alex-levesque.com

MTHE 237 - Lecture Notes

Differential Equations for Engineering Science

Prof. Thomas Barthelmé • Fall 2025 • Queen's University

Contents

#Math

1 Differential Equations Overview

[[09-02 Mathematical modelling]] [[09-03 Terminology and Classification]] [[09-05 Initial Value Problems]] [[09-09 IVPs and the Existence and Uniqueness Theorem cont'd]] [[09-10 Separable Equations]] [[09-12 Separable Equations cont'd]] [[09-16 Variable Coefficients]] [[09-17 Exact Equations]] [[09-19 Second order linear differential equations]] [[09-23 Homogeneous equations with constant coefficients]] [[09-24 Homogeneous equations with constant coefficients cont'd]] [[09-26 Mechanical and electrical vibrations]] [[10-01 Method of undetermined coefficients]] [[10-03 The phenomenon of resonance]] [[10-07 Method of variation of parameters]] [[10-08 Introduction to the Laplace transform]] [[10-10 Basic applications of the Laplace transform]] [[Future]] [[Solving ODEs Cheat Sheet]]

#Math

2 Mathematical modelling

2.1 How to solve

In order to provide a mathematical model describing physical phenomena, we can follow the following steps:

1. Fix our dependent and independent variables and set a frame of references for their measures
2. Choose convenient units of measurements
3. Find the underlying principle governing the motion of the object
4. Rewrite the above relation in terms of the variable we have chosen at step 1
5. Find a solution by integrating both sides
6. Add side conditions to eliminate constants and to find a **unique solution**

Example

$$\begin{aligned}
F &= ma \\
a &= \frac{dv}{dt} = \frac{d^2h}{dt^2} \\
\frac{d^2h}{dt^2} &= -g \\
v(t) = \frac{dh}{dt} &= -\int g dt = -gt + c_1 \\
h(t) = \int v(t)dt &= -\frac{1}{2}gt^2 + c_1t + c_2 \\
h(0) &= 1, v(0) = 0 \\
h(t) &= -\frac{1}{2}gt^2 + 1
\end{aligned}$$

2.2 Population Models

Given some internal and external conditions: $\frac{dp}{dt}$ = growth rate - death rate

Death rate = 0

$\frac{dp}{dt} = k_1 p$, where $p(0) = p_0$, and $k_1 > 0$ is the proportionality factor for the growth rate

$$\begin{aligned}
\frac{1}{p}dp &= kdt \\
\ln p &= kt + C \\
p(t) &= Ce^{kt}
\end{aligned}$$

2.3 Malthusian and competitors

A Malthusian model is a general model for population with rates k_1 and k_2 proportional to p

We can consider more factors to the death rate, called **competitors**, with two-party interactions modelled by $\frac{p(p-1)}{2}$

$\frac{dp}{dt} = k_1 p - k_3 \frac{p(p-1)}{2}$

We can rearrange the terms to find an equation in the form of a **logistic model**

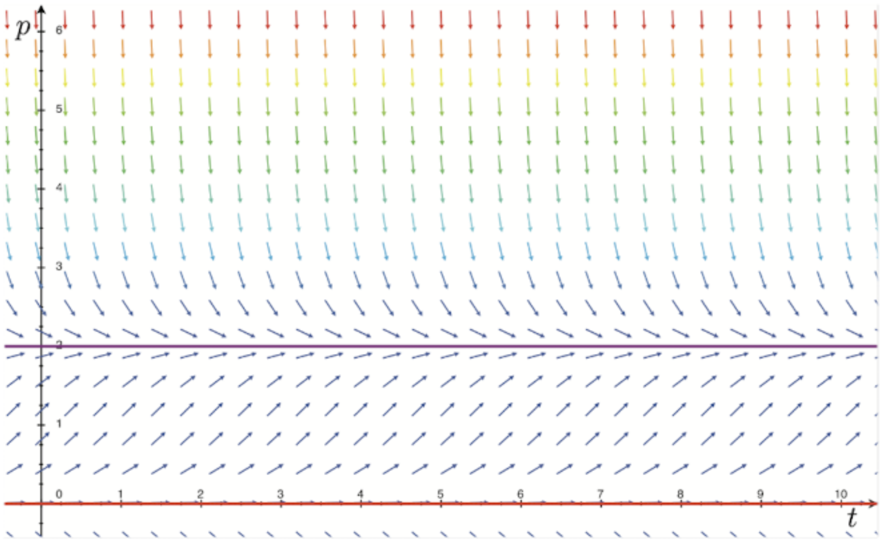
$\frac{dp}{dt} = -\frac{k_3}{2}\left(p^2 - \left(q + \frac{2k_1}{k_3}\right)p\right)$

which takes the form

$\frac{dp}{dt} = -Ap(p - p_1)$, where $A = \frac{k_3}{2}$ and $p_1 = 1 + \frac{2k_1}{k_3}$

2.4 Direction Fields

For $\frac{dp}{dt} = p(2-p)$, $\frac{dp}{dt} > 0 \Leftrightarrow 0 < p(t) < 2$



$p = 0, 2$ are equilibria [[09-24 Conservative Vector Fields]]

#Math

3 Terminology and Classification

A differential equation is an equation involving an independent variable (e.g., x, t), a dependent variable, and derivatives of the dependent variables

3.1 Examples

$y' = y + x$, y is dependent and x is independent

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x - 2y$, where x, y and independent and the function $u(x, y)$ is dependent

3.2 Types of differential equations

A differential equation with derivatives with respect to one variable is an ODE

A differential equation with partial derivatives with respect to two or more variables is a PDE

3.3 Ordinary differential equations

The general form of an ODE is given by $F(x, y, y', \ldots, y^{(n)}) = 0$

The **order** of a differential equation is the order of the highest derivative appearing in the equation

An ODE is **linear** if: 1. $y, y', y'', \ldots$ appear only to the first power 2. No product like yy' exist 3. Coefficients $a_i(x)$ can be constants or functions of x, but not of y

A linear ODE of order n has the form: $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0(x)y = f(x)$

3.4 Confirming a solution of an ODE

A function $\phi(x)$ is a solution of an n-th order ODE on some interval $I = (a, b)$ if 1. ϕ is n-times differentiable on I 2. It satisfies the ODE for every $x \in I$

Consider $y'' + \frac{2}{x^2}y = 0$ and the function $\phi(x) = x^2 - \frac{1}{x}$

We can confirm that its derivatives are continuous for all $x \neq 0$ and satisfy the ODE

Thus, the function ϕ is a solution of the given ODE in $(0, \infty)$

3.4.1 Superposition principle for linear ODEs

[[Superposition principle]]

Say we have an n-th order linear **homogeneous** ODE $a_n(x)y^{(n)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0(x)y = 0$ and suppose $\phi_1(x), \phi_2(x), \ldots, \phi_k(x)$ are solutions.

The theorem says any linear combination $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_k\phi_k(x)$ is also a solution in I for any choice of arbitrary constants, and the set of solutions is **closed under linear combinations**

These $\phi_k(x)$ solutions, unlike the next example, form a **vector space**, closed under linear combinations

Now, consider a **non-homogeneous** ODE when $f(x) \neq 0$, the linear combination property fails, and only holds in the homogeneous case

#Math
Existence and Uniqueness Theorem

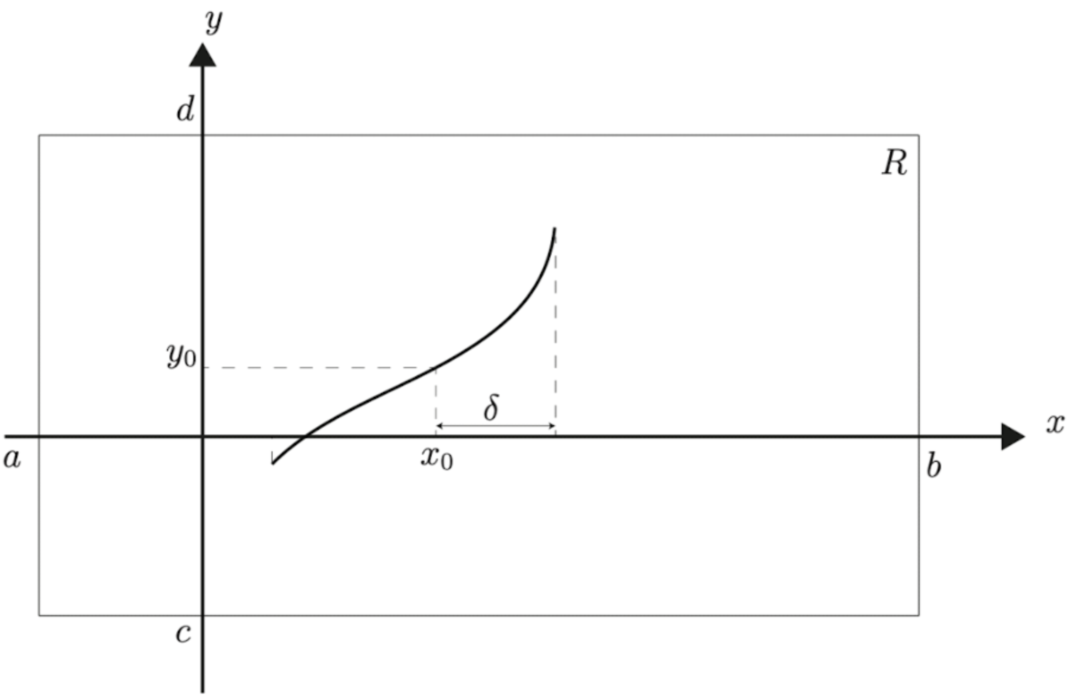
[Existence and Uniqueness Theorem]

Consider the IVP $y' = f(x, y)$ and $y(x_0) = y_0$

If f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $R = \{(x, y) \in \mathbf{R^2} : a < x < b, c < y < d\}$, that contains the point (x_0, y_0), then the IVP admits a unique solutions $\phi : (x_0 - \delta, x_0 + \delta) \to \mathbf{R}$ for some $\delta > 0$

If $f(x, y)$ is continuous, then at least one solution exists. Continuity of f is enough for *existence*, but not for *uniqueness*

If $f(x, y)$ is continuous *and* $\frac{\partial f}{\partial y}$ is continuous, then this is enough to show that *uniqueness*



Intuition:

In the original equation, this of it like a slope field that tells you the slope of the solution curve at each point $(x, y), f(x, y)$

If f is continuous, there are no gaps in the slope field. That means you can follow the slope arrows starting at origin to trace out a path

If $\frac{\partial f}{\partial y}$ is also continuous, then nearby paths cannot cross each other. Because the slope field is well defined, you cannot have two different marbles starting at the same point and taking different paths

The rectangle R is the safe zone where f behaves nicely

δ is how far you can move horizontally from x_0 while staying inside the safe zone

Inside $(x_0 - \delta, x_0 + \delta)$, the slope field is well-behaved enough to guarantee a single, smooth solution curve

3.5 Example

Use the existence and uniqueness theorem to show that the following IVP admits a unique solution

Given: $y' = 3x + \sin y, \quad y(0) = 1$

We know that $y' = f(x, y) = 3x + \sin y$, and each component of $f(x, y)$ is continuous for all $(x, y) \in \mathbb{R}^2$

At $(0, 1)$, both $f(0, 1)$ and $\frac{\partial f}{\partial y}(0, 1)$ are continuous

Therefore, the IVP admits a unique local solution around $x = 0$

#Math

4 Initial Value Problems

Given the n-th order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ in some interval I, the following two side conditions can be appended to the equation

1. **Initial Conditions:** dependent variable and all its derivatives up to order $n-1$ are specified at the same point $x_0 \in I$
2. **Boundary Conditions:** only applicable to PDEs

The problem of finding a solution to an ODE in an interval I containing x_0 and *such that the initial conditions are satisfied* is called **initial boundary value problem (IVP)**

4.1 Solving ODEs from implicit relations (template)

Example: given the implicit relation $x_2 + y^2 = 1, x \in (-1, 1)$, we want to see if $y(x)$ is a solution to the ODE $y' = -\frac{x}{y}$

1. **Differentiate w.r.t. dependent variable**

$2x + 2yy' = 0$

2. **Solve for y'**

$y' = -\frac{x}{y}$, this is the ODE

3. **Solve implicit relation explicitly**

Rearrange for y: $y(x) = \pm\sqrt{1 - x^2}$

4. **Verify candidates**

For $y(x) = +\sqrt{1 - x^2}$, $y' = \frac{-x}{\sqrt{1-x^2}}$, which matches the original ODE For $y(x) = -\sqrt{1 - x^2}$, $y' = \frac{x}{\sqrt{1-x^2}}$, which also satisfies the ODE

5. **Apply initial conditions**

$y_1(x) = -\sqrt{1 - x^2} \Rightarrow y_1(0) = -1$ is not valid $y_2(x) = \sqrt{1 - x^2} \Rightarrow y_2(0) = 1$ is valid

$\therefore$ the unique solution to the IVP is $y(x) = \sqrt{1 - x^2}$

#Math

5 IVPs and the Existence and Uniqueness Theorem cont'd

If one of uniqueness or existence fails, then $\phi(x)$ is not a solution to the ODE

5.1 Uniqueness fails

In $f(x, y) = \frac{y}{x}$, f is undefined at $(0, 0)$, but solutions $y(x) = 0$ and $y(x) = x$ hold and satisfy the IVP with $y(0) = 0$, thus $y(x) = cx$ is a solution for any constant c

This means that there are infinitely many solutions, so **uniqueness fails**

5.2 Existence fails

Now suppose $y(0) = 1$, when $x = 0$ this gives $y(0) = 0$, so **existence fails**

5.3 Alternate form

Given a 1st order linear ODE $a_1(x)y' + a_0(x)y = f(x)$, we can always put it in the form $y'(x) = P(x)y + Q(x)$ by denoting $P(x) = -\frac{a_0(x)}{a_1(x)}, Q(x) = \frac{f(x)}{a_1(x)}$

Thus, if $(x_0, y_0) \in \{(x, y) : P(x) \text{ and } Q(x) \text{ are continuous}\}$

Moreover, I can be chosen as the largest interval containing x_0 and such that P, Q are both continuous on I, giving a global, not local, solution

5.4 Picard Iterations

Picard iterations are a method to actually construct the solution that the Existence and Uniqueness Theorem guarantees.

We start at step 0, and iterate integration to construct a sequence of approximations (one per step) that should converge to the unique solution of the IVP

We can rewrite an IVP as an integral equation:

$y' = f(x, y), y(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$, where we don't know $y(t)$

We can approximate $y(x)$ step by step:

step 0: $\phi_0(x) = y_0$

step 1: $\phi_1(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s, \phi_0(s))\ ds$

step 2: $\phi_2(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s, \phi_1(s))\ ds$

$\vdots$

step n: $\phi_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f(s, \phi_{n-1}(s))\ ds$

$\vdots$

If, at some step of the Picard iteration, the new approximation $\phi_{k+1}(x)$ turns out to be exactly the same as the previous one $\phi_k(x)$, then we've already reached the solution

Formally, if there exists $k \in \mathbf{N}$ s.t. $\phi_{k+1}(x) = \phi_k(x)$, then the solution would be given by $\phi_k(x) = y_0 + \int_{x_0}^{x} f(s, \phi_k(s))ds$

Example:

$y' = 2t(1+y), y(0) = 0$

Step 0: $\phi_0(t) = 0$

Step 1: $y'(0) = 2t, \phi_1(t) = \int_0^t 2sds = t^2$

Step 2: $\phi_2(t) = \int_0^t 2s(1+t^2)ds = t^2 + \frac{t^4}{2}$

Step 3: $\phi_3(t) = \int_0^t 2s\left(1 + t^2 + \frac{t^4}{2}\right) ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2*3}$

Step n: $\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} = \Sigma_{k=1}^{n} \frac{t^{2k}}{k!}$

Hence, the sequence of functions converges if and only if the infinites series given above converges.

We can apply the ratio test to find this:

$\frac{|t^{2(k+1)}}{(k+1)!}\frac{k!}{k+1}| \to 0$ as $k \to \infty$

So, the series converges for every t, and $\phi(t) = \Sigma_{k=1}^{\infty} \frac{t^{2k}}{k!}$

#Math

6 Separable Equations

6.1 Definitions

General solution: The family of all possible solutions to the ODE, usually containing arbitrary constants

Explicit solution: A solution where the dependent variable (say y) is written explicitly as a function of the independent variable (say x)

Implicit solution: A solution where the relationship between x and y is given as an equation but not solved explicitly for y

6.2 Solutions of some ODEs

A linear ODE with constant coefficients has the form $y' = ay + b$, where a and b are real constants

We can rewrite this ODE in the equivalent form

$\frac{1}{y+\frac{b}{a}}\frac{dy}{dt} = a$

By integrating both sides with respect to t, we get

$\ln|y + \frac{b}{a}| = t + C \Leftrightarrow |y + \frac{b}{a}| = e^C e^{at} \Leftrightarrow y(t) = -\frac{b}{a} \pm ce^{at}$, where C and c are arbitrary constants

The general solution of an ODE determines an infinite family of curves called **integral curves**, where each curve corresponds to a different value of c. Imposing the initial condition, we are selecting the *unique curve* that passes through the initial point (t_0, y_0)

Example

Consider the classic drag proportional to velocity first-order linear ODE $m\frac{dv}{dt} = mg - \gamma v$

Put in standard form

$\frac{dv}{dt} = -\frac{\gamma}{m}v + g$

Rewrite into an equivalent form given

$\frac{1}{y+\frac{b}{a}}\frac{dy}{dt} = a$, where $a = -\frac{\gamma}{m}, b = g$

The general solution has the form $v(t) = -\frac{b}{a} + Ce^{at}$

Plug back in

$v(t) = \frac{mg}{\gamma} + Ce^{-(\gamma/m)t}$, where C is an arbitrary constant

Fix C with an initial condition

$v(0) = v_0$ $v_0 = \frac{mg}{\gamma} + C \Rightarrow C = v_0 - \frac{mg}{\gamma}$

Explicit solution by plugging back C

$v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right) e^{-\gamma t/m}$

6.3 Separable equations

We were able to solve the previous ODE because we were able to rewrite it in an equivalent form by **separating the variables**, thus the ODE falls in the class of *separable equations*

Definition: Consider the 1st order ODE $y' = f(x, y)$, if $f(x) = g(x)p(x)$, where g and p depends only on x and y, respectively, then the ODE is said to be **separable**

Consider the separable ODE on some interval $I = (a, b)$ $\frac{dy}{dx} = g(x)p(y)$ $(*)$

We have two cases:

1. If $p(y) \equiv 0$, then $y(x) = c$ constant function on I because the RHS is always 0
2. If $p(y) \neq 0$, then $(*)$ can be rewritten as $\frac{1}{p(y)}\frac{dy}{dx} = g(x)$

Let $H = H(y)$ and $G = G(x)$ be the antiderivatives of $\frac{1}{p(y)}$ and $g(x)$, respectively, from the chain rule we obtain $\frac{dH(y(x))}{dx} = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{p(y)}\frac{dy}{dx} = \frac{dG}{dx}$

Then, since their derivatives are equal, the functions must differ by a constant: $H(y(x)) = G(x) + c \Leftrightarrow \int \frac{1}{p(y)} dy = \int g(x)dx + c$

This is an implicit formula for the solution to the ODE.

For an explicit solution, we apply the **inverse function** of H to both sides: $H(y) = G(x) + C$ $H^{-1}(H(y)) = H^{-1}(G(x) + C)$ $y = H^{-1}(G(x) + C)$

Remark: finding the inverse function depends on H

#Math

7 Separable Equations cont'd

7.1 Example (implicit solution)

Consider $y' = \frac{x}{y\exp(x+2y)}$

It can be rewritten as

$ye^{2y}y' = xe^{-x}$

It can then be separated and integrated

$\int ye^{2y}dy = \int xe^{-x}dy$

$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -(x+1)e^{-x} \Leftrightarrow e^{2y}(2y-1) = -4e^{-x}(x+1) + c$

In this case, it is not possible to write y as an explicit function of x. We stop at an **implicit solution**

7.2 Example (explicit solution)

$y' = (1+y^2)\tan x$

It can be rewritten as

$\frac{1}{1+y^2}\frac{dy}{dx} = \tan x$

It can then be separated and integrated

$\int \frac{1}{1+y^2}dy = \int \tan x dx \Leftrightarrow \arctan[y(x)] = -\ln|\cos x| + c$

Integration of $\tan x$

$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du$ where $u = \cos x$ and $du = -\sin x dx$

$-\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$

#Math

8 Variable Coefficients

Consider a 1st order linear ODE with variable coefficients

$a_1(x)y' + a_0(x)y = b(x) \Leftrightarrow y' + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)} \Leftrightarrow \frac{dy}{dx} + P(x)y = Q(x)$

If $a_0(x) = a_1'(x)$, then the equation becomes $\frac{d}{dx}(a_1(x))y = b(x)$

Definition: An integrating factor is a special function that we multiply through a first-order linear ODE to make it easier to solve. Its purpose is to turn the left-hand side of the equation into the derivative of a product, so we can integrate directly

8.1 General Solution

Let $\frac{dy}{dx} + P(x)y = Q(x)$

1. We multiply through by an *integrating factor*

$\mu(x) = e^{\int P(x)dx}$

2. Multiply the whole equation by $\mu(x)$

$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \Leftrightarrow \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$

3. Integrate both sides

$\mu(x)y = \int \mu(x)Q(x)dx + C$

4. Solve for $y(x)$

$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$

General Solution:

$y(x) = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C\right)$

8.2 Example 1

Consider $y' + y = 1$, where $P(x) = Q(x) = 1$

Integrating factor: $e^{\int P(x)dx} = e^x$

Use general solution: $y(x) = e^{-x}\left(\int e^x dx + c\right)$

Solve: $y(x) = 1 + ce^{-x}$

8.3 Example 2

Consider $y' + \frac{3}{x}y = 3x - 2, y(1) = 1$, where $P(x) = \frac{3}{x}, Q(x) = 3x - 2$

Integrating factor: $\mu(x) = e^{3\ln x} = x^3$

Rewrite ODE: $\frac{d}{dx}[\mu y] = \mu Q \Leftrightarrow \frac{d}{dx}[x^3 y(x)] = 3x^4 - 2x^3$

Integrate on both sides: $x^3 y(x) = \frac{3}{5}x^5 - \frac{2}{4}x^4$

Solve: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{c}{x^3}$

Use initial conditions to find C: $y(1) = 1 = \frac{3}{5} - \frac{1}{2} + C \Leftrightarrow C = \frac{9}{10}$

Final Solution: $y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10x^3}$

#Math

9 Exact Equations

We start with a first-order ODE: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

which is equivalent to $M(x,y)dx + N(x,y)dy = 0$

Now, suppose there exists a *potential function* $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$, and $\frac{\partial F}{\partial y} = N(x,y)$

Then, if F exists, then the ODE is called exact.

Along any solution curve $y(x)$, if we compute the derivative of $F(x,y(x))$, we get:

$\frac{d}{dx}F(x,y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$

But notice that the right-hand side is exactly the same as $M(x,y) + N(x,y)\frac{dy}{dx}$

Since our ODE says that equals zero, it follows that $\frac{d}{dx}F(x,y(x)) = 0$

That means $F(x,y)$ is constant along solution curves, i.e. $F(x,y) = c$, which is the implicit solution of the ODE

9.1 Example 1

Consider $y' = -\frac{2xy^2+1}{2x^2y}$, where we can rearrange to get $M(x,y) = 2xy^2 - 1$ and $N(x,y) = 2x^2y$

Integrate an equation wrt x:

$\int(2xy^2 - 1)dx = x^2y^2 - x + g(y)$

The last term $g(y)$ appears instead of constant c because c can depend on y since we are integrating wrt x

Now, differentiate the latter function wrt y:

$\frac{\partial F}{\partial y} = 2x^2y + g'(y) = 2x^2y$, where the last term appears from $N(x,y) = 2x^2y$

We see that $g'(y) = 0$, so $g(y) = c$. Therefore, the given ODE is exact and the general solution is implicitly defined by $x^2y^2 - x = c$ for any arbitrary constant c

9.2 Example 2

Consider $3xy + y^2 + (x^2 + xy)y' = 0$

Integrate $M(x,y)$ wrt x: $F(x,y) = \frac{3}{2}x^2y + xy^2 + g(y)$

Differentiate F wrt y: $\frac{\partial F}{\partial y} = \frac{3}{2}x^2 + 2xy + g'(y) = x^2 + xy$

Rearrange for $g'(y) : g'(y) = -\frac{1}{2}x^2 - xy$, which cannot hold because the RHS depends on both variables x and y, while g is only a function of y. Hence, there is no F satisfying for the given ODE, which is then *not exact*

9.3 Clairaut's Theorem (Test for exactness)

An ODE is *exact* in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

Get

$$\begin{cases} \frac{\partial F}{\partial y} = N(x, y) \\ \frac{\partial F}{\partial x} = M(x, y) \end{cases}$$

Since M and N are differentiable wrt x and y, then

$$\begin{cases} \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \end{cases}$$

[[Clairaut's Theorem]] states that if $F(x, y)$ has continuous second order partial derivatives, then $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $(x, y) \in \mathbf{R}$

#Math

10 Second order linear differential equations

Let $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$, where the coefficients are functions of x only and are continuous on some open interval $I = (a, b)$

If the ODE $= 0$, we call it homogeneous.

The superposition principle also applies to second order ODEs.

10.1 Wronskian

[Representation Theorem] [Wronskian]

A second order ODE needs two initial conditions, and they must not be multiples of one another. Else, solutions y_1, y_2 are linearly dependent on the interval I

We can say that a 2nd order ODE is homogeneous if:

Let y_1, y_2 be solutions on the interval of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. If at some point $x_0 \in (a, b)$ these two solutions satisfy

$\det \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$

and call it **Wronskian of y_1 and y_2 at x_0**

10.2 Representation Theorem

Let $y_1(x), y_2(x)$ be two solutions on the interval (a, b) of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

If at some $x_0 \in (a, b)$, these two solutions satisfy: $W[y_1(x_0), y_2(x_0)] \neq 0$, then $y_1(x)$ and $y_2(x)$ are linearly independent solutions on (a, b)

#Math

11 Homogeneous equations with constant coefficients

Consider $ay'' + by' + cy = 0$, where $a, b, c \in \mathbf{R}$

If $\phi(x)$ is a solution to the ODE, then $\phi''(x) = -\frac{b}{a}\phi'(x) - \frac{c}{a}\phi(x)$ for all $x \in \mathbf{R}$

This means that the second derivative is a linear combination of the lower order derivatives

If we have a trial function (educated guess for the solution) $y(x) = e^{rx}$, then we get $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$

Into the ODE, we get $\cancel{e^{rx}}(ar^2 + br + c) = 0$, so e^{rx} is a solution $\Leftrightarrow$ r is a root of the 2nd order polynomial. Thus, r must be a solution to the *characteristic equation* $ar^2+br+c = 0$

11.1 Case 1: $b^2 - 4ac > 0$

If so, then there exists roots $r_1, r_2 \in \mathbf{R}$ with $r_1 \neq r_2$

r_1 cannot equal r_2 because $W[e^{r_1x}, e^{r_2x}] = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$

Hence, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ are linearly independent solutions, and by the [Representation Theorem], $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$, where constants are arbitrary and can be found by imposing initial conditions

Example:

Consider $2y'' + 7y' - 4y = 0$

Characteristic equation: $2r_2 + 7r - 4 = 0$

Confirm determinant > 0: $r_{1,2} = \frac{-7\pm\sqrt{49+32}}{4} = -4, \frac{1}{2}$

General solution: $y(x) = c_1e^{-4x} + c_2e^{x/2}$

Impose initial conditions if given

11.2 Case 2: $b_2 - 4ac = 0$

If so, then there exist two roots $r_1, r_2 \in \mathbf{R}$ with $r_1 = r_2$

In this case $c = \frac{b^2}{4a}$ and roots $r_1 = r_2 = -\frac{b}{2a}$. Because we essentially have one root, then the solution $y_1(x) = \exp\left(-\frac{b}{2a}x\right)$ exists

By the [Representation Theorem], $y(x) = c_1y_1(x) + c_2y_2(x)$. Now that we have $y_1(x)$, how do we find $y_2(x)$ such that $\frac{y_2(x)}{y_1(x)} \neq$ constant

We use the *Method of Reduction of Order*, which works on second order linear homogeneous ODEs with variable coefficients $y'' + p(x)y' + q(x)y = 0$

If $\phi_1(x)$ is a solution to the ODE above, and we want to find $\phi_2(x)$ such that they are linear independent on an interval, we want $\phi_2(x) = g(x)\phi_1(x)$ where $g(x)$ is an unknown function to be found to such that the ODE is satisfied by $\phi_2(x)$

Take $\phi_2(x)$ and take the second derivative: $\phi_2'' = g';\phi_1(x) + 2g'\phi_1' + g\phi_1''(x)$

Replacing in the ODE, we get $\Leftrightarrow \phi_1 v' + (2\phi_1' + p(x)\phi_1)v = 0$, where $v(x) := g'(x)$

We can then solve for $v(x)$ because the corresponding ODE is linear and separable

$g(x) = \int v(x)dx = \int \frac{\exp\left(-\int p(x)dx\right)}{\phi_1^2(x)}dx$

Example:

Find a second solution to $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ for $x > 0$ given a solution $y_1(x) = x^{-1/2}\sin x$

Let $y_2(x) = v(x)y_1(x)$, then $y_2'' = v''y_1 + 2v'y_1' + vy_1''$

Goal: find $v(x)$. Replacing the latter in the ODE, we find that

$x^2y_1v'' + v'(2x^2y_1' + xy_1) = 0$ because the y term is the same as the ODE, and it drops out

We can divide further by x^2y_1: $v'' + \left(2\frac{y_1'}{y_1} + \frac{1}{x}\right) = 0$

Now we need $\frac{y_1'}{y_1}$. Since $y_1'(x) = \frac{\sqrt{x}\cos x - \frac{\sin x}{2\sqrt{x}}}{x}$, then the former is $\frac{\cos x}{\sin x} - \frac{1}{2x}$, which can be simplified to $2\frac{\cos x}{\sin x}$

Subbing back, we get that $v(x)$ has to satisfy the following equation $v'' + 2\frac{\cos x}{\sin x}v' = 0$

Reduce order again: Setting $w(x) := v'(x)$, then $w' = -2\frac{\cos x}{\sin x}w \Rightarrow \frac{w'}{w} = -2\cot x$

Integrate: $\ln|w| = -2\ln(\sin x) + C$, $w = v'(x) = \frac{C_1}{\sin^2 x}$

Integrating to get $v(x)$: $v(x) = -C_1\cot x + C_2$

Build the solution: $y_2(x) = v(x)y_1(x) = (-C_1\cot x + C_2)\frac{\sin x}{\sqrt{x}} \Rightarrow y_2(x) = \frac{\cos x}{\sqrt{x}}$

11.3 Case 3: $b_2 - 4ac < 0$

In this case, we have two roots $r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a} =: \alpha \pm i\beta$

Hence, we find two *complex-valued* linearly independent solutions $y_1, y_2 : \mathbf{R} \to \mathbf{C}$ $y_1(x) = \exp[(\alpha + i\beta)x], y_2(x) = \exp[(\alpha - i\beta)x]$

We would like to find two *real-valued linearly independent solutions*

Important:

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Also, if the ODE has real coefficients and the complex-valued function $\phi = u(x) + iv(x)$ is a solution, then real-valued function $u(x) = \mathrm{Re}(\phi(x))$ and complex-valued function $v(x) = \mathrm{Im}(\phi(x))$ are solutions of the same equation

Back to Case 3, from Euler's formula we get: $y_{1,2}(x) = e^{\alpha x}[\cos(\beta x) \pm i\sin(\beta x)]$

Also, be the second remark above, we can conclude that also the functions below are also solutions: $Y_1, Y_2 : \mathbf{R} \to \mathbf{R}$ $Y_1(x) = \mathrm{Re}(y_1(x)) = \mathrm{Re}(y_2(x)) = e^{\alpha x}\cos(\beta x)$ and $Y_2(x) = \mathrm{Im}(y_1(x)) = -\mathrm{Im}(y_2(x)) = e^{\alpha x}\sin(\beta x)$

$W[Y_1(x), Y_2(x)] =$

$=$

Moreover, we can check if these are linearly independent via their [Wronskian]:

Thus, the solutions above are linearly independent (real-valued) solutions of $ay_2'' + by_2' + cy_2 = 0$

Moreover, every other solution takes the form $y(x) = c_1Y_1(x) + c_2Y_2(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Example:

Consider $y'' - y' + y = 0, y(0) = 1, y'(0) = -2$

Characteristic equation $r^2 - r + 1 = 0$

Solution: $r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

- Linearly independent solutions of the ODE:

$$y_1(x) = \mathrm{e}^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right), \qquad y_2(x) = \mathrm{e}^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

- General solution:

$$y(x) = \mathrm{e}^{x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

Imposing initial conditions, we get

$y(x) = e^{x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{5\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

#Math

12 Mechanical and electrical vibrations

A **damped mass-spring oscillator** is a physical system constituted by a mass m attached to an elastic spring with stiffness constant k and subject to friction $F_f(t) = -\gamma \frac{dy}{dt}$, where $\gamma \geq 0$ is the damping coefficient

The equation of motion is given by $my'' + \gamma y' + ky = F(t) = 0$ with no external force applied to the body

1. **If $\gamma \neq 0$ and $\gamma^2 - 4mk \geq 0$, the system is said to be overdamped**

$r_1, r_2 > 0, y \to 0, x \to \infty$

$r_1 = 0, r_2 < 0, |y| \to \infty, x \to \infty$

2. **If $\gamma \neq 0$ and $\gamma^2 - 4mk < 0$, the system is said to be underdamped**

$\alpha > 0, |y| \to \infty$, y oscillates to $\infty, x \to \infty$

$\alpha < 0, |y| \to 0$, y oscillates to $0, x \to \infty$

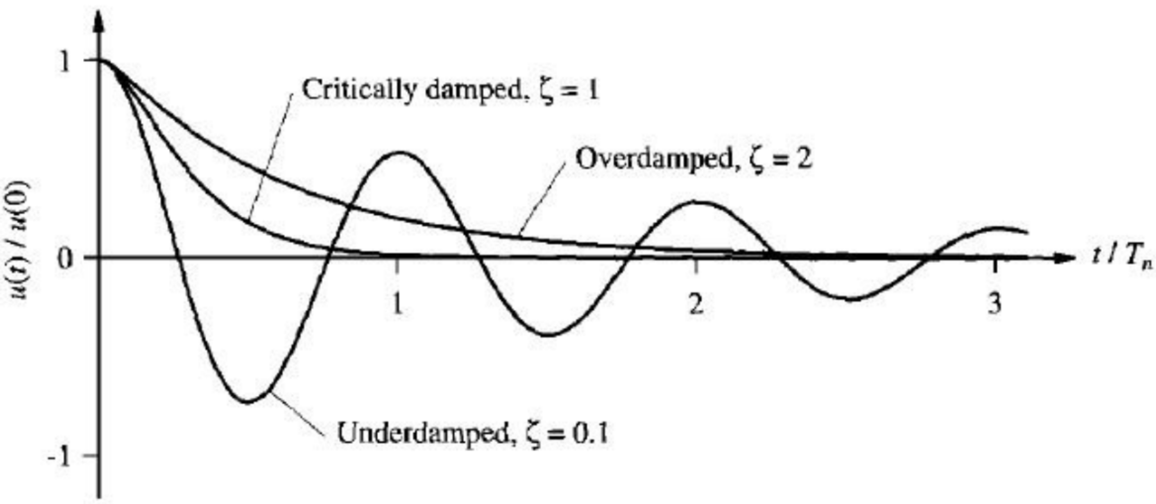
$\alpha = 0$, y is periodic, no limit

3. **If $\gamma = 0$, the system is said to be undamped**

$r_1 > 0, |y| \to \infty$ exponentially. $x \to \infty$

$r_1 < 0, y \to 0, x \to \infty$

$r_1 = 0, |y| \to \infty$ linearly. $x \to \infty$



#Math

13 Method of undetermined coefficients

Consider the IVP $y'' + p(x)y' + q(x)y = f(x), y(x_0) = Y_0, y'(x_0) = Y_1$

This IVP can be proved with the [Existence and Uniqueness Theorem], and an interval $I = (a, b)$ can be found

Furthermore, the [[Superposition principle]] applies, where solutions ϕ_1 and ϕ_2 forms another solution $\phi(x) = c_1\phi_1 + c_2\phi_2$, which solves for $y'' + p(x)y' + q(x)y = c_1 f_1(x) + c_2 f_2(x)$

13.1 Particular Solutions

Let $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ be a solution to a homogeneous equation, and $y_p(x)$ be a solution of the nonhomogeneous equation

Then, $y(x) = y_c(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

The solution $y_c(x)$ is called the **complementary solution**, whereas the solution $y_p(x)$ is called the **particular solution**

13.2 Method of Undetermined Coefficients

Considering a nonhomogeneous ODE, find $y_p(x)$

Example: $y'' - 3y' - 4y = 3e^{2x}$

Take the homogeneous version and find roots $r_1 = 4, r_2 = -1$

Find the complementary solution by looking for a function of the form $y_p(x) = Ae^{2x}$

Plug in for y, get derivatives, and get $A = -\frac{1}{2}$

General solution: $y(x) = y_c(x) + y_p(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}$

Example: $y'' - 3y' - 4y = 2\sin x$

We complementary homogeneous solution is $y_c(x) = c_1 e^{4x} + c_2 e^{-x}$

A solution to the nonhomogeneous solution can be guessed by $y_p(x) = A\cos x + B\sin x$

We always use both sine and cosine terms for a right side like $\sin x$ **or** $\cos x$ because their derivatives produce each other

We then plug in the derivatives and equate coefficients

$$\begin{cases} \text{cosine: } -5A - 3B = 0 \\ \text{sine: } 3A - 5B - 2 = 0 \end{cases}$$

We can solve for A, B and get a general solution: $y(x) = c_1 e^{4x} + c_2 e^{-x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x$

Example: $y'' - 3y' - 4y = 4x^2 - 1$

$y_c(x) = c_1 e^{4x} + c_2 e^{-x}$

For a polynomial right hand side, we try a particular solution of the same order $(Ax^2 + Bx + C)$

We then plug in the derivatives and equate coefficients

$$\begin{cases} \text{x squared:} & 2A - 3B - 4C + 1 = 0 \\ \text{x:} & -6A - 4B = 0 \\ \text{constant:} & A + 1 = 0 \end{cases}$$

$f(x)$	$y_p(x)$
$P_n(x) := a_n x^n + \cdots + a_1 x + a_0$	$x^s[A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0]$ Three possibilities for s: $s = 0$ if $r = 0$ is not a solution of the characteristic equation $s = 1$ if $r = 0$ is only one solution of the characteristic equation $s = 2$ if $r = 0$ is double solution of the characteristic equation
$P_n(x)\mathrm{e}^{\alpha x}$	$x^s[A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0]\mathrm{e}^{\alpha x}$ Three possibilities for s: $s = 0$ if $r = \alpha$ is not a solution of the characteristic equation $s = 1$ if $r = \alpha$ is only one solution of the characteristic equation $s = 2$ if $r = \alpha$ is double solution of the characteristic equation
$P_n(x)\mathrm{e}^{\alpha x}\cos(\beta x)+$ $+P_m(x)\mathrm{e}^{\alpha x}\sin(\beta x)$	$x^s[A_k x^k + A_{k-1}x^{k-1} + \cdots + A_1 x + A_0]\mathrm{e}^{\alpha x}\cos(\beta x)+$ $+x^s[B_k x^k + B_{k-1}x^{k-1} + \cdots + B_1 x + B_0]\mathrm{e}^{\alpha x}\sin(\beta x)$ $k = \max\{n, m\}$ Two possibilities for s: $s = 0$ if $r = \alpha + i\beta$ is not a solution of the characteristic equation $s = 1$ if $r = \alpha + i\beta$ is a solution of the characteristic equation

#Math

14 The phenomenon of resonance

Consider the undamped mass-spring oscillator

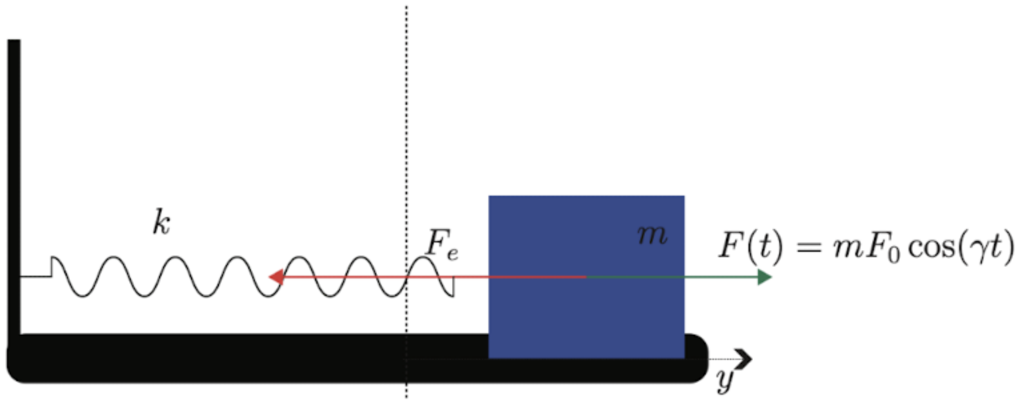


Figure 3.7: Forced mass-spring oscillator.

General ODE:

$$y'' + \omega^2 y = F_0 \cos(\gamma t), \quad \omega^2 = \frac{k}{m}$$

- ω: Natural frequency (system's own oscillation)
- γ: Forcing frequency (external input)

General Solution (Non-Resonant):

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + y_p(t)$$

Try particular solution:

$$y_p(t) = A \cos(\gamma t) + B \sin(\gamma t)$$

After differentiating and substituting:

$$A(\omega^2 - \gamma^2) \cos(\gamma t) + B(\omega^2 - \gamma^2) \sin(\gamma t) = F_0 \cos(\gamma t)$$

So:

$$A = \frac{F_0}{\omega^2 - \gamma^2}, \quad B = 0$$

Final non-resonant solution (for $\omega \neq \gamma$):

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \cos(\gamma t)$$

14.1 Resonance and Generalization

Resonance occurs when $\omega = \gamma$, i.e., the forcing matches the system's own frequency. This breaks the usual method since the input matches a homogeneous solution.

- **General principle:** When the forcing function is also a solution of the homogeneous equation, *multiply your usual particular solution guess by t for linear independence.*
- For resonance ($\omega = \gamma$), trial:

$$y_p(t) = [A \cos(\omega t) + B \sin(\omega t)]t$$

Solving gives:

$$A = 0, \quad B = \frac{F_0}{2\omega}$$

So the resonance particular solution is:

$$y_p(t) = \frac{F_0}{2\omega} t \sin(\omega t)$$

14.2 Examples

Example: $y'' - 3y' - 4y = -8e^{-x}$

$y_c(x) = c_1 e^{4x} + c_2 e^{-x}$

Notice the forcing term $f(x) = -8e^{-x}$ is a constant multiple of $c_2 e^{-x}$, thus we need a particular solution of the form Axe^{-x}

We then plug in the derivatives and get the general solution $y(x) = c_1 e^{4x} + c_2 e^{-x} + \frac{8}{5} xe^{-x}$

Example: $y'' - 2y' + y = (x + 1)e^x$

$y_c(x) = (c_1 + c_2 x)e^x$

If we plug in the RHS, the DE will equal 0

Then, we look for a particular solution of the form $y_p(x) = x^2(Ax + B)e^x$

Example: $2y'' + 3y' + y = x^2 + 3 \sin x$

In this case, we can develop a particular solution from the superposition principle because the forcing term is an addition: $f(x) = x^2 + 3 \sin x$

Thus $y_p(x) = y_{p,1}(x) + y_{p,2}(x)$, where the former is a particular solution to ODE $= x^2$ and the latter is a particular solution to $ODE = 3 \sin x$

#Math

15 Method of variation of parameters

Find a solution to $y'' + 4y = 3\csc x$, but $f(x)$ cannot be expressed as an l.c. of our common terms from the previous note

We can use **variation of parameters**, which works for any possible type of forcing term

Consider general form ODE $y'' + a(x)y' + b(x)y = f(x)$, and consider the particular solution of the form $y_p(x) = v_1(x)y_1(x) + v_2y_2(x)$

Get $y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$, and assume that $v_1'y_1 + v_2'y_2 = 0$

Then, $y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$

We can impose that to the ODE and get that $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution iff v_1 and v_2 satisfy:

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = f(x) \end{cases}$$

where $W[y_1(x), y_2(x)] \neq 0$

We then find the following solution of the system via Cramer's rule: $v_1(x) = \int \frac{-f(x)y_2(x)}{W[y_1(x),y_2(x)]}dx$ $v_2(x) = \int \frac{f(x)y_1(x)}{W[y_1(x),y_2(x)]}dx$

15.1 Examples

Example 1

Consider $xy'' - (x+2)y' + 2y = x^3$ in $(0, \infty)$, or in standard form: $y'' - \frac{x+2}{2}y' + \frac{2}{x}y = x^2$

and two linearly independent solutions of the corresponding homogeneous equation $y_1(x) = e^x, y_2(x) = x^2 + 2x + 2$

Find a particular solution by the method of variation of parameters. $y_p(x) = v_1y_1 + v_2y_2$, where v_1 and v_2 must satisfy Cramer's rule

$$\begin{cases} v_1'e^x + v_2'(x^2 + 2x + 2) = 0 \\ v_1'e^x + v_2'(2x + 2) = x^2 \end{cases} \Leftrightarrow \begin{cases} v_1' = -(x^2 + 2x + 2)e^{-x}v_2' \\ v_2' = -1 \end{cases} \Leftrightarrow \begin{cases} v_1 = (x^2 + 4x + 6)e^{-x} \\ v_2 = -x \end{cases}$$

General solution: $y(x) = c_1e^x + c_2(x^2 + 2x + 2) - x^3 - 3x^2 - 6x - 6$

Example 2

Consider $y'' + y = \tan x + 3x - 1$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Homogeneous equation: $y''+y = 0$ with linearly independent solutions $y_1(x) = \cos x, y_2(x) = \sin x$

Particular solution: $y_p(x) = y_{p,1}(x) + y_{p,2}(x)$, where $y_{p,1}$ is a particular solution of $y'' + y = \tan x$ $y_{p,2}$ is a particular solution of $y'' + y = 3x - 1$

We can find $y_{p,2}$ via undetermined coefficients $= 3x - 1$

We find $y_{p,1}$ with variation of parameters $= v_1 y_1 + v_2 y_2$, where $v_1 v_2$ must satisfy Cramer's rule

$$\begin{cases} v_1' \cos x + v_2' \sin x = 0 \\ -v_1' \sin x + v_2' \cos x = \tan x \end{cases} \Rightarrow \begin{cases} v_1' = -v_2' \tan x \text{ and replace in the second equation} \\ -v_1' \sin x + v_2' \cos = \tan x \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = -v_2' \tan x \\ v_2' = \sin x \text{ and back substitute in the first equation} \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = -\frac{\sin^2 x}{\cos x} \\ v_2' = \sin x \end{cases} \Rightarrow \begin{cases} v_1 = \sin x - \ln|\sec x + \tan x| + \cancel{k_1} \\ v_2 = -\cos x + \cancel{k_2} \end{cases} \Rightarrow y_{p,1} = -\ln|\sec x + \tan x| \cos x$$

#Math

16 Introduction to the Laplace transform

The Laplace transform is a mathematical tool that converts a function of time $f(t)$ into $F(s)$. It is used to simplify differential equations into algebraic ones.

Definition:

[[Laplace Transform]]

$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dx$:

- t is the original variable
- s is a new variable
- e^{-st} is a decaying exponential that "weights" the signal over time

When the transform is applied, you move from the time domain (functions of t), to the **frequency domain** (functions of s), where s helps us analyze how a system responds to different "frequencies" and how it behaves over time

Note: $\int_0^\infty e^{-st} f(t)\, dt := \lim\limits_{N\to\infty} \int_0^N e^{-st} f(t)\, dt$ If this limit exists, then the integral is said to converge to that limiting value.

The Laplace transform is an integral operator: $\mathcal{L} : f \to \mathcal{L}\{f\}$

The Laplace transform is a linear operator: $\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$

General idea: The general idea in using the Laplace transform to solve a differential equation is as follows:

1. Use the relation defined above to transform an IVP for an unknown function f in the $t-$domain into a simpler, algebraic problem for F in the $s-$domain
2. Solve this algebraic problem to find F
3. Recover the desired function f from its transform F. This last step is known "inverting the transform."

Application: If $f(t) = 1$, then the Laplace transform is $\int_0^\infty e^{-st}\, dt = \frac{1}{s}$ for $s > 0$

16.1 Jump discontinuity

A Laplace transform can not be defined for functions with jump discontinuities

Definition: A function $f(t)$ on $[a, b]$ is said to have a *jump discontinuity* at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0 but the following limits exists and are finite: $\lim\limits_{t\to t_0^-}$ and $\lim\limits_{t\to t_0^+}$

16.2 Hyperboloid trig functions

[[Hyperboloid Trigonometric Functions]]

$\sinh(t) = \frac{e^t - e^{-t}}{2}, \cosh(t) = \frac{e^t + e^{-t}}{2}$

For example, if we take $\mathcal{L}\{\cosh t\}(s)$, we can: $\mathcal{L}\left\{\frac{e^t+e^{-t}}{2}\right\}(s) = \frac{1}{2}\mathcal{L}\{e^t\}(s) + \frac{1}{2}\mathcal{L}\{e^{-t}\}(s)$, and solve in terms of s

#Math

17 Properties of Laplace transform

17.1 Piecewise Continuity

Definition: A function $f(t)$ is *piecewise continuous* on $[0, \infty]$ if, for any interval $[0, N]$, you can split it into a finite number of subintervals where $f(t)$ is continuous (except possibly for jump discontinuities)

Example:

Given a piecewise function:

$$\begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & t > 10 \end{cases}$$

For the integral of $\mathcal{L}\{f\}(s)$, we can break up our values of $f(t)$ into three integrals with their respective t bounds, and solve in terms of s

17.2 Exponential Order Condition

A function must not grow too fast. Specifically, there must be constants $M > 0, T > 0$ and a such that:

for all $t \geq T : |f(t)| \leq Me^{at}$

This means that, after some point T, $f(t)$ is always less than or equal to an exponential function Me^{at}. So, $f(t)$ can't explode faster than an exponential.

17.3 Convergence of the Laplace transform

Therefore, combining both theorems, a Laplace transform $\mathcal{L}\{f\}(s)$ exists (i.e. converges) for all $s > a$ if

1. f is piecewise continuous on $[0, \infty]$
2. There exist real constant T, a and M with $T, M > 0$ such that $|f(t)| \leq Me^{at}$ for all $t \geq T$

Example: $\mathcal{L}\{e^{at}\}, t \geq 0$

$F(s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty$

If $s > a$: We see that $e^{\text{negative exponential } \times \infty}$ goes to zero, therefore it **converges**

If $s \leq a$: We see that $e^{\text{positive exponential } \times \infty}$ goes to infinity, therefore it **diverges**

17.4 Elementary Laplace transforms

These are some elementary Laplace transforms

$f(t)$	$\mathscr{L}\{f\}(s)$
e^{at}	$\dfrac{1}{s-a}$ for $s > a$
t^n, $n \in \mathbb{N}$	$\dfrac{n!}{s^{n+1}}$ for $s > 0$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$ for $s > 0$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$ for $s > 0$

17.5 Theorem: Laplace transform of 1st order derivative

Suppose that f is continuous on $[0, \infty]$, f' is piecewise continuous on $[0, \infty]$, and There exists real constants T, a and M with $T, M > 0$ such that $|f(t)| \leq Me^{at}$ for all $t \geq T$

Then, there exists $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

By induction, there exists $\mathcal{L}\{f^{(n)}\}$ for $s > a$ and it is given by $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Example: $f(t) = \sin(3t)$ $f'(t) = 3\cos(3t)$ $f(0) = \sin 0 = 0$ $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\sin(3t)\}(s) = \frac{3}{s^2+9}$

Method 1: $\mathcal{L}\{f'(t)\}(s) = \mathcal{L}\{3\cos(3t)\}(s) = 3 \cdot \frac{s}{s^2+9}$

Method 2: Use property $\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f(0)$

We get $= s \cdot \frac{3}{s^2+9} - 0 = \frac{3s}{s^2+9}$

#Math

18 Solving IVPs: the inverse Laplace transform

Consider $y'' - y = -t$ with $y(0) = 0, y'(0) = 1$

Taking the Laplace transform of both sides, we obtain: $\mathcal{L}\{y''\}(s) - \mathcal{L}\{y\}(s) = -\mathcal{L}\{t\}(s) \Rightarrow s^2\mathcal{L}\{y''\}(s) - \cancel{sy(0)} - \cancel{y'(0)} - \mathcal{L}\{y\}(s) = -\frac{1}{s^2} \Rightarrow \mathcal{L}\{y(t)\}(s) = \frac{1}{s^2} = \mathcal{L}\{f\}(s)$

19 How to Solve an ODE — Step by Step

19.1 General Concepts

- **ODE form:** $F(x, y, y', \dots, y^{(n)}) = 0$
- **General solution:** family with arbitrary constants.
- **Particular solution:** found using initial/boundary conditions.
- **Explicit solution:** $y = f(x)$.
- **Implicit solution:** equation involving x and y, not isolated.

19.2 Existence and Uniqueness Theorem

1. Write the IVP in Standard Form

- Make sure your equation is in the form:

$y' = f(x, y), \quad y(x_0) = y_0$

2. Check Continuity of $f(x, y)$

- Is $f(x, y)$ continuous near (x_0, y_0)?

Look for division by zero, square roots of negatives, or other undefined operations

If $f(x, y)$ is a ratio of polynomials, the only condition to be imposed is that its denominator must be nonzero

3. Compute $\frac{\partial f}{\partial y}$ and check continuity

4. State Your Conclusion

- If both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0), the IVP admits a **unique local solution**.

- If either is not continuous, uniqueness or existence may fail at that point.

Tip: Always check for points where the denominator could be zero or where the function is not defined. That's where continuity can fail!

19.3 First Order ODEs

19.3.1 1. Separation of Variables

Form: $\frac{dy}{dx} = g(x)\,p(y)$

Steps:

1. Rearrange: $\frac{1}{p(y)}\,dy = g(x)\,dx$
2. Integrate both sides.
3. Get implicit solution: $H(y) = G(x) + C$

4. Solve for y if possible.

Example: $\frac{dy}{dx} = x \cdot y$

- Step 1: $\frac{1}{y} dy = x\, dx$
- Step 2: $\int \frac{1}{y} dy = \int x\, dx$
- Step 3: $\ln |y| = \frac{x^2}{2} + C$
- Step 4: $y = Ae^{x^2/2}$ where $A = e^C$.

19.3.2 2. Integrating Factor (Variable Coefficients)

Form: $y' + P(x)y = Q(x)$

Steps:

1. Put to standard form
2. Find $p(x)$ and compute integrating factor $\mu(x) = e^{\int P(x)dx}$
3. Multiply ODE by μ: $\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$, LHS should be $\frac{d}{dx}[\mu(x)y]$
4. Integrate: $\mu y = \int \mu Q\, dx + C$
5. Solve for $y(x) = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x)dx + C \right)$

Example: $y' - 2y = e^{3x}$

- Step 1: $\mu(x) = e^{\int -2dx} = e^{-2x}$
- Step 2: Multiply: $e^{-2x}y' - 2e^{-2x}y = e^x$
- Step 3: $\frac{d}{dx}[e^{-2x}y] = e^x$
- Integrate: $e^{-2x}y = e^x + C$
- Step 4: $y = e^{3x} + Ce^{2x}$.

19.3.3 3. Exact Equations

Form: $M(x, y)dx + N(x, y)dy = 0$ with exactness condition $M_y = N_x$.

Steps:

1. Compute M_y and N_x — check $M_y = N_x$.
2. Find potential F with $F_x = M$ and $F_y = N$.
3. Solution: $F(x, y) = C$.

Example: $(2xy)dx + (x^2)dy = 0$

- Step 1: $M = 2xy,\ N = x^2.\ M_y = 2x,\ N_x = 2x$
- Step 2: Integrate M w.r.t x: $F = x^2y + h(y)$. $F_y = x^2 + h'(y)$ must equal $N = x^2$ $\to h'(y) = 0 \to h$ constant.
- Step 3: $F = x^2y + C$.

19.3.4 4. Picard Iteration (Approximation)

Form: $y' = f(x, y),\ y(x_0) = y_0$

Steps:

1. Write integral form: $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$.
2. Iterate: $\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt$.
3. Continue until sequence converges.

Example: $y' = y,\ y(0) = 1$

- $\phi_0(x) = 1$
- $\phi_1(x) = 1 + \int_0^x 1\ dt = 1 + x$
- $\phi_2(x) = 1 + \int_0^x (1+t)dt = 1 + x + x^2/2$
- Continues toward e^x.

19.4 Second Order ODEs

19.4.1 1. Characteristic Equation (Constant Coefficients Homogeneous)

Form: $ay'' + by' + cy = 0$

Steps:

1. Compute characteristic polynomial: $ar^2 + br + c = 0$
2. Discriminant $\Delta = b^2 - 4ac$.
3. Solve for r and write general solution:
    - **Distinct real:** $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
    - **Repeated:** $y = (C_1 + C_2 x)e^{rx}$
    - **Complex:** $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Example: $y'' - y = 0$

- $r^2 - 1 = 0 \to r = 1$.
- Solution: $y = C_1 e^x + C_2 e^{-x}$.

Example: $y'' + 4y' + 5 = 0$

- $r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2}, r = -2 \pm i$
- Solution: $y(x) = e^{-2x}(c_1 \cos x + c_2 \sin x)$

19.4.2 2. Method of Undetermined Coefficients

Form: nonhomogeneous $ay'' + by' + cy = f(x)$

Steps:

1. Solve homogeneous for y_c.
2. Guess y_p based on $f(x)$.
3. Adjust guess if it matches y_c (multiply by x as needed).
4. Plug in, solve coefficients.

Example: $y'' - y = x$

- Step 1: Homogeneous $\to y_c = C_1 e^x + C_2 e^{-x}$.

- Step 2: For $f(x) = x$, guess $y_p = Ax + B$.
- Step 3: No overlap → plug into ODE: $0 - (Ax + B) = x \to A = -1, B = 0$.
- Solution: $y = C_1 e^x + C_2 e^{-x} - x$.

19.4.3 3. Variation of Parameters

Form: $y'' + P(x)y' + Q(x)y = R(x)$

Steps:

1. Solve y_c using characteristic method.
2. Use formula: $u_1' = -\frac{y_2 R}{W}$, $u_2' = \frac{y_1 R}{W}$.
3. Integrate u_1', u_2', then $y_p = u_1 y_1 + u_2 y_2$.

Example: $y'' - y = e^{2x}$

- Step 1: Homogeneous: $y_c = C_1 e^x + C_2 e^{-x}$.
- Step 2: Take $y_1 = e^x, y_2 = e^{-x}$, $W = -2$.
- $u_1' = -\frac{e^{-x} e^{2x}}{-2} = -\frac{e^x}{-2} = \frac{e^x}{2}$.
- Integrate: $u_1 = \frac{e^x}{2}$.
- $u_2' = \frac{e^x e^{2x}}{-2} = -\frac{e^{3x}}{2}$, integrate: $u_2 = -\frac{e^{3x}}{6}$.
- Step 3: $y_p = \frac{e^x}{2} e^x - \frac{e^{3x}}{6} e^{-x} = \frac{e^{2x}}{2} - \frac{e^{2x}}{6} = \frac{e^{2x}}{3}$.

Solution: $y = C_1 e^x + C_2 e^{-x} + \frac{e^{2x}}{3}$.

19.4.4 Particular Solution Guess Table

Forcing Function	Standard Guess	If matches homogeneous?
Ax^n	polynomial deg n	multiply by x or higher
Ae^{kx}	Ae^{kx}	multiply by x or higher
$A\sin(kx), B\cos(kx)$	$A\cos(kx) + B\sin(kx)$	multiply by x or higher
$x^r e^{kx}$	$(Ax^r + \dots)e^{kx}$	multiply by x^m to ensure independence

---

Mnemonic: If your guess is a solution to the homogeneous part (same root/frequency), multiply by x until it's independent.

This step-by-step sheet now gives you both the road map and examples to guide you through ODE solving.