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MTHE 237 - Lecture Notes
Differential Equations for Engineering Science

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1 Introduction and Mathematical modelling

A differential equation is a relationship between a function and its derivatives. It tells you how a quantity changes, not directly what the quantity is. Therefore, a differential equation relates *rates of change*

An ODE describes *how* a function changes, e.g. $\frac{dy}{dx} = 3x^2$. This tells you the slope of y at any point x. It doesn't tell you which curve $y(x)$ you're on, just the family of all curves that follow that slope rule, by a constant C, e.g. $y = x^3 + C$

An IVP is an ODE plus an initial condition that picks out a specific curve from that family, e.g. $\frac{dy}{dx} = 3x^2, y(0) = 5$, and solving gives $y = x^3 + 5$, which is a *specific member of the family of solutions*

In order to provide a mathematical model describing physical phenomena, we can follow the following steps:

1. Fix our dependent and independent variables and set a frame of references for their measures
2. Choose convenient units of measurements
3. Find the underlying principle governing the motion of the object
4. Rewrite the above relation in terms of the variable we have chosen at step 1
5. Find a solution by integrating both sides
6. Add side conditions to eliminate constants and to find a **unique solution**

Example

$$\begin{aligned}
F &= ma \\
a &= \frac{dv}{dt} = \frac{d^2h}{dt^2} \\
\frac{d^2h}{dt^2} &= -g \\
v(t) = \frac{dh}{dt} = -\int g dt &= -gt + c_1 \\
h(t) = \int v(t)dt &= -\frac{1}{2}gt^2 + c_1 t + c_2 \\
h(0) &= 1, v(0) = 0 \\
h(t) &= -\frac{1}{2}gt^2 + 1
\end{aligned}$$

1.1 Population, Malthusian, and Competitor Models

Given some internal and external conditions: $\frac{dp}{dt} =$ growth rate - death rate

Death rate = 0

$\frac{dp}{dt} = k_1 p$, where $p(0) = p_0$, and $k_1 > 0$ is the proportionality factor for the growth rate

$$\begin{aligned}\frac{1}{p}dp &= kdt \\ \ln p &= kt + C \\ p(t) &= Ce^{kt}\end{aligned}$$

A Malthusian model is a general model for population with rates k_1 and k_2 proportional to p

We can consider more factors to the death rate, called **competitors**, with two-party interactions modelled by $\frac{p(p-1)}{2}$

$\frac{dp}{dt} = k_1 p - k_3 \frac{p(p-1)}{2}$

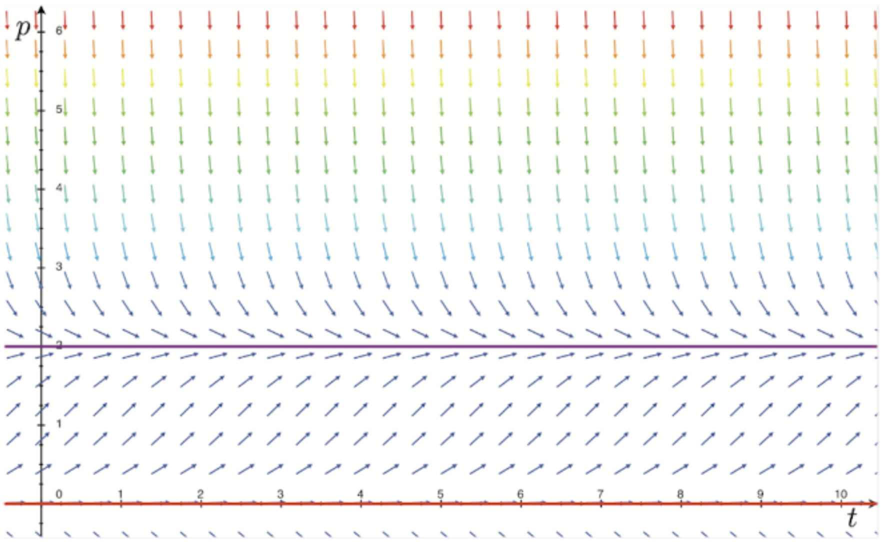
We can rearrange the terms to find an equation in the form of a **logistic model**

$\frac{dp}{dt} = -\frac{k_3}{2}\left(p^2 - \left(q + \frac{2k_1}{k_3}\right)p\right)$

which takes the form

$\frac{dp}{dt} = -Ap(p - p_1)$, where $A = \frac{k_3}{2}$ and $p_1 = 1 + \frac{2k_1}{k_3}$

For $\frac{dp}{dt} = p(2-p)$, $\frac{dp}{dt} > 0 \Leftrightarrow 0 < p(t) < 2$



$p = 0, 2$ are equilibria

2 Terminology and Classification

General solution: The family of all possible solutions to the ODE, usually containing arbitrary constants

Explicit solution: A solution where the dependent variable (say y) is written explicitly as a function of the independent variable (say x)

Implicit solution: A solution where the relationship between x and y is given as an equation but not solved explicitly for y

Examples

$y' = y + x$, y is dependent and x is independent

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x - 2y$, where x, y and independent and the function $u(x, y)$ is dependent

2.1 Ordinary Differential Equations

A differential equation with derivatives with respect to one variable is an ODE

The general form of an ODE is given by $F(x, y, y', \ldots, y^{(n)}) = 0$

The **order** of a differential equation is the order of the highest derivative appearing in the equation

An ODE is **linear** if: 1. $y, y', y'', \ldots$ appear only to the first power 2. No product like yy' exist 3. Coefficients $a_i(x)$ can be constants or functions of x, but not of y

A linear ODE of order n has the form: $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0(x)y = f(x)$

2.2 Confirming a solution of an ODE

A function $\phi(x)$ is a solution of an n-th order ODE on some interval $I = (a, b)$ if 1. ϕ is n-times differentiable on I 2. It satisfies the ODE for every $x \in I$

Consider $y'' + \frac{2}{x^2}y = 0$ and the function $\phi(x) = x^2 - \frac{1}{x}$

We can confirm that its derivatives are continuous for all $x \neq 0$ and satisfy the ODE

Thus, the function ϕ is a solution of the given ODE in $(0, \infty)$

2.2.1 Superposition principle for linear ODEs

Say we have an n-th order linear **homogeneous** ODE $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0(x)y = 0$ and suppose $\phi_1(x), \phi_2(x), \ldots, \phi_k(x)$ are solutions.

The theorem says any linear combination $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_k\phi_k(x)$ is also a solution in I for any choice of arbitrary constants, and the set of solutions is **closed under linear combinations**

These $\phi_k(x)$ solutions, unlike the next example, form a **vector space**, closed under linear combinations

Now, consider a **non-homogeneous** ODE when $f(x) \neq 0$, the linear combination property fails, and only holds in the homogeneous case

3 Existence and Uniqueness Theorem

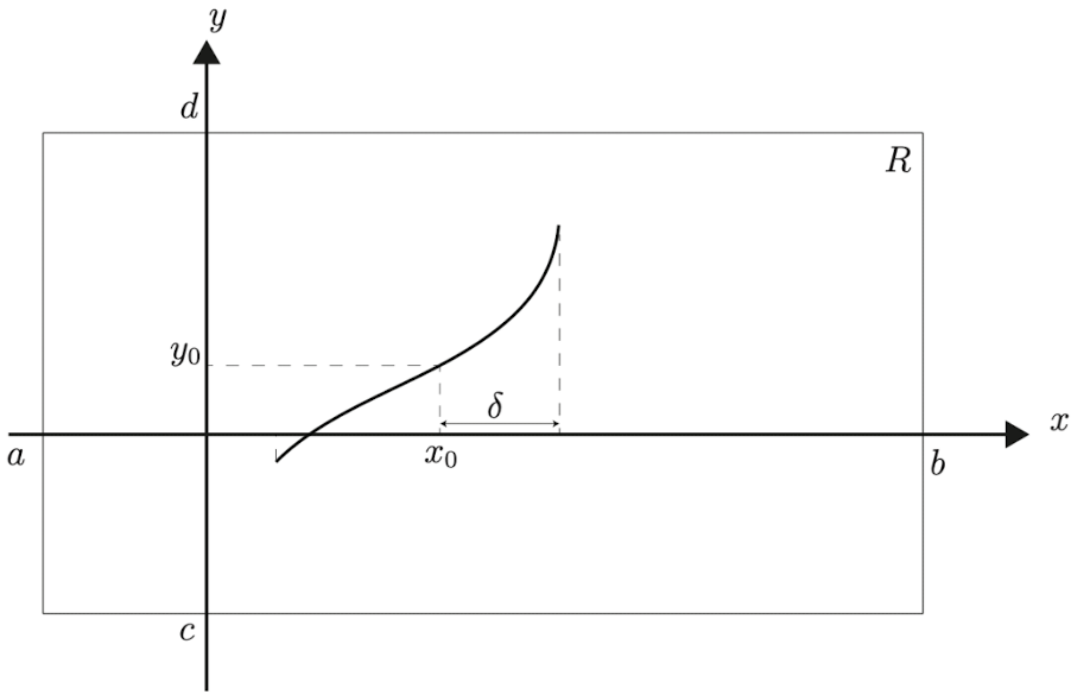
"Does a solution exist that passes through (x_0, y_0)? And if so, is it the only one?"

Consider the IVP $y' = f(x, y)$ and $y(x_0) = y_0$

If f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $R = \{(x, y) \in \mathbf{R^2} : a < x < b, c < y < d\}$, that contains the point (x_0, y_0), then the IVP admits a unique solution $\phi : (x_0 - \delta, x_0 + \delta) \to \mathbf{R}$ for some $\delta > 0$. Therefore, this guarantees a local solution $\phi(x)$ in some interval $x \in (x_0 \pm \delta)$, and there's one and only one smooth path that passes through (x_0, y_0)

If $f(x, y)$ is continuous, then at least one solution exists. Continuity of f is enough for *existence*, but not for *uniqueness*

If $f(x, y)$ is continuous *and* $\frac{\partial f}{\partial y}$ is continuous, then this is enough to show that *uniqueness*



Intuition:

In the original equation, this of it like a slope field that tells you the slope of the solution curve at each point (x, y), $f(x, y)$

If f is continuous, there are no gaps in the slope field. That means you can follow the slope arrows starting at origin to trace out a path

If $\frac{\partial f}{\partial y}$ is also continuous, then nearby paths cannot cross each other. Because the slope field is well defined, you cannot have two different marbles starting at the same point and taking different paths

The rectangle R is the safe zone where f behaves nicely

δ is how far you can move horizontally from x_0 while staying inside the safe zone

Inside $(x_0 - \delta, x_0 + \delta)$, the slope field is well-behaved enough to guarantee a single, smooth solution curve

Example

Use the existence and uniqueness theorem to show that the following IVP admits a unique solution

Given: $y' = 3x + \sin y, \quad y(0) = 1$, we know that $y' = f(x,y) = 3x + \sin y$, and each component of $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$. At $(0,1)$, both $f(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ are continuous

Therefore, the IVP admits a unique local solution around $x = 0$

If one of uniqueness or existence fails, then $\phi(x)$ is not a solution to the ODE

3.1 Failed Cases

Uniqueness fails

In $f(x,y) = \frac{y}{x}$, f is undefined at $(0,0)$, but solutions $y(x) = 0$ and $y(x) = x$ hold and satisfy the IVP with $y(0) = 0$, thus $y(x) = cx$ is a solution for any constant c

This means that there are infinitely many solutions, so **uniqueness fails**

Existence fails

Now suppose $y(0) = 1$, when $x = 0$ this gives $y(0) = 0$, so **existence fails**

3.2 Alternate form

Given a 1st order linear ODE $a_1(x)y' + a_0(x)y = f(x)$, we can always put it in the form $y'(x) = P(x)y + Q(x)$ by denoting $P(x) = -\frac{a_0(x)}{a_1(x)}, Q(x) = \frac{f(x)}{a_1(x)}$

Thus, if $(x_0, y_0) \in \{(x,y) : P(x) \text{ and } Q(x) \text{ are continuous}\}$

Moreover, I can be chosen as the largest interval containing x_0 and such that P, Q are both continuous on I, giving a global, not local, solution

3.3 Picard Iterations

Picard iterations are a method to actually construct the solution that the Existence and Uniqueness Theorem guarantees.

We start at step 0, and iterate integration to construct a sequence of approximations (one per step) that should converge to the unique solution of the IVP

We can rewrite an IVP as an integral equation:

$y' = f(x,y), y(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$, where we don't know $y(t)$

We can approximate $y(x)$ step by step:

step 0: $\phi_0(x) = y_0$

step 1: $\phi_1(x) = y_0 + \int_{x_0}^{x} f(s, \phi_0(s))\ ds$

step 2: $\phi_2(x) = y_0 + \int_{x_0}^{x} f(s, \phi_1(s))\ ds$

$\vdots$

step n: $\phi_n(x) = y_0 + \int_{x_0}^{x} f(s, \phi_{n-1}(s))\ ds$

$\vdots$

If, at some step of the Picard iteration, the new approximation $\phi_{k+1}(x)$ turns out to be exactly the same as the previous one $\phi_k(x)$, then we've already reached the solution

Formally, if there exists $k \in \mathbf{N}$ s.t. $\phi_{k+1}(x) = \phi_k(x)$, then the solution would be given by $\phi_k(x) = y_0 + \int_{x_0}^{x} f(s, \phi_k(s))ds$

Example:

$y' = 2t(1+y), y(0) = 0$

Step 0: $\phi_0(t) = 0$

Step 1: $y'(0) = 2t, \phi_1(t) = \int_0^t 2sds = t^2$

Step 2: $\phi_2(t) = \int_0^t 2s(1+t^2)ds = t^2 + \frac{t^4}{2}$

Step 3: $\phi_3(t) = \int_0^t 2s\left(1 + t^2 + \frac{t^4}{2}\right)ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2*3}$

Step n: $\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} = \Sigma_{k=1}^{n} \frac{t^{2k}}{k!}$

Hence, the sequence of functions converges if and only if the infinites series given above converges.

We can apply the ratio test to find this:

$\frac{|t^{2(k+1)}}{(k+1)!}\frac{k!}{k+1}| \to 0$ as $k \to \infty$

So, the series converges for every t, and $\phi(t) = \Sigma_{k=1}^{\infty} \frac{t^{2k}}{k!}$

4 Initial Value Problems

Given the n-th order ODE $F(x, y, y', \dots, y^{(n)}) = 0$ in some interval I, the following two side conditions can be appended to the equation

1. **Initial Conditions:** dependent variable and all its derivatives up to order $n-1$ are specified at the same point $x_0 \in I$
2. **Boundary Conditions:** only applicable to PDEs

The problem of finding a solution to an ODE in an interval I containing x_0 and *such that the initial conditions are satisfied* is called **initial boundary value problem (IVP)**

4.1 Solving ODEs from implicit relations

Example: given the implicit relation $x_2 + y^2 = 1, x \in (-1, 1)$, we want to see if $y(x)$ is a solution to the ODE $y' = -\frac{x}{y}$

1. **Differentiate w.r.t. dependent variable**

$2x + 2yy' = 0$

2. **Solve for y'**

$y' = -\frac{x}{y}$, this is the ODE

3. **Solve implicit relation explicitly**

Rearrange for y: $y(x) = \pm\sqrt{1 - x^2}$

4. **Verify candidates**

For $y(x) = +\sqrt{1 - x^2}$, $y' = \frac{-x}{\sqrt{1-x^2}}$, which matches the original ODE For $y(x) = -\sqrt{1 - x^2}$, $y' = \frac{x}{\sqrt{1-x^2}}$, which also satisfies the ODE

5. **Apply initial conditions**

$y_1(x) = -\sqrt{1 - x^2} \Rightarrow y_1(0) = -1$ is not valid $y_2(x) = \sqrt{1 - x^2} \Rightarrow y_2(0) = 1$ is valid

$\therefore$ the unique solution to the IVP is $y(x) = \sqrt{1 - x^2}$

5 Separable Equations

Definition: Consider the 1st order ODE $y' = f(x, y)$, if $f(x) = g(x)p(x)$, where g and p depends only on x and y, respectively, then the ODE is said to be **separable**

Consider the separable ODE on some interval $I = (a, b)$ $\frac{dy}{dx} = g(x)p(y)$ $(*)$

We have two cases:

1. If $p(y) \equiv 0$, then $y(x) = c$ constant function on I because the RHS is always 0
2. If $p(y) \neq 0$, then $(*)$ can be rewritten as $\frac{1}{p(y)}\frac{dy}{dx} = g(x)$

Let $H = H(y)$ and $G = G(x)$ be the antiderivatives of $\frac{1}{p(y)}$ and $g(x)$, respectively, from the chain rule we obtain $\frac{dH(y(x))}{dx} = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{p(y)}\frac{dy}{dx} = \frac{dG}{dx}$

Then, since their derivatives are equal, the functions must differ by a constant: $H(y(x)) = G(x) + c \Leftrightarrow \int \frac{1}{p(y)}dy = \int g(x)dx + c$

This is an implicit formula for the solution to the ODE.

For an explicit solution, we apply the **inverse function** of H to both sides: $H(y) = G(x) + C$ $H^{-1}(H(y)) = H^{-1}(G(x) + C)$ $y = H^{-1}(G(x) + C)$

Remark: finding the inverse function depends on H

5.1 Examples

Example 1

Consider $y' = \frac{x}{y\exp(x+2y)}$. It can be rewritten as $ye^{2y}y' = xe^{-x}$. It can then be separated and integrated $\int ye^{2y}dy = \int xe^{-x}dy$.

$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -(x + 1)e^{-x} \Leftrightarrow e^{2y}(2y - 1) = -4e^{-x}(x + 1) + c$

In this case, it is not possible to write y as an explicit function of x. We stop at an **implicit solution**

Example 2

$y' = (1 + y^2)\tan x$. It can be rewritten as $\frac{1}{1+y^2}\frac{dy}{dx} = \tan x$. It can then be separated and integrated

$\int \frac{1}{1+y^2}dy = \int \tan x dx \Leftrightarrow \arctan[y(x)] = -\ln|\cos x| + c$

Integration of $\tan x$**

$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du$ where $u = \cos x$ and $du = -\sin x dx$

$-\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$

6 Variable Coefficients (Integrating Factor)

Consider a 1st order linear ODE with variable coefficients

$a_1(x)y' + a_0(x)y = b(x) \Leftrightarrow y' + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)} \Leftrightarrow \frac{dy}{dx} + P(x)y = Q(x)$

If $a_0(x) = a_1'(x)$, then the equation becomes $\frac{d}{dx}(a_1(x))y = b(x)$

Definition: An integrating factor is a special function that we multiply through a first-order linear ODE to make it easier to solve. Its purpose is to turn the left-hand side of the equation into the derivative of a product, so we can integrate directly

6.1 General Solution

Let $\frac{dy}{dx} + P(x)y = Q(x)$

1. We multiply through by an *integrating factor*

$\mu(x) = e^{\int P(x)dx}$

2. Multiply the whole equation by $\mu(x)$

$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \Leftrightarrow \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$

3. Integrate both sides

$\mu(x)y = \int \mu(x)Q(x)dx + C$

4. Solve for $y(x)$

$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$

General Solution:

$y(x) = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$

6.2 Examples

Example 1: Consider $y' + y = 1$, where $P(x) = Q(x) = 1$

Integrating factor: $e^{\int P(x)dx} = e^x$

Use general solution: $y(x) = e^{-x}\left(\int e^x dx + c\right)$

Solve: $y(x) = 1 + ce^{-x}$

Example 2: Consider $y' + \frac{3}{x}y = 3x - 2, y(1) = 1$, where $P(x) = \frac{3}{x}, Q(x) = 3x - 2$

Integrating factor: $\mu(x) = e^{3\ln x} = x^3$

Rewrite ODE: $\frac{d}{dx}[\mu y] = \mu Q \Leftrightarrow \frac{d}{dx}[x^3y(x)] = 3x^4 - 2x^3$

Integrate on both sides: $x^3y(x) = \frac{3}{5}x^5 - \frac{2}{4}x^4$

Solve: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{c}{x^3}$

Use initial conditions to find C: $y(1) = 1 = \frac{3}{5} - \frac{1}{2} + C \Leftrightarrow C = \frac{9}{10}$

Final Solution: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10x^3}$

7 Exact Equations

We start with a first-order ODE: $M(x,y)+N(x,y)\frac{dy}{dx}=0$ which is equivalent to $M(x,y)dx+N(x,y)dy=0$

Now, suppose there exists a *potential function* $F(x,y)$ such that $\frac{\partial F}{\partial x}=M(x,y)$, and $\frac{\partial F}{\partial y}=N(x,y)$. Then, if F exists, then the ODE is called exact.

Along any solution curve $y(x)$, if we compute the derivative of $F(x,y(x))$, we get: $\frac{d}{dx}F(x,y(x))=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}$

But notice that the right-hand side is exactly the same as $M(x,y)+N(x,y)\frac{dy}{dx}$

Since our ODE says that equals zero, it follows that $\frac{d}{dx}F(x,y(x))=0$

That means $F(x,y)$ is constant along solution curves, i.e. $F(x,y)=c$, which is the implicit solution of the ODE

7.1 Level Curve Concept

Let $M(x,y)dx+N(x,y)dy=0$ (1), and its exact when $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

This means that there exists some function $F(x,y)$ such that: $dF=F_x dx+F_y dy=Mdx+Ndy$ However, $F(x,y)$ is unique up to an additive constant, therefore it defines a *family of level curves.*

Equation (1) basically says that $dF=0$ along any solution curve, meaning F stays constant as you move along the curve. Because F is constant, we're staying at the same "height" or at one of the **level curves** $F(x,y)=C$

7.2 Examples

Example 1:

Consider $y'=-\frac{2xy^2+1}{2x^2y}$, where we can rearrange to get $M(x,y)=2xy^2-1$ and $N(x,y)=2x^2y$

Integrate an equation wrt x:

$\int(2xy^2-1)dx=x^2y^2-x+g(y)$

The last term $g(y)$ appears instead of constant c because c can depend on y since we are integrating wrt x

Now, differentiate the latter function wrt y:

$\frac{\partial F}{\partial y}=2x^2y+g'(y)=2x^2y$, where the last term appears from $N(x,y)=2x^2y$

We see that $g'(y)=0$, so $g(y)=c$. Therefore, the given ODE is exact and the general solution is implicitly defined by $x^2y^2-x=c$ for any arbitrary constant c

Example 2:

Consider $3xy+y^2+(x^2+xy)y'=0$

Integrate $M(x,y)$ wrt x: $F(x,y)=\frac{3}{2}x^2y+xy^2+g(y)$

Differentiate F wrt y: $\frac{\partial F}{\partial y} = \frac{3}{2}x^2 + 2xy + g'(y) = x^2 + xy$

Rearrange for $g'(y) : g'(y) = -\frac{1}{2}x^2 - xy$, which cannot hold because the RHS depends on both variables x and y, while g is only a function of y. Hence, there is no F satisfying for the given ODE, which is then *not exact*

7.3 Clairaut's Theorem

We use Clairaut's theorem to test for exactness

An ODE is *exact* in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

Get

$$\begin{cases} \frac{\partial F}{\partial y} = N(x, y) \\ \frac{\partial F}{\partial x} = M(x, y) \end{cases}$$

Since M and N are differentiable wrt x and y, then

$$\begin{cases} \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \end{cases}$$

[Clairaut's Theorem] states that if $F(x, y)$ has continuous second order partial derivatives, then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

8 Second Order Linear Differential Equations

Let $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$, where the coefficients are functions of x only and are continuous on some open interval $I = (a, b)$

If the ODE $= 0$, we call it homogeneous.

The superposition principle also applies to second order ODEs.

8.1 Existence and Uniqueness for second-order

Let $\frac{d^2y}{dx^2} = f(x, y, y')$. We now need two initial conditions to pin down a unique solution: $y(x_0) = y_0, y'(x_0) = y'_0$, where the former gives the starting position and the latter gives the starting slope

Example: $y'' = y - x$, $y(0) = 1, y'(0) = 0$

Here: $f(x, y, y') = y - x$, $\frac{\partial f}{\partial y} = 1$, $\frac{\partial f}{\partial y'} = 0$. Both are continuous everywhere on $\mathbb{R}^3$, so the conditions of the theorem are satisfied

8.2 Wronskian

A second order ODE needs two initial conditions, and they must not be multiples of one another. Else, solutions y_1, y_2 are linearly dependent on the interval I

We can say that a 2nd order ODE is homogeneous if:

Let y_1, y_2 be solutions on the interval of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. If at some point $x_0 \in (a, b)$ these two solutions satisfy

$\det \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} \neq 0$

and call it **Wronskian of y_1 and y_2 at x_0**

8.3 Representation Theorem

Let $y_1(x), y_2(x)$ be two solutions on the interval (a, b) of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

If at some $x_0 \in (a, b)$, these two solutions satisfy: $W[y_1(x_0), y_2(x_0)] \neq 0$, then $y_1(x)$ and $y_2(x)$ are linearly independent solutions on (a, b)

9 2nd Order Homogeneous DEs With Constant Coefficients

Consider $ay'' + by' + cy = 0$, where $a, b, c \in \mathbf{R}$

If $\phi(x)$ is a solution to the ODE, then $\phi''(x) = -\frac{b}{a}\phi'(x) - \frac{c}{a}\phi(x)$ for all $x \in \mathbf{R}$

This means that the second derivative is a linear combination of the lower order derivatives

If we have a trial function (educated guess for the solution) $y(x) = e^{rx}$, then we get $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$

Into the ODE, we get $\cancel{e^{rx}}(ar^2 + br + c) = 0$, so e^{rx} is a solution $\Leftrightarrow$ r is a root of the 2nd order polynomial. Thus, r must be a solution to the *characteristic equation* $ar^2 + br + c = 0$

9.1 Case 1: $b^2 - 4ac > 0$

If so, then there exists roots $r_1, r_2 \in \mathbf{R}$ with $r_1 \neq r_2$

r_1 cannot equal r_2 because $W[e^{r_1x}, e^{r_2x}] = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$

Hence, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ are linearly independent solutions, and by the [Representation Theorem], $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$, where constants are arbitrary and can be found by imposing initial conditions

Example:

Consider $2y'' + 7y' - 4y = 0$

Characteristic equation: $2r_2 + 7r - 4 = 0$

Confirm determinant > 0: $r_{1,2} = \frac{-7\pm\sqrt{49+32}}{4} = -4, \frac{1}{2}$

General solution: $y(x) = c_1e^{-4x} + c_2e^{x/2}$

Impose initial conditions if given

9.2 Case 2: $b_2 - 4ac = 0$

If so, then there exist two roots $r_1, r_2 \in \mathbf{R}$ with $r_1 = r_2$

In this case $c = \frac{b^2}{4a}$ and roots $r_1 = r_2 = -\frac{b}{2a}$. Because we essentially have one root, then the solution $y_1(x) = \exp\left(-\frac{b}{2a}x\right)$ exists

9.3 Case 3: $b_2 - 4ac < 0$

In this case, we have two roots $r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a} =: \alpha \pm i\beta$

Hence, we find two *complex-valued* linearly independent solutions $y_1, y_2 : \mathbf{R} \to \mathbf{C}$ $y_1(x) = \exp[(\alpha + i\beta)x], y_2(x) = \exp[(\alpha - i\beta)x]$

We would like to find two *real-valued linearly independent solutions*

Also, if the ODE has real coefficients and the complex-valued function $\phi = u(x) + iv(x)$ is a solution, then real-valued function $u(x) = \mathrm{Re}(\phi(x))$ and complex-valued function $v(x) = \mathrm{Im}(\phi(x))$ are solutions of the same equation

Back to Case 3, from Euler's formula we get: $y_{1,2}(x) = e^{\alpha x}[\cos(\beta x) \pm i \sin(\beta x)]$

Also, be the second remark above, we can conclude that also the functions below are also solutions: $Y_1, Y_2 : \mathbf{R} \to \mathbf{R}$ $Y_1(x) = \mathrm{Re}(y_1(x)) = \mathrm{Re}(y_2(x)) = e^{\alpha x}\cos(\beta x)$ and $Y_2(x) = \mathrm{Im}(y_1(x)) = -\mathrm{Im}(y_2(x)) = e^{\alpha x}\sin(\beta x)$

Moreover, we can check if these are linearly independent via their Wronskian:

$$\begin{aligned} W[Y_1(x), Y_2(x)] &= \det \begin{bmatrix} \mathrm{e}^{\alpha x}\cos(\beta x) & \mathrm{e}^{\alpha x}\sin(\beta x) \\ \alpha \mathrm{e}^{\alpha x}\cos(\beta x) - \beta \mathrm{e}^{\alpha x}\sin(\beta x) & \alpha \mathrm{e}^{\alpha x}\sin(\beta x) + \beta \mathrm{e}^{\alpha x}\cos(\beta x) \end{bmatrix} \\ &= \mathrm{e}^{\alpha x} \det \begin{bmatrix} \cos(\beta x) & \sin(\beta x) \\ \alpha\cos(\beta x) - \beta\sin(\beta x) & \alpha\sin(\beta x) + \beta\cos(\beta x) \end{bmatrix} = \beta \neq 0. \end{aligned}$$

Thus, the solutions above are linearly independent (real-valued) solutions of $ay_2'' + by_2' + cy_2 = 0$

Moreover, every other solution takes the form $y(x) = c_1 Y_1(x) + c_2 Y_2(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$

Example:

Consider $y'' - y' + y = 0, y(0) = 1, y'(0) = -2$

Characteristic equation $r^2 - r + 1 = 0$

Solution: $r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

- Linearly independent solutions of the ODE:

$$y_1(x) = \mathrm{e}^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), \qquad y_2(x) = \mathrm{e}^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

- General solution:

$$y(x) = \mathrm{e}^{x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

Imposing initial conditions, we get $y(x) = e^{x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{5\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

10 Method of Reduction of Order

By the Representation Theorem, $y(x) = c_1y_1(x) + c_2y_2(x)$. Now that we have $y_1(x)$, how do we find $y_2(x)$ such that $\frac{y_2(x)}{y_1(x)} \neq$ constant

We use the *Method of Reduction of Order*, which works on second order linear homogeneous ODEs with variable coefficients $y'' + p(x)y' + q(x)y = 0$

If $\phi_1(x)$ is a solution to the ODE above, and we want to find $\phi_2(x)$ such that they are linear independent on an interval, we want $\phi_2(x) = g(x)\phi_1(x)$ where $g(x)$ is an unknown function to be found to such that the ODE is satisfied by $\phi_2(x)$

Take $\phi_2(x)$ and take the second derivative: $\phi_2'' = g'; \phi_1(x) + 2g'\phi_1' + g\phi_1''(x)$

Replacing in the ODE, we get $\Leftrightarrow \phi_1 v' + (2\phi_1' + p(x)\phi_1)v = 0$, where $v(x) := g'(x)$

We can then solve for $v(x)$ because the corresponding ODE is linear and separable

$g(x) = \int v(x)dx = \int \frac{\exp\left(-\int p(x)dx\right)}{\phi_1^2(x)}dx$

Example:

Find a second solution to $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ for $x > 0$ given a solution $y_1(x) = x^{-1/2}\sin x$

Let $y_2(x) = v(x)y_1(x)$, then $y_2'' = v''y_1 + 2v'y_1' + vy_1''$

Goal: find $v(x)$. Replacing the latter in the ODE, we find that

$x^2y_1v'' + v'(2x^2y_1' + xy_1) = 0$ because the y term is the same as the ODE, and it drops out

We can divide further by x^2y_1: $v'' + \left(2\frac{y_1'}{y_1} + \frac{1}{x}\right) = 0$

Now we need $\frac{y_1'}{y_1}$. Since $y_1'(x) = \frac{\sqrt{x}\cos x - \frac{\sin x}{2\sqrt{x}}}{x}$, then the former is $\frac{\cos x}{\sin x} - \frac{1}{2x}$, which can be simplified to $2\frac{\cos x}{\sin x}$

Subbing back, we get that $v(x)$ has to satisfy the following equation $v'' + 2\frac{\cos x}{\sin x}v' = 0$

Reduce order again: Setting $w(x) := v'(x)$, then $w' = -2\frac{\cos x}{\sin x}w \Rightarrow \frac{w'}{w} = -2\cot x$

Integrate: $\ln|w| = -2\ln(\sin x) + C$, $w = v'(x) = \frac{C_1}{\sin^2 x}$

Integrating to get $v(x)$: $v(x) = -C_1\cot x + C_2$

Build the solution: $y_2(x) = v(x)y_1(x) = (-C_1\cot x + C_2)\frac{\sin x}{\sqrt{x}} \Rightarrow y_2(x) = \frac{\cos x}{\sqrt{x}}$

11 Mechanical and Electrical Vibrations

A **damped mass-spring oscillator** is a physical system constituted by a mass m attached to an elastic spring with stiffness constant k and subject to friction $F_f(t) = -\gamma\frac{dy}{dt}$, where $\gamma \geq 0$ is the damping coefficient

The equation of motion is given by $my'' + \gamma y' + ky = F(t) = 0$ with no external force applied to the body

1. **If $\gamma \neq 0$ and $\gamma^2 - 4mk \geq 0$, the system is said to be overdamped**

$r_1, r_2 > 0, y \to 0, x \to \infty$

$r_1 = 0, r_2 < 0, |y| \to \infty, x \to \infty$

2. **If $\gamma \neq 0$ and $\gamma^2 - 4mk < 0$, the system is said to be underdamped**

$\alpha > 0, |y| \to \infty$, y oscillates to $\infty, x \to \infty$

$\alpha < 0, |y| \to 0$, y oscillates to $0, x \to \infty$

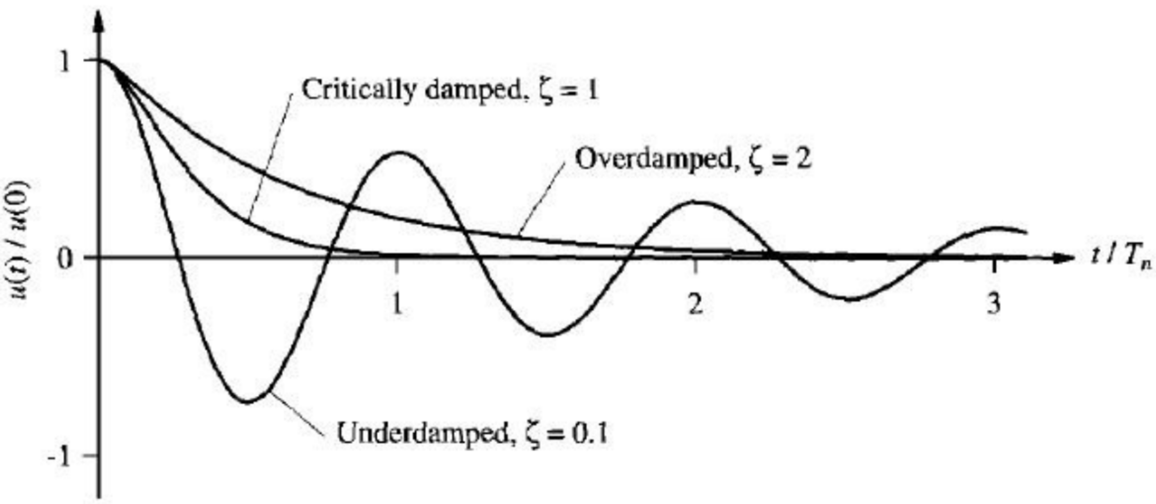
$\alpha = 0$, y is periodic, no limit

3. **If $\gamma = 0$, the system is said to be undamped**

$r_1 > 0, |y| \to \infty$ exponentially. $x \to \infty$

$r_1 < 0, y \to 0, x \to \infty$

$r_1 = 0, |y| \to \infty$ linearly. $x \to \infty$



12 Method of Undetermined Coefficients

Consider the IVP $y'' + p(x)y' + q(x)y = f(x), y(x_0) = Y_0, y'(x_0) = Y_1$

This IVP can be proved with the Existence and Uniqueness Theorem, and an interval $I = (a, b)$ can be found

Furthermore, the Superposition principle applies, where solutions ϕ_1 and ϕ_2 forms another solution $\phi(x) = c_1\phi_1 + c_2\phi_2$, which solves for $y'' + p(x)y' + q(x)y = c_1f_1(x) + c_2f_2(x)$

12.1 Particular Solutions

Let $y_c(x) = c_1y_1(x) + c_2y_2(x)$ be a solution to a homogeneous equation, and $y_p(x)$ be a solution of the nonhomogeneous equation

Then, $y(x) = y_c(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

The solution $y_c(x)$ is called the **complementary solution**, whereas the solution $y_p(x)$ is called the **particular solution**

12.2 Method of Undetermined Coefficients

Considering a nonhomogeneous ODE, find $y_p(x)$

Example: $y'' - 3y' - 4y = 3e^{2x}$

Take the homogeneous version and find roots $r_1 = 4, r_2 = -1$

Find the complementary solution by looking for a function of the form $y_p(x) = Ae^{2x}$

Plug in for y, get derivatives, and get $A = -\frac{1}{2}$

General solution: $y(x) = y_c(x) + y_p(x) = c_1e^{4x} + c_2e^{-x} - \frac{1}{2}e^{2x}$

Example: $y'' - 3y' - 4y = 2\sin x$

We complementary homogeneous solution is $y_c(x) = c_1e^{4x} + c_2e^{-x}$

A solution to the nonhomogeneous solution can be guessed by $y_p(x) = A\cos x + B\sin x$

We always use both sine and cosine terms for a right side like $\sin x$ **or** $\cos x$ because their derivatives produce each other

We then plug in the derivatives and equate coefficients

$$\begin{cases} \text{cosine: } -5A - 3B = 0 \\ \text{sine: } 3A - 5B - 2 = 0 \end{cases}$$

We can solve for A, B and get a general solution: $y(x) = c_1e^{4x} + c_2e^{-x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x$

Example: $y'' - 3y' - 4y = 4x^2 - 1$

$y_c(x) = c_1e^{4x} + c_2e^{-x}$

For a polynomial right hand side, we try a particular solution of the same order $(Ax^2 + Bx + C)$

We then plug in the derivatives and equate coefficients

$$\begin{cases} \text{x squared:} & 2A - 3B - 4C + 1 = 0 \\ \text{x:} & -6A - 4B = 0 \\ \text{constant:} & A + 1 = 0 \end{cases}$$

$f(x)$	$y_p(x)$
$P_n(x) := a_n x^n + \cdots + a_1 x + a_0$	$x^s[A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0]$ Three possibilities for s: $s = 0$ if $r = 0$ is not a solution of the characteristic equation $s = 1$ if $r = 0$ is only one solution of the characteristic equation $s = 2$ if $r = 0$ is double solution of the characteristic equation
$P_n(x)\mathrm{e}^{\alpha x}$	$x^s[A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0]\mathrm{e}^{\alpha x}$ Three possibilities for s: $s = 0$ if $r = \alpha$ is not a solution of the characteristic equation $s = 1$ if $r = \alpha$ is only one solution of the characteristic equation $s = 2$ if $r = \alpha$ is double solution of the characteristic equation
$P_n(x)\mathrm{e}^{\alpha x}\cos(\beta x) +$ $+P_m(x)\mathrm{e}^{\alpha x}\sin(\beta x)$	$x^s[A_k x^k + A_{k-1} x^{k-1} + \cdots + A_1 x + A_0]\mathrm{e}^{\alpha x}\cos(\beta x) +$ $+x^s[B_k x^k + B_{k-1} x^{k-1} + \cdots + B_1 x + B_0]\mathrm{e}^{\alpha x}\sin(\beta x)$ $k = \max\{n, m\}$ Two possibilities for s: $s = 0$ if $r = \alpha + i\beta$ is not a solution of the characteristic equation $s = 1$ if $r = \alpha + i\beta$ is a solution of the characteristic equation

13 The Phenomenon of Resonance and Linear Independent Solutions

Consider the undamped mass-spring oscillator

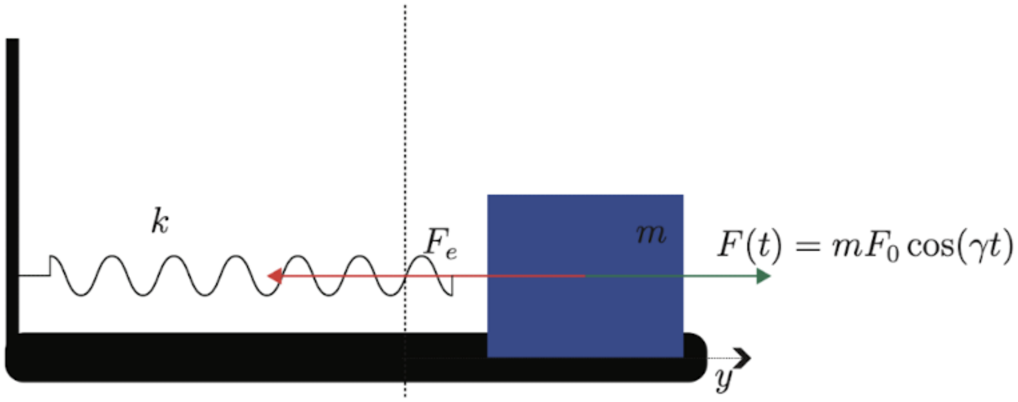


Figure 3.7: Forced mass-spring oscillator.

General ODE:

$$y'' + \omega^2 y = F_0 \cos(\gamma t), \quad \omega^2 = \frac{k}{m}$$

- ω: Natural frequency (system's own oscillation)
- γ: Forcing frequency (external input)

General Solution (Non-Resonant):

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + y_p(t)$$

Try particular solution:

$$y_p(t) = A\cos(\gamma t) + B\sin(\gamma t)$$

After differentiating and substituting:

$$A(\omega^2 - \gamma^2)\cos(\gamma t) + B(\omega^2 - \gamma^2)\sin(\gamma t) = F_0 \cos(\gamma t)$$

So:

$$A = \frac{F_0}{\omega^2 - \gamma^2}, \quad B = 0$$

Final non-resonant solution (for $\omega \neq \gamma$):

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2}\cos(\gamma t)$$

13.1 Resonance and Generalization

Resonance occurs when $\omega = \gamma$, i.e., the forcing matches the system's own frequency. This breaks the usual method since the input matches a homogeneous solution.

- **General principle:** When the forcing function is also a solution of the homogeneous equation, *multiply your usual particular solution guess by t for linear independence.*
- For resonance ($\omega = \gamma$), trial:

$$y_p(t) = [A\cos(\omega t) + B\sin(\omega t)]t$$

Solving gives:

$$A = 0, \quad B = \frac{F_0}{2\omega}$$

So the resonance particular solution is:

$$y_p(t) = \frac{F_0}{2\omega} t\sin(\omega t)$$

13.2 Examples

Example: $y'' - 3y' - 4y = -8e^{-x}$

$y_c(x) = c_1e^{4x} + c_2e^{-x}$

Notice the forcing term $f(x) = -8e^{-x}$ is a constant multiple of c_2e^{-x}, thus we need a particular solution of the form Axe^{-x}

We then plug in the derivatives and get the general solution $y(x) = c_1e^{4x} + c_2e^{-x} + \frac{8}{5}xe^{-x}$

Example: $y'' - 2y' + y = (x + 1)e^x$

$y_c(x) = (c_1 + c_2x)e^x$

If we plug in the RHS, the DE will equal 0

Then, we look for a particular solution of the form $y_p(x) = x^2(Ax + B)e^x$

Example: $2y'' + 3y' + y = x^2 + 3\sin x$

In this case, we can develop a particular solution from the superposition principle because the forcing term is an addition: $f(x) = x^2 + 3\sin x$

Thus $y_p(x) = y_{p,1}(x) + y_{p,2}(x)$, where the former is a particular solution to ODE $= x^2$ and the latter is a particular solution to $ODE = 3\sin x$

14 Method of Variation of Parameters

Find a solution to $y'' + 4y = 3\csc x$, but $f(x)$ cannot be expressed as an l.c. of our common terms from the previous note

We can use **variation of parameters**, which works for any possible type of forcing term

Consider general form ODE $y'' + a(x)y' + b(x)y = f(x)$, and consider the particular solution of the form $y_p(x) = v_1(x)y_1(x) + v_2y_2(x)$

Get $y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$, and assume that $v_1'y_1 + v_2'y_2 = 0$

Then, $y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$

We can impose that to the ODE and get that $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution iff v_1 and v_2 satisfy:

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = f(x) \end{cases}$$

where $W[y_1(x), y_2(x)] \neq 0$

We then find the following solution of the system via Cramer's rule:

$v_1(x) = \int \frac{-f(x)y_2(x)}{W[y_1(x),y_2(x)]}dx$, $v_2(x) = \int \frac{f(x)y_1(x)}{W[y_1(x),y_2(x)]}dx$

Example 1

Consider $xy'' - (x+2)y' + 2y = x^3$ in $(0, \infty)$, or in standard form: $y'' - \frac{x+2}{2}y' + \frac{2}{x}y = x^2$

and two linearly independent solutions of the corresponding homogeneous equation $y_1(x) = e^x, y_2(x) = x^2 + 2x + 2$

Find a particular solution by the method of variation of parameters. $y_p(x) = v_1y_1 + v_2y_2$, where v_1 and v_2 must satisfy Cramer's rule

$$\begin{cases} v_1'e^x + v_2'(x^2+2x+2) = 0 \\ v_1'e^x + v_2'(2x+2) = x^2 \end{cases} \Leftrightarrow \begin{cases} v_1' = -(x^2+2x+2)e^{-x}v_2' \\ v_2' = -1 \end{cases} \Leftrightarrow \begin{cases} v_1 = (x^2+4x+6)e^{-x} \\ v_2 = -x \end{cases}$$

General solution: $y(x) = c_1e^x + c_2(x^2+2x+2) - x^3 - 3x^2 - 6x - 6$

Example 2

Consider $y'' + y = \tan x + 3x - 1$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Homogeneous equation: $y''+y = 0$ with linearly independent solutions $y_1(x) = \cos x, y_2(x) = \sin x$

Particular solution: $y_p(x) = y_{p,1}(x) + y_{p,2}(x)$, where $y_{p,1}$ is a particular solution of $y'' + y = \tan x$ $y_{p,2}$ is a particular solution of $y'' + y = 3x - 1$

We can find $y_{p,2}$ via undetermined coefficients $= 3x - 1$

We find $y_{p,1}$ with variation of parameters $= v_1y_1 + v_2y_2$, where v_1v_2 must satisfy Cramer's rule

$$\begin{cases} v_1' \cos x + v_2' \sin x = 0 \\ -v_1' \sin x + v_2' \cos x = \tan x \end{cases} \Rightarrow \begin{cases} v_1' = -v_2' \tan x \text{ and replace in the second equation} \\ -v_1' \sin x + v_2' \cos = \tan x \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = -v_2' \tan x \\ v_2' = \sin x \text{ and back substitute in the first equation} \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = -\frac{\sin^2 x}{\cos x} \\ v_2' = \sin x \end{cases} \Rightarrow \begin{cases} v_1 = \sin x - \ln|\sec x + \tan x| + \cancel{k_1} \\ v_2 = -\cos x + \cancel{k_2} \end{cases} \Rightarrow y_{p,1} = -\ln|\sec x + \tan x| \cos x$$

15 Introduction to the Laplace transform

The Laplace transform is a mathematical tool that converts a function of time $f(t)$ into $F(s)$. It is used to simplify differential equations into algebraic ones.

Definition:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dx$$

where t is the original variable, s is a new variable, and e^{-st} is a decaying exponential that "weights" the signal over time

When the transform is applied, you move from the time domain (functions of t), to the **frequency domain** (functions of s), where s helps us analyze how a system responds to different "frequencies" and how it behaves over time

Note: $\int_0^\infty e^{-st} f(t)\, dt := \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt$ If this limit exists, then the integral is said to converge to that limiting value.

The Laplace transform is an integral operator and a linear operator (i.e., $\mathcal{L} : f \to \mathcal{L}\{f\}$, and $\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$)

General idea:

The general idea in using the Laplace transform to solve a differential equation is as follows:

1. Use the relation defined above to transform an IVP for an unknown function f in the $t-$domain into a simpler, algebraic problem for F in the $s-$domain
2. Solve this algebraic problem to find F
3. Recover the desired function f from its transform F. This last step is known "inverting the transform."

Application:

If $f(t) = 1$, then the Laplace transform is $\int_0^\infty e^{-st}\, dt = \frac{1}{s}$ for $s > 0$

16 Properties of Laplace transform

16.1 Piecewise Continuity

Definition: A function $f(t)$ is *piecewise continuous* on $[0, \infty]$ if, for any interval $[0, N]$, you can split it into a finite number of subintervals where $f(t)$ is continuous (except possibly for jump discontinuities)

Example:

Given a piecewise function:

$$\begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & t > 10 \end{cases}$$

For the integral of $\mathcal{L}\{f\}(s)$, we can break up our values of $f(t)$ into three integrals with their respective t bounds, and solve in terms of s

Jump discontinuity

A Laplace transform can not be defined for functions with jump discontinuities

Definition: A function $f(t)$ on $[a, b]$ is said to have a *jump discontinuity* at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0 but the following limits exists and are finite: $\lim\limits_{t \to t_0^-}$ and $\lim\limits_{t \to t_0^+}$

16.2 Theorem: Laplace transform of 1st order derivative

Suppose that f is continuous on $[0, \infty]$, f' is piecewise continuous on $[0, \infty]$, and There exists real constants T, a and M with $T, M > 0$ such that $|f(t)| \leq Me^{at}$ for all $t \geq T$

Then, there exists $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

By induction, there exists $\mathcal{L}\{f^{(n)}\}$ for $s > a$ and it is given by $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

16.2.1 Exponential Order Condition

A function must not grow too fast. Specifically, there must be constants $M > 0, T > 0$ and a such that:

for all $t \geq T : |f(t)| \leq Me^{at}$

This means that, after some point T, $f(t)$ is always less than or equal to an exponential function Me^{at}. So, $f(t)$ can't explode faster than an exponential.

16.2.2 Convergence of the Laplace transform

Combining both theorems, a Laplace transform $\mathcal{L}\{f\}(s)$ exists (i.e. converges) for all $s > a$ if

1. f is piecewise continuous on $[0, \infty]$
2. There exist real constant T, a and M with $T, M > 0$ such that $|f(t)| \leq Me^{at}$ for all $t \geq T$

16.3 Useful Properties

$\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$

Examples:

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$ for all $s > a + T$

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^k}\right\}(t) = \frac{t^{k-1}e^{at}}{(k-1)!}$

Example: $\mathcal{L}\{e^{at}\}, t \geq 0$

$F(s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty$

If $s > a$: We see that $e^{\text{negative exponential }\times\infty}$ goes to zero, therefore it **converges**

If $s \leq a$: We see that $e^{\text{positive exponential }\times\infty}$ goes to infinity, therefore it **diverges**

Example: $f(t) = \sin(3t)$ $f'(t) = 3\cos(3t)$ $f(0) = \sin 0 = 0$ $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\sin(3t)\}(s) = \frac{3}{s^2+9}$

Method 1: $\mathcal{L}\{f'(t)\}(s) = \mathcal{L}\{3\cos(3t)\}(s) = 3 \cdot \frac{s}{s^2+9}$

Method 2: Use property $\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f(0)$

We get $= s \cdot \frac{3}{s^2+9} - 0 = \frac{3s}{s^2+9}$

17 Method of Laplace Transform

The method of Laplace transform:

1. Take the Laplace transform of both sides of the DE
2. Use the properties of Laplace transform and the initial conditions to obtain an equation for $\mathcal{L}\{y\}$
3. Solve the corresponding equation for $\mathcal{L}\{y\}$
4. Calculate the inverse Laplace transform yielding the solution of the IVP
5. A posteriori analysis: check the regularity (i.e. continuity and differentiability) of the solution, and check that the initial conditions are satisfied

Advantages to the Laplace Transform:

1. Solve IVPs without finding a general solution
2. Solve ODEs with forcing functions having jump discontinuities
3. Solve certain linear ODEs with variable coefficients
4. Solve certain integral equation

17.1 Examples

Short Example: Find $\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\}$

$\frac{2}{s^2+3s-4} = \frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1} = \frac{(A+B)s+(4B-A)}{(s+4)(s-1)}$

$$\Rightarrow \begin{cases} A+B=0 \\ 4B-A=2 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{2}{5} \\ B=\frac{2}{5} \end{cases}$$

Hence, $-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}(t) + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = -\frac{2}{5}e^{-4t} + \frac{2}{5}e^{t}$, using: $\mathcal{L}\{e^{at}\}(t) = \frac{1}{s-a}$

Advanced Inverse Example: Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t)$

Identify $F(s-1) = \frac{1}{(s-1)^2}$, so $a=1$ and $F(s) = \frac{1}{2}$

Consider the property: $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t t$

Advanced Partial Fraction Example: Find $\mathcal{L}^{-1}\left\{\frac{6s^2+28}{(s^2-2s+5)(s+2)}\right\}$

Rewrite s^2-2s+5 as $(s-1)^2+2^2$. This is helpful because it matches forms involving shifted s

Partial Fractions, match numerator with denominator form: $\frac{A(s-1)+2B}{(s-1)^2+2^2}$ and $\frac{C}{s+2}$

Solving for coefficients, we get $A=2, B=3, C=4$

Then:

$\mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+2^2}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{2s}{s^2+2^2}\right\} = e^t \cdot 2\cos(2t)$

for $\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^2+2^2}\right\}$, get numerator equal to a from denominator $\Rightarrow \mathcal{L}^{-1}\left\{3 \cdot \frac{2}{(s-1)^2+2^2}\right\} = 3e^t\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = 3e^t \sin(2t)$

$\mathcal{L}^{-1}\left\{\frac{4}{s+2}\right\} = 4e^{-2t}$

Therefore, $2e^t\cos(2t) + 3e^t\sin(2t) + 4e^{-2t}$

17.2 Time Shifting Example

Full Example: Consider the IVP $y'' + 2y' + y = 4e^{\pi-t}, y(\pi) = 2, y'(\pi) = -1$

Since the initial conditions are given at $t = \pi$, we need to shift the IVP from $t = 0$ to $t = \pi$, and let $\tau = t - \pi$

Now, we can rewrite the forcing term $f(t) = 4e^{\pi-t}$ as $4e^{\pi-t=-\tau} = 4e^{-\tau}$ initial conditions as $y(\tau = 0) = 2, y'(\tau = 0) = -1$

Apply Laplace transform to the IVP with τ as the time variable: $\mathcal{L}\{y''(\tau)\} + 2\mathcal{L}\{y'(\tau)\} + \mathcal{L}\{y(\tau)\} = \mathcal{L}\{4e^{-\tau}\}$

Apply properties, where $Y(s)$ is the Laplace transform, or image, of $y(t)$: $s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{4}{s+1}$

Substitute initial values: $s^2Y(s) - 2s + 1 + 2sY(s) - 4 + Y(s) = \frac{4}{s+1}$

Group terms: $(s+1)^2Y(s) = \frac{4}{s+1} + 2s + 3$

Rearrange: $Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}$

Apply inverse Laplace transforms: $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^3}\right\} = 2\tau^2 + e^{-\tau}$ $\mathcal{L}^{-1}\left\{\frac{2s+3}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)} + \frac{1}{(s+1)^2}\right\} = 2e^{-\tau} + \tau e^{-\tau}$

Final solution in τ: $y(\tau) = 2\tau^2e^{-\tau} + 2e^{-\tau} + \tau e^{-\tau} = e^{-\tau}(2\tau^2 + \tau + 2)$

Change back to t: $y(t) = e^{-(t-\pi)}(2(t-\pi)^2 + (t-\pi) + 2)$

18 Transform of Discontinuous Functions

Definition: The unit step function $u(t)$, also known as the **Heaviside step function** is:

$$u(t) := \begin{cases} 0 & \text{for some } t < 0 \\ 1 & \text{for some } t > 0 \end{cases}$$

We can shift the argument and multiply with a function:

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{for some } t < a \\ f(t-a) & \text{for some } t > a \end{cases}$$

In this argument, the function f is turned on at a

18.1 Rectangular Window Function

Definition: For any two real numbers $a < b$, the **rectangular window function** $\prod_{a,b}$ is defined by:

$$\prod_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a < t < b \\ 0 & \text{for } t > b \end{cases}$$

The function gets "turned on" at $u(t-a)$, and switches from the previous function f_0 to the new function f_1 by the difference $f_1 - f_0$. They can then be multiplied and added to the function $f(t)$, by $u(t-a)(f_1 - f_0)$

Example: The function

$$f(t) = \begin{cases} -t & \text{for } t < 0 \\ 6 & \text{for } -1 < t < 1 \\ \cos t & \text{for } t > 1 \\ 3 & \text{for } t > 6 \end{cases}$$

The function starts at $-t$ until -1, where it switches to 6: $-t + u(t+1)(-(-t)+6)$

Then, the function switches to $\cos t$ at $t = 1$: $u(t-1)(-6+\cos t)$

Then, the function switches to 3 at $t = 6$: $u(t-6)(-\cos t + 3)$

Together: $f(t) = -t + u(t+1)(t+6) + u(t-1)(\cos t - 6) + u(t-6)(3 - \cos t)$ Alternatively: $f(t) = -t\prod_{-\infty,-1}(t) + 6\prod_{-1,1}(t) + \cos t\prod_{1,6}(t) + 3u(t-6)$

18.2 Solving discontinuous functions

Example 1:

Find the Laplace transform of:

$$f(t) = \begin{cases} \sin t & 0 \le t \le \frac{\pi}{4} \\ \sin t + \cos\left(t - \frac{\pi}{4}\right) & t > \frac{\pi}{4} \end{cases}$$

We agree that

$$f(t) = \sin t \prod_0^{\pi/4}(t) + \left(\sin t + \cos\left(t - \frac{\pi}{4}\right)\right) \prod_{\frac{\pi}{4}}^{\infty}(t)$$

Into unit step functions:

$$f(t) = \sin t \left(u(t-0) - u\left(t - \frac{\pi}{4}\right)\right) + \left(\sin t + \cos\left(t - \frac{\pi}{4}\right)\right) u\left(t - \frac{\pi}{4}\right)$$

Simplify:

$$f(t) = \sin t + \cos\left(t - \frac{\pi}{4}\right) u\left(t - \frac{\pi}{4}\right)$$

Take Laplace:

$$\boxed{F(s) = \frac{1}{\sin^2 t + 1} + e^{-s\pi/4} \frac{\cos t}{\cos^2 t + 1}}$$

Example 2:

Consider $y'' + 4y = g(t), \quad y(0) = y'(0) = 0$, where

$$g(t) = \begin{cases} 0 & \text{for } t < 5 \\ \frac{t-5}{5} & \text{for } 5 < t < 10 \\ 1 & \text{for } t \ge 10 \end{cases}$$

Considering jump discontinuous at $5, 10$:

$$g(t) = \frac{t-5}{5} \prod_{5,10}(t) + u(t-10) = \frac{t-5}{5} u(t-5) - \frac{t-10}{5} u(t-10)$$

Take the Laplace transform of both sides of the ODE:

$$s^2Y(s) + 4Y(s) = \frac{1}{5}\mathcal{L}\{(t-5)u(t-5)\}(s) - \frac{1}{5}\mathcal{L}\{(t-10)u(t-10)\}(s) = \frac{1}{5}\mathcal{L}\{t\}(s)[e^{-5s} - e^{-10s}]$$

$\Leftrightarrow Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)} = \frac{e^{-5s} - e^{-10s}}{5} H(s)$, where $H(s) := \frac{1}{s^2(s^2+4)}$

Then take the inverse Laplace transform:

$$y(t) = \frac{1}{5}[\mathcal{L}^{-1}\{e^{-5s}H(s)\}(t) - \mathcal{L}^{-1}\{e^{-10s}H(s)\}(t)]$$

$= \frac{1}{5}[h(t-5)u(t-5) - h(t-10)u(t-10)]$

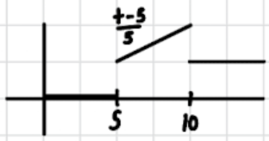
by partial fractions

$$h(\tau) = \mathcal{L}^{-1}\{H(s)\}(\tau) = \frac{\tau}{4} - \frac{1}{8}\sin(2\tau)$$

Detailed:

$y'' + 4y = g(t)$, $y(0) = y'(0) = 0$

$$g(t) = \begin{cases} 0 & \text{for } t < 5 \\ \frac{t-5}{5} & \text{for } 5 < t < 10 \\ 1 & \text{for } t \geq 10 \end{cases}$$



$$g(t) = 0 + u(t-5)\cdot\frac{t-5}{5} + u(t-10)\cdot\left(1 - \frac{t-5}{5}\right)$$

$$= \frac{t-5}{5}\left(u(t-5) - u(t-10)\right) + u(t-10)$$

$$= \frac{t-5}{5}u(t-5) - \frac{t-5}{5}u(t-10) + u(t-10)$$

$$= \frac{t-5}{5}u(t-5) - u(t-10)\left[\frac{t-10}{5}\right]$$

$\mathcal{L}$: $s^2Y(s) - sy(0) - y'(0) + 4Y(s) = s^2Y(s) + 4Y(s)$

RHS $\mathcal{L}$: $\mathcal{L}\{u(t-5)\frac{t-5}{5}\}(s) - \mathcal{L}\{u(t-10)\frac{t-10}{5}\}(s)$

$$= \frac{1}{5}\mathcal{L}\{(t-5)u(t-5)\}(s) - \frac{1}{5}\mathcal{L}\{(t-10)u(t-10)\}(s)$$

$$= \frac{1}{5}\mathcal{L}\{t\}e^{-5s} - \frac{1}{5}\mathcal{L}\{t\}e^{-10s}$$

$$= \frac{1}{5}e^{-5s}\cdot\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$$

$$= 1/5s^2\left(e^{-5s} - e^{-10s}\right)$$

LHS = RHS: $s^2Y(s) + 4Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}$

$Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}$, or $Y(s) = \frac{e^{-5s} - e^{-10s}}{5}H(s)$, where $H(s) = 1/s^2(s^2+4)$

Therefore, our sol'n to the IVP is $y(t) = \frac{1}{5}\left[\mathcal{L}^{-1}\{e^{-5s}H(s)\}(t) - \mathcal{L}^{-1}\{e^{-10s}H(s)\}(t)\right]$

$= \frac{1}{5}\left(h(t-5)u(t-5) - h(t-10)u(t-10)\right)$, where $h(t) = \mathcal{L}^{-1}\{H(s)\}(t) = \frac{t}{4} - \frac{\sin 2t}{8}$

18.3 Useful Properties

If $a \geq 0$, then $\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$, and $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t-a)$

If $0 < a < b$, then $\mathcal{L}\left\{\prod_{a,b}\right\}(s) = \frac{e^{-as} - e^{-sb}}{s}$

Also, $e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt$, and we can start the integral at 0 by:

$e^{-as}F(s) = \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt = \mathcal{L}\{u(t-a)f(t-a)\}$

19 Convolution Integral

Consider $y'' + y = g(t), \quad y(0) = 0, y'(0) = 0$

By taking the Laplace transform of both sides, we get $Y(s) = \frac{1}{s^2+1}G(s)$

In the general case where $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}G(s)\right\}(t)$, we can use an new operation "$*$" such that:

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) * \mathcal{L}^{-1}\{G(s)\}(t)$

Note: $\mathcal{L}^{-1}\{F(s)G(s)\} \neq \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}$

19.1 Convolution of Two Functions

Let $f(t)$ and $g(t)$ be piecewise continuous functions $[0, +\infty]$. The **convolution** of f and g is defined by:

$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)\, d\tau$

Therefore, when $y(0) = 0$ and $y'(0) = 0$, we can solve an IVP with arbitrary forcing term $g(t)$ without knowing $\mathcal{L}\{g(t)\}$. This is useful when $g(t)$ is complicated.

Example: Let $f(t) = t$ and $g(t) = t^2$, then

$(f * g)(t) = t * t^2 = \int_0^t (t-\tau)\tau^2\, d\tau = \frac{t^4}{12}$, and not $t \cdot t^2 = t^3$

Properties:

1. $f * g = g * f$
2. $f * (g + h) = f * g + f * h$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0$

Use Cases:

$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}(t) = e^{at} * e^{bt}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}(t) = \frac{1}{a}\mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}(t) = \frac{1}{a}t * \sin(at) = \frac{1}{a}\int_0^t (t-\tau)\sin(a\tau)\, d\tau$

$= \frac{at - \sin(at)}{a^3}$

19.2 Transfer and Impulse Response

Consider the general homogeneous IVP $ay'' + by' + cy = g(t)$, for $t > 0, \quad y(0) = y'(0) = 0$

We find that, by taking the Laplace transform of both sides: $Y(s) = \frac{G(s)}{as^2+bs+c}$

By taking inverse Laplace:

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\} * g(t) \Rightarrow y(t) = (h * g)(t)$

where $h(t) := \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\}(t)$

The function

$$H(s) := \frac{1}{as^2 + bs + c} \text{ is called the transfer function}$$

The function

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}(t) \text{ is called the impulse response function}$$

19.3 Nonzero Initial Conditions

How does this change if $y(0) \neq 0, y'(0) \neq 0$?

Consider the previous IVP with initial conditions $y(0) = Y_0, y'(0) = Y_1$,

admits the solution

$$y(t) = (h * g)(t) + y_k(t)$$

where the first term is the solution to the nonhomogeneous IVP with $y(0) = y'(0) = 0$ and the second term is the solution to the homogeneous IVP with $y(0) = Y_0, y'(0) = Y_1$

Example: $y'' + y = t, \quad y(0) = 2, y'(0) = -1$

1. Particular solution (zero initial conditions)

Laplace: $(s^2 + 1)Y(s) = \frac{1}{s^2} \Rightarrow Y(s) = \frac{1}{s^2(s^2+1)}$

Inverse: $y_p(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \Rightarrow y_p(t) = t - \sin t$

2. Homogeneous solution (nonzero initial conditions)

General solution: $y_k(t) = A\cos t + B\sin t$

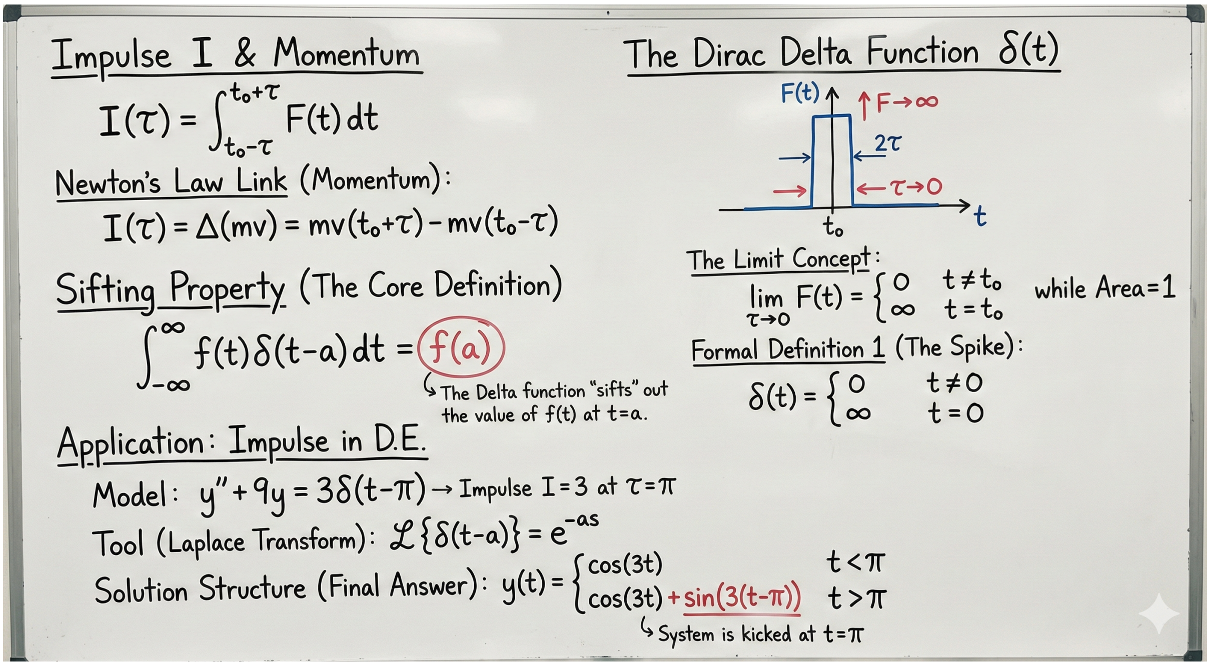
Use initial conditions to solve for $A = 2, B = -1 \Rightarrow y_k(t) = 2\cos t - \sin t$

3. Full solution

$\boxed{y(t) = [t - \sin t] + [2\cos t - \sin t]}$

$y(t) = t + 2\cos t - 2\sin t$

20 Impulse Functions and the Dirac Delta Function



Many phenomena have an *impulsive nature*, where a force $\mathfrak{F}(t)$ is applied in the time interval $[t_0 - \tau, t_0 + \tau]$ for a very small τ. The impulse $\mathfrak{I}$ due to the force $\mathfrak{F}$ is defined as:

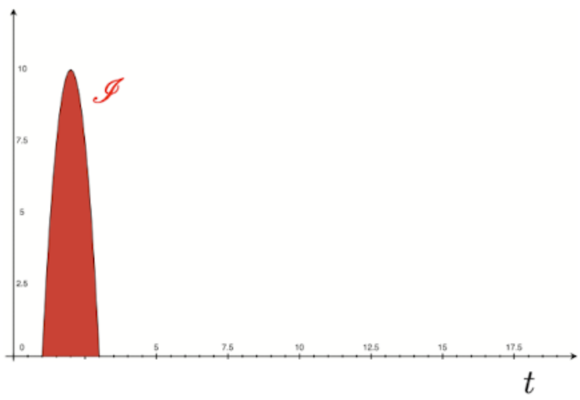
$$\mathfrak{I}(\tau) := \int_{t_0-\tau}^{t_0+\tau} \mathfrak{F}(t)\,dt \equiv \int_{-\infty}^{\infty} \mathfrak{F}(t)\,dt$$

By Newtons' second law, $\mathfrak{F} = m\frac{dv}{dt}$, therefore:

$$\mathfrak{I}(\tau) = \int_{t_0-\tau}^{t_0-\tau} m\frac{dv}{dt}\,dt = mv(t_0+\tau) - mv(t_0-\tau)$$

20.1 Dirac Delta Function

Consider the impulse $\mathfrak{I}$ on the following graph, where its peak is at t_0, and width from the centre is $[t_0 - \tau, t_0 + \tau]$



If $\mathfrak{I}$ is consistent (i.e., $\mathfrak{I}(\tau) \equiv 1$), when $\tau \to 0$, for the area ($\mathfrak{I}$) to stay $= 1$, $\mathfrak{F} \to \infty$. Then,

$$\lim_{n\to\infty} \mathfrak{F}_n(t) = \begin{cases} 0 & \text{if } t \neq t_0 \\ \infty & \text{if } t = t_0 \end{cases} \text{ and } 1 = \lim_{n\to\infty} \mathfrak{I}(\tau_n) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \mathfrak{F}_n(t)\, dt$$

In essence, the sequence $\mathfrak{F}_n$ behaves more and more like a spike at t_0 that concentrates all its area there while becoming negligible everywhere else, mimicking the Dirac delta function $\delta(t - t_0)$

Definition: The **Dirac delta function** δ is characterized by the following two properties:

1. $$\delta(t) = \begin{cases} 0 & t \neq 0 \\ +\infty & t = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

We can also shift the argument and define

$$\delta(t-a) = \begin{cases} 0 & t \neq 0 \\ +\infty & t = 0 \end{cases}, \int_{-\infty}^{\infty} f(t)\delta(t-a)\, dt = f(a)$$

20.2 Laplace Transform of $\delta(t - a)$

$$\mathcal{L}\{\delta(t-a)\} = e^{sa}$$

Example: $y'' + 9y = 3\delta(t - \pi), \quad y(0) = 1, y'(0) = 0$

This models an undamped spring, originally at rest at position $y_0 = 1$, then hit at $\tau = \pi$ by an impulse $\mathfrak{I} = 3$

Laplace: $(s^2 + 9)Y(s) = s\cancel{y(0)}(= 1) + 3e^{-\pi s}$

Inverse: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{s^2+9}\right\}(t) = cos3t + u(t-\pi)\sin(3(t-\pi))$ Therefore:

$$y(t) = \begin{cases} \cos(3t) & \text{for } t < \pi \\ \cos(3t) + \sin(3(t-\pi)) & \text{for } t > \pi \end{cases}$$

Before the impulse ($t < \pi$): the system behaves like a normal oscillator with the given initial conditions

After the impulse ($t > \pi$): the impulse “kicks” the oscillator, causing it to suddenly gain energy and shift its motion, reflected by the extra $+\sin(3(t - \pi))$

21 Systems of First-Order Linear Differential Equations

21.1 Transform Into First-Order ODEs

What happens when we have multiple interacting first-order ODEs (systems)?

Common systems of differential equations arise in the evolution of competing species in population dynamics, interconnected fluid tanks for mixing problems, and mass-spring systems.

Remark: We can take any n-th order (linear and nonlinear) ODE, $y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$, and transform it into a system of n first-order ODEs:

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\ldots \\ x_n &= y^{(n-1)} \end{aligned}$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_2 \\ \ldots \\ x_n' = F(t, x_1, x_2, \ldots, x_n) \end{cases}$$

Define: $\underline{x}(t) := (x_i(t))_{i=1,\ldots,n}$, $\underline{F}(t, \underline{x}) = (F_i(t, x_1(t), \ldots, x_n(t)))_{i=1,\ldots,n}$, and $\underline{x}(t_0) = \underline{x_0} \in \mathbb{R}^n$

We then obtain the general **initial value problem** for a system of n differential equations

$$\underline{x}' = \underline{F}(t, \underline{x}), \quad \underline{x}(t_0) = \underline{x}_0$$

A **general system** looks like:

$x_1' = ax_1 + bx_2$, with x_1, x_2 being functions of time $x_2' = cx_1 + dx_2$, the independent variable is t, and dependent are x_1, x_2

We can write this in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x' = Ax$$

Let $x(t) = ve^{\lambda t}$, where v is a 2×1 vector. Then:

$$\lambda ve^{\lambda t} = Ave^{\lambda t} \Rightarrow Av = \lambda v$$

where λ are the eigenvalues of A and v are the eigenvectors

To compute λ, solve $\det(A - \lambda I) \Rightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0 \quad \star$

21.2 Existence and Uniqueness Theorem for systems

Consider $\underline{x}' = \underline{F}(t, \underline{x}), \quad \underline{x}(t_0) = \underline{x}_0$

Assume that the following two conditions are satisfied:

(EU1) The functions F_i and $\frac{\partial F_i}{\partial x_j}$, for all $i, j = 1, \ldots, n$ are continuous in a "$(n+1)$-th dimensional rectangle" $R = \{(t, x_1, \ldots, x_n) : \alpha < t < \beta, \alpha_k < x_k < \beta_k$ for $k = 1, 2, \ldots, n\} \subset \mathbb{R}^{n+1}$

(EU2) $(t_0, x_0) \in R$

Then there exists $h > 0$ and there exists a unique solution $\underline{x(t)}$, defined on $(t_0 - h, t_0 + h)$, of the IVP

Definition: An **affine function** is a function that combines a linear transformation with a translation. For real variables, an affine function has the form: $f(x) = Ax + b$

Definition: A **linear system of n first-order ODEs** is given by $\underline{x}' = \underline{A}(t)\underline{x} + \underline{b}(t)$

21.2.1 Existence and Uniqueness Theorem for linear systems

If the functions $a_{i,j}(t)$ and $b_i(t)$, for $i, j = 1, 2, \ldots, n$ are continuous on some open interval $I = (\alpha, \beta)$ and $t_0 \in I$, then there exists a unique solution $\underline{x}(t)$ to the IVP

$$\underline{x}' = \underline{A}(t)\underline{x} + \underline{b}(t), \quad \underline{x}(t_0) = \underline{x}_0$$

for any choice of initial data $\underline{x}_0 \in \mathbb{R}^n$. Moreover, the solution is defined for every $t \in I = (\alpha, \beta)$

21.2.2 Superposition principle

A linear system $\underline{x}' = \underline{A}(t)\underline{x} + \underline{b}(t)$ is said to be **homogeneous** if $\underline{b}(t) = \underline{0}$ for all t

Theorem: Let $\underline{y}$ and $\underline{z}$ be two solutions of the homogeneous linear system $\underline{x}' = \underline{A}(t)\underline{x}$. Then, for any two real constants $c_1, c_2 \in \mathbb{R}$, the vector function

$$\underline{x}(t) = c_1\underline{y}(t) + c_2\underline{z}(t) \text{ is also a solution}$$

22 Linear Independence of Systems

Definition: $m-$vector functions $\underline{v_1}(t), \ldots, \underline{v}_m(t)$ are said to be **linearly dependent** on an open interval I if:

$$c_1\underline{v_1}(t) + \cdots + c_m\underline{v}_m(t) = \underline{0} \text{ for all } t \in I$$

If not $= \underline{0}$, they are said to be **linearly independent on** I

Proposition: If there exists $t_0 \in I$ such that the matrix having $\underline{v}_1, \ldots, \underline{v}_m$ as column satisfies

$$\det \begin{bmatrix} v_{1,1}(t_0) \ldots v_{m,1}(t_0) \\ \ldots \\ v_{1,m}(t_0) \ldots v_{3,m}(t_0) \end{bmatrix} \neq 0 \text{ then those vectors are linearly independent on } I$$

Remark: The inverse of the above proposition does not hold true.

22.1 Linear homogeneous systems

Consider the linear homogeneous system of n first-order ODEs

$$\underline{x}' = \underline{A}(t)\underline{x}, \text{ with } \underline{A}(t) = (a_{i,j}(t))_{i,j=1,\ldots,n} \text{ continuous on some open interval } I$$

and $n-$solutions $\underline{v}_1(t), \ldots, \underline{v}_n(t)$. Then the following statements are equivalent:

1. $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent on I
2. The Wronskian $\neq 0$ for every $t \in I$
3. There exists $t_0 \in I$ such that $W[\underline{v}_1, \ldots, \underline{v}_n](t_0) \neq 0$

In words, the above proposition says that for solutions to homogeneous linear systems of 1st order ODEs, the Wronskian is either identically zero or never zero

22.2 General Solution

Theorem: (Representations of solutions to linear homogeneous systems of 1st order ODEs)

Let $\underline{x}_1(t), \ldots, \underline{x}_n(t)$ be $n-$linearly independent solutions of

$$\underline{x}' = \underline{A}(t)\underline{x} \text{ on some interval } I$$

Then, every other solution on I can be determined by

$$\underline{x}(t) = c_1\underline{x}_1(t) + \cdots + c_n\underline{x}_n(t)$$

and called the **general solution** for arbitrary constants $c_n \in \mathbb{R}$. These constants can be determined by imposing initial conditions.

23 Generalization of Solutions

Our goal is to solve $x'(t) = Ax(t)$ with A being the $n \times m$ matrix

Remark: Suppose $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ is diagonal, then $x' = Ax$ can be written as:

$$\begin{cases} x' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \\ x_3' = \lambda_3 x_3 \end{cases} \Rightarrow \begin{cases} x_1 = c_1 e^{\lambda_1 t} \\ \vdots \\ x_n = c_n e^{\lambda_n t} \end{cases}$$

So, the general solution is

$$x(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n e^{\lambda_n (t)} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = c_1 e^{\lambda_1 t} u_1 \cdots + c_n e^{\lambda_n t} u_n$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues and $u_1, \dots, u_n$ are eigenvectors

Plugging into the equation: $x(t) = e^{\lambda t} u$ is a solution $\underline{u} = 0$ if and only if $\det(\underline{A} - \lambda \underline{I}) = 0$

$$\begin{aligned} \lambda e^{\lambda t} u &= A(e^{\lambda t} u) \\ \lambda u &= Au \\ A = \lambda, \lambda \text{ is an eigenvalue with eigenvector } u \end{aligned}$$

Lemma: $\underline{x} = e^{\lambda t} \underline{u}$ is a solution of $\underline{x}' = \underline{Ax}$ if and only if λ is an eigenvalue of $\underline{A}$ and $\underline{u}$ is the corresponding eigenvector

Theorem: Let $\underline{A}$ be a $n \times n$ constant matrix. If $\underline{u}_1, \dots, \underline{u}_n$ are $n-$linearly independent eigenvectors of $\underline{A}$ corresponding to $n-$real eigenvalues $r_1, \dots, r_n$, then:

$$\underline{x}_1(t) = e^{\lambda_1 t} \underline{u}_1, \dots, \underline{x}_n = e^{\lambda_n t} \underline{u}_n \text{ are n-linearly independent solutions of } \underline{x}' = \underline{Ax}$$

Example: Consider $A = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$

There are two eigenvalues $r_1 = 1$ and $r_2 = 4$. Eigenvector corresponding to $r_1 = 1$ is $\underline{u}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and eigenvector corresponding to $r_2 = 4$ is $\underline{u}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. Then,

$x_1 = e^{\lambda} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $x_2 = e^{4\lambda} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ are linearly independent solutions of

$$\underline{x}' = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix} \underline{x}$$

23.1 Case: Distinct Real Roots

Example: $x_1' = x_1 + x_2$, $x_2' = 4x_1 + x_2$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Try $x(t) = ve^{\lambda t}$, we then get $Av = \lambda v$

Find lambdas: $\det(a - \lambda I) = \lambda^2 - 2\lambda - 3 = 0$ (by equation $\star$). We then get $\lambda_1 = -1, \lambda_2 = 3$

$\lambda_1 = -1$: $(A - \lambda_1 I)v_1 = 0$, therefore:

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_{1,1} \\ v_{2,1} \end{bmatrix} = 0 \Rightarrow 2v_{1,1} + v_{2,1} = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\lambda_2 = 3$: $(A - \lambda_2 I)v_2 = 0$, therefore:

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_{1,2} \\ v_{2,2} \end{bmatrix} = 0 \Rightarrow -2v_{1,2} + v_{2,2} = 0 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General Solution: by superposition:

$$x(t) = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} \Rightarrow \begin{cases} x_1 = c_1e^{-t} + c_2e^{3t} \\ x_2 = -2c_1e^{-t} + 2c_2e^{3t} \end{cases}$$

23.2 Case: Complex Conjugate Roots

If matrix A has complex conjugate eigenvalues $\alpha \pm i\beta$, then the corresponding eigenvectors are also complex conjugate

Two linearly independent real-valued solutions to $x' = Ax$ with complex conjugate eigenvalues are given by

$$x_1(t) = e^{\alpha t}[\cos(\beta t)a - \sin(\beta t)b]$$
$$x_2(t) = e^{\alpha t}[\sin(\beta t)a + \cos(\beta t)b]$$

Example: Consider the linear system where $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

Eigenvalues with corresponding eigenvectors are:

$$\begin{cases} r_1 = 1 & u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ r_2 = 1 + i & u_2 = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{a}} + i \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\underline{b}} \\ r_3 = 1 - i & u_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{cases}$$

Then, three linearly independent solutions are given by

$$x_1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2(t) = e^t (\cos t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix})$$

$$x_3(t) = e^t (\sin t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix})$$

24 Repeated Eigenvalues

Firstly we define:

algebraic multiplicity: $m_a(r) :=$ the number of times r as root of $p(r) = \det(A - rI)$

geometric multiplicity: $m_g(r) :=$ number (dimension) of corresponding LI eigenvectors (or Dim of eigenspace)

General, $m_a(r) \geq m_g(r)$. For example, the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \text{ has repeated eigenvalue } r = 2, m_a = 2, m_g = 1$$

In cases as such, we need to find a second solution of the form $x_2(t) = \xi t e^{rt} + \eta e^{rt}$

$$x_2'(t) = \xi(e^{rt} + tre^{rt}) + re^{rt}\eta$$

$$Ax_2 = te^{rt}A\xi + e^{rt}A\eta$$

$$x_2'(t) = Ax_2 \Rightarrow \begin{cases} (A - rI)\xi = 0 \\ (A - rI)\eta = \xi \quad \star \end{cases}$$

Similarly, for $m_a = 3$, we need a third solution of the form $x_3(t) = \frac{t^2}{2}e^{2t}\xi + te^{2t}\eta + e^{2t}\zeta$

Example: Solving $x' = Ax$ for a repeated eigenvalues problem

Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

First we find that $r = 2$ with $m_a(r) = 2$ (meaning it is a double root)

Then, we find the eigenvector $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ by $(A - 2I)u = 0$

Solution 1: $x_1 = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution 2: $x_2 = (\xi t + \eta)e^{2t}$

By $\star$ and letting $\xi = u$: $(A - 2I)\eta = u$:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To solve, we get $\eta = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Solution: $x(t) = c_1e^{rt}v + c_2e^{rt}(tv + w)$

$x(t) = c_1e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left[(te^{2t})\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]$

25 Stability of Autonomous Systems

Can we understand the behaviour of systems of ODEs without solving them? Let's look at a differential equation in a small region, locally, for a linear part of the nonlinear equation

Consider the system:

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

Note that t is the independent variable, and the given functions f and g do not depend explicitly on the independent variable

Definition: Systems of ODEs that do not depend on the independent variable are called **autonomous systems**

We can develop a parametric representation of a curve in the xy-plane as:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

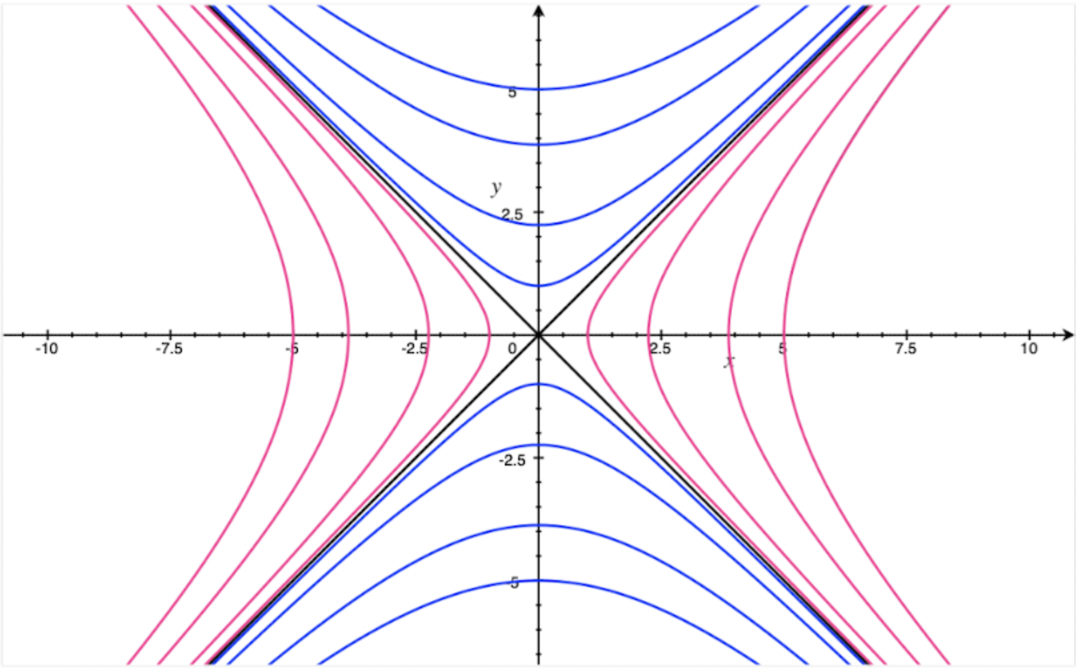
and by the chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{g}{f}$. The latter ODE is called the **phase plane equation**. Solving this is hard, but critical points (oftentimes the origin $(0,0)$) are solutions, called **equilibria**

Example: Consider the linear homogeneous system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x \end{cases} \qquad \text{Here, f = y, g = x}$$

We can find one equilibrium at $(0,0)$, and try to say something about the behaviour without solving the system

The phase plane equation is $\frac{dy}{dx} = \frac{g}{f} = \frac{x}{y}$, we can solve this DE: $y^2 - x^2 = c$. Therefore, the trajectories are hyperbolas



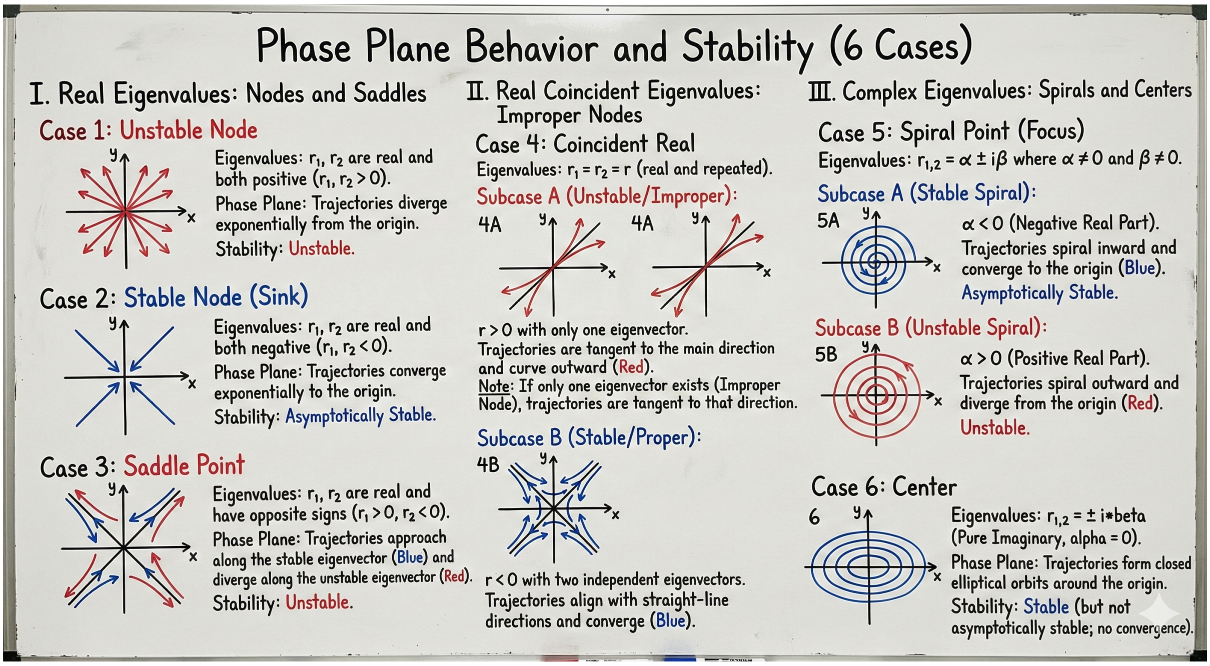
Definition: A **linear autonomous system in the plane** has the form

$$\begin{cases} x'(t) = a_{11}x + a_{12}y + b_1 \\ y'(t) = a_{21}x + a_{22}y + b_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Making the change of variables: $x_1 := a_{11}x + a_{12}y + b_1$, $x_2 := a_{21}x + a_{22}y + b_2$ a linear autonomous system can be rewritten as:

$$\begin{cases} x_1'(t) = a_{11}x_1 + a_{12}x_2 \\ x_2'(t) = a_{21}x_1 + a_{22}x_2 \end{cases} \Leftrightarrow x' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x$$

26 Phase Plane Behaviour



Stable System: All nearby solutions remain close to the equilibrium point for all time.

Unstable System: Some nearby solutions eventually move away from the equilibrium point.

Asymptotically Stable System: Solutions starting near the equilibrium approach and converge to the equilibrium as time increases.

Proper node: Real repeated eigenvalue with two independent eigenvectors; trajectories align with straight-line directions.

Improper node: Real repeated eigenvalue with only one eigenvector; some trajectories curve and are tangent to the main direction.

Saddle point: Real eigenvalues with opposite signs; trajectories approach along one direction and depart along another.

Spiral (focus): Complex eigenvalues with nonzero real part; trajectories spiral inward (stable) or outward (unstable).

Center: Pure imaginary eigenvalues; trajectories form closed orbits around equilibrium; no convergence.

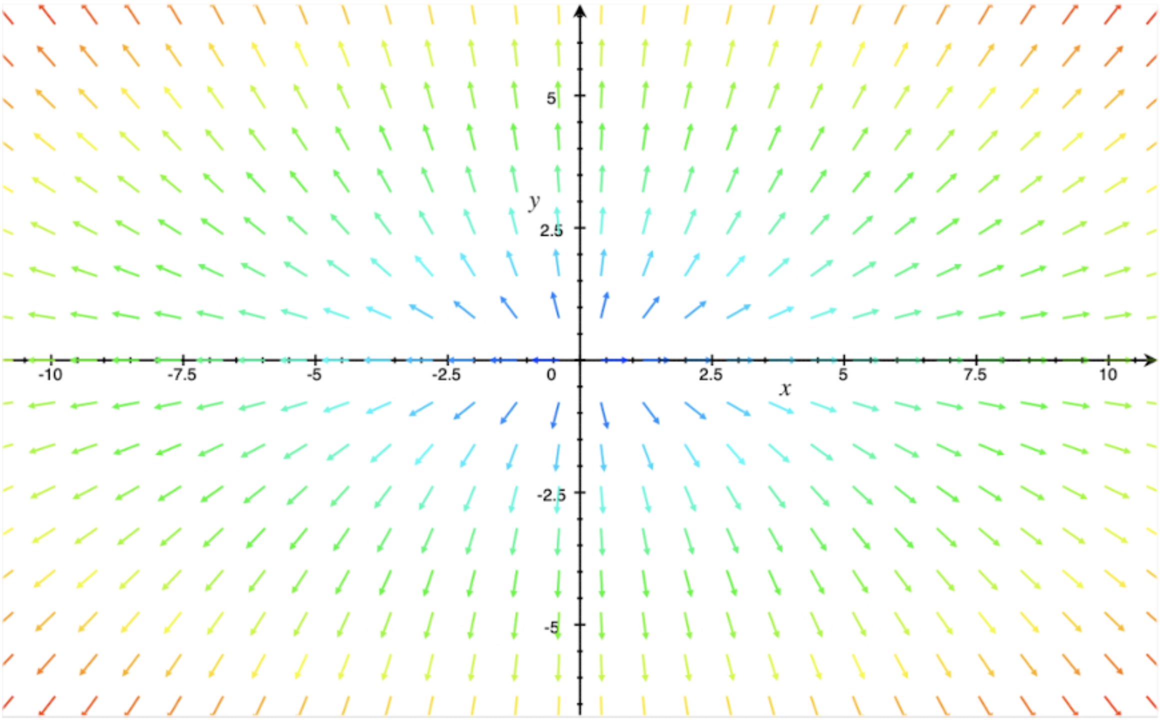
There are 6 cases of long-time behaviour of solutions in the phase plane:

26.1 Stability Cases

Case 1: $r_1 \neq r_2$ real and $r_1, r_2 > 0$

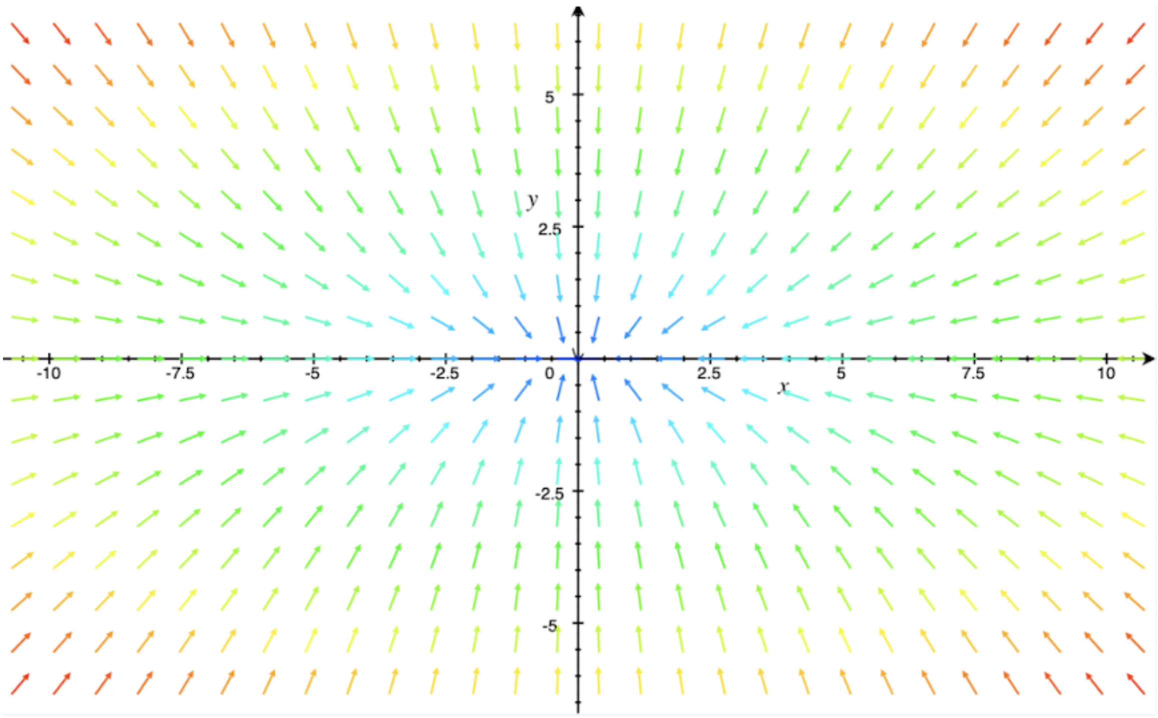
- both eigenvalues are positive real numbers
- the phase plane shows an unstable improper node
- trajectories diverge exponentially from the origin

- the equilibrium point is unstable



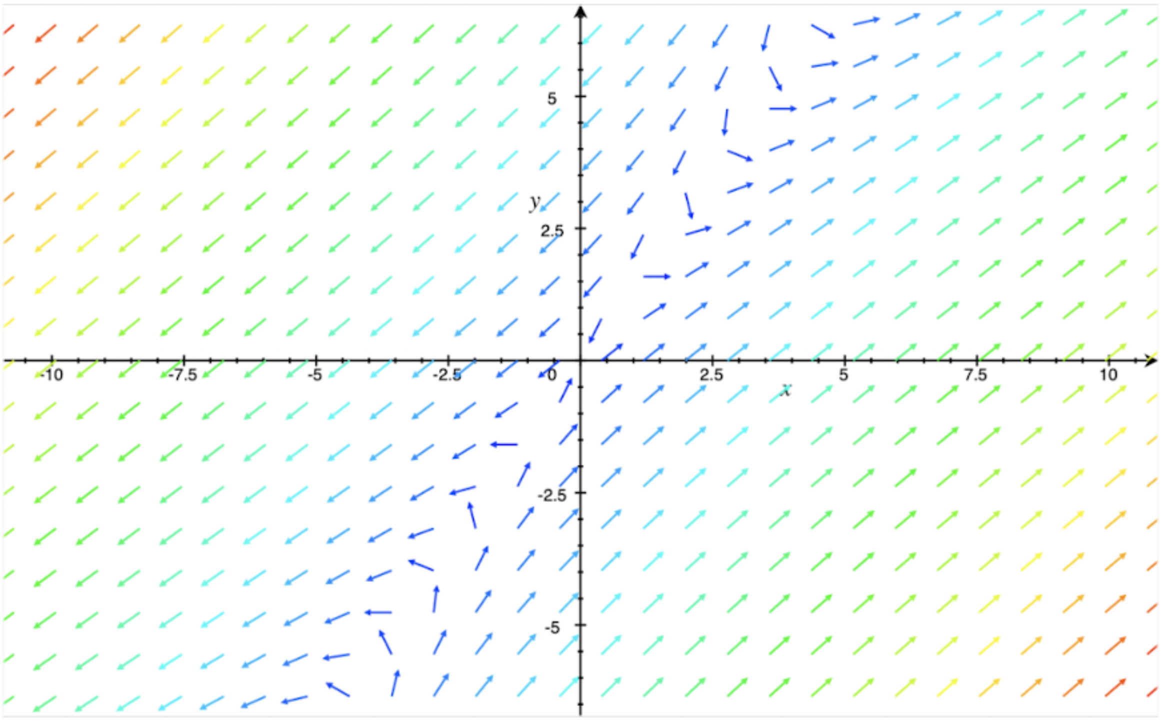
Case 2: $r_1 \neq r_2$ **real and** $r_1, r_2 < 0$

- both eigenvalues are negative real numbers
- the phase plane shows a stable node (sink)
- trajectories converge exponentially to the origin
- the equilibrium point is asymptotically stable



Case 3: $r_1 \neq r_2$ **real and opposite signs**

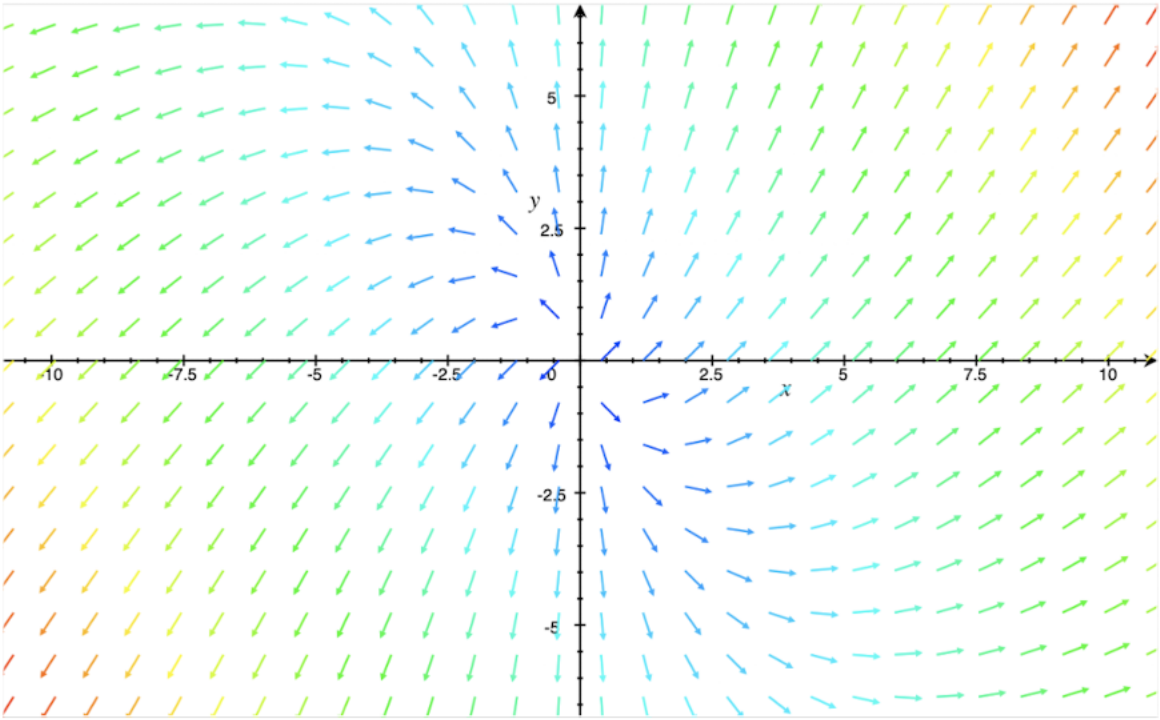
- one eigenvalue is positive, the other negative
- the origin is a saddle point and is unstable
- trajectories approach zero along the eigenvector corresponding to the negative eigenvalue and diverge along the positive eigenvalue direction



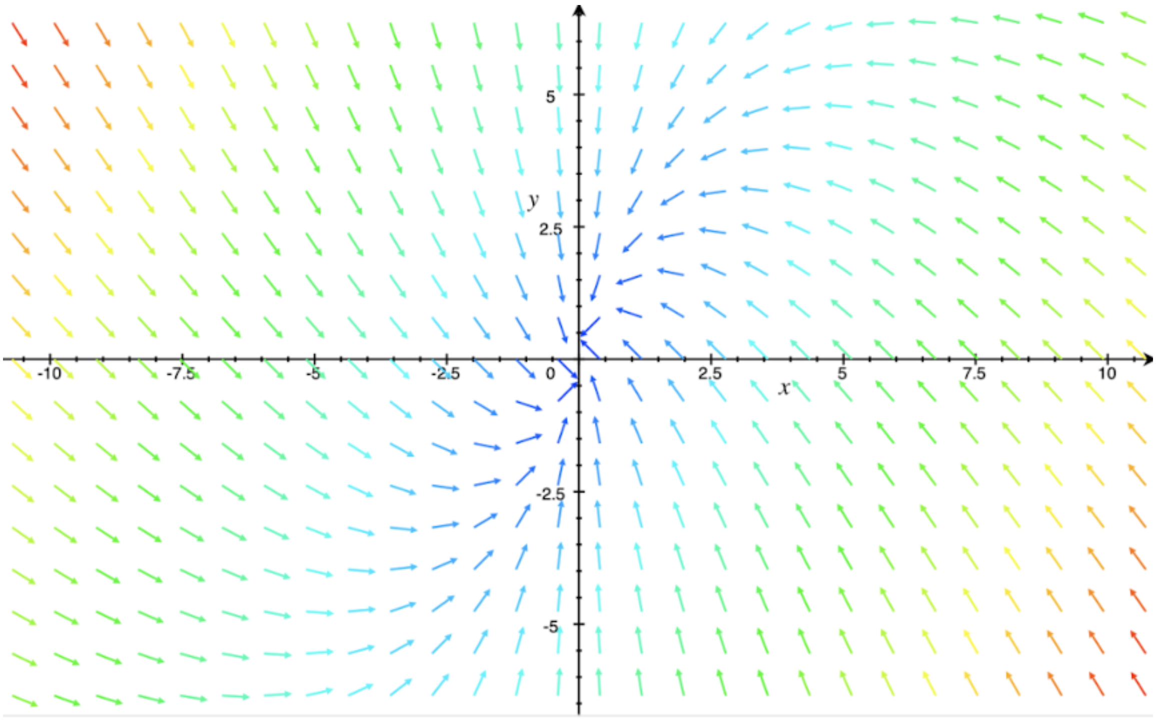
Case 4: $r_1 = r_2$ **real**

- eigenvalues coincide and are real
- Cases:
  - If $r > 0$, solutions move away from origin
  - If $r < 0$, solutions converge toward origin
  - If $r > 0$ with only one eigenvector, solutions behave as in $\star$
  - If $r < 0$ with only one eigenvectors, solutions behave as in ∇
- stability depends on the sign of the eigenvalue; negative for stable node, positive for unstable node

$\star$



∇

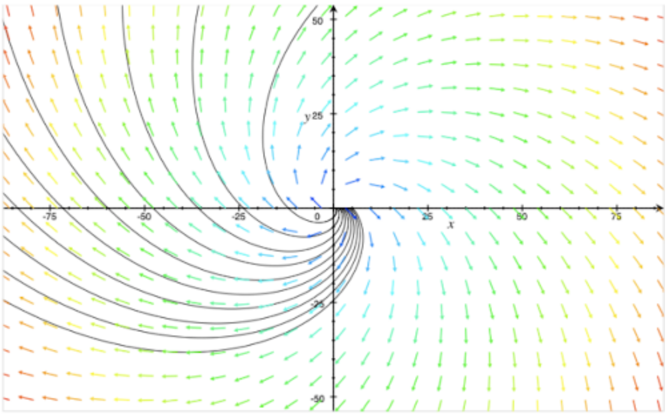


Case 5: $r_{1,2} = \alpha \pm i\beta$ **with** $\alpha, \beta \in \mathbb{R} \setminus \{(0,0)\}$

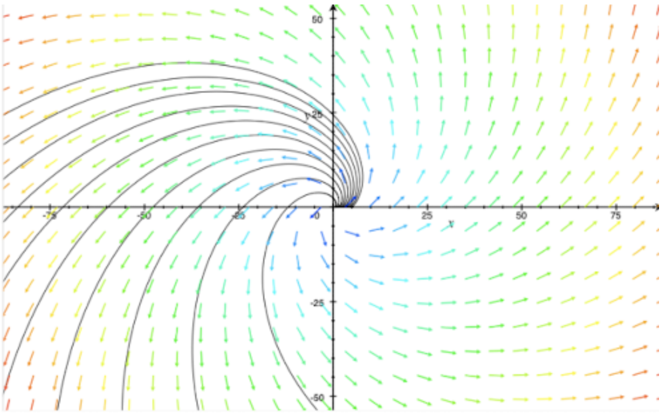
For the general system $x' = Ax$ with $A = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$, we can solve the system with $\rho(t) = \rho(0)e^{\lambda t}$, where $\rho^2 = x^2 + y^2$ in polar coordinates

- the phase plane shows a spiral point at $(0, 0)$
- it is asymptotically stable if $\alpha > 0$, unstable if $\beta > 0$
- If $\beta > 0$, then θ decreases as time increases, and vice versa

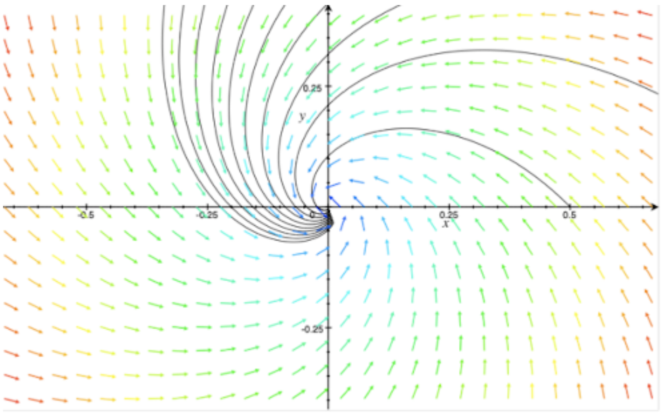
- If $\alpha > 0$, then solutions diverge, and vice versa



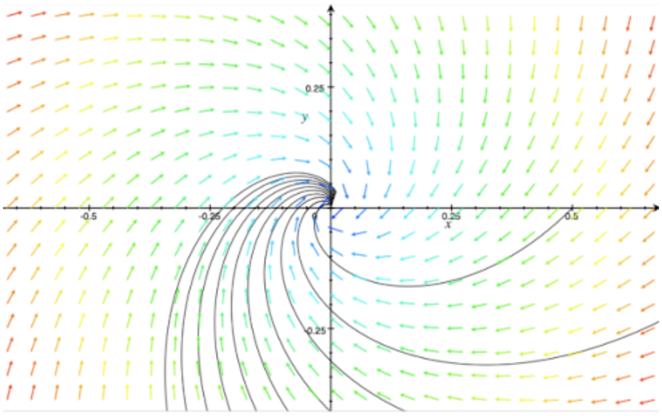
Trajectories for $\alpha > 0, \beta > 0$



Trajectories for $\alpha > 0, \mu < 0$



Trajectories for $\alpha < 0, \mu < 0$



Trajectories for $\alpha < 0, \beta > 0$

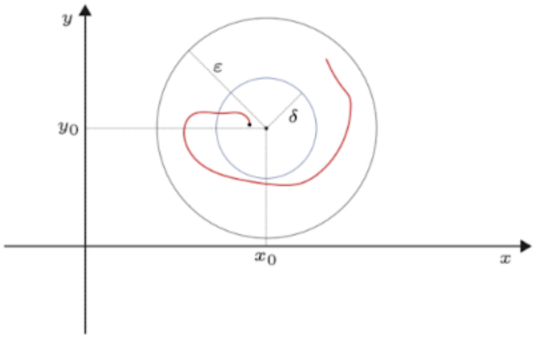
Case 6: $r_{1,2} = \pm i\beta$ **with** $\beta \neq 0$

The trajectories are periodic with period $T = \frac{2\pi}{\beta}$ and bounded. $(0, 0)$ is a stable center point

27 Autonomous Systems & Stability

Critical point: A point $\underline{x_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$ is a **critical point** of the autonomous system $x' = F(x)$ if and only if $F(x_0) = 0$. For a linear system $x' = Ax$, a critical point is a solution where the RHS is zero: $Ax = 0$. Ex: the origin is always a critical point because $A \cdot 0 = 0$

Stable point: A critical point x_0 is said to be **stable** if and only if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $x(t)$ of $x' = F(x)$ satisfying at $t = 0$



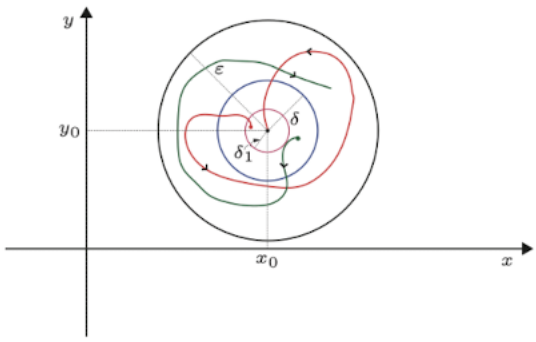
Unstable point: A critical point x_0 is said to be **unstable** if it is not stable, i.e., $\exists\varepsilon > 0$ such that for every $\delta > 0$, $\exists\tau > 0$ and a solution $x(t)$ of $x' = F(x)$ with $||x(t) - x_0|| < \delta$ such that

$$||x(\tau) - x_0|| > \varepsilon$$

Asymptotically stable point: A critical point x_0 is said to be **asymptotically stable** if the following two properties are satisfied:

1. x_0 is stable
2. $\exists\delta_1 > 0$ such that if a solution $x(t)$ of $x' = F(x)$ satisfies $||x(t) - x_0|| < \delta_1$, then

$$\lim_{t\to+\infty} x(t) = x_0$$



Theorem: Let $A \in R^{2\times 2}$ with $\det A \neq 0$ and let r_1, r_2 be the eigenvalues of A. For a linear system with constant coefficients $x' = Ax$, the critical point $(0, 0)$ is

- asymptotically stable if r_1 and r_2 are both real and negative, or complex conjugate with negative real part
- stable but not asymptotically stable if r_1 and r_2 are purely imgainary
- unstable if r_1 and r_2 are both real and positive or with opposite sign, or complex conjugate with positive real part

Definition: The point (x_0, y_0) is said to be **isolated critical point** for the autonomous system $x' = F(x)$ if there exists a circle centred at (x_0, y_0) within which there are no other critical points.

Lemma: If an autonomous system $x' = F(x)$ admits a finite number of critical points, then each critical point is isolated. Contrarily, if we find ∞ critical points (ex: the line of critical points $y = -x$), each critical point is not isolated as any circle centred at $(0, 0)$ will contain other critical points.

28 Locally Linear Systems

Definition: Consider the autonomous system $x' = Ax + g(x)$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad g(x) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

The autonomous system is said to be **locally linear in a neighbourhood of the critical point** $(0,0)$ if:

- $F(x,y)$ and $G(x,y)$ are continuous in a disk around $(0,0)$
- $(0,0)$ is an isolated critical point of $x' = Ax + g(x)$
- $\det A = ad - bc \neq 0$ (i.e., $(0,0)$) is an isolated critical point for the linear system $x' = Ax$
- $$\lim_{||x|| \to 0} \frac{||g(x)||}{||x||} = 0$$

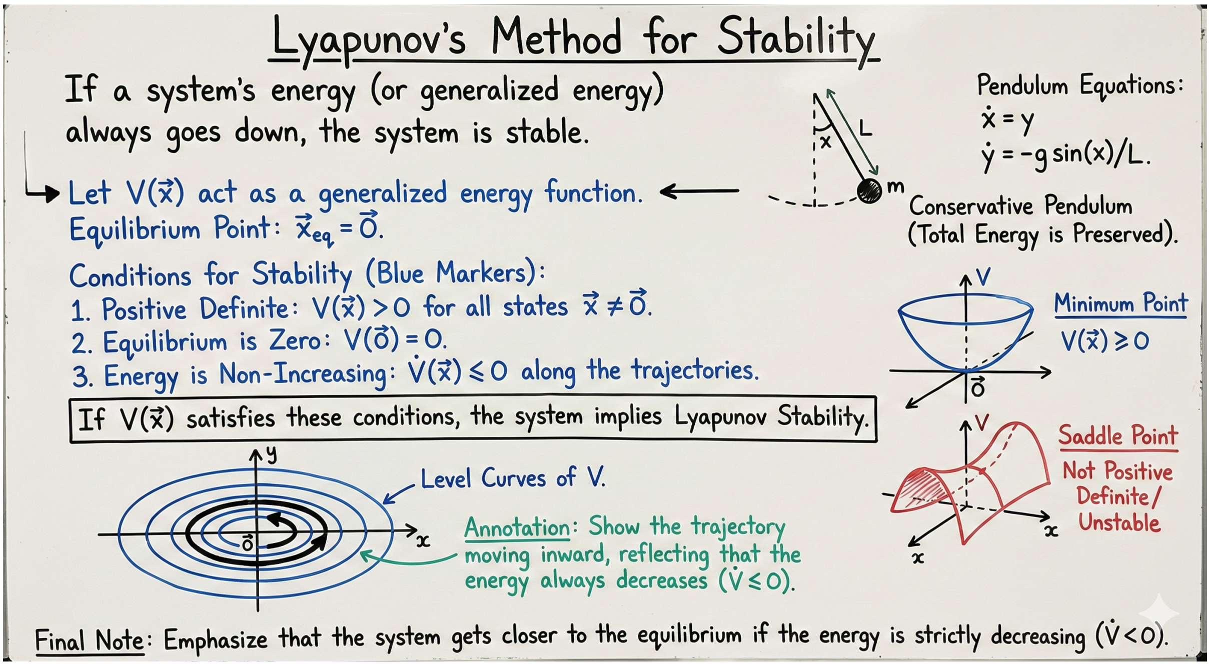
Example: Consider the autonomous system:

$$\begin{cases} x' = x - x^2 - xy \\ y' = \frac{1}{2}y - \frac{3}{4}xy - \frac{1}{4}y^2 = 0 \end{cases} \quad \Leftrightarrow \underbrace{x' = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}_{\text{linear part}} x + \underbrace{\begin{bmatrix} -x^2 - xy \\ -\frac{3}{4}xy - \frac{1}{4}y^2 \end{bmatrix}}_{\text{non-linear part}}$$

- $F = -x^2 - xy$ and $G = -\frac{3}{4}xy - \frac{1}{4}y^2$ are continuous everywhere, and in particular in any disk around $(0,0)$
- $(0,0)$ is an isolated critical point of the system since the nonlinear system admits only a finite number of critical points
- $\det A = \frac{1}{2} \neq 0$, therefore the only solution to $Ax = 0$ is $x = 0$ (the origin). So, the origin is an isolated critical point for the linear system
- By switching to polar coordinates, we get that $F(x,y) \Rightarrow F(\theta, r) = -r^2(\cos^2\theta + \cos\theta\sin\theta)$. Also, $|| \begin{bmatrix} x \\ y \end{bmatrix} || = \sqrt{x^2 + y^2} = r$Then, as $r \to 0$, $\frac{F(\theta,r)}{||x||} = 0$. We can do this similarly for $G(x,y)$

Eigenvalues of $\underline{A}$	*Stability of* $(0,0)$
$r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$, r_1, $r_2 > 0$	*unstable improper node*
$r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$, r_1, $r_2 < 0$	*asymptotically stable improper node*
$r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$, *opposite sign*	*unstable saddle point*
$r_1 = r_2 \in \mathbb{R}$, r_1, $r_2 > 0$	*unstable either proper or improper node*
$r_1 = r_2 \in \mathbb{R}$, r_1, $r_2 < 0$	*asymptotically stable either proper or improper node*
$r_{1,2} = \lambda \pm i\mu$ $\lambda, \mu \in \mathbb{R}$, $\lambda > 0$	*unstable spiral point*
$r_{1,2} = \lambda \pm i\mu$ $\lambda, \mu \in \mathbb{R}$, $\lambda < 0$	*asymptotically stable spiral point*
$r_{1,2} = i\mu$ $\lambda, \mu \in \mathbb{R}$	*indeterminate*

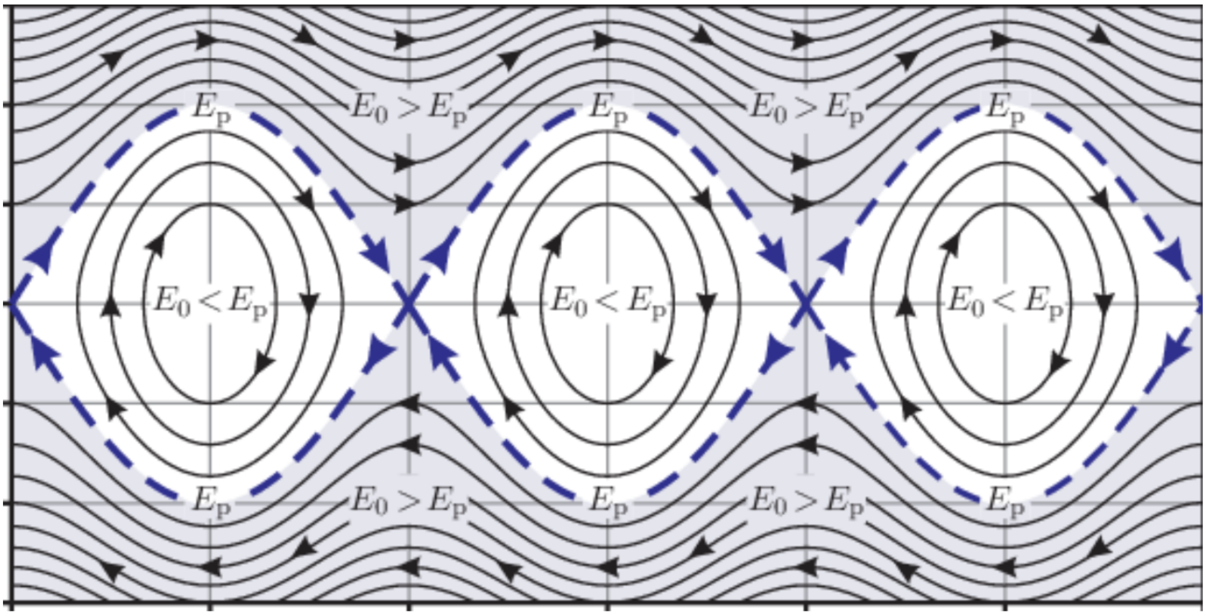
29 Lyapunov's Method



We know that a conservative pendulum has its total energy preserved (because of no friction and no external forces).

$$
\begin{cases} x' = y \\ y' = g \sin x / L \end{cases}
$$

We find that this system has stable centre points.



Lyapunov realized that if you can find any function that behaves like energy, the same logic holds as if you modelled the real system. If we let $V(x)$ act as a generalized energy, we want:

1. $V(x) > 0$ for all states that are not the equilibrium
2. $V(0) = 0$, meaning the equilibrium has zero energy
3. $\dot{V}(x) \leq 0$, meaning the energy is never increasing along the trajectories of the system

If the energy always goes down, the system gets closer and closer to the equilibrium. This directly implies Lyapunov stability.

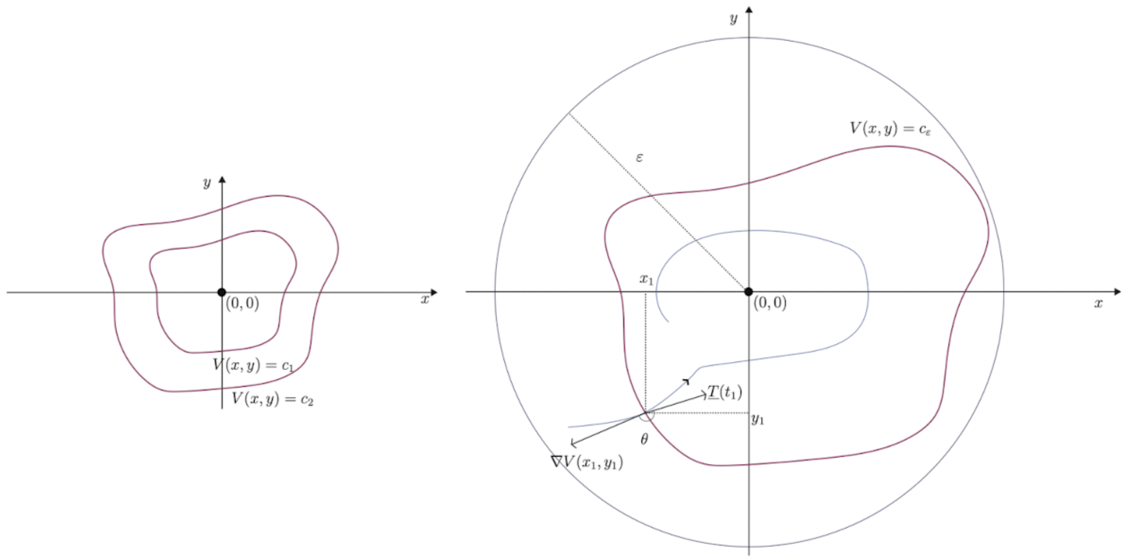
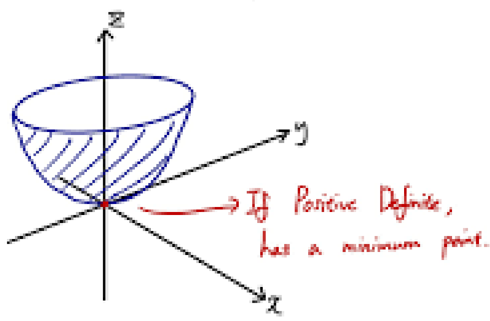


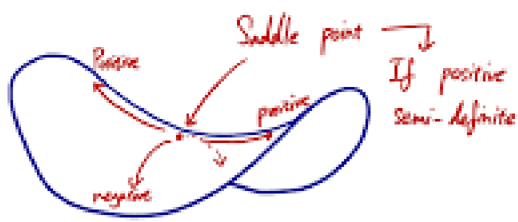
Figure 6.11: Right Figure: Level curves of $V(x, y)$. Left Figure: Geometric interpretation of Lyapunov's Theorem.

Definitions:

- V is positive definite on D if $V(0, 0) = 0$ and $V(x, y) > 0 \forall (x, y) \in D \setminus \{(0, 0)\}$
- V is positive semi-definite on D if $V(0, 0) = 0$ and $V(x, y) \geq 0$
- V is negative definite on D if $-V$ is positive definite
- V is negative semi-definite on D if $-V$ is positive semi-definite

Geometric Interpretation





Lyapunov Stability Theorem:

Let $(0,0)$ be an isolated critical point of the system $x' = f(x,y), y' = g(x,y)$ and let $V(x,y)$ be a function defined on domain D containing $(0,0)$ where we suppose that $V > 0$, then:

1) if $\dot{V} < 0$ on D then $(0,0)$ is asymptotically stable
2) if $\dot{V} \leq 0$ on D, then $(0,0)$ is stable

Lyapunov Instability Theorem:

Assume $(0,0)$ is isolated critical point $\in D$ of the previous system, and V is defined on D and $V(0,0) = 0$, then:

1) Suppose $\forall$ disk around $(0,0)\exists(x,y)$ s.t. $V(x,y) \geq 0$ and $\dot{V} > 0$. Then $(0,0)$ is unstable
2) Suppose $\forall$ disk around $(0,0), \exists(x,y)$ s.t. $V(x,y) < 0$ and $\dot{V} < 0$, then $(0,0)$ is unstable

Example: $\begin{cases} x' = -x^3/2 + 2xy^3, & \text{is not locally linear near } (0,0) \\ y' = -y^3 & \text{because } J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$

$\exists a, b$ s.t. $V(x,y) = ax^2 + by^2$ is a Lyapunov function

- $V(0,0) = 0$
- Assume $a, b > 0$, then $V(x,y) > 0$

Show that $\dot{V} < 0$ if $b > 2a > 0$

$$\dot{V} = \partial V/\partial x \left(-x^3/2 + 2xy^3\right) + \partial V/\partial y \left(-y^3\right)$$

By computing derivatives, we get:

$$\dot{V} = -ax^4 + 4ax^2y^3 - 2by^4$$

Let $u = x^2$, we get: $\dot{V} = -au^2 + 4auy^3 - 2by^4$

Since $-au^2$ is negative, the polynomial points downwards and to have $\dot{V} < 0$, the discriminant D must be < 0

Compute: $b^2 - 4ac \Rightarrow \left(4ay^3\right)^2 - 4(-a)\left(-2by^4\right)$

$$D = 16a^2y^6 - 8aby^4 \Rightarrow 8ay^4\left(2ay^2 - b\right)$$

Then, $D < 0$ if and only if:

$$2ay^2 - b < 0 \Rightarrow b > 2a > 0$$

30 How to Solve an ODE — Step by Step

30.1 General Concepts

- **ODE form:** $F(x, y, y', \dots, y^{(n)}) = 0$
- **General solution:** family with arbitrary constants.
- **Particular solution:** found using initial/boundary conditions.
- **Explicit solution:** $y = f(x)$.
- **Implicit solution:** equation involving x and y, not isolated.

30.2 Existence and Uniqueness Theorem

1. Write the IVP in Standard Form

- Make sure your equation is in the form:

$y' = f(x, y), \quad y(x_0) = y_0$

2. Check Continuity of $f(x, y)$

- Is $f(x, y)$ continuous near (x_0, y_0)?

Look for division by zero, square roots of negatives, or other undefined operations

If $f(x, y)$ is a ratio of polynomials, the only condition to be imposed is that its denominator must be nonzero

3. Compute $\frac{\partial f}{\partial y}$ and check continuity

4. State Your Conclusion

- If both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0), the IVP admits a **unique local solution**.

- If either is not continuous, uniqueness or existence may fail at that point.

Tip: Always check for points where the denominator could be zero or where the function is not defined. That's where continuity can fail!

30.3 First Order ODEs

30.3.1 1. Separation of Variables

Form: $\frac{dy}{dx} = g(x)\,p(y)$

Steps:

1. Rearrange: $\frac{1}{p(y)}\,dy = g(x)\,dx$
2. Integrate both sides.
3. Get implicit solution: $H(y) = G(x) + C$

4. Solve for y if possible.

Example: $\frac{dy}{dx} = x \cdot y$

- Step 1: $\frac{1}{y} dy = x\,dx$
- Step 2: $\int \frac{1}{y} dy = \int x\,dx$
- Step 3: $\ln|y| = \frac{x^2}{2} + C$
- Step 4: $y = Ae^{x^2/2}$ where $A = e^C$.

30.3.2 2. Integrating Factor (Variable Coefficients)

Form: $y' + P(x)y = Q(x)$

Steps:

1. Put to standard form
2. Find $p(x)$ and compute integrating factor $\mu(x) = e^{\int P(x)dx}$
3. Multiply ODE by μ: $\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$, LHS should be $\frac{d}{dx}[\mu(x)y]$
4. Integrate: $\mu y = \int \mu Q\,dx + C$
5. Solve for $y(x) = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x)dx + C \right)$

Example: $y' - 2y = e^{3x}$

- Step 1: $\mu(x) = e^{\int -2dx} = e^{-2x}$
- Step 2: Multiply: $e^{-2x}y' - 2e^{-2x}y = e^x$
- Step 3: $\frac{d}{dx}[e^{-2x}y] = e^x$
- Integrate: $e^{-2x}y = e^x + C$
- Step 4: $y = e^{3x} + Ce^{2x}$.

30.3.3 3. Exact Equations

Form: $M(x,y)dx + N(x,y)dy = 0$ with exactness condition $M_y = N_x$.

Steps:

1. Compute M_y and N_x — check $M_y = N_x$.
2. Find potential F with $F_x = M$ and $F_y = N$.
3. Solution: $F(x,y) = C$.

Example: $(2xy)dx + (x^2)dy = 0$

- Step 1: $M = 2xy,\ N = x^2.\ M_y = 2x,\ N_x = 2x$
- Step 2: Integrate M w.r.t x: $F = x^2y + h(y)$. $F_y = x^2 + h'(y)$ must equal $N = x^2$ $\rightarrow h'(y) = 0 \rightarrow h$ constant.
- Step 3: $F = x^2y + C$.

30.3.4 4. Picard Iteration (Approximation)

Form: $y' = f(x,y),\ y(x_0) = y_0$

Steps:

1. Write integral form: $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$.
2. Iterate: $\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt$.
3. Continue until sequence converges.

Example: $y' = y,\ y(0) = 1$

- $\phi_0(x) = 1$
- $\phi_1(x) = 1 + \int_0^x 1\ dt = 1 + x$
- $\phi_2(x) = 1 + \int_0^x (1+t)dt = 1 + x + x^2/2$
- Continues toward e^x.

30.4 Second Order ODEs

30.4.1 1. Characteristic Equation (Constant Coefficients Homogeneous)

Form: $ay'' + by' + cy = 0$

Steps:

1. Compute characteristic polynomial: $ar^2 + br + c = 0$
2. Discriminant $\Delta = b^2 - 4ac$.
3. Solve for r and write general solution:
    - **Distinct real:** $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
    - **Repeated:** $y = (C_1 + C_2 x)e^{rx}$
    - **Complex:** $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Example: $y'' - y = 0$

- $r^2 - 1 = 0 \to r = 1$.
- Solution: $y = C_1 e^x + C_2 e^{-x}$.

Example: $y'' + 4y' + 5 = 0$

- $r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2}, r = -2 \pm i$
- Solution: $y(x) = e^{-2x}(c_1 \cos x + c_2 \sin x)$

30.4.2 2. Method of Undetermined Coefficients

Form: nonhomogeneous $ay'' + by' + cy = f(x)$

Steps:

1. Solve homogeneous for y_c.
2. Guess y_p based on $f(x)$.
3. Adjust guess if it matches y_c (multiply by x as needed).
4. Plug in, solve coefficients.

Example: $y'' - y = x$

- Step 1: Homogeneous $\to y_c = C_1 e^x + C_2 e^{-x}$.

- Step 2: For $f(x) = x$, guess $y_p = Ax + B$.
- Step 3: No overlap → plug into ODE: $0 - (Ax + B) = x \to A = -1, B = 0$.
- Solution: $y = C_1e^x + C_2e^{-x} - x$.

30.4.3 3. Variation of Parameters

Form: $y'' + P(x)y' + Q(x)y = R(x)$

Steps:

1. Solve y_c using characteristic method.
2. Use formula: $u_1' = -\frac{y_2R}{W}$, $u_2' = \frac{y_1R}{W}$.
3. Integrate u_1', u_2', then $y_p = u_1y_1 + u_2y_2$.

Example: $y'' - y = e^{2x}$

- Step 1: Homogeneous: $y_c = C_1e^x + C_2e^{-x}$.
- Step 2: Take $y_1 = e^x, y_2 = e^{-x}$, $W = -2$.
- $u_1' = -\frac{e^{-x}e^{2x}}{-2} = -\frac{e^x}{-2} = \frac{e^x}{2}$.
- Integrate: $u_1 = \frac{e^x}{2}$.
- $u_2' = \frac{e^xe^{2x}}{-2} = -\frac{e^{3x}}{2}$, integrate: $u_2 = -\frac{e^{3x}}{6}$.
- Step 3: $y_p = \frac{e^x}{2}e^x - \frac{e^{3x}}{6}e^{-x} = \frac{e^{2x}}{2} - \frac{e^{2x}}{6} = \frac{e^{2x}}{3}$.

Solution: $y = C_1e^x + C_2e^{-x} + \frac{e^{2x}}{3}$.

30.4.4 Particular Solution Guess Table

Forcing Function	Standard Guess	If matches homogeneous?
Ax^n	polynomial deg n	multiply by x or higher
Ae^{kx}	Ae^{kx}	multiply by x or higher
$A\sin(kx), B\cos(kx)$	$A\cos(kx) + B\sin(kx)$	multiply by x or higher
x^re^{kx}	$(Ax^r + \dots)e^{kx}$	multiply by x^m to ensure independence

---

Mnemonic: If your guess is a solution to the homogeneous part (same root/frequency), multiply by x until it's independent.

30.5 Laplace Transform

Definition:

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

Where:

- t: original (time) variable
- s: new (complex frequency) domain variable

Why Laplace?

- Converts ODEs (differential equations) into algebraic equations in the s-domain, which are typically easier to solve.
- Handles initial conditions naturally.
- Powerful for discontinuous/impulse forcing functions.

Main Steps ("Method Note"):

1. **Take the Laplace transform** of both sides of the DE, using linearity and Laplace rules for derivatives.
2. **Substitute initial conditions** to simplify the resulting algebraic equation for $Y(s) = \mathcal{L}\{y(t)\}(s)$.
3. **Solve for** $Y(s)$ (algebraic manipulation; often involves partial fraction decomposition).
4. **Invert the Laplace transform** to return to the time domain: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Core Properties:

- **Linearity**: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
- **First Derivative Rule**: $\mathcal{L}\{y'\} = sY(s) - y(0)$
- **Second Derivative**: $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$
- **Shifting**: $\mathcal{L}\{e^{at}f(t)\} = F(s - a)$
- **Step Function (Heaviside)**: $\mathcal{L}\{u_c(t)f(t - c)\} = e^{-cs}F(s)$

Existence (Conditions to Use Laplace):

- $f(t)$ is "piecewise continuous."
- "Exponential order": $|f(t)| < Me^{at}$ for some $M, a, t > T$
- See for details in your lecture notes.

Common Transforms:

$$\begin{aligned}
1 &\longrightarrow \frac{1}{s}, \quad s > 0 \\
t^n &\longrightarrow \frac{n!}{s^{n+1}}, \; s > 0 \\
e^{at} &\longrightarrow \frac{1}{s - a}, \quad s > a \\
\sin(\omega t) &\longrightarrow \frac{\omega}{s^2 + \omega^2} \\
\cos(\omega t) &\longrightarrow \frac{s}{s^2 + \omega^2}
\end{aligned}$$

Solving ODEs with Laplace ("Full Algorithm"):

1. **Given:** $y'' + ay' + by = f(t)$, with $y(0)$, $y'(0)$.
2. **Transform each term:** e.g., y'' becomes $s^2Y(s) - sy(0) - y'(0)$.
3. **Insert initial conditions** (from the problem statement).
4. **Solve for** $Y(s)$.
5. **Invert** using tables or partial fractions.

Special Case: Time Shift (Discontinuous Inputs):

- Use Heaviside unit step and shifting properties.
- "Window functions" help encode functions defined piecewise in time .

Laplace Transform: Summary Table

Function $f(t)$	Laplace $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$\delta(t-c)$	e^{-cs}
$f(t-a)u_a(t)$	$e^{-as}F(s)$