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MTHE 237 - Lecture Notes

Differential Equations for Engineering Science

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Contents

1 Mathematical modelling & direction field

1.1 How to solve

In order to provide a mathematical model describing physical phenomena, we can follow the following steps:

1. Fix our dependent and independent variables and set a frame of references for their measures
2. Choose convenient units of measurements
3. Find the underlying principle governing the motion of the object
4. Rewrite the above relation in terms of the variable we have chosen at step 1
5. Find a solution by integrating both sides
6. Add side conditions to eliminate constants and to find a **unique solution**

1.1.1 Example

$$F = ma$$
$$a = \frac{dv}{dt} = \frac{d^2h}{dt^2}$$
$$\frac{d^2h}{dt^2} = -g$$
$$v(t) = \frac{dh}{dt} = -\int gdt = -gt + c_1$$
$$h(t) = \int v(t)dt = -\frac{1}{2}gt^2 + c_1t + c_2$$
$$h(0) = 1, v(0) = 0$$
$$h(t) = -\frac{1}{2}gt^2 + 1$$

1.2 Population Models

Given some internal and external conditions: $\frac{dp}{dt}$ = growth rate - death rate

1.2.1 Death rate = 0

$\frac{dp}{dt} = k_1p$, where $p(0) = p_0$, and $k_1 > 0$ is the proportionality factor for the growth rate

$$\frac{1}{p}dp = kdt$$
$$\ln p = kt + C$$
$$p(t) = Ce^{kt}$$

1.3 Malthusian and competitors

A Malthusian model is a general model for population with rates k_1 and k_2 proportional to p

We can consider more factors to the death rate, called **competitors**, with two-party interactions modelled by $\frac{p(p-1)}{2}$

$\frac{dp}{dt} = k_1 p - k_3 \frac{p(p-1)}{2}$

We can rearrange the terms to find an equation in the form of a **logistic model**

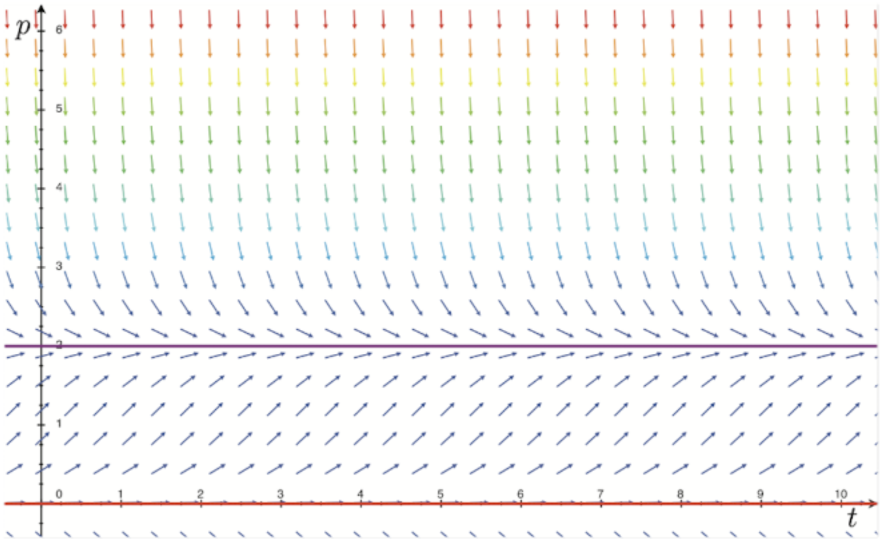
$\frac{dp}{dt} = -\frac{k_3}{2}\left(p^2 - \left(q + \frac{2k_1}{k_3}\right)p\right)$

which takes the form

$\frac{dp}{dt} = -Ap(p - p_1)$, where $A = \frac{k_3}{2}$ and $p_1 = 1 + \frac{2k_1}{k_3}$

1.4 Direction Fields

For $\frac{dp}{dt} = p(2-p)$, $\frac{dp}{dt} > 0 \Leftrightarrow 0 < p(t) < 2$



$p = 0, 2$ are equilibria

2 Terminology & classification of differential equations

A differential equation is an equation involving an independent variable (e.g., x, t), a dependent variable, and derivatives of the dependent variables

2.1 Examples

$y' = y + x$, y is dependent and x is independent

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x - 2y$, where x, y and independent and the function $u(x, y)$ is dependent

2.2 Types of differential equations

A differential equation with derivatives with respect to one variable is an ODE

A differential equation with partial derivatives with respect to two or more variables is a PDE

2.3 Ordinary differential equations

The general form of an ODE is given by $F(x, y, y', \dots, y^{(n)}) = 0$

The **order** of a differential equation is the order of the highest derivative appearing in the equation

An ODE is **linear** if: 1. $y, y', y'', \dots$ appear only to the first power 2. No product like yy' exist 3. Coefficients $a_i(x)$ can be constants or functions of x, but not of y

A linear ODE of order n has the form: $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 x\frac{dy}{dx} + a_0(x)y = f(x)$

2.4 Confirming a solution of an ODE

A function $\phi(x)$ is a solution of an n-th order ODE on some interval $I = (a, b)$ if 1. ϕ is n-times differentiable on I 2. It satisfies the ODE for every $x \in I$

Consider $y'' + \frac{2}{x^2}y = 0$ and the function $\phi(x) = x^2 - \frac{1}{x}$

We can confirm that its derivatives are continuous for all $x \neq 0$ and satisfy the ODE

Thus, the function ϕ is a solution of the given ODE in $(0, \infty)$

2.4.1 Superposition principle for linear ODEs

Say we have an n-th order linear **homogeneous** ODE $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 x\frac{dy}{dx} + a_0(x)y = 0$ and suppose $\phi_1(x), \phi_2(x), \dots, \phi_k(x)$ are solutions.

The theorem says any linear combination $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_k\phi_k(x)$ is also a solution in I for any choice of arbitrary constants, and the set of solutions is **closed under linear combinations**

These $\phi_k(x)$ solutions, unlike the next example, form a **vector space**, closed under linear combinations

Now, consider a **non-homogeneous** ODE when $f(x) \neq 0$, the linear combination property fails, and only holds in the homogeneous case

3 Initial value problems and the Existence and Uniqueness Theorem for 1st order ODEs

Given the n-th order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ in some interval I, the following two side conditions can be appended to the equation

1. **Initial Conditions:** dependent variable and all its derivatives up to order $n-1$ are specified at the same point $x_0 \in I$
2. **Boundary Conditions:** only applicable to PDEs

The problem of finding a solution to an ODE in an interval I containing x_0 and *such that the initial conditions are satisfied* is called **initial boundary value problem (IVP)**

3.1 Solving ODEs from implicit relations (template)

Example: given the implicit relation $x_2 + y^2 = 1, x \in (-1, 1)$, we want to see if $y(x)$ is a solution to the ODE $y' = -\frac{x}{y}$

1. **Differentiate w.r.t. dependent variable**

$2x + 2yy' = 0$

2. **Solve for y'**

$y' = -\frac{x}{y}$, this is the ODE

3. **Solve implicit relation explicitly**

Rearrange for y: $y(x) = \pm\sqrt{1 - x^2}$

4. **Verify candidates**

For $y(x) = +\sqrt{1 - x^2}$, $y' = \frac{-x}{\sqrt{1-x^2}}$, which matches the original ODE For $y(x) = -\sqrt{1 - x^2}$, $y' = \frac{x}{\sqrt{1-x^2}}$, which also satisfies the ODE

5. **Apply initial conditions**

$y_1(x) = -\sqrt{1 - x^2} \Rightarrow y_1(0) = -1$ is not valid $y_2(x) = \sqrt{1 - x^2} \Rightarrow y_2(0) = 1$ is valid

$\therefore$ the unique solution to the IVP is $y(x) = \sqrt{1 - x^2}$

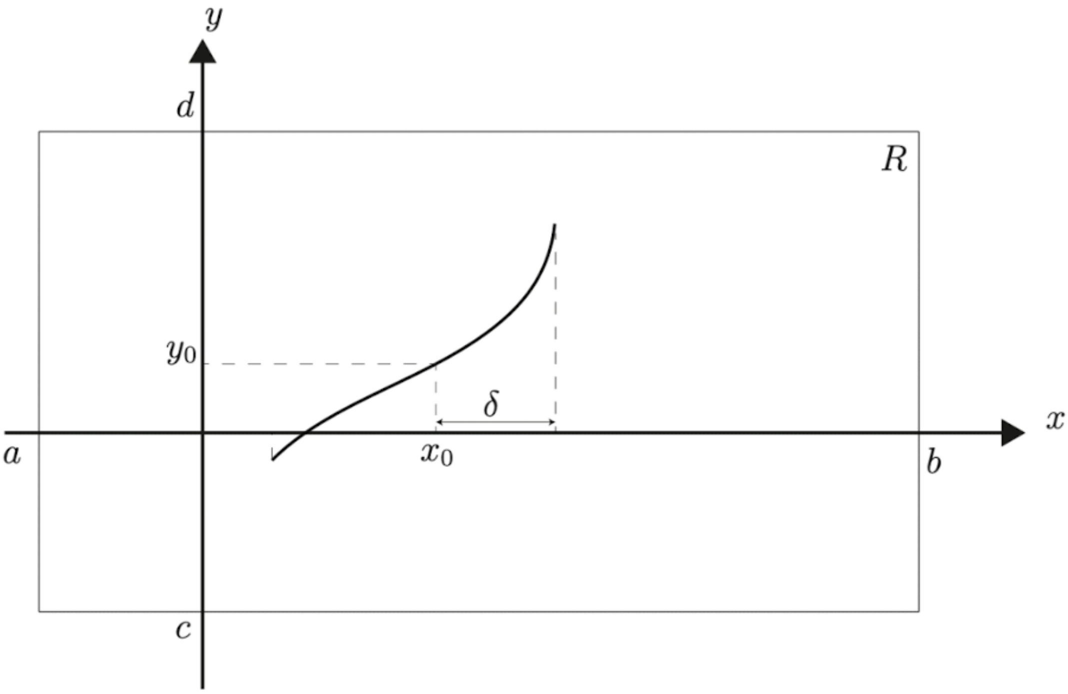
3.2 Theorem: Existence and Uniqueness of Solutions to the 1st order IVP, or Picard-Lindelof, or Cauchy-Lipshitz

Consider the IVP $y' = f(x, y)$ and $y(x_0) = y_0$

If f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $R = \{(x, y) \in \mathbf{R^2} : a < x < b, c < y < d\}$, that contains the point (x_0, y_0), then the IVP admits a unique solutions $\phi : (x_0 - \delta, x_0 + \delta) \to \mathbf{R}$ for some $\delta > 0$

If $f(x, y)$ is continuous, then at least one solution exists. Continuity of f is enough for *existence*, but not for *uniqueness*

If $f(x, y)$ is continuous *and* $\frac{\partial f}{\partial y}$ is continuous, then this is enough to show that *uniqueness*



Intuition:

In the original equation, this of it like a slope field that tells you the slope of the solution curve at each point (x, y), $f(x, y)$

If f is continuous, there are no gaps in the slope field. That means you can follow the slope arrows starting at origin to trace out a path

If $\frac{\partial f}{\partial y}$ is also continuous, then nearby paths cannot cross each other. Because the slope field is well defined, you cannot have two different marbles starting at the same point and taking different paths

The rectangle R is the safe zone where f behaves nicely

δ is how far you can move horizontally from x_0 while staying inside the safe zone

Inside $(x_0 - \delta, x_0 + \delta)$, the slope field is well-behaved enough to guarantee a single, smooth solution curve

4 Initial value problems and the Existence and Uniqueness Theorem for 1st order ODEs (cont.)

If one of uniqueness or existence fails, then $\phi(x)$ is not a solution to the ODE

4.1 Uniqueness fails

In $f(x,y) = \frac{y}{x}$, f is undefined at $(0,0)$, but solutions $y(x) = 0$ and $y(x) = x$ hold and satisfy the IVP with $y(0) = 0$, thus $y(x) = cx$ is a solution for any constant c

This means that there are infinitely many solutions, so **uniqueness fails**

4.2 Existence fails

Now suppose $y(0) = 1$, when $x = 0$ this gives $y(0) = 0$, so **existence fails**

4.3 Alternate form

Given a 1st order linear ODE $a_1(x)y' + a_0(x)y = f(x)$, we can always put it in the form $y'(x) = P(x)y + Q(x)$ by denoting $P(x) = -\frac{a_0(x)}{a_1(x)}, Q(x) = \frac{f(x)}{a_1(x)}$

Thus, if $(x_0, y_0) \in \{(x,y) : P(x) \text{ and } Q(x) \text{ are continuous}\}$

Moreover, I can be chosen as the largest interval containing x_0 and such that P, Q are both continuous on I, giving a global, not local, solution

4.4 Picard Iterations

Picard iterations are a method to actually construct the solution that the Existence and Uniqueness Theorem guarantees.

We start at step 0, and iterate integration to construct a sequence of approximations (one per step) that should converge to the unique solution of the IVP

We can rewrite an IVP as an integral equation:

$y' = f(x,y), y(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$, where we don't know $y(t)$

We can approximate $y(x)$ step by step:

step 0: $\phi_0(x) = y_0$

step 1: $\phi_1(x) = y_0 + \int_{x_0}^{x} f(s, \phi_0(s))\ ds$

step 2: $\phi_2(x) = y_0 + \int_{x_0}^{x} f(s, \phi_1(s))\ ds$

$\vdots$

step n: $\phi_n(x) = y_0 + \int_{x_0}^{x} f(s, \phi_{n-1}(s))\ ds$

$\vdots$

If, at some step of the Picard iteration, the new approximation $\phi_{k+1}(x)$ turns out to be exactly the same as the previous one $\phi_k(x)$, then we've already reached the solution

Formally, if there exists $k \in \mathbf{N}$ s.t. $\phi_{k+1}(x) = \phi_k(x)$, then the solution would be given by $\phi_k(x) = y_0 + \int_{x_0}^{x} f(s, \phi_k(s))ds$

Example:

$y' = 2t(1+y), y(0) = 0$

Step 0: $\phi_0(t) = 0$

Step 1: $y'(0) = 2t, \phi_1(t) = \int_0^t 2sds = t^2$

Step 2: $\phi_2(t) = \int_0^t 2s(1+t^2)ds = t^2 + \frac{t^4}{2}$

Step 3: $\phi_3(t) = \int_0^t 2s\left(1 + t^2 + \frac{t^4}{2}\right) ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2*3}$

Step n: $\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} = \Sigma_{k=1}^{n} \frac{t^{2k}}{k!}$

Hence, the sequence of functions converges if and only if the infinites series given above converges.

We can apply the ratio test to find this:

$\frac{|t^{2(k+1)}}{(k+1)!} \frac{k!}{k+1}| \to 0$ as $k \to \infty$

So, the series converges for every t, and $\phi(t) = \Sigma_{k=1}^{\infty} \frac{t^{2k}}{k!}$

5 Tutorial - September 10, 2025

5.1 Direction field

$y' = \frac{dy}{dx} = f(x, y), f : \mathbf{RxR} \to \mathbf{R}$

The direction field is a 2-dimensional vector field that shows the direction of the solution at every point in a region

$< dx, dy >=< 1, \frac{dy}{dx} > dx$

If $y' > 0$, draw an up right arrow (blue in figure) If $y' = 0$, draw a horizontal arrow (black in figure) If $y' < 0$, draw a down-right arrow (red in figure)

When working with an ODE, it tells us the equation of the slope at any given point. If we set $y' = 0$, we get a new formula. When we plot this formula, we know that at each point, $y' = 0$, so the slope is 0

5.1.1 Example

$y' = \frac{dy}{dx} = y - x^2 + 1$

Draw the direction field $\frac{dy}{dx} \Leftrightarrow y = x^2 - 1$

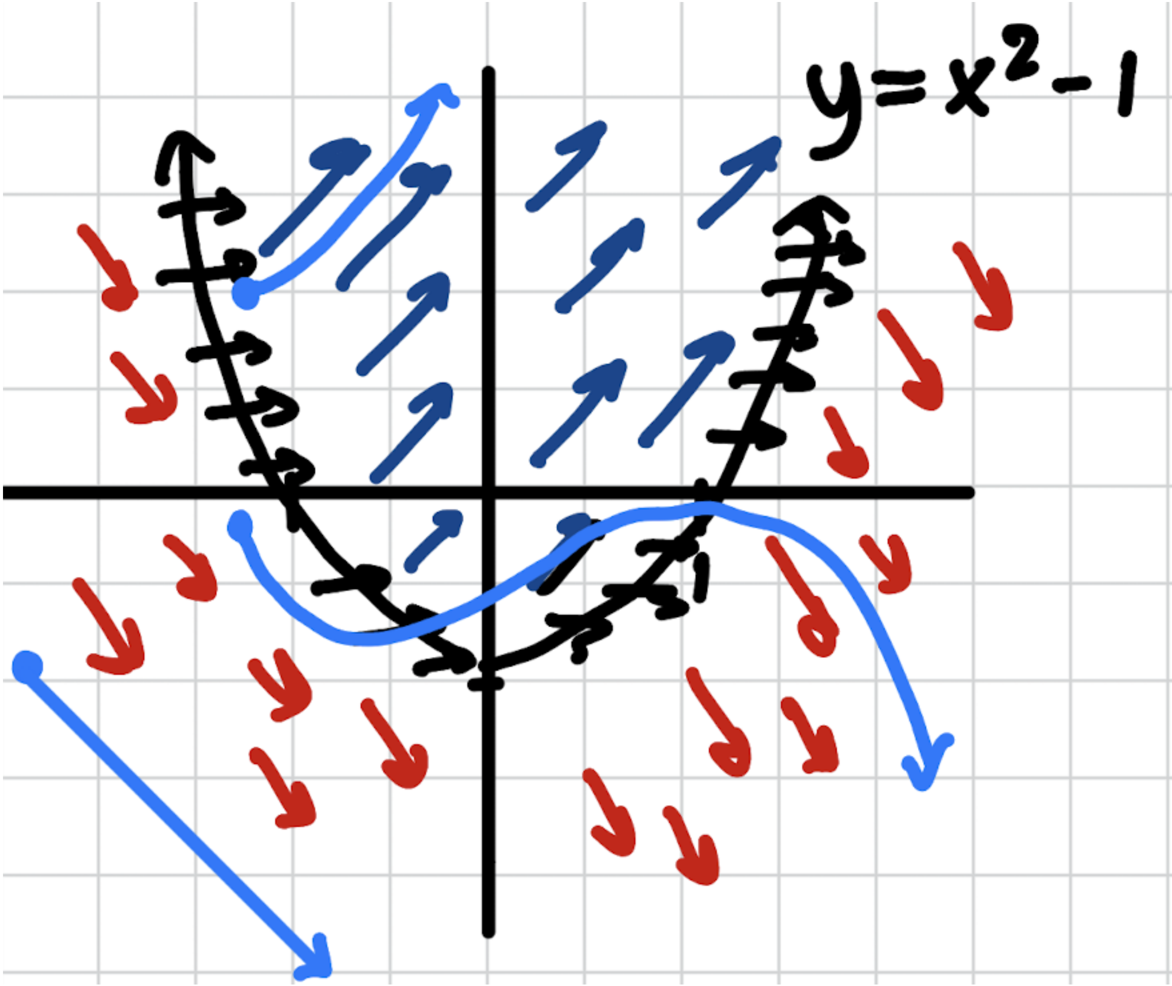
$\frac{dy}{dx} > 0 \Leftrightarrow y > x^2 - 1$ (up-right arrow, slope is upward) $\frac{dy}{dx} = 0 \Leftrightarrow y = x^2 - 1$ (flat arrow, slope is flat) $\frac{dy}{dx} = 0 \Leftrightarrow y < x^2 - 1$ (down-right arrow, slope is downward)

Initial point/condition

$y(x_0) = y_0, (x_0, y_0) \in \mathbf{R^2}$, light blue in figure

b) Behaviour as $x \to +\infty$

The solution becomes unbounded. Either $y \to +\infty$ or $y \to -\infty$ depending on the initial condition



5.2 Solution of a differential equation

Recall: Given $F(x, y, y', y'', \ldots, y^{(n)}) = 0$, a function $\phi = \phi(x)$ is a solution in some interval $I = (a, b)$ if and only if a) $\phi', \phi'', \ldots, \phi^{(n)}$ exists on I and, b) satisfy the differential equation for all $x \in I$

5.2.1 Example 1

$y' = y - x^2 + 1$. Is $\phi(x) = x^2 + 1$ a solution on $I = (-\infty, \infty) = \mathbf{R}$.

Condition (a) is satisfied since $\phi'(x) = 2x$ exists on $\mathbf{R}$. However, $\phi'(x) = 2x \neq 0 = \phi(x) - x^2 + 1$, so (b) is not satisfied

5.2.2 Example 2

$y' - 2xy = 1$. Verify that $\phi(x) = e^{x^2} \int_0^x e^{-t^2} dt + e^{x^2}$ is a solution on $\mathbf{R} = (-\infty, \infty)$

Use the Fundamental Theorem of Calculus to solve this problem

Let $g(x) = \int_0^x e^{-t^2}$. By FTC I, $g'(x) = e^{-x^2}$

a) $\phi'(x)$ exists on $I = (-\infty, \infty)$ by FTC I

b) Satisfy?

$\phi(x) = e^{x^2}g(x) + e^{x^2}$ $\phi'(x) = e^{x^2}e^{-x^2} + g(x)2xe^{x^2} + 2xe^{x^2}$

Check $\phi'(x) - 2x\phi(x) = 1$

$1 + 2xe^{x^2}g(x) + 2xe^{x^2} - 2x(e^{x^2}g(x) + e^{x^2}) = 1$, $\therefore$ it satisfies

5.2.2.1 FTC If f is continuous on an interval $[a, b]$, and we define a function $F(x) = \int_a^x f(t)dt, a \leq x \leq b$, then F is differentiable on (a, b), and $F'(x) = f(x)$

If f is continuous on $[a, b]$, and F is any antiderivative of f (so $F'(x) = f(x)$), then $\int_a^b f(x)dx = F(b) - F(a)$

They say that (1) integration undoes differentiation and (2) definite integrals can be computed using antiderivatives

5.3 Existence and Uniqueness Theorem

Given the IVP

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

If f and $\frac{\partial f}{\partial y}$ are continuous in $R = \{(x, y) \in \mathbf{R^2} : a < x < b, c < y < d\}$, such that $x_0, y_0 \in \mathbf{R}$, then IVP has a unique solution $\phi : (x_0 - \delta, x_0 + \delta) \to \mathbf{R}$ for some $\delta > 0$

5.3.1 Example

$y' = \sqrt{y}, y(0) = y_0 \geq 0$, $f(x, y) = \sqrt{y}$ is continuous for $x \in \mathbf{R}$ and $y \geq 0$

$\frac{\partial f(x,y)}{\partial y} = \frac{1}{2\sqrt{y}}$ is continuous for $x \in \mathbf{R}$ and $y > 0$

Define the rectangle R $= \mathbf{R}x(0, +\infty)$

If $y(0) = y_0 > 0$, so the IVP has a unique solution and it lies in the rectangle

If $y(0) = y_0 = 0$, no conclusion can be drawn from the existence and uniqueness theorem (not unique)

6 Separable Equations

6.1 Definitions

General solution: The family of all possible solutions to the ODE, usually containing arbitrary constants

Explicit solution: A solution where the dependent variable (say y) is written explicitly as a function of the independent variable (say x)

Implicit solution: A solution where the relationship between x and y is given as an equation but not solved explicitly for y

6.2 Solutions of some ODEs

A linear ODE with constant coefficients has the form $y' = ay + b$, where a and b are real constants

We can rewrite this ODE in the equivalent form

$\frac{1}{y+\frac{b}{a}} \frac{dy}{dt} = a$

By integrating both sides with respect to t, we get

$\ln |y + \frac{b}{a}| = t + C \Leftrightarrow |y + \frac{b}{a}| = e^C e^{at} \Leftrightarrow y(t) = -\frac{b}{a} \pm ce^{at}$, where C and c are arbitrary constants

The general solution of an ODE determines an infinite family of curves called **integral curves**, where each curve corresponds to a different value of c. Imposing the initial condition, we are selecting the *unique curve* that passes through the initial point (t_0, y_0)

6.2.1 Example

Consider the classic drag proportional to velocity first-order linear ODE $m\frac{dv}{dt} = mg - \gamma v$

Put in standard form

$\frac{dv}{dt} = -\frac{\gamma}{m} v + g$

Rewrite into an equivalent form given

$\frac{1}{y+\frac{b}{a}} \frac{dy}{dt} = a$, where $a = -\frac{\gamma}{m}, b = g$

The general solution has the form $v(t) = -\frac{b}{a} + Ce^{at}$

Plug back in

$v(t) = \frac{mg}{\gamma} + Ce^{-(\gamma/m)t}$, where C is an arbitrary constant

Fix C with an initial condition

$v(0) = v_0$ $v_0 = \frac{mg}{\gamma} + C \Rightarrow C = v_0 - \frac{mg}{\gamma}$

Explicit solution by plugging back C

$v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right) e^{-\gamma t/m}$

6.3 Separable equations

We were able to solve the previous ODE because we were able to rewrite it in an equivalent form by **separating the variables**, thus the ODE falls in the class of *separable equations*

Definition: Consider the 1st order ODE $y' = f(x,y)$, if $f(x) = g(x)p(x)$, where g and p depends only on x and y, respectively, then the ODE is said to be **separable**

Consider the separable ODE on some interval $I = (a,b)$ $\frac{dy}{dx} = g(x)p(y)$ $(*)$

We have two cases:

1. If $p(y) \equiv 0$, then $y(x) = c$ constant function on I because the RHS is always 0
2. If $p(y) \neq 0$, then $(*)$ can be rewritten as $\frac{1}{p(y)}\frac{dy}{dx} = g(x)$

Let $H = H(y)$ and $G = G(x)$ be the antiderivatives of $\frac{1}{p(y)}$ and $g(x)$, respectively, from the chain rule we obtain $\frac{dH(y(x))}{dx} = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{p(y)}\frac{dy}{dx} = \frac{dG}{dx}$

Then, since their derivatives are equal, the functions must differ by a constant: $H(y(x)) = G(x) + c \Leftrightarrow \int \frac{1}{p(y)}dy = \int g(x)dx + c$

This is an implicit formula for the solution to the ODE.

For an explicit solution, we apply the **inverse function** of H to both sides: $H(y) = G(x) + C$ $H^{-1}(H(y)) = H^{-1}(G(x) + C)$ $y = H^{-1}(G(x) + C)$

Remark: finding the inverse function depends on H

7 Separable Equations (cont.)

7.1 Example (implicit solution)

Consider $y' = \frac{x}{y\exp(x+2y)}$

It can be rewritten as

$ye^{2y}y' = xe^{-x}$

It can then be separated and integrated

$\int ye^{2y}dy = \int xe^{-x}dy$

$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -(x+1)e^{-x} \Leftrightarrow e^{2y}(2y-1) = -4e^{-x}(x+1) + c$

In this case, it is not possible to write y as an explicit function of x. We stop at an **implicit solution**

7.2 Example (explicit solution)

$y' = (1+y^2)\tan x$

It can be rewritten as

$\frac{1}{1+y^2}\frac{dy}{dx} = \tan x$

It can then be separated and integrated

$\int \frac{1}{1+y^2}dy = \int \tan x dx \Leftrightarrow \arctan[y(x)] = -\ln|\cos x| + c$

7.2.1 Integration of $\tan x$

$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du$ where $u = \cos x$ and $du = -\sin x dx$

$-\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$

8 Variable Coefficients (Integration Factor Method)

Consider a 1st order linear ODE with variable coefficients

$a_1(x)y' + a_0(x)y = b(x) \Leftrightarrow y' + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)} \Leftrightarrow \frac{dy}{dx} + P(x)y = Q(x)$

If $a_0(x) = a_1'(x)$, then the equation becomes $\frac{d}{dx}(a_1(x))y = b(x)$

Definition: An integrating factor is a special function that we multiply through a first-order linear ODE to make it easier to solve. Its purpose is to turn the left-hand side of the equation into the derivative of a product, so we can integrate directly

8.1 General Solution

Let $\frac{dy}{dx} + P(x)y = Q(x)$

1. We multiply through by an *integrating factor*

$\mu(x) = e^{\int P(x)dx}$

2. Multiply the whole equation by $\mu(x)$

$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \Leftrightarrow \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$

3. Integrate both sides

$\mu(x)y = \int \mu(x)Q(x)dx + C$

4. Solve for $y(x)$

$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$

General Solution:

$y(x) = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$

8.2 Example 1

Consider $y' + y = 1$, where $P(x) = Q(x) = 1$

Integrating factor: $e^{\int P(x)dx} = e^x$

Use general solution: $y(x) = e^{-x}\left(\int e^x dx + c\right)$

Solve: $y(x) = 1 + ce^{-x}$

8.3 Example 2

Consider $y' + \frac{3}{x}y = 3x - 2, y(1) = 1$, where $P(x) = \frac{3}{x}, Q(x) = 3x - 2$

Integrating factor: $\mu(x) = e^{3\ln x} = x^3$

Rewrite ODE: $\frac{d}{dx}[\mu y] = \mu Q \Leftrightarrow \frac{d}{dx}[x^3y(x)] = 3x^4 - 2x^3$

Integrate on both sides: $x^3y(x) = \frac{3}{5}x^5 - \frac{2}{4}x^4$

Solve: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{c}{x^3}$

Use initial conditions to find C: $y(1) = 1 = \frac{3}{5} - \frac{1}{2} + C \Leftrightarrow C = \frac{9}{10}$

Final Solution: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10x^3}$

9 Exact Equations

We start with a first-order ODE: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

which is equivalent to $M(x,y)dx + N(x,y)dy = 0$

Now, suppose there exists a *potential function* $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$, and $\frac{\partial F}{\partial y} = N(x,y)$

Then, if F exists, then the ODE is called exact.

Along any solution curve $y(x)$, if we compute the derivative of $F(x,y(x))$, we get:

$\frac{d}{dx}F(x,y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$

But notice that the right-hand side is exactly the same as $M(x,y) + N(x,y)\frac{dy}{dx}$

Since our ODE says that equals zero, it follows that $\frac{d}{dx}F(x,y(x)) = 0$

That means $F(x,y)$ is constant along solution curves, i.e. $F(x,y) = c$, which is the implicit solution of the ODE

9.1 Example 1

Consider $y' = -\frac{2xy^2+1}{2x^2y}$, where we can rearrange to get $M(x,y) = 2xy^2 - 1$ and $N(x,y) = 2x^2y$

Integrate an equation wrt x:

$\int(2xy^2 - 1)dx = x^2y^2 - x + g(y)$

The last term $g(y)$ appears instead of constant c because c can depend on y since we are integrating wrt x

Now, differentiate the latter function wrt y:

$\frac{\partial F}{\partial y} = 2x^2y + g'(y) = 2x^2y$, where the last term appears from $N(x,y) = 2x^2y$

We see that $g'(y) = 0$, so $g(y) = c$. Therefore, the given ODE is exact and the general solution is implicitly defined by $x^2y^2 - x = c$ for any arbitrary constant c

9.2 Example 2

Consider $3xy + y^2 + (x^2 + xy)y' = 0$

Integrate $M(x,y)$ wrt x: $F(x,y) = \frac{3}{2}x^2y + xy^2 + g(y)$

Differentiate F wrt y: $\frac{\partial F}{\partial y} = \frac{3}{2}x^2 + 2xy + g'(y) = x^2 + xy$

Rearrange for $g'(y) : g'(y) = -\frac{1}{2}x^2 - xy$, which cannot hold because the RHS depends on both variables x and y, while g is only a function of y. Hence, there is no F satisfying for the given ODE, which is then *not exact*

9.3 Clairaut's Theorem (Test for exactness)

An ODE is *exact* in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

Get

$$\begin{cases} \frac{\partial F}{\partial y} = N(x, y) \\ \frac{\partial F}{\partial x} = M(x, y) \end{cases}$$

Since M and N are differentiable wrt x and y, then

$$\begin{cases} \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \end{cases}$$

Clairaut's Theorem states that if $F(x, y)$ has continuous second order partial derivatives, then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

10 Second order linear differential equations

CREATE CHEAT SHEET TO SOLVE 1ST ORDER ODES

Let $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$, where the coefficients are functions of x only and are continuous on some open interval $I = (a, b)$

If the ODE $= 0$, we call it homogeneous.

The superposition principle also applies to second order ODEs.

10.1 Wronskian

A second order ODE needs two initial conditions, and they must not be multiples of one another. Else, solutions y_1, y_2 are linearly dependent on the interval I

We can say that a 2nd order ODE is homogeneous if:

Let y_1, y_2 be solutions on the interval of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. If at some point $x_0 \in (a, b)$ these two solutions satisfy

$$\det \begin{matrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{matrix} \neq 0$$

and call it **Wronskian of y_1 and y_2 at x_0**

10.2 Representation Theorem

Let $y_1(x), y_2(x)$ be two solutions on the interval (a, b) of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

If at some $x_0 \in (a, b)$, these two solutions satisfy: $W[y_1(x_0), y_2(x_0)] \neq 0$, then $y_1(x)$ and $y_2(x)$ are linearly independent solutions on (a, b)

11 Homogeneous equations with constant coefficients

Consider $ay'' + by' + cy = 0$, where $a, b, c \in \mathbf{R}$

If $\phi(x)$ is a solution to the ODE, then $\phi''(x) = -\frac{b}{a}\phi'(x) - \frac{c}{a}\phi(x)$ for all $x \in \mathbf{R}$

This means that the second derivative is a linear combination of the lower order derivatives

If we have a trial function (educated guess for the solution) $y(x) = e^{rx}$, then we get $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$

Into the ODE, we get $\cancel{e^{rx}}(ar^2 + br + c) = 0$, so e^{rx} is a solution $\Leftrightarrow$ r is a root of the 2nd order polynomial. Thus, r must be a solution to the *characteristic equation* $ar^2 + br + c = 0$

11.1 Case 1: $b^2 - 4ac > 0$

If so, then there exists roots $r_1, r_2 \in \mathbf{R}$ with $r_1 \neq r_2$

r_1 cannot equal r_2 because $W[e^{r_1x}, e^{r_2x}] = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$

Hence, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ are linearly independent solutions, and by the Representation Theorem, $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$, where constants are arbitrary and can be found by imposing initial conditions

Example:

Consider $2y'' + 7y' - 4y = 0$

Characteristic equation: $2r_2 + 7r - 4 = 0$

Confirm determinant > 0: $r_{1,2} = \frac{-7\pm\sqrt{49+32}}{4} = -4, \frac{1}{2}$

General solution: $y(x) = c_1e^{-4x} + c_2e^{x/2}$

Impose initial conditions if given

11.2 Case 3: $b_2 - 4ac < 0$

In this case, we have two roots $r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a} =: \alpha \pm i\beta$

Hence, we find two *complex-valued* linearly independent solutions $y_1, y_2 : \mathbf{R} \to \mathbf{C}$ $y_1(x) = \exp[(\alpha + i\beta)x], y_2(x) = \exp[(\alpha - i\beta)x]$

We would like to find two *real-valued linearly independent solutions*

Important:

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Also, if the ODE has real coefficients and the complex-valued function $\phi = u(x) + iv(x)$ is a solution, then real-valued function $u(x) = \mathrm{Re}(\phi(x))$ and complex-valued function $v(x) = \mathrm{Im}(\phi(x))$ are solutions of the same equation

Back to Case 3, from Euler's formula we get: $y_{1,2}(x) = e^{\alpha x}[\cos(\beta x) \pm i\sin(\beta x)]$

Also, be the second remark above, we can conclude that also the functions below are also solutions: $Y_1, Y_2 : \mathbf{R} \to \mathbf{R}$ $Y_1(x) = \mathrm{Re}(y_1(x)) = \mathrm{Re}(y_2(x)) = e^{\alpha x}\cos(\beta x)$ and $Y_2(x) = \mathrm{Im}(y_1(x)) = -\mathrm{Im}(y_2(x)) = e^{\alpha x}\sin(\beta x)$

$$W[Y_1(x), Y_2(x)] = \mathrm{d}$$

$$= \mathrm{e}$$

Moreover, we can check if these are linearly independent via their Wronskian:

Thus, the solutions above are linearly independent (real-valued) solutions of $ay_2'' + by_2' + cy_2 = 0$

Moreover, every other solution takes the form $y(x) = c_1Y_1(x) + c_2Y_2(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$

Example:

Consider $y'' - y' + y = 0, y(0) = 1, y'(0) = -2$

Characteristic equation $r^2 - r + 1 = 0$

Solution: $r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

- Linearly independent solutions of the ODE:

$$y_1(x) = \mathrm{e}^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), \qquad y_2(x) = \mathrm{e}^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

- General solution:

$$y(x) = \mathrm{e}^{x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

Imposing initial conditions, we get

$y(x) = e^{x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{5\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

12 Homogeneous equations with constant coefficients (cont.)

12.1 Case 2: $b_2 - 4ac = 0$

If so, then there exist two roots $r_1, r_2 \in \mathbf{R}$ with $r_1 = r_2$

In this case $c = \frac{b^2}{4a}$ and roots $r_1 = r_2 = -\frac{b}{2a}$. Because we essentially have one root, then the solution $y_1(x) = \exp\left(-\frac{b}{2a}x\right)$ exists

By the Representation Theorem, $y(x) = c_1 y_1(x) + c_2 y_2(x)$. Now that we have $y_1(x)$, how do we find $y_2(x)$ such that $\frac{y_2(x)}{y_1(x)} \neq$ constant

We use the *Method of Reduction of Order*, which works on second order linear homogeneous ODEs with variable coefficients $y'' + p(x)y' + q(x)y = 0$

If $\phi_1(x)$ is a solution to the ODE above, and we want to find $\phi_2(x)$ such that they are linear independent on an interval, we want $\phi_2(x) = g(x)\phi_1(x)$ where $g(x)$ is an unknown function to be found to such that the ODE is satisfied by $\phi_2(x)$

Take $\phi_2(x)$ and take the second derivative: $\phi_2'' = g'; \phi_1(x) + 2g'\phi_1' + g\phi_1''(x)$

Replacing in the ODE, we get $\Leftrightarrow \phi_1 v' + (2\phi_1' + p(x)\phi_1)v = 0$, where $v(x) := g'(x)$

We can then solve for $v(x)$ because the corresponding ODE is linear and separable

$g(x) = \int v(x)dx = \int \frac{\exp\left(-\int p(x)dx\right)}{\phi_1^2(x)}dx$

Example:

Find a second solution to $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ for $x > 0$ given a solution $y_1(x) = x^{-1/2}\sin x$

Let $y_2(x) = v(x)y_1(x)$, then $y_2'' = v''y_1 + 2v'y_1' + vy_1''$

Goal: find $v(x)$. Replacing the latter in the ODE, we find that

$x^2y_1v'' + v'(2x^2y_1' + xy_1) = 0$ because the y term is the same as the ODE, and it drops out

We can divide further by x^2y_1: $v'' + \left(2\frac{y_1'}{y_1} + \frac{1}{x}\right) = 0$

Now we need $\frac{y_1'}{y_1}$. Since $y_1'(x) = \frac{\sqrt{x}\cos x - \frac{\sin x}{2\sqrt{x}}}{x}$, then the former is $\frac{\cos x}{\sin x} - \frac{1}{2x}$, which can be simplified to $2\frac{\cos x}{\sin x}$

Subbing back, we get that $v(x)$ has to satisfy the following equation $v'' + 2\frac{\cos x}{\sin x}v' = 0$

Reduce order again: Setting $w(x) := v'(x)$, then $w' = -2\frac{\cos x}{\sin x}w \Rightarrow \frac{w'}{w} = -2\cot x$

Integrate: $\ln|w| = -2\ln(\sin x) + C$, $w = v'(x) = \frac{C_1}{\sin^2 x}$

Integrating to get $v(x)$: $v(x) = -C_1\cot x + C_2$

Build the solution: $y_2(x) = v(x)y_1(x) = (-C_1\cot x + C_2)\frac{\sin x}{\sqrt{x}} \Rightarrow y_2(x) = \frac{\cos x}{\sqrt{x}}$

13 Mechanical and electrical vibrations

A **damped mass-spring oscillator** is a physical system constituted by a mass m attached to an elastic spring with stiffness constant k and subject to friction $F_f(t) = -\gamma \frac{dy}{dt}$, where $\gamma \geq 0$ is the damping coefficient

The equation of motion is given by $my'' + \gamma y' + ky = F(t) = 0$ with no external force applied to the body

1. **If $\gamma \neq 0$ and $\gamma^2 - 4mk \geq 0$, the system is said to be overdamped**

$r_1, r_2 > 0, y \to 0, x \to \infty$

$r_1 = 0, r_2 < 0, |y| \to \infty, x \to \infty$

2. **If $\gamma \neq 0$ and $\gamma^2 - 4mk < 0$, the system is said to be underdamped**

$\alpha > 0, |y| \to \infty$, y oscillates to $\infty, x \to \infty$

$\alpha < 0, |y| \to 0$, y oscillates to $0, x \to \infty$

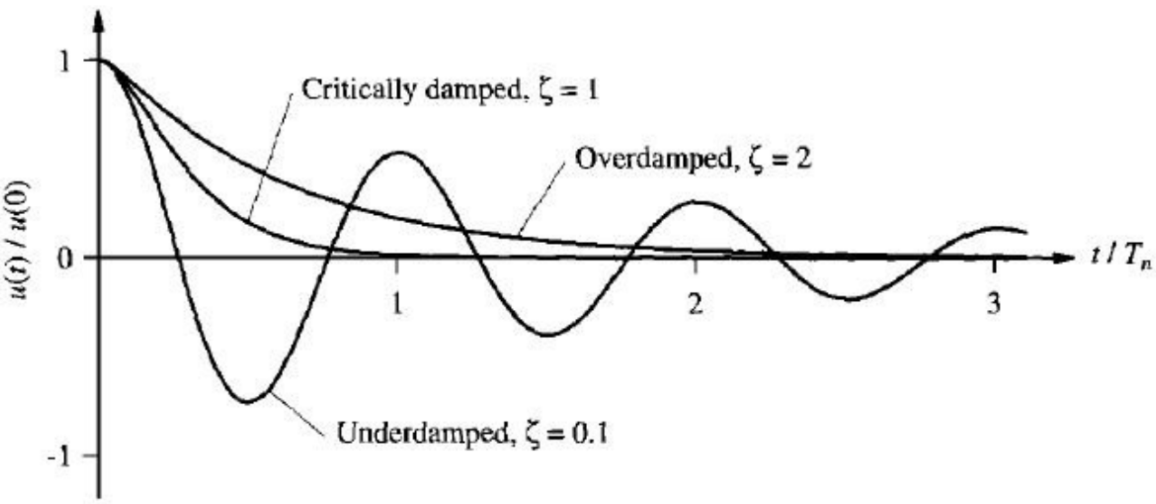
$\alpha = 0$, y is periodic, no limit

3. **If $\gamma = 0$, the system is said to be undamped**

$r_1 > 0, |y| \to \infty$ exponentially. $x \to \infty$

$r_1 < 0, y \to 0, x \to \infty$

$r_1 = 0, |y| \to \infty$ linearly. $x \to \infty$



14 How to Solve an ODE

14.1 General Setup

ODE form: $F(x, y, y', y', \ldots, y^{(n)} = 0$

General solution: Family of solutions with arbitrary constants.

Particular solution: Obtained by applying initial/boundary conditions.

Explicit solution: $y = f(x)$.

Implicit solution: Relation between x and y, not fully solved for y.

14.2 First Order

14.2.1 Separation of Variables

Form: $\frac{dy}{dx} = g(x)p(y)$

Steps:

1. Rearrange: $\frac{1}{p(y)}dy = g(x)dx$
2. Integrate both sides
3. Get implicit solution $H(y) = G(x) + C$
4. Solve for y if possible (explicit solution)

14.2.2 Variable Coefficients

Form: $y' = P(x)y = Q(x)$

Steps:

1. Compute integrating factor $\mu(x) = e^{\int P(x)dx}$
2. Multiply ODE by $\mu(x)$: $\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$
3. Integrate both sides: $\mu(x)y = \int \mu(x)Q(x)dx + C$
4. Solve for $y(x)$

14.2.3 Exact Equations

Form: $M(x, y) + N(x, y)y' = 0$ or $M(x, y)dx + N(x, y)dy = 0$

Condition (Clairaut's Test): $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Steps:

1. Check exactness with Clairaut's test
2. Find potential function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$
3. Implicit solution: $F(x, y) = C$

14.2.4 Picard Iteration (Approximate Solutions)

Form: For IVP $y' = f(x, y), y(x_0) = 0$

Steps:

- Rewrite as integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$
- Approximate iteratively $\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt$
- Sequence converges to the solution

14.3 Second Order

Case 1: $\Delta > 0$

General solution: $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: $\Delta = 0$

General solution: $y(x) = (c_1 + c_2)e^{(-b/2a)x}$

From a known y_1: $y_2 = y_1 \int \frac{\exp\left(-\int P dx\right)}{y_1^2(x)} dx$ where P is from $y'' + Py' + Qy = 0$

Case 3: $\Delta < 0$

General solution: $y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$ where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$