) = V$, then span$(Tv_1, \ldots, Tv_n) =$ range(T)).

2.3 Topic 3: Null Space, Range, & Rank-Nullity

Focus: Direct sums involving null/range, proving subspace equality ($A \subseteq B$ and $B \subseteq A$), Rank-Nullity Theorem.

Question 3.1 (Subspace Equality via Double Inclusion):

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $T(x, y, z) = (x - y, y - z, z - x)$.

(a) Find a basis for null(T).

(b) Let $S = \text{span}((1, 1, 1))$. Prove that null$(T) = S$ by showing that $S \subseteq \text{null}(T)$ and null$(T) \subseteq S$.

Question 3.2 (Rank-Nullity & Direct Sums):

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, V)$ be a linear operator.

(a) State the Rank-Nullity Theorem (Fundamental Theorem of Linear Maps).

(b) Suppose that range$(T) \cap \text{null}(T) = \{0\}$. Prove that $V = \text{range}(T) \oplus \text{null}(T)$.

(Hint: Use the Rank-Nullity theorem to show that the dimension of the sum equals the dimension of V, then use the property of direct sums).

3 Quiz 2 Practice 2

3.1 Topic 1: Direct Sums & Dimension Formulas

Prof's Hint: "Understand the equivalence between the intersection being zero and the sum being direct."

Question 1.1 (Concrete Intersection - Practice for the Exam):

Consider the vector space $\mathbb{R}^4$. Let U and W be the following subspaces:

$$U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y = 0 \text{ and } z = 2w\}$$

$$W = \operatorname{span}((1, -1, 0, 0), (0, 0, 1, 1))$$

(a) Find a basis for U and state $\dim U$.

(b) Find a basis for W and state $\dim W$.

(c) Find a basis for $U \cap W$. (Hint: Set a general vector in W equal to a general vector in U, or plug the generic vector of W into the equations for U).

(d) Is $U + W$ a direct sum? Explain using the intersection result.

Question 1.2 (The Equivalence Proof):

Let U and W be subspaces of a finite-dimensional vector space V.

Prove that $U + W$ is a direct sum if and only if $\dim(U + W) = \dim U + \dim W$.

(Note: This directly tests the "equivalence" your professor mentioned. You will likely need to use the full dimension formula $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ *to prove this).*

3.2 Topic 2: Definitions & Basic Proofs

Prof's Hint: "Comfortable proving basic facts that follow directly from these definitions (injective, surjective, basis)."

Question 2.1 (Isomorphisms & Basis):

Let V and W be finite-dimensional vector spaces and let $T \in \mathcal{L}(V, W)$ be an isomorphism (injective and surjective).

Prove that if $(v_1, \ldots, v_n)$ is a basis of V, then $(Tv_1, \ldots, Tv_n)$ is a basis of W.

(Hint: You must prove two things: that the list is linearly independent (uses injectivity) and that it spans W (uses surjectivity).)

Question 2.2 (Injectivity via Null Space):

Let $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if $\operatorname{null}(T) = \{0\}$.

*(Note: This is a fundamental "definition check" in Axler. You need to prove the implication in both directions:

1. If T is injective, then only 0 maps to 0.
2. If the only thing that maps to 0 is 0, then $T(u) = T(v) \Rightarrow u = v$.)*

3.3 Topic 3: Null Space, Range & Subspace Equality

Prof's Hint: "Prove that two subspaces are equal ($A = B$) by showing $A \subseteq B$ and $B \subseteq A$."

Question 3.1 (Subspace Equality Challenge):

Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be the linear map defined by $T(p(x)) = x \cdot p(x)$.

Let S be the subspace defined by $S = \{q(x) \in \mathcal{P}_3(\mathbb{R}) \mid q(0) = 0\}$.

(a) Find the range of T.

(b) Prove that $\text{range}(T) = S$ using the double inclusion method.

(i.e., Show every polynomial in the range has a zero constant term, and show every polynomial with a zero constant term is in the range).

Question 3.2 (Rank-Nullity & Intersection):

Let V be a finite-dimensional vector space and let $T \in \mathcal{L}(V, V)$.

Suppose that $\text{range}(T) = \text{null}(T)$.

(a) Prove that $\dim V$ must be an even number.

(b) Give an example of such a map T on $\mathbb{R}^2$.

(Hint: Use the Rank-Nullity Theorem $\dim V = \dim \textit{null}(T) + \dim \textit{range}(T)$ *and substitute the equality given)*