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MTHE 280 - Lecture Notes
Advanced Calculus

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Contents

1 Introduction to Multivariable Functions

A function $f(x, y)$ is a rule that assigns to every element x a unique element y, and is denoted by $f : x \to y$, where x is the domain of f and y is the codomain of f

Example

$f : \mathbf{N} \to \mathbf{R}, f(x) = 2x$

In this case, every value of f is even and does not take the whole codomain

We introduce the range, a subset of the codomain, $range(f) \subseteq codomain(f)$

1.1 Properties of functions

One-one/[[Injective]]

$f : X \to Y$ if $x_1, x_2 \in X, f(x_1) = f(x_2)$

Onto/[[Surjective]]

$f : X \to Y$ is onto if for every $y \in Y$, there exists some $x \in X$ such that $f(x) = y$

In this case, codomain = range

Bijective

if $f : x \to y$ is both one-one and onto, it is bijective

Scalar-valued

Consider $f : x \to y$ where $x \subseteq \mathbf{R}$ and $y \subseteq \mathbf{R}$, $n, m \in \mathbf{N}$

When the codomain is just $\mathbf{R}$, the function is called a Scalar-valued function

Example

$f : \mathbf{R^2} \to \mathbf{R}$ where $f(x, y) = \sqrt{x^2 + y^2}$

This returns the length of a 2D vector, which is a scalar

Vector-valued

A vector-valued function has codomain $\mathbf{R^n}$ where $n > 1, n \in N$

Example

$f : \mathbf{R} \to \mathbf{R^2}$, $f(x) = (\cos x, \sin x)$

1.2 Identify domain and codomain

Examples

$f(x) = \ln x$, domain = $(0, \infty)$, codomain = $\mathbf{R}$

$f(x) = \sqrt{2 - x}$, domain = $(-\infty, 2]$, codomain = $(0, \infty)$

$f(x, y) = (\sqrt{1 - x^2 - y^2}, \ln(y + 1), x^2 + y^2)$

1: $x^2 + y^2 = 1$ 2: $y > -1$

domain: $\{(x, y) \in \mathbf{R^2} : x^2 + y^2 \leq 1, y > -1\}$

2 Level Curves and Contours

Level Curve

Given a scalar-valued function, the level curve at height c is the curve in $\mathbf{R^2}$ s.t. $f(x, y) = c$

Or, the level curve at height $c = \{(x, y) \in \mathbf{R^2} | f(x, y) = c\}$

[[Level Curve]]

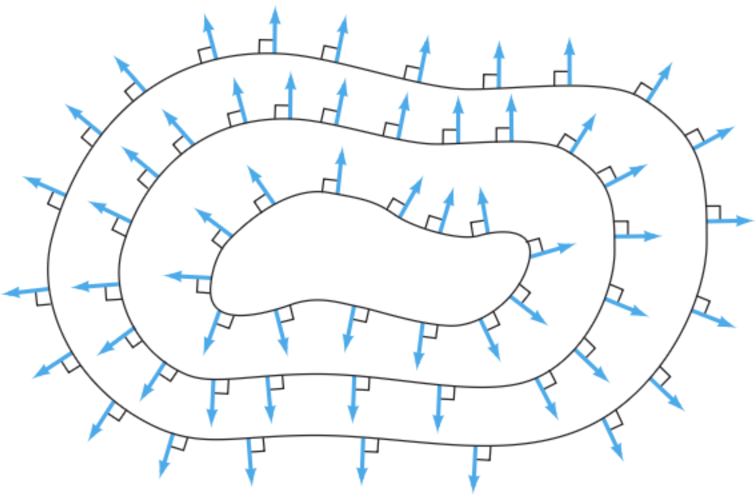


Figure 3.31 A gradient vector field $\mathbf{F} = \nabla f$. Equipotential lines are shown where f is constant.

Contour

The contour curve at height c is the collection of points (x, y, z) s.t. $z = f(x, y) = c$

Or, $\{(x, y, z) \in \mathbf{R^3} | z = f(x, y) = c\}$

The projection of the contour is the level curve

Section

A section of a surface by a place is just the intersection of the surface with that plane

3 Limits of a function

General form: $f : \mathbf{R} \to \mathbf{R}$

$\lim_{x \to a} f(x) = L, \therefore f(x)$ tends to L as x tends to a

3.1 L'Hospital's Rule

If we have a case where we are evaluating a limit and we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can use $\lim_{x \to a} \frac{f'(x)}{g'(x)}$

Why?: The ratio $\frac{f(x)}{g(x)}$ near a depends not only on the values of f and g, but on how fast they approach 0 or ∞

3.2 Limits in two variables

Let $f : \mathbf{R^2} \to \mathbf{R}$, $\lim_{(x,y) \to (a,b)} f(x, y)$

The Line $y = mx$ trick

All paths approaching point (e.g. $(0, 0)$) must give the same value

A simple test path is a straight line mx through the origin, and plug $f(x, y) \to f(x, mx)$

If the result depends on m, the limit does not exist

Does Exist Example

$$\lim_{(x,mx) \to (0,0)} \frac{x^2}{x^2 + y^4}$$

$$\lim_{x \to 0} \frac{x^2}{x^2 + m^4 x^4}$$

$$\lim_{x \to 0} \frac{1}{1 + m^4 x^2} = 1 \therefore \text{limit exists}$$

Does Not Exist Example

$$\lim_{(x,y) \to (0,0)} \frac{1}{1 + m^2} = \frac{x^2}{x^2 + m^2 x^2} = \frac{1}{1 + m^2} \therefore \text{limit does not exist}$$

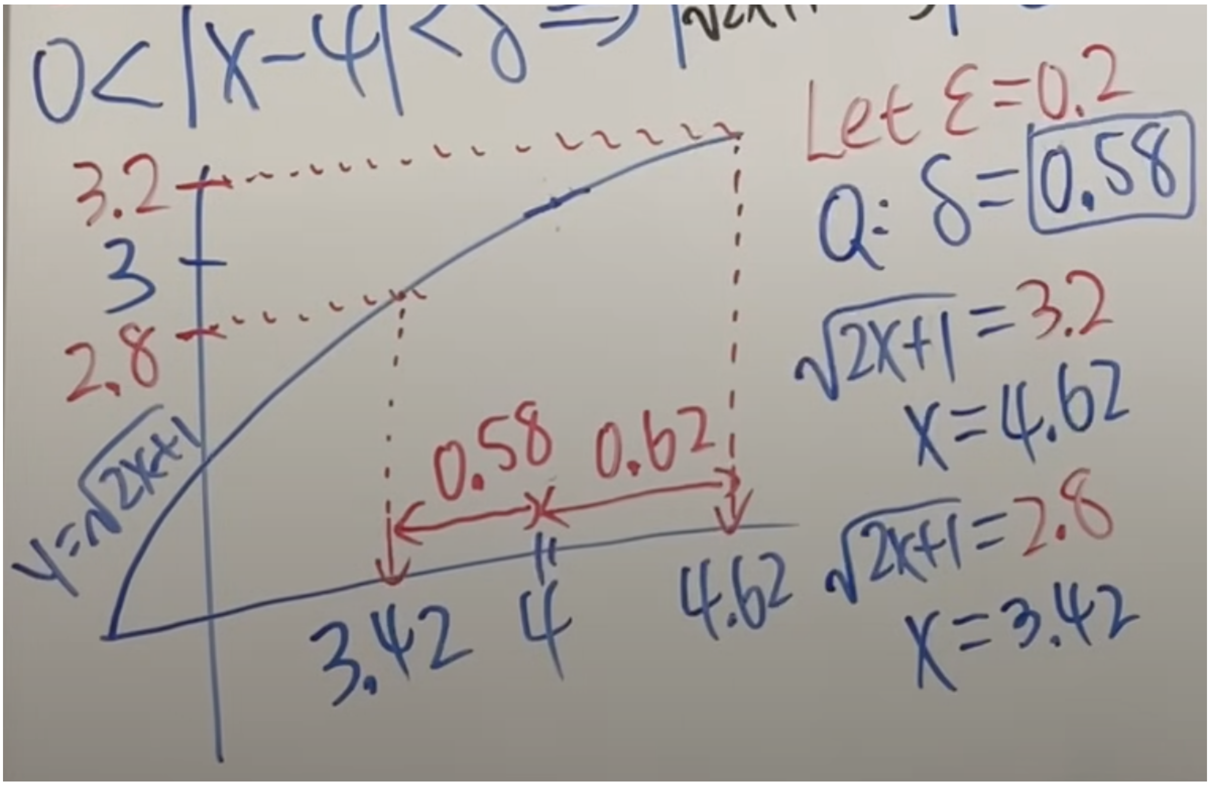
3.3 Epsilon-delta definition of a limit

$\lim_{x \to a} f(x) = L$ means $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example: we know that $\lim_{x \to 4} \sqrt{2x + 1} = 3$ by plugging in 4 into the continuous function

To prove this, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - 4| < \delta \Rightarrow |\sqrt{2x + 1} - 3| < \delta$

If x is near 4, of a distance less than δ, then the corresponding value of the function is near the limit $L = 3$, of a distance ε



3.3.1 General solution process

Proof: Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x-a|| < \delta$, then $|f(x)-L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim\limits_{x \to a} f(x) = L$

Triangle Inequality

It says: $|a + b| \leq |a| + |b|$

Order Trick

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0$, lim is likely to exist when order is ≥ 1, here it is 1

Simplify Trick

We can: $\frac{3|x|y^2}{x^2+y^2} \leq \frac{3|x|y^2}{y^2} = 3|x|$

We can also: $|x| \leq \sqrt{x^2 + y^2}$

Linear combination of coordinate differences

$|a(x - a) + b(y - b)| \leq |a||x - a| + |b||y - b| \leq (|a| + |b|)||\mathbf{x} - \mathbf{a}||.$

3.4 When to use either strategy

We use the epsilon-delta proof to rigorously prove that a limit exists (or equals some value)

We take the limit along lines, parabolas, or curves to test whether a limit exists, or to guess its value. It is useful when you are not sure if the limit exists.

3.5 $\varepsilon - \delta$ for vector-valued functions

Let $F : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}, \vec{a} \in U$

We write $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{L}, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $||F(\vec{x}) - \vec{L}|| < \varepsilon$ if $||\vec{x} - \vec{a}|| < \delta$

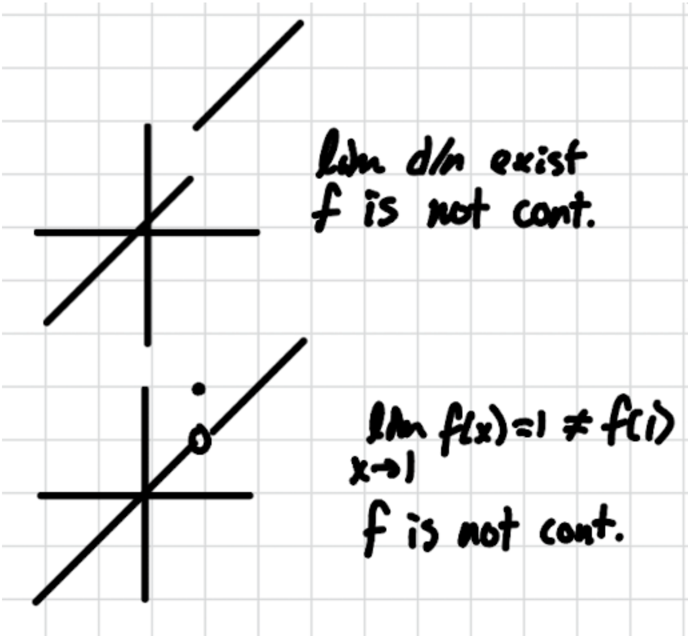
Ex: does $\lim_{(x,y) \to (0,0)} (\frac{3xy^2}{x^2+y^2}, \frac{e^x + \cos y}{x^2+y^2+1})$ exist?

We know that the first component does. For the second component, both the numerator and the denominator are continuous at $(0,0)$, thus we can plug in that point and get that the limit approaches 2

4 Continuity and its properties

4.1 Continuity of single variable functions

Let $f : A \to R, a \in A$. f is continuous if (1) $\lim\limits_{x \to a} f(x)$ exists and (2) $\lim\limits_{x \to a} f(x) = f(a)$



4.2 Continuity of multivariable functions

Let $f : U(\subseteq \mathbf{R^n} \to \mathbf{R}$ and $\vec{a} \in U$. f is continuous at $\vec{a}$ if (1) $\lim\limits_{\vec{x} \to \vec{a}} F(\vec{x})$ exists and (2) $\lim\limits_{\vec{x} \to \vec{z}} F(\vec{x}) = F(\vec{a})$

4.3 Properties of continuity (scalar- and vector-valued functions)

Suppose that f and g are continuous at $\vec{a} \in U$

1. $f + g$ is continuous at $\vec{a}$
2. $f * g$ is continuous at $\vec{a}$
3. $\frac{f}{g}$ is continuous at $\vec{a}$ if $g(\vec{a}) \neq 0$

Further:

1. $\lim\limits_{\vec{x} \to \vec{a}} (f + g)(\vec{x}) = f(\vec{a}) + g(\vec{a})$
2. $\lim\limits_{\vec{x} \to \vec{a}} (f * g)(x) = f(\vec{a})g(\vec{a})$
3. $\lim\limits_{\vec{x} \to \vec{a}} \left(\frac{f}{g}\right)(\vec{x}) = \frac{f(\vec{a})}{g(\vec{a})}$ if $g(\vec{a}) \neq 0$

Example:

$$f(x) = \begin{cases} \frac{3xy^2}{x^2+y^2}, (x,y) \neq (0,0), \\ \\ a, (x,y) = (0,0). \end{cases}$$

For which values of a is F continuous?

We know that the first component is continuous everywhere, except possible at $(0, 0)$

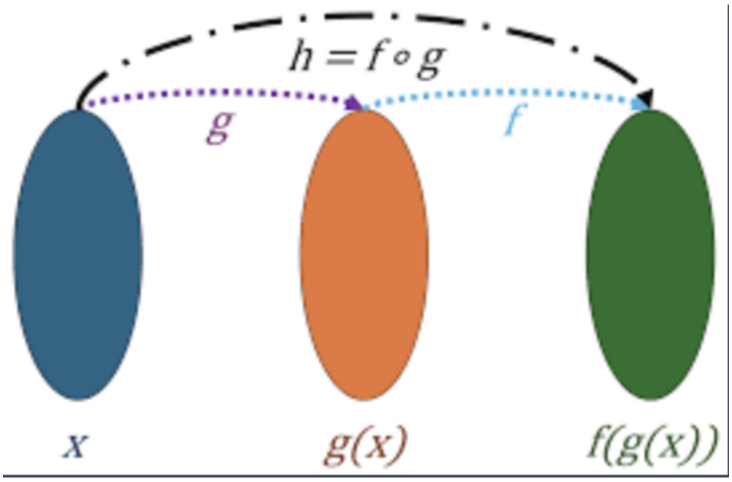
For continuity at $(0, 0)$, we need the limit of F at $(0, 0) =$ a, which is equivalent to saying that the continuous function $F(0, 0) = a$

That means we need to compute the first term's limit while approaching $(0, 0)$, which is $= 0$

$\therefore a = 0$

4.4 Composition of two continuous functions

If: 1. g is continuous at $x = a$, and 2. f is continuous at $g(a)$, then $f \circ g$ is continuous at a, where $f(g(x)) \to f(g(a))$



5 Differentiation

5.1 The derivative

f is differentiable at c if $\lim_{h\to c} \frac{f(x+h)-f(c)}{h}$ exists. If the limit exists, then it is denoted by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(h)}{h}$, where $f'(x)$ captures the rate of change of f near c

If $f(c)$ exists, we can draw a tangent line at c, and its slope is $f'(c)$

5.2 Partial Differentiation

f is partially differentiable wrt x at (a,b) if $\lim_{x\to a} \frac{f(a+h,h)-f(a,h)}{h}$ exists. If exists: $\frac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$

5.3 Tangent plane visualized

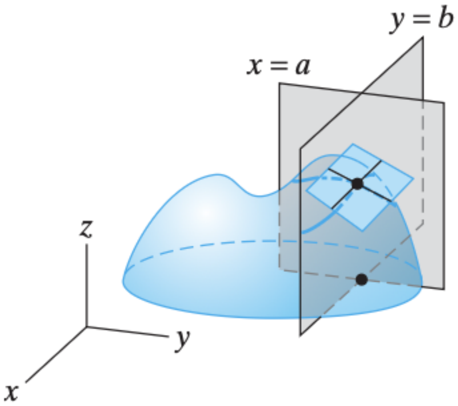


Figure 2.51 The tangent plane at $(a, b, f(a,b))$ contains the lines tangent to the curves formed by intersecting the surface $z = f(x,y)$ by the planes $x = a$ and $y = b$.

5.4 Directional derivative

The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point p in the direction of a vector $\vec{v}$ is the rate at which f changes at p as you move in the direction of $\vec{v}$

$D_{\vec{v}} f(p) = \nabla f(p) \cdot \vec{v}$

For vector valued functions, we can compute using the Jacobian $D_{\vec{v}} f(p) = DH(p) \cdot \vec{v}$

Definition: The directional derivative of f at $\vec{a} = (a,b)$ in the direction of $\vec{v}$ is given by $D_{\vec{v}} f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}$, if it exists

Example: let $f(x,y) = x^2y - 3x$, $D_{\vec{v}} f(0,0) =?$ where $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$D_{\vec{v}}f(0,0) = \lim\limits_{h\to 0} \frac{f(0,0)+h\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)-f(0,0)}{h}$

Simplify, then plug in h

$= -\frac{3}{\sqrt{2}}$

5.5 Multivariable differentiability at (a, b)

Definition: $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable at (a, b) if $\exists h(x, y) = f(a, b) + f_x(a, b) + f_y(a, b)$

1. $f_x(a, b)$ and $f_y(a, b)$ exists
2. $\exists \mathbf{R} f'(a)$ s.t. $\lim\limits_{h\to 0} \frac{f(x)-h(x,y)}{|x-a|} = 0$, where $h(x, y)$ is the equation of the tangent plane (or line) $f(a, b) + f_x(a, b)(x - a) + f_y(a, b))y - b$

How?

Single variable differentiability is defined by $f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$

We can rearrange to emphasize linear approximation: $\lim\limits_{x\to a} \frac{f(x)-[f(a)+f'(a)(x-a)]}{x-a} = 0$

This is saying that the function is differentiable at a if it can be approximated by th linear function $h(x, y)$ with error smaller than order $|x - a|$

Multivariable differentiability is now as follows $\lim\limits_{(x,y)\to(a,b)} \frac{f(x,y)-h(x,y)}{||((x,y)-(a,b)||} = 0$

6 Types of Points

An **open ball** in $\mathbf{R^n}$ with centre at $\vec{a} \in \mathbf{R^n}$ and radius $r : B(\vec{a}, r)$. The ball is open, meaning that the boundary points are not included

Definition: A point $\vec{a}$ is an **interior point** of a set A if there exists an open ball $B_\varepsilon(\vec{a})$, for some $\varepsilon > 0$, such that $B_\varepsilon(\vec{a}) \subseteq A$. So, the open ball lies entirely inside the set, without touching its complement

Definition: A **boundary point** is a point $\vec{a}$ such that every open ball $B_\varepsilon(\vec{a})$, no matter how small $\varepsilon > 0$ is, intersects the function and its complement (not the function)

Essentially, an open ball is all points strictly inside a certain radius form the centre, not including the edge. The interior points are inside the open ball, and boundary points are on the edge.

A set $U \subseteq \mathbf{R^n}$ is called open if every point of U is an interior point

7 Gradients, More Derivatives, and the Jacobian

7.1 Gradient

The gradient of a scalar function is a vector that collects all the partial derivatives of f with respect to each variable:

$\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_n})$

At a specific point, the gradient becomes:

$\nabla f(\vec{a}) = (f_{x_1}(\vec{a}), \ldots, f_{x_n}(\vec{a}))$

This vector points in the direction of the steepest increase of f and its magnitude gives the rate of increase

The difference vector:

$\vec{x} - \vec{a} = (x_1 - a_1, \ldots, x_n - a_n)$

The linear approximation of f near $\vec{a}$ can be written as:

$\nabla f(\vec{a})(\vec{x} - \vec{a}) = f_{x_1}(\vec{a})(x_1 - a_1) + \cdots + f_{x_n}(\vec{a})(x_n - a_n)$

Example:

Let $f(x, y) = xy^2 + e^{xy}$, find the gradient at $(0, 0)$

$f_x = y^2 + ye^{xy}, f_y = 2yx + xe^{xy}$

$\nabla f = (f_x, f_y) = (y^2 + ye^{xy}, 2xy + xe^{xy})\ \nabla f(0, 0) = (0, 0)$

Dot product of two vectors

If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + \cdots + a_nb_n$

7.2 Derivative Matrix

Let $U \subseteq \mathbf{R^n}$ and $f : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}$

$f = (f_1, f_2, \ldots, f_m)$

Let $f(x, y) = (x^2, x + y)$

$f_1(x) = x^2, f_2(x) = x + y$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \ldots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

This matrix is called the matrix of partial derivatives of f, otherwise called the Derivative Matrix or the **Jacobian Matrix**. Essentially, the derivative is a linear map, and in coordinates it is built from the partial derivatives

Example:

Let $f(x, y) = (xy, y^2 \sin x, x^3e^y)$, find the derivative matrix

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{matrix} = \begin{matrix} y, x \\ y^2 \cos x, 2y \sin x \\ 3x^2 e^y, x^3 e^y \end{matrix}$$

7.3 Differentiability in higher dimensions $f : U \to \mathbf{R^m}$

f is differentiable if: - $Df(\vec{a})$ exists - Tangent plane $h : \mathbf{R^n} \to \mathbf{R^m}$, $h(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$, where $Df(\vec{a})(\vec{x} - \vec{a})$ is a matrix multiplication, satisfies $\lim\limits_{\vec{x} \to \vec{a}} \frac{||f(\vec{x}) - h(\vec{x})||}{||\vec{x} - \vec{a}||} = 0$, which is hard to use

This is why we introduce the following theorems:

7.3.1 Theorems for higher-dimension differentiability

Theorem 1:

If $f = (f_1, f_2, \ldots, f_m)$, then f is differentiable at $\vec{a} \Leftrightarrow f_1, f_2, \ldots, f_m$ is differentiable at $\vec{a}$

Theorem 2:

If $f = (f_1, f_2, \ldots, f_m)$ and all partials $\frac{\partial f_i}{\partial x_j}$, as $i, j, \ldots, i_m, j_m$, are continuous then f is differentiable

Example:

$f(x, y) = (x^2 y, e^y \sin x)$ is differentiable because all of its partial derivatives are continuous

Theorem 3:

If f is differentiable at $\vec{a}$, then directional derivatives can be computed using: $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

If f is differentiable at $\vec{a}$, then $D_{\vec{v}} f(\vec{a}) = Df(\vec{a})\vec{v}$ where $Df(\vec{a})\vec{v}$ is a matrix multiplication

Example:

$f(x, y) = (e^x y, x^2 y)$, find rate of change of f at $(1, 2)$ in direction $\vec{v} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$Df = \begin{matrix} e^x y, & e^x \\ 2xy, & x^2 \end{matrix}, \; Df(1, 2) = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix}$$

$$Df(1, 2)\vec{v} = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix} \cdot \begin{matrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{matrix} = \begin{matrix} e + \frac{\sqrt{3}}{2} e \\ 2 + \frac{\sqrt{3}}{2} \end{matrix}$$

7.4 Properties of Differentiability

Let $F : \mathbf{R^n} \to \mathbf{R}, G : \mathbf{R^n} \to \mathbf{R}$ be differentiable at $\vec{a}$

- $F + G$ is differentiable at $\vec{a}$
- $F \cdot G$ is differentiable at $\vec{a}$
- If $G(\vec{a}) \neq 0$, $\frac{F}{G}$ is differentiable at $\vec{a}$
- If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $\frac{d}{dx}(g \circ f) = g'(f(a)) * f'(a)$

- The graph of a function is the set $\{(x, y, f(x, y)) \in \mathbf{R^3} : (x, y) \in \text{ domain}\}$
- If f_x, f_y, f_{xy}, f_{yx} are continuous, then $f_{xy} = f_{yx}$

8 Differentiability in $\mathbb{R}^3$

8.1 Chain Rule in Composition

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$, where the RHS is a matrix multiplication

Example: $F(x, y) = (x^2y, e^{3x})$ and $G(x, y) = (x + y, xy, \sin(2x - y))$

Find: $D(G \circ F)(1, 1)$, where $(1, 1) = (\vec{a})$

Apply the chain rule equation and get $= DG(1, e^3)DF(1, 1)$

$DF = \begin{bmatrix} 2xy & x^2 \\ 3e^{3x} & 0 \end{bmatrix}$ and $DG = \begin{bmatrix} 1 & 1 \\ y & x \\ 2\cos(2x - y) & -\cos(2x - y) \end{bmatrix}$

$DF(1, 1) = \begin{bmatrix} 2 & 1 \\ 3e^3 & 0 \end{bmatrix}$ and $DG(1, e^3) = \begin{bmatrix} 1 & 1 \\ e^3 & 1 \\ 2\cos(2 - e^3) & -\cos(2 - e^3) \end{bmatrix}$

Now, $D(G \circ F)(1, 1) = \begin{bmatrix} 2 + 3e^3 & 1 \\ 5e^3 & e^3 \\ 4\cos(2 - e^3) - 3e^3\cos(2 - e^3) & 2\cos(2 - e^3) \end{bmatrix}$

8.2 Polar Coordinate Examples

$x = r\cos\theta, y = r\sin\theta$

$DH(r, \theta) = DG(r\cos\theta, r\sin\theta)DF(r, \theta)$

$DH(r, \theta) = \frac{\partial G}{\partial x}\cos\theta + \frac{\partial G}{\partial y}\sin\theta - \frac{\partial G}{\partial x}r\sin\theta + \frac{\partial G}{\partial y}\cos\theta$

Example: Find DH

With a given $r, \theta, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$, we can find $DH(r, \theta)$ through the chain rule

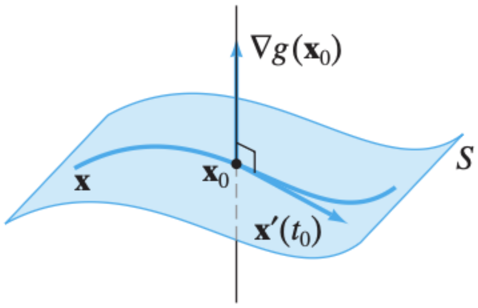
Example: Find DG

With a given $r, \theta, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial \theta}$, we can find DG with: $\left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right] = \left[\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right] \cdot DF^{-1}$

9 Applications of the Gradient

9.1 Gradients and level curves

If we have a level curve for the function x^2+y^2, so $f(x,y)=c=x^2+y^2$, then the gradient ∇F is always perpendicular to the tangent plane to the level curve



Thus, the equation of the tangent plane is given by $\nabla F \cdot (\vec{x}-\vec{a})=0, \forall \vec{x}$ on tangent plane, where $\vec{a}$ is the fixed reference vector

Example: Find equation of tangent plane given the function and the reference vector

$f(x,y)=x^2y+ye^x$ at $(0,1,-1)$

Isolate and get the gradient: $f(x,y,z)=z-x^2y+ye^x$ $\nabla F=(-2xy+ye^x,-x^2+e^x,1)$ $\nabla F(0,1,-1)=(1,1,1)$

$(1,1,1)\cdot(x-0,y-1,z+1)=0 \therefore x+y+z=0$

9.2 Magnitude of ∇F

Consider the directional derivative $D_{\vec{v}}f(\vec{a})=\nabla f(\vec{a})\cdot\vec{v}$

In what direction does the function increase the most?

If θ is the angle between $\vec{v}$ and teh gradient vector $\nabla f(\vec{a})$, then we have:

$D_{\vec{v}}f(\vec{a})=||\nabla f(a)||||\vec{v}||\cos\theta=||\nabla f(\vec{a})||\cos\theta$ because the magnitude of the unit vector $\vec{v}=1$

Thus, the max ROC is at $\theta=0,=||\nabla f(\vec{a})||$

The min ROC is at $\theta=\pi,=-||\nabla f(\vec{a})||$ and is opposite to $\nabla f(\vec{a})$

9.2.1 Example

Given $f(x,y)=3\sin xy, \vec{a}=(1,\pi)$ find: 1. direction of max ROC, value of ROC at $f(\vec{a})$, and direction of tangent to the level curve at $\vec{a}$

1. Get gradient, plug in point, $\therefore$ max ROC is in the direction of gradient
2. Get magnitude of gradient at point, $\therefore$ this is the max ROC
3. ∇f is perpendicular to tangent line to the level curve at $(1,\pi)$. Find $\vec{v}\perp(-3\pi,-3)$

10 Conservative Vector Fields

A vector field is conservative if $\exists f : U \to \mathbf{R}$ such that $F = \nabla f$

The function f is called a potential function of F

Example: $F(x, y) = (2x, 2y)$

Thus, if $F = \nabla f$ and the potential function $f(x, y) = x^2 + y^2$, then $F(x, y)$ is conservative and f is the potential function

10.1 Test for conservative

Function $G(x, y, z)$ is conservative if

$$\begin{array}{ccccccccccc} (G_1)_y & = & (G_2)_x & \qquad & (G_2)_z & = & (G_3)_y & \qquad & (G_1)_z & = & (G_3)_x \\ \parallel & & & & \parallel & & \parallel & & \parallel & & \parallel \\ F_{xy} & = & F_{yx} & & F_{yz} & = & F_{zy} & & F_{xz} & = & F_{zy} \end{array}$$

10.2 Reconstruct a potential function given its gradient

Find $\nabla f = (f_x, f_y, f_z) = g = (g_1, g_2, g_3)$

1. Integrate g_1 wrt x

$f(x, y, z) = \int g_1 dx + h(y, z)$

2. Differentiate wrt y, set equal to g_2, solve for $h(y, z)$ by integrating wrt y and get a $k(z)$ term
3. Differentiate wrt z, set equal to g_3, solve for $k(z)$ up to constant C
4. Assemble final $f(x, y, z) + C$

11 Parametrization and Class

Definition of Path: a continuous function $f : I \to \mathbf{R^n}$ where $I \in \mathbf{R}$ is on the interval $[a, b]$

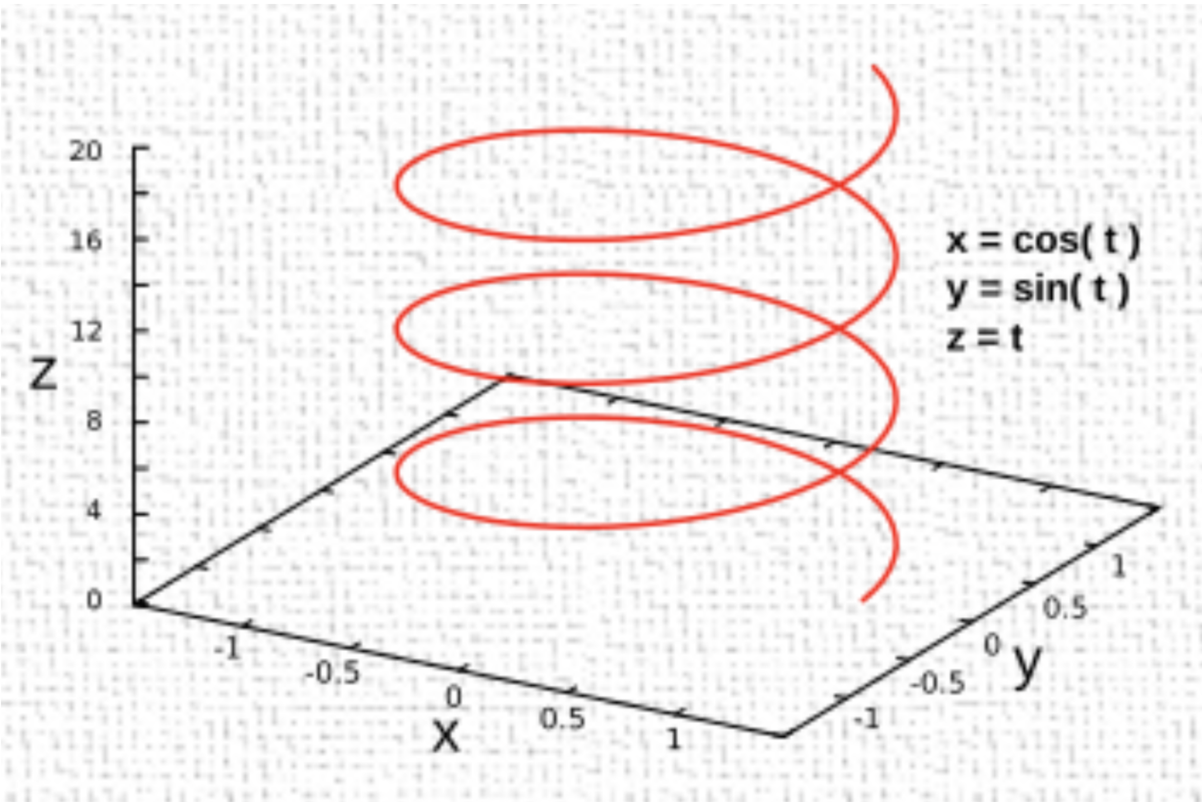
11.1 Parametrization

Ti parametrize a function means to express it in terms of one or more new variables, called parameters, instead of directly in terms of the original variables. Oftentimes, we introduce a variable t that "traces out" the function as it changes

[[Trigonometric Parametrization]]: Use cos and sin when parametrizing a circle, an ellipse $(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1)$, or a super-ellipse $(x^{2/n} + y^{2/n} = 1)$

$f(a)$ = starting point of f, $f(b)$ = end point of f

The Im of the path, denoted by $f(I)$ is called the curve in $\mathbf{R^2}$ and f is a parametrization of C



Important result: Parametrization is not unique

$f(t) = (\cos t, \sin t)$ and $g(t) = (t, \sqrt{1 - t^2})$ have the same curve $\text{Im}(f) = \text{Im}(g)$

11.2 Class

Example: $y^2 = x^3$

Parametrized: $f(t) = (t, t^{3/2}) \to f'(t) = \left(1, \frac{3}{2} \cdot \sqrt{t}\right) \to f'' = \left(0, \frac{3}{4} \cdot \frac{1}{\sqrt{t}}\right)$, which is not defined at $t = 0$

$\therefore f$ is of class C^1 and not C^2

12 Arc Length, Divergence, and Curl

12.1 Arc Length

[Arc length] from a to b, with $f : I \to \mathbb{R}^m$, and c is a curve in f:

$L(f) = \int_a^b ||f'(t)||\, dx$

Method: get parametrization $f(t)$, get speed, then integrate w.r.t. bounds

12.2 Divergence of a vector field

Denoted by $Div(f) = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$, it can measure net mass flow or flux density

If $Div(f) > 0$, consider the field as a source, flowing out If $Div(f) < 0$, consider the field as a sink, flows in

12.3 Curl of a vector field

$Curl(F) = \nabla \times F = \begin{vmatrix} i & -j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$Curl(F) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

$\begin{array}{cc} y & 2 \\ z & 3 \\ x & 1 \\ y & 2 \end{array}$ $\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right.$

13 Identities of Operations in $\mathbb{R}^3$

Scalar field: $f(x, y, z)$ Vector field: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

∇f inputs a scalar field and outputs a vector field

$\nabla \cdot F$ inputs a vector field $\vec{F}$ and outputs a scalar field

$\nabla \times F$ inputs a vector field $\vec{F}$ and outputs a vector field

13.1 Identities

The curl of a gradient, $\nabla \times (\nabla f) = \vec{0}$, gradient fields are irrotational

The divergence of a curl, $\nabla \cdot (\nabla \times \vec{F}) = 0$, curl fields have no net source

The divergence of a gradient, $\nabla \cdot (\nabla f)$ is the Laplacian, Δf, a scalar field

The curl of a divergence, $\nabla \times (\nabla \cdot \vec{F})$ is undefined, divergence can't input a scalar field

The gradient of a curl, $\nabla(\nabla \times \vec{F})$ is undefined, gradient can't input a vector field

The curl of a curl, $\nabla \times (\nabla \times (F)) = \nabla(Div(F)) - \nabla^2 F$, and is defined in $\mathbb{R}^3$

G is conservative if $\exists f : U \to \mathbb{R}$ such that $G = \nabla F$, where F is the potential function

The dot product of two vector fields, e.g. $F \cdot G$, is a scalar field defined by $\mathbb{R}^3 \to \mathbb{R}$

If $G : U(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$, so (G_1, G_2). If $Curl(G) = 0$, then G is conservative

If $G : U(\subseteq \mathbb{R}^3) \to \mathbb{R}^3$, if G is the curl of some vector field, then div=0

14 Special Domains and Conservative Functions

A space is path-connected if, for any two points x, y in a space X, there exists a continuous path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$

Let $U \subseteq \mathbb{R}^3$ be an open set. U is simply connected if:

1. U is connected (any two points can be connected by a path)
2. Every loop inside U can be shrunk continuously to a point inside U

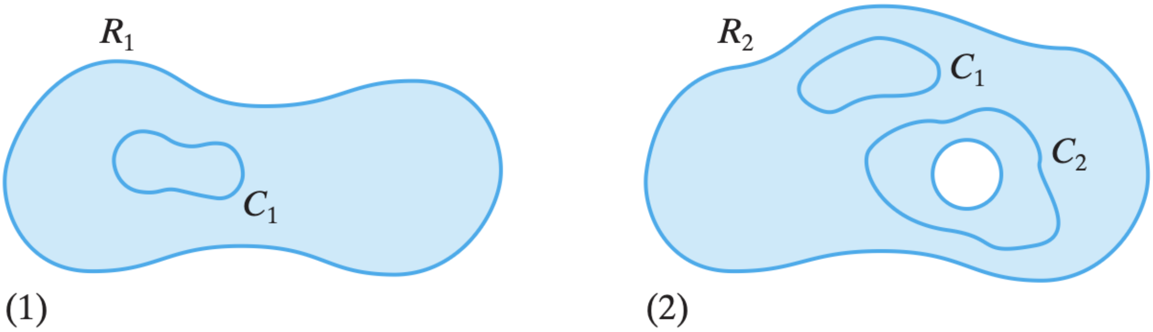


Figure 6.35 (1) The region $R_1 \subset \mathbf{R}^2$ is simply-connected: All points surrounded by any simple, closed curve in R_1 lie in R_1. (2) In contrast, R_2 is not simply-connected: Although the curve C_1 encloses points that lie in R_2, the curve C_2 surrounds a hole. Hence, C_2 cannot be continuously shrunk to a point while remaining in R_2.

If we let $U \subseteq \mathbb{R}^n$ be a simply connected open set, and $F : U \to \mathbb{R}^n$ be a vector field, then f is conservative if and only if $Curl(f) = 0$

Example:

Let $G(x,y,z) = (y^2, 2xy + z, y - \sin z)$, is G conservative? If so, find the potential function f such that $G = \nabla f$

Domain$(G) = \mathbb{R}^3$, simply connected, and $Curl(G) = (1-1, 0, 0-2y-2y) = 0$, thus G is conservative

Let $(G_1, G_2, G_3) = (F_x, F_y, F_z)$

$F_x = y^2 \Rightarrow \int F_x dx = xy^2 + g(y,z)$

$F_y = 2xy + z \Rightarrow \frac{\partial F(x,y,z)}{\partial y} = 2xy + \frac{\partial g(y,z)}{\partial y} = 2xy + z \Rightarrow \frac{\partial g(y,z)}{\partial y} = z$

$g(y,z) = \int z dy = yz + h(z) \Rightarrow F(x,y,z) = xy^2 + yz + h(z)$

$F_z = y - \sin z \Rightarrow \frac{\partial F(x,y,z)}{\partial z} = y + \frac{dh(z)}{dz} = y - \sin z \Rightarrow \frac{dh(z)}{dz} = -\sin z$

$h(z) = \int -\sin z dz = \cos z + C$

$\therefore F(x,y,z) = xy^2 + yz + \cos z$

15 Riemann Sums

15.1 Single-variable Integration

Let $f[a, b] \to \mathbb{R}$ be a function

$\int_a^b f(x)\, dx$ represents the area under the curve

We partition $[a, b]$ into subintervals for **Riemann sums**

Area under $f \approx$ sum of area of rectangles, $A = f(\xi_i)\delta x_i, \Delta x_i = (a_i - a_{i-1}), \xi \in [a_{i-1}, a_i]$

A is integrable on $[a, b]$ if $\lim\limits_{\Delta x_i \to 0} \int \Sigma_{i=1}^n f(\xi_i)\Delta x_i$ exists

15.2 How to integrate functions of two variables

Let $f : [a, b] \times [c, d] \to \mathbb{R}$

$\Delta x_i = a_i - a_{i-1}, \Delta y_j = c_j - c_{j-1}$

V of partitions $= lbh = f(_\xi)\Delta x_i \Delta y_j$

$Vol(A) \approx \Sigma_{i=1}^n \Sigma_{j=1}^m f(\xi_i)\Delta x_i \Delta y_j$

f is integrable over $[a, b] \times [c, d]$ if $\lim\limits_{\Delta x_i \text{ and } \Delta y_j \to 0} \Sigma_{i=1}^n \Sigma_{j=1}^m f(\xi_i)\Delta x_i \Delta y_j$ exists, and is denoted by $\iint_{[a,b]\times[c,d]} f dA$

If f is continuous over $[a, b] \times [c, d]$, then it is integrable

Fubini's Theorem: let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous

Then, $\iint_{[a,b]\times[c,d]} = \int_c^d \int_a^b f(x, y)\, dx\, dy$ and can be reversed

16 Domains in Integration

When integrating over rectangle R such that $\iint_R f(x)dx, Domain(f) = R$

[[Type 1 Region]]:

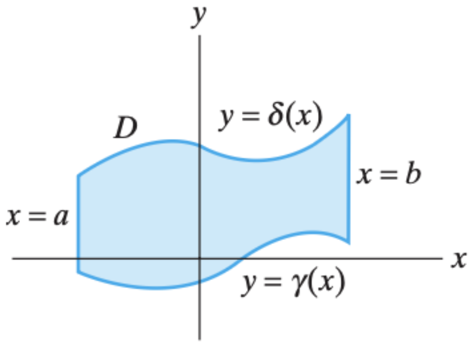
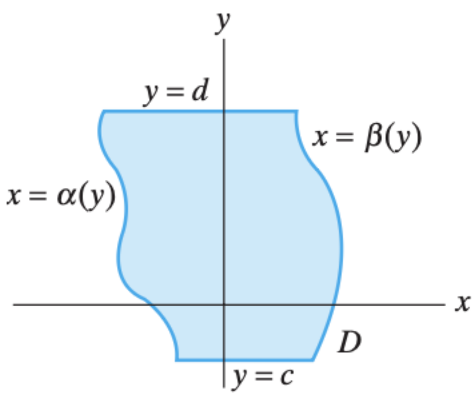


Figure 5.22 A type 1 elementary region.

$D = \{(x, y)|\gamma(x) \leq y \leq \delta(x), a \leq x \leq b\}$, where γ and δs are continuous on $[a, b]$

It is necessary to integrate wrt y first, because x is "uncertain"

[[Type 2 Region]]:



$D = \{(x, y)|\alpha(y) \leq x \leq \beta(y), c \leq y \leq d\}$

It is necessary to integrate wrt x first, because y is "uncertain"

17 Center of Mass

The center of mass of a region $D \subset \mathbb{R}^2$, where $\delta(x, y)$ is density per unit volume

We can get CoM$=(\bar{x}, \bar{y})$:

$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ and $\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$

Example:

Let D be the lamina represented by the unit disk with $\delta(x, y) = 1$, what is the center of mass?

$\iint_D x dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, dy \, dx = 0$ (with u substitution, or notice it is an odd function are we are integrating from $-a$ to a)

Similarly, $\iint_D y dA = \int_{-1}^{1} \int_{-(\sqrt{1-x^2})}^{\sqrt{1-x^2}} y \, dy \, dx = 0$ for the same reasons as above

Therefore, $(\bar{x}, \bar{y}) = (0, 0)$

18 Change of Variables

Any point (x, y) in a circle may be written as $(r\cos\theta, r\sin\theta)$, $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$

These points are on the $r - \theta$ plane, and we call the shape it makes D^*

By changing the variables, the integral becomes $= \iint_{D^*} f \cdot$ "factor" dA

If we take, for example, $x = x(u, v), y = y(u, v)$, the integral is: $\iint_D f(x, y)dA = \iint f(x(u, v), y(u, v))|\frac{\partial(x,y)}{\partial(u,v)}|dudv$, and the Jacobian inside this integral is $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Example:

D is between $y = x, y = 3x, xy = 2, xy = 5$ and in Quarter 1, compute $\iint_D xy^3 dxdy$

Change: $u = xy, v = \frac{y}{x}$, then isolate for y and x:

$u \cdot v = xy \cdot \frac{y}{x} = y^2 \Rightarrow y = \sqrt{uv}$ $\frac{u}{v} = \frac{xy}{\left(\frac{y}{x}\right)} = x^2 \Rightarrow x = \frac{\sqrt{u}}{\sqrt{v}}$

Jacobian: $\frac{1}{2\sqrt{u}}\frac{1}{\sqrt{v}} \cdot \sqrt{u}\frac{1}{2\sqrt{v}} - \left(\frac{1}{2\sqrt{u}}\frac{1}{v} \cdot \sqrt{u}\left(-\frac{1}{2}v^{-3/2}\right)\right) = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}$

$\iint_{D^*} \frac{\sqrt{u}}{\sqrt{v}}(\sqrt{uv})^3 \cdot \frac{1}{2v}dudv = \int_1^3 \int_2^5 \frac{u^2}{2}\, du\, dv = \frac{117}{3}$

19 Triple Integrals and Complicated Regions

Let $R = [a, b] \times [c, d] \times [p, q]$ be a cube in $\mathbb{R}^3$ Let $f : R \to \mathbb{R}$ be a continuous function

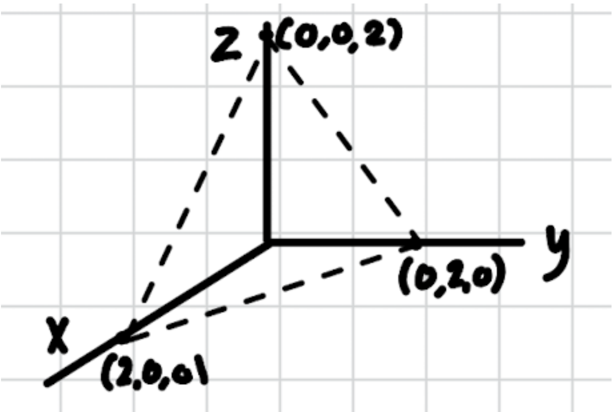
Fubini's theorem states that $\int\int\int_R f dV = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$

Example: $\int_0^5 \int_3^4 \int_1^2 xy^2z^3\, dx\, dy\, dz$

Through simple integration, we get $37 \cdot \frac{625}{8}$

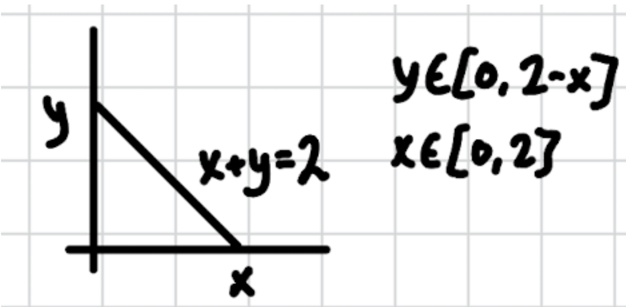
19.1 Complicated Regions

Let $\int\int\int_D f dV$, $D = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x + y + z \leq 2\}$, where $x, y, z \geq 0$ states that the region D lies in Q1



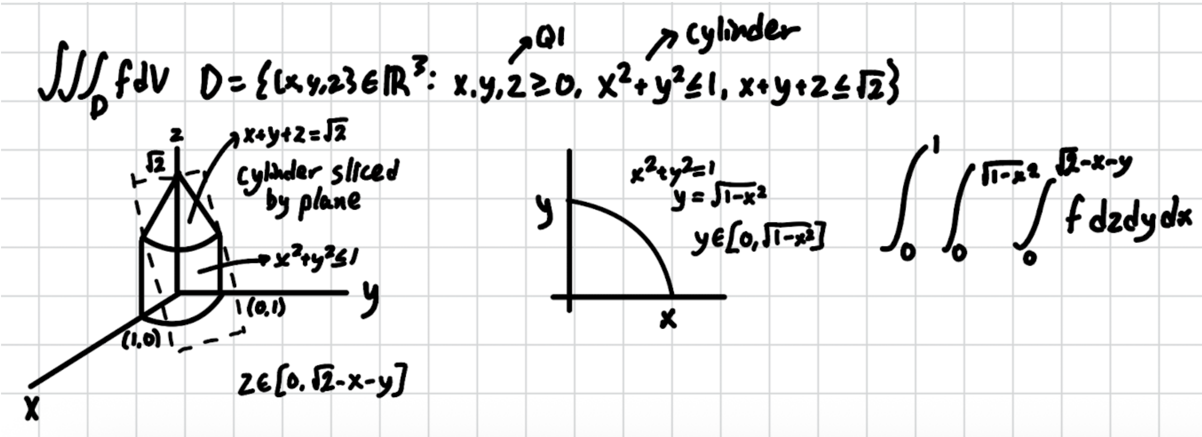
We see that $z \in [0, 2 - x - y]$ by $z = 2 - x - y$

By setting $z = 0$, we see that



These are the variables we will put into the bounds of our integral

See a cylindrical region example below



Full Example: Let $D = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x^2 + y^2 \leq 1, x + y + z \leq \sqrt{2}\}$

Rearrange for z: $z = \sqrt{2} - x - y$

By the cylinder, we can rearrange for y: $y = \sqrt{1 - x^2}$

Therefore, $Vol(D) = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{2}-x-y} 1\, dz\, dy\, dz$

By integration, and the cosine rule $\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$, we get the volume: $= \frac{3\pi\sqrt{2}+7}{12}$

20 Cylindrical and Spherical Coordinates

20.1 Cone

A cone has the equation $z^2 = x^2 + y^2$

Let $D = \{(x, y, z) \in \mathbb{R}^3 : -9 \leq x \leq 9\}$

This tells us that $z \in [\sqrt{x^2 + y^2}, 9]$, because the upper x bound 9 is the outermost portion of the cone, which is also the topmost point of the cone

By setting $z = 0$, we see its projection as a circle, with $x \in [-9, 9]$

Therefore, $y \in [-\sqrt{9^2 - x^2}, \sqrt{9^2 - x^2}]$

20.2 Cylindrical

Consider a cone with coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

and bounds $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$

By similar triangles, we get $\tan\phi = \frac{a}{h} = \frac{r}{z}$, therefore $z = \frac{hr}{a}$

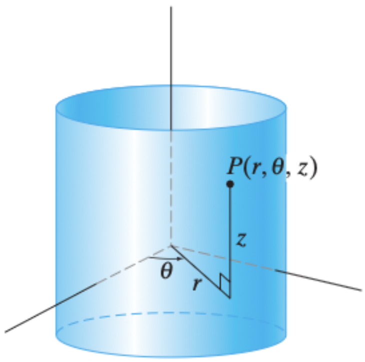


Figure 1.90 Locating a point P, using cylindrical coordinates.

We can set the integral $Vol(D) = \int_0^{2\pi} \int_0^a \int_{\frac{hr}{a}}^h f(x(r, \theta, z), y(r, \theta, z), z(r, \theta, z)) \cdot |\frac{\partial(x,y,z)}{\partial(r,\theta,z)}| \, dz \, dr \, d\theta$, where the Jacobian is simply r

By integration, we get $\frac{a^2 h\pi}{3}$

20.3 Spherical Coordinates

By the image below, we see that $r = \rho\sin\phi$

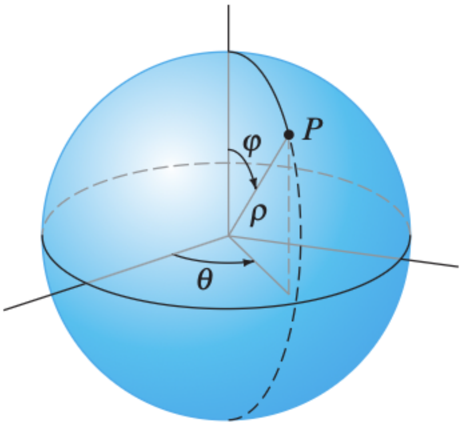


Figure 1.97 Locating the point P, using spherical coordinates.

Therefore, spherical coordinates are: $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$

Note: a sphere are boundary points only, a ball are interior and boundary points

Example: Compute the volume of a ball of radius a

The Jacobian of spherical coordinates is $\rho^2 \sin\phi$

$Vol(D) = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

By integration, we get $\frac{4\pi a^3}{3}$

21 Scalar and Vector Line Integrals

Integration over a line is a way of integrating a function along a curve or path in space, rather than over an axis.

21.1 Scalar Line Integrals

Let a curve C be in a space, and it is described by $\gamma(t) = (x(t), y(t))$, $\gamma : [a, b] \to \mathbb{R}$

Then the line integral of a scalar function $f(x, y)$ along γ is:

$\int_\gamma f(x, y)\, ds := \int_a^b f(\gamma(t))||\gamma'(t)||\, dt$

Example: Compute the line integral of $f(x, y, z) = xy + z$ along the curve $\gamma(t) = (\cos t, \sin t, t), \gamma : [0, 2\pi] \to \mathbb{R}^3$

$\gamma'(t) = (-\sin t, \cos t, 1), ||\gamma'(t)|| = \sqrt{2}, f(\gamma(t)) = \cos t \sin t + 1$

By definition, $\int_\gamma f\, ds = \int_0^{2\pi} \sqrt{2}(\cos t \sin t + 1)\, dt$

By integration, $= 2\sqrt{2}\pi^2$

21.1.1 Line integral depends solely on the image of parameterization

For example, the half circle above the $x-$axis has the parameterizations:

$\gamma : [0, \pi] \to \mathbb{R}^2, \gamma(t) = (\cos t, \sin t)$ and $\delta : [-1, 1] \to \mathbb{R}^2, \delta(t) = (t, \sqrt{1 - t^2})$

By finding the line integral of both parameterizations, we get $\frac{2}{3}$ for both, hence why the line integral is independent of the parametrization

Therefore, line integrals of scalar functions depend solely on the curve' image and not the specific way you move along it (parameterization)

21.2 Vector Line Integrals

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 path, and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field

Definition: $\int_\gamma F \cdot ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$

Example: let $\gamma : [0, \pi] \to \mathbb{R}^2$ be $\gamma(t) = (-\cos t, \sin t)$, compute $\int_\gamma F \cdot ds$

Find $F(\gamma(t))$, then $\gamma'(t)$, compute, and get $\frac{2}{3} + \frac{\pi}{2}$

Example 2: Try with a different parameterization $\delta(t) = (\cos t, \sin t)$, which is the opposite direction to γ from the previous example

Notice that the result is opposite from last time, therefore $\int_\gamma F\, ds = -\int_\delta F \cdot ds$

Notice that the difference between vector and scalar line integrals is that the form has $ds = \gamma'(t)dt$, which is the **vector differential of position** along the curve

21.2.1 Tangential component

The vector line integral can be rewritten as a scalar line integral involving the *tangential component* of F:

The **unit tangent vector** at t:

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$$

Therefore, we may represent $\int_C F \cdot ds$ as $\int_a^b F(\gamma(t)) \cdot T(t)|\gamma'(t)|\, dt$

At each point, we are projecting F onto the direction of the curve (i.e., measuring how much F "pushes" along the curve), and integrating this along the entire curve

22 Circulation and the Fundamental Theorem of Line Integrals

An important interpretation of the vector line integral occurs when γ is a closed path. In this circumstance, the quantity $\int_\gamma F \cdot ds$ is called the **circulation** of F along x.

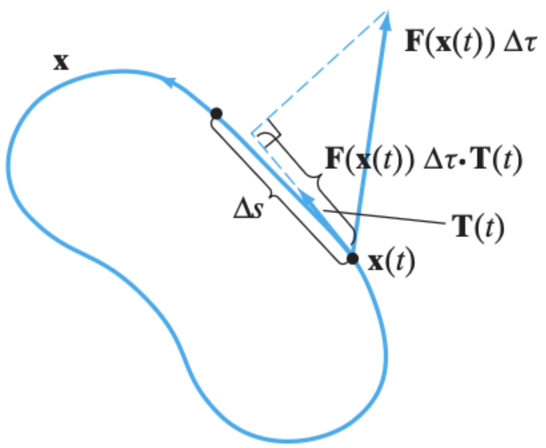


Figure 6.6 The amount of fluid transported tangentially along a segment of the closed path **x** is approximately $(\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T}(t))\,\Delta s$.

Sometimes, line integrals are path dependent, and sometimes independent. For example, for $F(x, y) = (x, y)$, and $c_1 = y = x^2, c_2 = y^2 = x$, this integral is path independent (i.e. only the start/end points matter, not the path to them).

Theorem: Let $D \subseteq \mathbb{R}^n$ be path connected and $F : D \to \mathbb{R}^n$ be a continuous vector field. Then F has path independent line integrals if and only if:

$$\oint_c F \cdot ds = 0$$

for any closed piecewise C^1 path on D

Fundamental Theorem of Calculus: Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function and F is continuous, then

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

22.1 Fundamental Theorem for Line Integrals

Let $F : D(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be a C^1 vector field. Suppose $F = \nabla f$ for some f. Then, for any path $\gamma : [a, b] \to \mathbb{R}^n$ inside D:

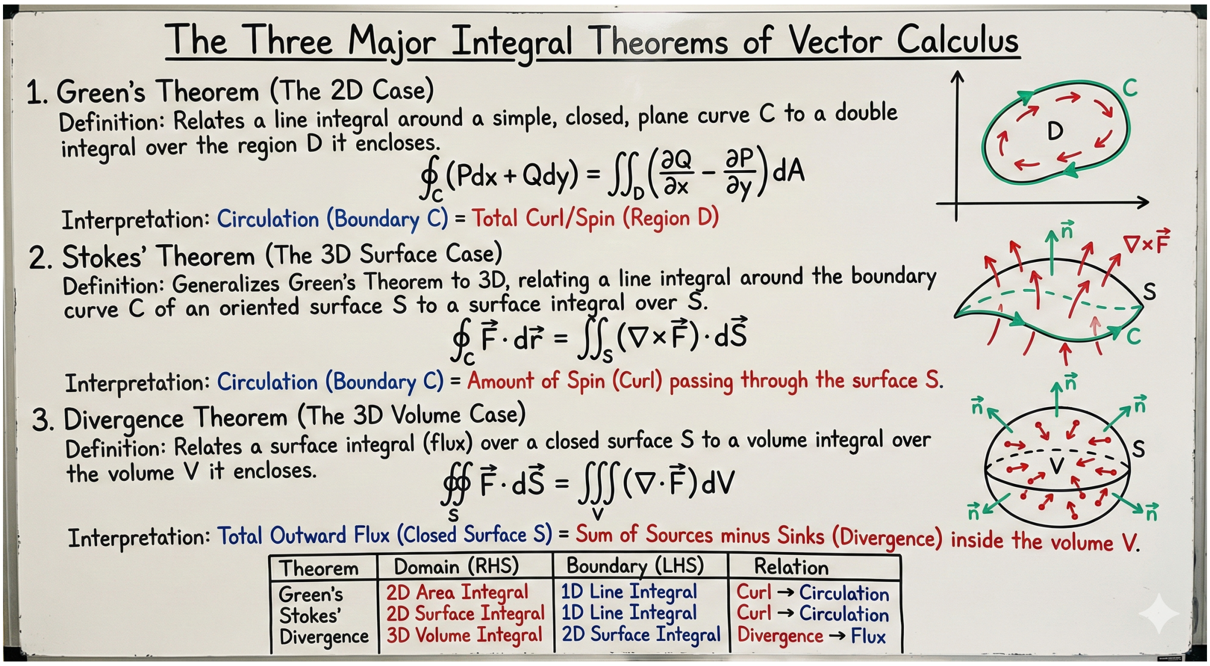
$$\int_\gamma F\,ds = f(B) - f(A)$$

Example: Let $f(x,y) = e^y \sin\left(\frac{\pi x}{2}\right) + 2x + 3y$ and $F = \nabla f$, $F = \left(\frac{\pi}{2} e^y \sin\left(\frac{\pi x}{2}\right) + 2, e^y \sin\left(\frac{\pi x}{2}\right) + 3\right)$. Let γ be a path from $(0,0) \to (1,1)$. Compute $\int_\gamma F \cdot ds$

Consider FTLI: $\int_\gamma F \cdot ds = F(B) - F(A)$ since $F = \nabla f$

$= f(1,1) - f(0,0) = e \sin\left(\frac{\pi}{2}\right) + 5 - 0 = e + 5$

23 Green's Theorem



Green's Theorem: Let C be a positively oriented (CCW), simple, closed, piecewise-smooth curve in the plane, and let D be the region enclosed by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region containing D, then:

$$\underbrace{\oint_C (P\,dx + Q\,dy)}_{\text{circulation around the boundary C}} = \underbrace{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\text{total curl (spin) over region R}}$$

Example: Let C be a circle with radius 2, and $F(x, y) = (14(1 + x^2)^{1/4} - 4y, e^{y^2} - 23\sin(\sin y) + x)$. Compute $\oint_C F ds$

Consider Green's Theorem, where $\frac{\partial F_2}{\partial x} = 1$ and $\frac{\partial F_2}{\partial y} = -4$. Then:

$$\oint_C F ds = \int\int_D (1 - -4) dxdy = 5 \int\int_D dxdy = 5\pi r^2 = \boxed{20\pi}$$

Note that $\int\int_D dxdy = \pi r^2$, i.e., the area of a circle

Example: Find area of the hyper-cycloid with boundary $x^{2/3} + y^{2/3} = 1$ with $F = \left(-\frac{y}{2}, \frac{x}{2}\right)$

Parameterize: $(x^{1/3})^2 + (y^{1/3})^2 = 1 \Rightarrow x = \cos^3 t, \quad y = \sin^3 t, 0 \le t \le 2\pi$

By the vector line integral equation, we get $\frac{3\pi}{8}$

Example: Let D be inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ above $x = 0$. Compute $\oint_C F ds$, where $F(x, y) = (0, xy)$

We know that $x \in [-3, 3]$, and $\frac{y^2}{4} < 1 - \frac{x^2}{9}$ by rearranging, $\Rightarrow y < 2\sqrt{1 - \frac{x^2}{9}}$

Also, $\frac{\partial F_2}{\partial x} = y$ and $\frac{\partial F_1}{\partial y} = 0$.

$$\oint_{\partial D} f\, ds = \int_{-3}^{3} \int_{0}^{2\sqrt{1-x^2/9}} y\, dy\, dx = 8$$

Remark: F must be C^1 throughout D, including the origin, for Green's Theorem to be applicable

24 Scalar and Vector Surface Integrals

Definition: Let D be a connected set $\in \mathbb{R}^3$, let $\gamma : D \to \mathbb{R}^3$ be a continuous and one-one function expect possibly on ∂D, let $S = \mathrm{Im}(\gamma)$, then S is a **surface** in $\mathbb{R}^3$ and γ is the parameterization of S

Definition: Consider that any point on $S : (s, t, f(s,t))$

Example: Parameterize $Graph(f)$ in the unit disk D where $f(x,y) = \sqrt{x^2+y^2}$ (this is a cone)

We know that $\gamma(s,t) = (s, t, \sqrt{x^2+y^2})$. Also consider that $\gamma(s,t) = (t\cos s, t\sin s, t) \quad 0 \leq t \leq 1$

24.1 Smooth Surface

Definition: S is smooth if the normal vector to two tangent lines at $\gamma(s,t,)$ is not equal to 0

Example: Take any point (s_0, t_0) on the surface, then we have two tangent vectors $T_s = \frac{\partial \gamma}{\partial s}(s_0,t_0)$ and $T_t = \frac{\partial \gamma}{\partial t}(s_0,t_0)$. We find the normal vector $N(s_0,t_0) = T_s \times T_t$

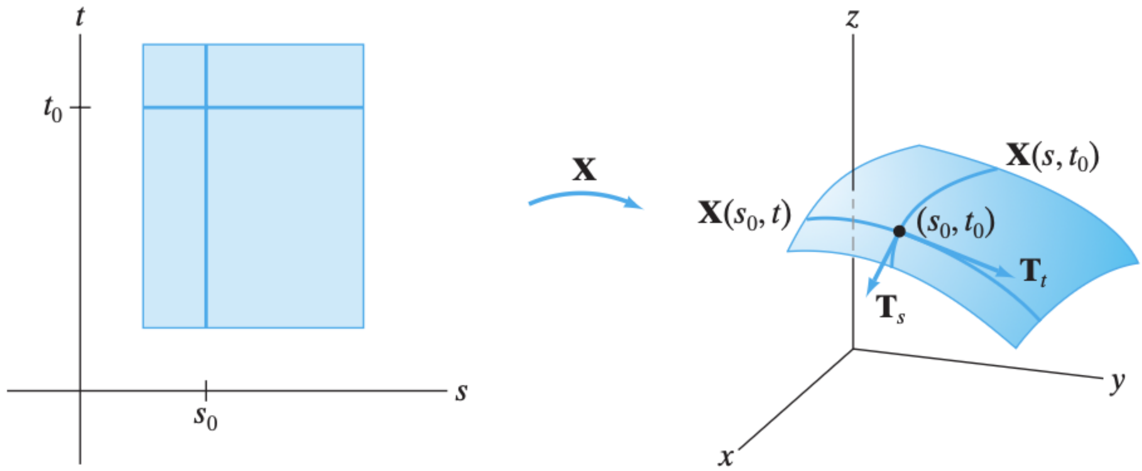


Figure 7.10 The tangent vectors $\mathbf{T}_s$ and $\mathbf{T}_t$ to the coordinate curves.

24.2 Scalar Surface Integrals

Definition: Let $X : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Let f be a continuous function whose domain includes $S = X(D)$. Then the **scalar surface integral** of f along X is denoted by:

$$\int\int_X f\, dS = \int\int_D f(X(s,t))||T_s \times T_t|| ds\, dt = \int\int_D f(X(s,t))||N(s,t)|| ds\, dt$$

Example: Compute SA of a sphere with $r = a$, $\gamma : [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$

$\gamma(\rho, \theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\theta)$, and we can find $||N|| = a^2 \sin\phi$

Then, Area(s)= $\int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi \, d\rho \, d\theta = 4\pi a^2$

Example: Compute SA of cone $z = \sqrt{x^2+y^2}$ over disk $x^2+y^2 \le 9$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

We can parametrize S: $\gamma(r,\theta) = (r\cos\theta, r\sin\theta, r)$, and find $||N|| = \sqrt{2}r$

SA of cone: $\int_0^{2\pi}\int_0^{3} \sqrt{2}r \, dr \, d\theta$

Example: Compute SA of a half cylinder

$\gamma : D \to \mathbb{R}^3, S = \text{lm}(\gamma)$ let $f : S \to \mathbb{R}$ be a continuous function then, the scalar surface integral of f over S is defined by:

$$\iint_\gamma f ds = \iint_0 f(\gamma(s,t)) \, \|T_s \times T_t\| \, dsdt$$

ex: let $S = \{(x,y,z) \in \mathbb{R}^3 \mid -1 \le x \le 1, y \ge 0, x^2+y^2 = 1\}$, compute $\iint_S f ds$, $f(x,y,z) = y$

Parameterize S: $x = \cos\theta, y = \sin\theta, z = r$. $\gamma : [0,\pi] \times [-1,1] \to \mathbb{R}^3, \gamma(\theta, r) = (\cos\theta, \sin\theta, r)$.

Then, we get the tangential components: $T_\theta = (-\sin\theta, \cos\theta, 0), T_r = (0,0,1)$.

$$\begin{vmatrix} i & -j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0) \Rightarrow ||T_\theta \times T_r|| = 1$$

$$\iint_\gamma f dS = \int_0^\pi \int_{-1}^{1} f(\gamma(\theta, r)) \, \|T_\theta \times T_\gamma\| \, drd\theta = 4$$

24.3 Vector Surface Integrals

$D \in \mathbb{R}^2$ and $\gamma : D \to \mathbb{R}^3$, let $S = 1$ m(r), a surface on $\mathbb{R}^3$, let $F : S \to \mathbb{R}^3$ be a continuous vector field, the vector surface integral is defined by:

$$\begin{aligned}\iint_s F \cdot ds &:= \iint_D F(\gamma(s,t)) \cdot (T_s \times T_t) \, dsdt \\ &= \iint_D F(\gamma(s,t)) \cdot N(s,t) dsdt\end{aligned}$$

Remark: Vector surface Integrals depend on the parametrization of γ

Oriented Surfaces:

let S be an $\mathbb{R}^3$ surface. It is oriented If we can define normal vectors at every point of S in a continuous way. i.e., two close-by vectors have the same directions

Example: let S be the graph of $z = 5 - x^2 - y^2$ over the rectangle $(x,y) \in [-1,1] \times [-1,1]$, oriented upwards, let $F(x,y,z) = (y, 0, z)$, compute $\iint_e F \cdot ds$

Parametrize S such that N is oriented upwards: $\gamma : [-1,1] \times [-1,1] \to \mathbb{R}^3 \Rightarrow \gamma(s,t) = (s,t,5-s^2-t^2)$

Then, we get the tangential components: $T_s = (1,0,-2s), T_t = (0,1,-2t)$

$$\begin{vmatrix} i & -j & k \\ 1 & 0 & -2s \\ 0 & 1 & -2t \end{vmatrix} = (2s, 2t, 1)$$

If we plug in $s = t = 0$, we see that $|T_s \times T_t| = (0,0,1)$ is pointed upwards

$$\begin{aligned} \int_s f \cdot ds &= \int_{-1}^{1} \int_{-1}^{1} F\left(s,t,s-s^2,c^2\right) \cdot (2s,2t,1) dsdt \\ &= \int_{-1}^{1} \int_{-1}^{1} \left(t,0,s-s^2-t^2\right) \cdot (2s,2,1) dsdt \\ &= \int_{-1}^{1} \int_{-1}^{1} 2st + s - s^2 - t^2 dsdt \\ &= \int_{-1}^{1} 2t + 5 - t^2 - 2/3 dt \\ &= 4 + s - 2/3 - 2/3 = 9 - 4/3 = 23/3 \end{aligned}$$

25 Stokes' Theorem

How can we generalize Green's Theorem in $\mathbb{R}^3$?

in $\mathbb{R}^2$:

$$\oint_{\delta D} F \cdot dS = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

in $\mathbb{R}^3$: points $(x, y) \in \mathbb{R}^2 \longrightarrow (x, y, 0) \in \mathbb{R}^3$

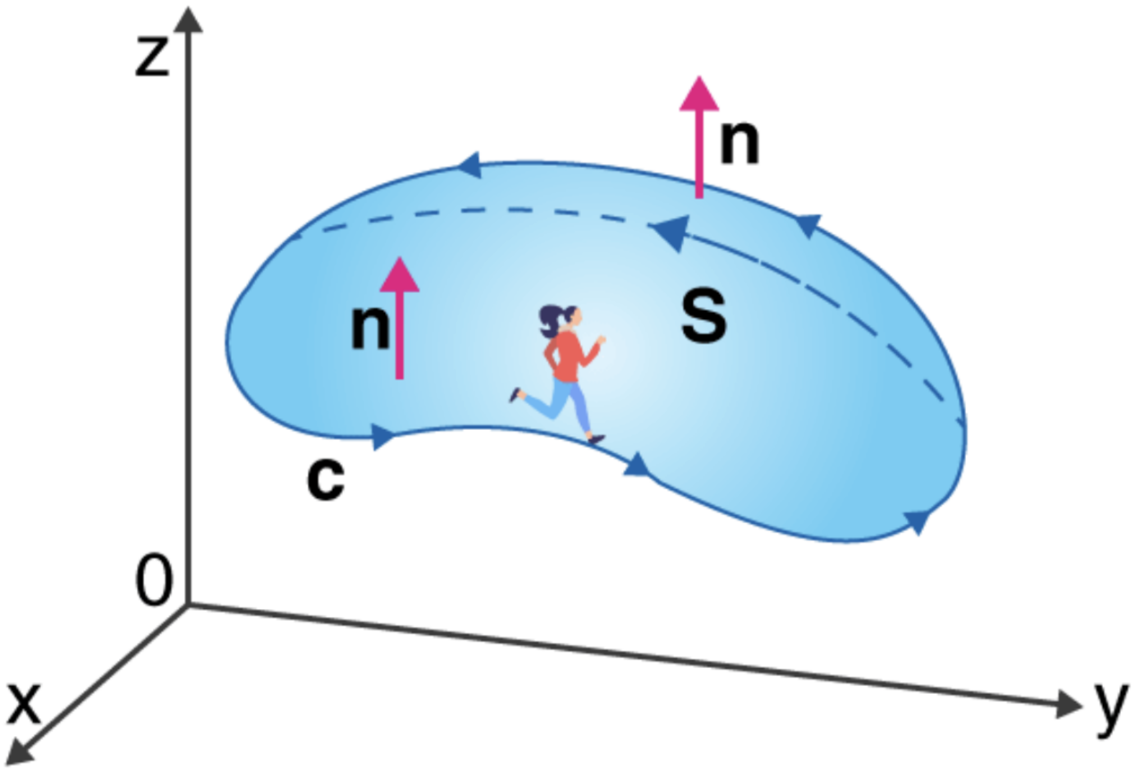
Generally: $F(x, y) = (F_1, F_2) \longrightarrow F^*(x, y, z) = (F_1, F_2, F_3)$. $Curl(F^*) = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

We can let $k = (0, 0, 1)$ and see that $Curl(F^*) \cdot k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Stokes' Theorem: Let S be an oriented, smooth surface in $\mathbb{R}^3$ with a positively oriented (right-hand rule), simple, closed, piecewise-smooth boundary curve $C = \delta S$. If $F(x, y, z)$ is a vector field with continuous partial derivatives on an open region containing S, then:

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{circulation around the boundary of S}} = \underbrace{\iint_S \nabla \times \vec{F} \cdot d\vec{S}}_{\text{amount of spin across the surface}}$$

Furthermore, the orientation of the surface is important. If you choose the upward normal on a disk, the boundary must be counterclockwise.



26 Divergence Theorem

Suppose $D \subset \mathbb{R}^3$ is a domain bounded by a simple closed surface S, and $\vec{F}$ is a vector field defined on all of D. Then,

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{total outward flux through the closed surface}} = \underbrace{\iiint_V \nabla \cdot \vec{F}\, dV}_{\text{sum of sources minus sinks inside the volume}}$$

Notes:

- simple := no self-intersections
- closed := no holes
- the surface must be oriented away from D (dS must include the normal vector $\vec{n}$ of the surface)

Example: The cone C is defined by $z^2 = x^2 + y^2$ where $z \geq 1$. $\vec{F} = \langle 3z \sin y, e^{z^2 x + \cos x}, z \rangle$. Compute the volume of the cone.

Note that the cone is not closed, we must add a plane T to close it, which then changes the divergence theorem's right hand side.

Find that $div(\vec{F}) = 0 + 0 + 1 = 1$. With the divergence theorem, we then find that:

$$Vol(D) = \iint_C \vec{F} \cdot \vec{n} dS + \iint_T \vec{F} \cdot \vec{n} dS$$

Example: Given a cylinder $T = \{x, y, z\} \in \mathbb{R}^3, x^2 + y^2 \leq 9, 0 \leq z \leq 2$, compute $\iint_{S_2} F \cdot dS$

By the divergence theorem, we find that $Div(x, y, z) = 1 + 1 + 1 = 3$, so we can:

$$\iiint_T Div(F) \cdot dS = \iiint_T 3dV = 3Vol(T) = 3\pi r^2 h = 3\pi 3^2 2 = 54\pi$$

Example: Similarly for a sphere and the vector field $F(x, y, z) = (2x + e^{y^2}, z^z + e^{(\log x)^2}, y^{e^y}$, we find that $Div(F) = 2 + 0 + 0 = 2$, so we can:

$$\iiint_T Div(F) \cdot dV = 2Vol(D) = 2 \cdot \frac{4\pi r^3}{3} = \frac{8\pi}{3}$$

Example (not closed volume):

let $S = \mathrm{Gr}(f), z = (1 - x^2 - y^2)\, e^{\left(-3x^2 - 3y^2\right.}$ oriented out, and $z \geq 0$

Compute $\iint_S F \cdot ds$, where $F(x, y, z) = \left(e^y \cos z, \sqrt{x^2 + 1} \sin z, x^2 + y^2 + 3\right)$

$z = [1 - (x^2 + y^2)]\, e^{1 - 3\left(x^2 + y^2\right)}$. To ensure that $z \geq 0, x^2 + y^2 \leq 1$

By the graph of the function, we can tell that it is not closed. We must add a unit disk S_1 and let $S_2 = S \cup S_1$

Now, the LHS: $\iint_{S_2} F \cdot dS = \iint_S F \cdot dS + \iint_{S_1} F \cdot dS$

RHS: $\text{Div}(F) = 0$, so $\iiint_T \text{Div}(F)dV = 0$

$$\begin{aligned}
\Rightarrow \iint_S F \cdot ds &= -\iint_{S_1} F \cdot ds \\
&= \int_0^{2\pi} \int_0^1 F(\gamma(r,\theta)) \cdot N dr d\theta \\
&= 2\pi \int_0^1 \left(\cdots, \cdots, r^2\cos^2\theta + r^2\sin^2\theta + 3\right) \cdot (0,0,-r) dr \\
&= 2\pi \int_0^1 \left(3r - r^2\right) dr \\
&= 5\pi/2
\end{aligned}$$

Example (converting SSI to VSI to use Divergence Theorem): Let S be the unit sphere. Compute the scalar surface integral $\iint_S (x^2 + y + z)dS$. The divergence theorem is applicable if we convert scalar to vector surface integral.

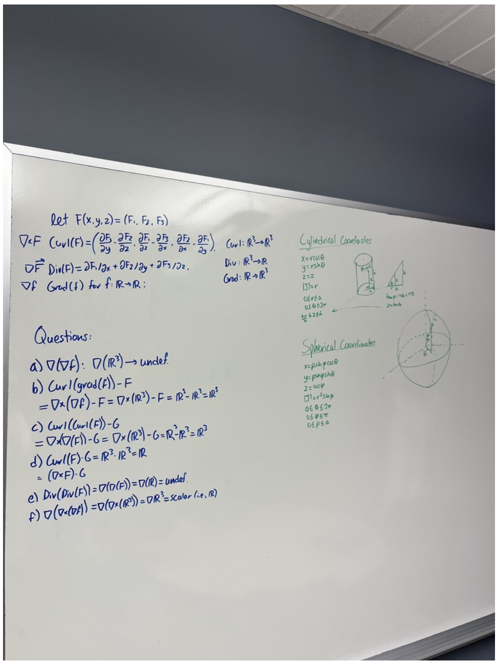
Converting, let $n = \frac{N(s,t)}{\|N(s,t\|}$ be the unit outward normal vector

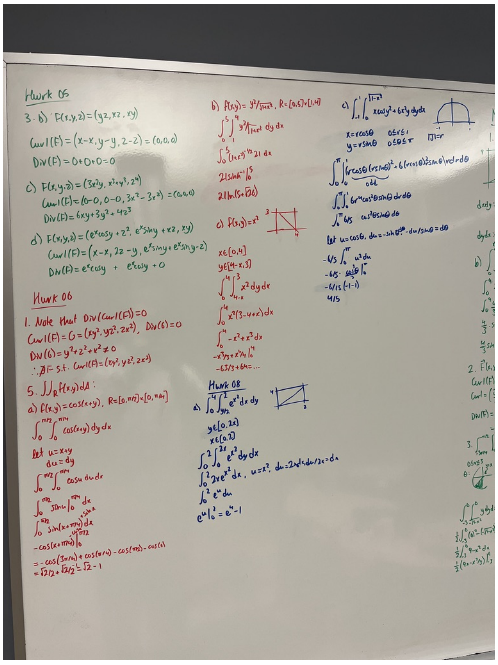
Then, we have $\iint [F(r(s,t)) \cdot n] \|N(s,t)\| ds dt$

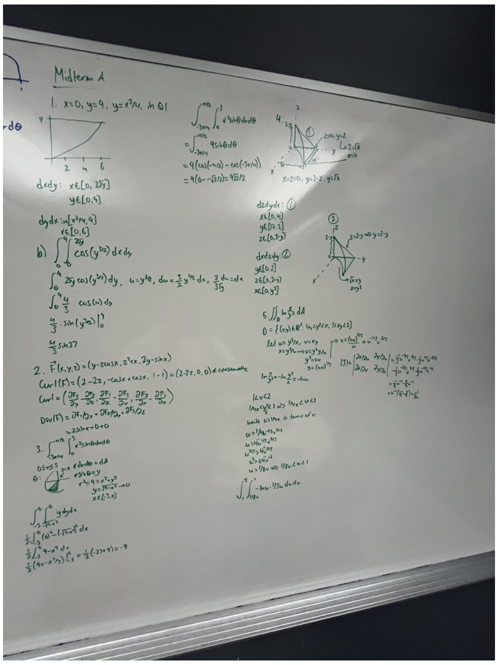
Now, find F such that $F \cdot n = x^2 + y + z, F \cdot (x,y,z) = x^2 + y + z$

Hence, take $F(x,y,z) = (x,1,1)$, and we now have $\iiint_S (x^2 + y + z)\, dS = \iint_S (x,1,1) \cdot dS = \iiint_T \text{div}(F) dv$

Get that $\text{div}(F) = 1$, then $\iiint_T \text{div}(F) dV = I\, \text{Vol}(0) = 4\pi/3$


let F(x,y,z) = (F1, F2, F3)
Cylindrical Coordinates
Spherical Coordinates
Questions:


Hwrk 05
Hwrk 06
Hwrk 08


Midterm A
1. x=0, y=9, y=x²/4, in Q1
dxdy: x∈[0, 2√y]
y∈[0,9]
2. F(x,y,z)=(y-2cosx, z²+x, 2y-sinx)
3.
4.
5.

27 Cheat Sheet

27.1 Delta-Epsilon

The condition $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ means our input point is inside the $\delta-$neighbourhood of (a, b)

The proof then shows that whenever the input point is that close to (a, b), the function value $f(x, y)$ lies in the $\varepsilon-$neighbourhood of the limit L: $|f(x, y) - L| < \varepsilon$

Proof: Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x - a|| < \delta$, then $|f(x) - L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim\limits_{x \to a} f(x) = L$

27.2 Disproving a Multivariable Limit

1. Prove with direct substitution

If you get a determinate value (like 5, 0, or ∞) and the function is built from continuous functions, you're done

If you get an indeterminate form like 0/0, proceed with next steps.

2. Disprove with two-path test

For a limit approaching $(0, 0)$, common paths to test include: axis paths (along x, let $y = 0$, vice-versa), linear paths $y = mx$ and the limit d/n exist if it depends on m, parabolic paths

3. Disprove with polar coordinates $x = r\cos\theta, y = r\sin\theta$

27.3 Partial Derivative

Definition of partial derivatives at a point

$\frac{\partial F}{\partial x}(0, 0) = \lim\limits_{h \to 0} \frac{F(h,0) - F(0,0)}{h}$

27.4 Derivative Matrix

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \ldots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Let $A = [a, b]$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae + bg & af + bh \end{bmatrix}$

Let A be of size $m \times m$ and B of size $p \times q$, then $C = A \times B$ has dimensions $m \times q$

27.5 Divergence and Curl

The divergence of F denoted by $\nabla \cdot F$ is $\mathbb{R}^3 \to \mathbb{R}$, measures the net rate of flow outward from a point, and is $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

The curl of F denoted by $\nabla \times F$ is $\mathbb{R}^3 \to \mathbb{R}^3$, measures the tendency to rotate or swirl around a point, and is $\nabla \times F =< \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} >$

The gradient of f denoted by ∇f is $\mathbb{R}^3 \to \mathbb{R}^3$ points in the direction of greatest increase of f, and its magnitude is the rate of increase, and is $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$

$\nabla \cdot (\nabla \times F) = 0$, or in words, the divergence of the curl of any vector field F is 0

27.6 Hyperbolic Functions

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Derivatives are the same as non-hyperbolic trig functions

$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$ $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1}(x) + C$ $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1}(x)$ $\int \frac{1}{1-x^2}dx = \tanh^{-1}(x) + C$

$\cosh^2(x) - \sinh^2(x) = 1$