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MTHE 280 - Lecture Notes

Advanced Calculus

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#Math

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#Math

2 Introduction to Multivariable Functions

A function $f(x, y)$ is a rule that assigns to every element x a unique element y, and is denoted by $f : x \to y$, where x is the domain of f and y is the codomain of f

Example

$f : \mathbf{N} \to \mathbf{R}, f(x) = 2x$

In this case, every value of f is even and does not take the whole codomain

We introduce the range, a subset of the codomain, $range(f) \subseteq codomain(f)$

2.1 Properties of functions

One-one/[[Injective]]

$f : X \to Y$ if $x_1, x_2 \in X, f(x_1) = f(x_2)$

Onto/[[Surjective]]

$f : X \to Y$ is onto if for every $y \in Y$, there exists some $x \in X$ such that $f(x) = y$

In this case, codomain = range

Bijective

if $f : x \to y$ is both one-one and onto, it is bijective

Scalar-valued

Consider $f : x \to y$ where $x \subseteq \mathbf{R}$ and $y \subseteq \mathbf{R}$, $n, m \in \mathbf{N}$

When the codomain is just $\mathbf{R}$, the function is called a Scalar-valued function

Example

$f : \mathbf{R^2} \to \mathbf{R}$ where $f(x, y) = \sqrt{x^2 + y^2}$

This returns the length of a 2D vector, which is a scalar

Vector-valued

A vector-valued function has codomain $\mathbf{R^n}$ where $n > 1, n \in N$

Example

$f : \mathbf{R} \to \mathbf{R^2}$, $f(x) = (\cos x, \sin x)$

2.2 Identify domain and codomain

Examples

$f(x) = \ln x$, domain = $(0, \infty)$, codomain = $\mathbf{R}$

$f(x) = \sqrt{2 - x}$, domain = $(-\infty, 2]$, codomain = $(0, \infty)$

$f(x, y) = (\sqrt{1 - x^2 - y^2}, \ln(y + 1), x^2 + y^2)$

1: $x^2 + y^2 = 1$ 2: $y > -1$

domain: $\{(x, y) \in \mathbf{R^2} : x^2 + y^2 \leq 1, y > -1\}$

#Math

3 Level Curves and Contours

Level Curve

Given a scalar-valued function, the level curve at height c is the curve in $\mathbf{R^2}$ s.t. $f(x, y) = c$

Or, the level curve at height $c = \{(x, y) \in \mathbf{R^2} | f(x, y) = c\}$

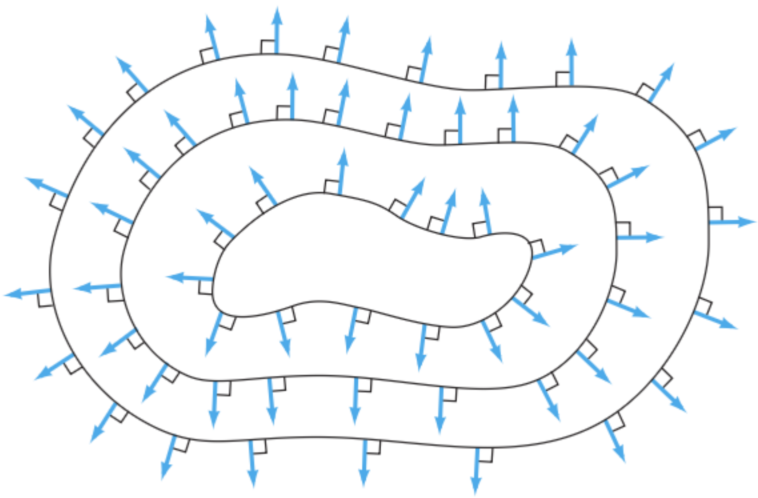


Figure 3.31 A gradient vector field $\mathbf{F} = \nabla f$. Equipotential lines are shown where f is constant.

Contour

The contour curve at height c is the collection of points (x, y, z) s.t. $z = f(x, y) = c$

Or, $\{(x, y, z) \in \mathbf{R^3} | z = f(x, y) = c\}$

The projection of the contour is the level curve

Section

A section of a surface by a place is just the intersection of the surface with that plane

#Math

4 Limits of a function

General form: $f : \mathbf{R} \to \mathbf{R}$

$\lim_{x\to a} f(x) = L, \therefore f(x)$ tends to L as x tends to a

4.1 L'Hospital's Rule

If we have a case where we are evaluating a limit and we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can use $\lim_{x\to a} \frac{f'(x)}{g'(x)}$

Why?: The ratio $\frac{f(x)}{g(x)}$ near a depends not only on the values of f and g, but on how fast they approach 0 or ∞

4.2 Limits in two variables

Let $f : \mathbf{R^2} \to \mathbf{R}$, $\lim_{(x,y)\to(a,b)} f(x, y)$

The Line $y = mx$ trick

All paths approaching point (e.g. $(0, 0)$) must give the same value

A simple test path is a straight line mx through the origin, and plug $f(x, y) \to f(x, mx)$

If the result depends on m, the limit does not exist

Does Exist Example

$$\lim_{(x,mx)\to(0,0)} \frac{x^2}{x^2 + y^4}$$

$$\lim_{x\to 0} \frac{x^2}{x^2 + m^4x^4}$$

$$\lim_{x\to 0} \frac{1}{1 + m^4x^2} = 1 \therefore \text{limit exists}$$

Does Not Exist Example

$$\lim_{(x,y)\to(0,0)} \frac{1}{1 + m^2} = \frac{x^2}{x^2 + m^2x^2} = \frac{1}{1 + m^2} \therefore \text{limit does not exist}$$

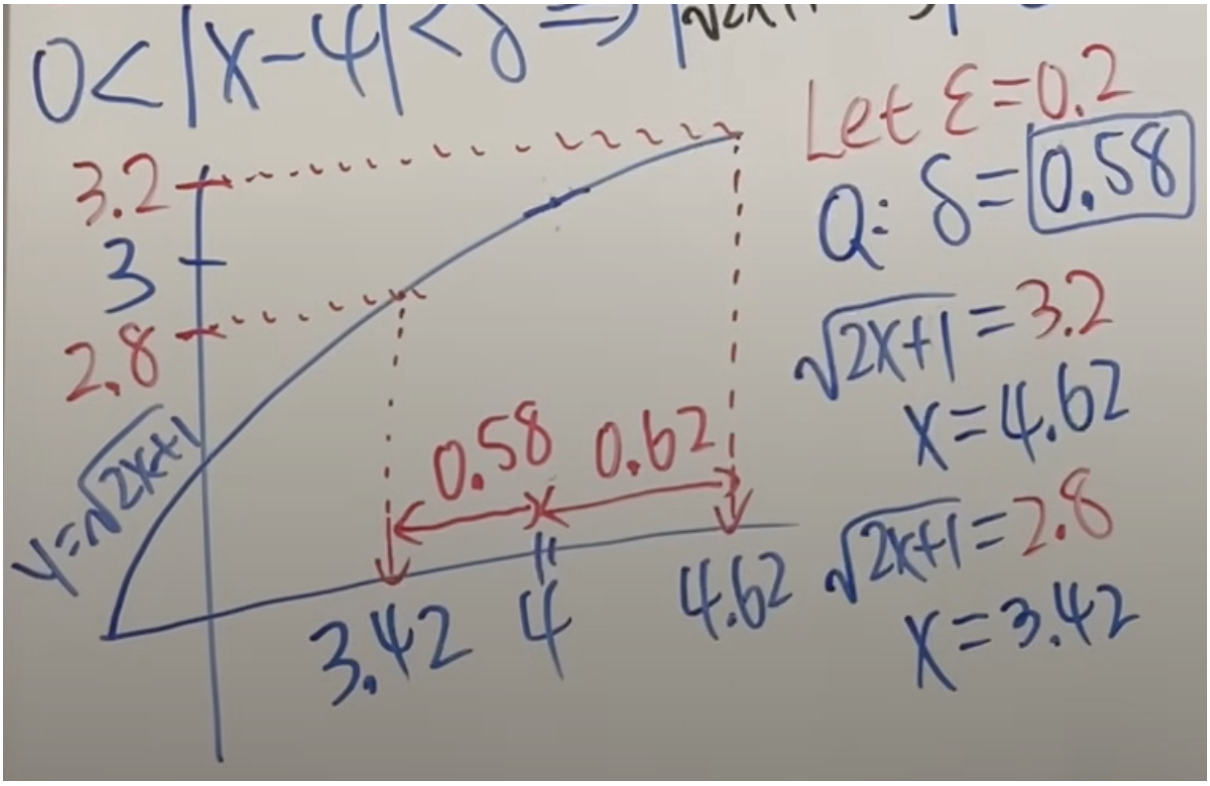
4.3 Epsilon-delta definition of a limit

$\lim_{x\to a} f(x) = L$ means $\forall\varepsilon > 0, \exists\delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example: we know that $\lim_{x\to 4} \sqrt{2x + 1} = 3$ by plugging in 4 into the continuous function

To prove this, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-4| < \delta \Rightarrow |\sqrt{2x+1} - 3| < \delta$

If x is near 4, of a distance less than δ, then the corresponding value of the function is near the limit $L = 3$, of a distance ε



4.3.1 General solution process

Proof: Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x-a|| < \delta$, then $|f(x)-L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim\limits_{x \to a} f(x) = L$

Triangle Inequality

It says: $|a + b| \leq |a| + |b|$

Order Trick

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0$, lim is likely to exist when order is ≥ 1, here it is 1

Simplify Trick

We can: $\frac{3|x|y^2}{x^2+y^2} \leq \frac{3|x|y^2}{y^2} = 3|x|$

We can also: $|x| \leq \sqrt{x^2 + y^2}$

Linear combination of coordinate differences

$|a(x-a) + b(y-b)| \leq |a||x-a| + |b||y-b| \leq (|a| + |b|)\|\mathbf{x} - \mathbf{a}\|$.

4.4 When to use either strategy

We use the epsilon-delta proof to rigorously prove that a limit exists (or equals some value)

We take the limit along lines, parabolas, or curves to test whether a limit exists, or to guess its value. It is useful when you are not sure if the limit exists.

4.5 $\varepsilon - \delta$ for vector-valued functions

Let $F : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}, \vec{a} \in U$

We write $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \vec{L}, \forall\varepsilon > 0, \exists\delta > 0$ s.t. $||F(\vec{x}) - \vec{L}|| < \varepsilon$ if $||\vec{x} - \vec{a}|| < \delta$

Ex: does $\lim_{(x,y)\to(0,0)} (\frac{3xy^2}{x^2+y^2}, \frac{e^x+\cos y}{x^2+y^2+1})$ exist?

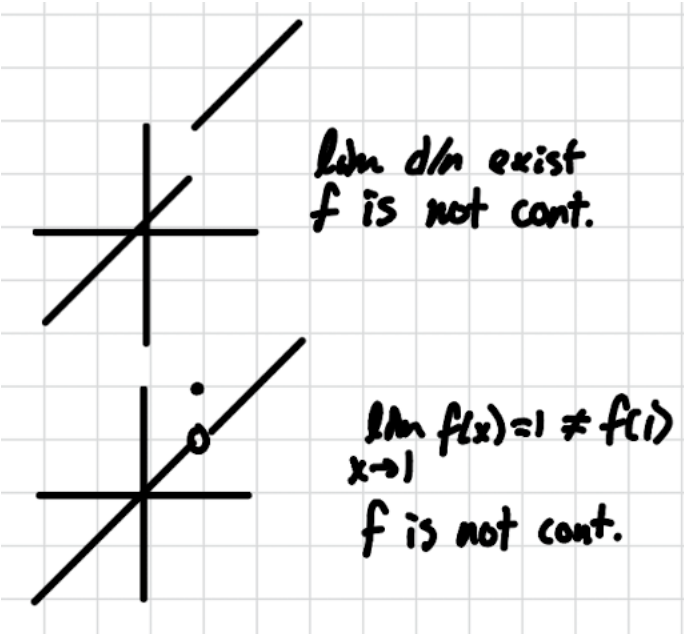
We know that the first component does. For the second component, both the numerator and the denominator are continuous at $(0, 0)$, thus we can plug in that point and get that the limit approaches 2

#Math

5 Continuity and its properties

5.1 Continuity of single variable functions

Let $f : A \to R, a \in A$. f is continuous if (1) $\lim_{x\to a} f(x)$ exists and (2) $\lim_{x\to a} f(x) = f(a)$



5.2 Continuity of multivariable functions

Let $f : U(\subseteq \mathbf{R^n} \to \mathbf{R}$ and $\vec{a} \in U$. f is continuous at $\vec{a}$ if (1) $\lim_{\vec{x}\to\vec{a}} F(\vec{x})$ exists and (2) $\lim_{\vec{x}\to\vec{z}} F(\vec{x}) = F(\vec{a})$

5.3 Properties of continuity (scalar- and vector-valued functions)

Suppose that f and g are continuous at $\vec{a} \in U$

1. $f + g$ is continuous at $\vec{a}$
2. $f * g$ is continuous at $\vec{a}$
3. $\frac{f}{g}$ is continuous at $\vec{a}$ if $g(\vec{a}) \neq 0$

Further:

1. $\lim_{\vec{x}\to\vec{a}}(f + g)(\vec{x}) = f(\vec{a}) + g(\vec{a})$
2. $\lim_{\vec{x}\to\vec{a}}(f * g)(x) = f(\vec{a})g(\vec{a})$
3. $\lim_{\vec{x}\to\vec{a}} \left(\frac{f}{g}\right)(\vec{x}) = \frac{f(\vec{a})}{g(\vec{a})}$ if $g(\vec{a}) \neq 0$

Example:

$$f(x) = \begin{cases} \frac{3xy^2}{x^2+y^2}, (x,y) \neq (0,0), \\ \\ a, (x,y) = (0,0). \end{cases}$$

For which values of a is F continuous?

We know that the first component is continuous everywhere, except possible at $(0,0)$

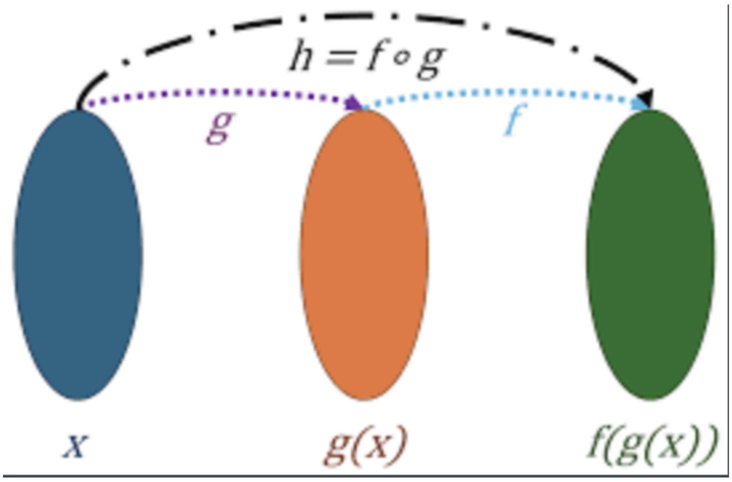
For continuity at $(0,0)$, we need the limit of F at $(0,0) =$ a, which is equivalent to saying that the continuous function $F(0,0) = a$

That means we need to compute the first term's limit while approaching $(0,0)$, which is $= 0$

$\therefore a = 0$

5.4 Composition of two continuous functions

If: 1. g is continuous at $x = a$, and 2. f is continuous at $g(a)$, then $f \circ g$ is continuous at a, where $f(g(x)) \to f(g(a))$



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6 Differentiation

6.1 The derivative

f is differentiable at c if $\lim_{h \to c} \frac{f(x+h)-f(c)}{h}$ exists. If the limit exists, then it is denoted by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(h)}{h}$, where $f'(x)$ captures the rate of change of f near c

If $f(c)$ exists, we can draw a tangent line at c, and its slope is $f'(c)$

6.2 Partial Differentiation

f is partially differentiable wrt x at (a,b) if $\lim_{x \to a} \frac{f(a+h,h)-f(a,h)}{h}$ exists. If exists: $\frac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$

6.3 Tangent plane visualized

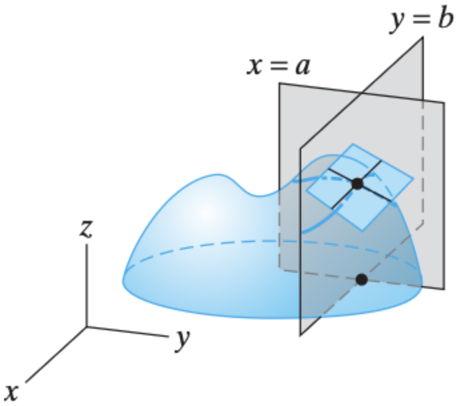


Figure 2.51 The tangent plane at $(a, b, f(a, b))$ contains the lines tangent to the curves formed by intersecting the surface $z = f(x, y)$ by the planes $x = a$ and $y = b$.

6.4 Directional derivative

The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point p in the direction of a vector $\vec{v}$ is the rate at which f changes at p as you move in the direction of $\vec{v}$

$D_{\vec{v}} f(p) = \nabla f(p) \cdot \vec{v}$

For vector valued functions, we can compute using the Jacobian $D_{\vec{v}} f(p) = DH(p) \cdot \vec{v}$

Definition: The directional derivative of f at $\vec{a} = (a, b)$ in the direction of $\vec{v}$ is given by $D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}$, if it exists

Example: let $f(x, y) = x^2y - 3x$, $D_{\vec{v}} f(0, 0) =?$ where $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$D_{\vec{v}} f(0, 0) = \lim_{h \to 0} \frac{f(0,0)+h\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)-f(0,0)}{h}$

Simplify, then plug in h

$= -\frac{3}{\sqrt{2}}$

6.5 Multivariable differentiability at (a, b)

Definition: $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable at (a, b) if $\exists h(x, y) = f(a, b) + f_x(a, b) + f_y(a, b)$

1. $f_x(a, b)$ and $f_y(a, b)$ exists
2. $\exists \mathbf{R} f'(a)$ s.t. $\lim_{h \to 0} \frac{f(x)-h(x,y)}{|x-a|} = 0$, where $h(x, y)$ is the equation of the tangent plane (or line) $f(a, b) + f_x(a, b)(x - a) + f_y(a, b))y - b$

How?

Single variable differentiability is defined by $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

We can rearrange to emphasize linear approximation: $\lim_{x \to a} \frac{f(x)-[f(a)+f'(a)(x-a)]}{x-a} = 0$

This is saying that the function is differentiable at a if it can be approximated by th linear function $h(x, y)$ with error smaller than order $|x - a|$

Multivariable differentiability is now as follows $\lim_{(x,y) \to (a,b)} \frac{f(x,y)-h(x,y)}{||((x,y)-(a,b)||} = 0$

#Math

7 Types of Points

An **open ball** in $\mathbf{R^n}$ with centre at $\vec{a} \in \mathbf{R^n}$ and radius $r : B(\vec{a}, r)$. The ball is open, meaning that the boundary points are not included

Definition: A point $\vec{a}$ is an **interior point** of a set A if there exists an open ball $B_\varepsilon(\vec{a})$, for some $\varepsilon > 0$, such that $B_\varepsilon(\vec{a}) \subseteq A$. So, the open ball lies entirely inside the set, without touching its complement

Definition: A **boundary point** is a point $\vec{a}$ such that every open ball $B_\varepsilon(\vec{a})$, no matter how small $\varepsilon > 0$ is, intersects the function and its complement (not the function)

Essentially, an open ball is all points strictly inside a certain radius form the centre, not including the edge. The interior points are inside the open ball, and boundary points are on the edge.

A set $U \subseteq \mathbf{R^n}$ is called open if every point of U is an interior point

#Math

8 Gradients, More Derivatives, and the Jacobian

8.1 Gradient

The gradient of a scalar function is a vector that collects all the partial derivatives of f with respect to each variable:

$\nabla f = (f_{x_1}, f_{x_2}, \dots, f_{x_n})$

At a specific point, the gradient becomes:

$\nabla f(\vec{a}) = (f_{x_1}(\vec{a}), \dots, f_{x_n}(\vec{a}))$

This vector points in the direction of the steepest increase of f and its magnitude gives the rate of increase

The difference vector:

$\vec{x} - \vec{a} = (x_1 - a_1, \dots, x_n - a_n)$

The linear approximation of f near $\vec{a}$ can be written as:

$\nabla f(\vec{a})(\vec{x} - \vec{a}) = f_{x_1}(\vec{a})(x_1 - a_1) + \cdots + f_{x_n}(\vec{a})(x_n - a_n)$

Example:

Let $f(x, y) = xy^2 + e^{xy}$, find the gradient at $(0, 0)$

$f_x = y^2 + ye^{xy}, f_y = 2yx + xe^{xy}$

$\nabla f = (f_x, f_y) = (y^2 + ye^{xy}, 2xy + xe^{xy})$ $\nabla f(0, 0) = (0, 0)$

Dot product of two vectors

If $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$

8.2 Derivative Matrix

Let $U \subseteq \mathbf{R^n}$ and $f : U(\subseteq \mathbf{R^n}) \to \mathbf{R^m}$

$f = (f_1, f_2, \dots, f_m)$

Let $f(x, y) = (x^2, x + y)$

$f_1(x) = x^2, f_2(x) = x + y$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \dots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \dots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

This matrix is called the matrix of partial derivatives of f, otherwise called the Derivative Matrix or the **Jacobian Matrix**. Essentially, the derivative is a linear map, and in coordinates it is built from the partial derivatives

Example:

Let $f(x, y) = (xy, y^2 \sin x, x^3 e^y)$, find the derivative matrix

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{matrix} = \begin{matrix} y, x \\ y^2 \cos x, 2y \sin x \\ 3x^2 e^y, x^3 e^y \end{matrix}$$

8.3 Differentiability in higher dimensions $f : U \to \mathbf{R^m}$

f is differentiable if: - $Df(\vec{a})$ exists - Tangent plane $h : \mathbf{R^n} \to \mathbf{R^m}$, $h(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$, where $Df(\vec{a})(\vec{x} - \vec{a})$ is a matrix multiplication, satisfies $\lim\limits_{\vec{x} \to \vec{a}} \frac{||f(\vec{x}) - h(\vec{x})||}{||\vec{x} - \vec{a}||} = 0$, which is hard to use

This is why we introduce the following theorems:

8.3.1 Theorems for higher-dimension differentiability

Theorem 1:

If $f = (f_1, f_2, \ldots, f_m)$, then f is differentiable at $\vec{a} \Leftrightarrow f_1, f_2, \ldots, f_m$ is differentiable at $\vec{a}$

Theorem 2:

If $f = (f_1, f_2, \ldots, f_m)$ and all partials $\frac{\partial f_i}{\partial x_j}$, as $i, j, \ldots, i_m, j_m$, are continuous then f is differentiable

Example:

$f(x, y) = (x^2 y, e^y \sin x)$ is differentiable because all of its partial derivatives are continuous

Theorem 3:

If f is differentiable at $\vec{a}$, then directional derivatives can be computed using: $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

If f is differentiable at $\vec{a}$, then $D_{\vec{v}} f(\vec{a}) = Df(\vec{a})\vec{v}$ where $Df(\vec{a})\vec{v}$ is a matrix multiplication

Example:

$f(x, y) = (e^x y, x^2 y)$, find rate of change of f at $(1, 2)$ in direction $\vec{v} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$Df = \begin{matrix} e^x y, & e^x \\ 2xy, & x^2 \end{matrix}, \; Df(1, 2) = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix}$$

$$Df(1, 2)\vec{v} = \begin{matrix} 2e, & e \\ 4, & 1 \end{matrix} \cdot \begin{matrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{matrix} = \begin{matrix} e + \frac{\sqrt{3}}{2} e \\ 2 + \frac{\sqrt{3}}{2} \end{matrix}$$

8.4 Properties of Differentiability

Let $F : \mathbf{R^n} \to \mathbf{R}, G : \mathbf{R^n} \to \mathbf{R}$ be differentiable at $\vec{a}$

- $F + G$ is differentiable at $\vec{a}$
- $F \cdot G$ is differentiable at $\vec{a}$

- If $G(\vec{a}) \neq 0, \frac{F}{G}$ is differentiable at $\vec{a}$
- If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $\frac{d}{dx}(g \circ f) = g'(f(a)) * f'(a)$
- The graph of a function is the set $\{(x, y, f(x, y)) \in \mathbf{R^3} : (x, y) \in \text{ domain}\}$
- If f_x, f_y, f_{xy}, f_{yx} are continuous, then $f_{xy} = f_{yx}$

#Math

9 Differentiability in $\mathbb{R}^3$

9.1 Chain Rule in Composition

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$, where the RHS is a matrix multiplication

Example: $F(x, y) = (x^2y, e^{3x})$ and $G(x, y) = (x + y, xy, \sin(2x - y))$

Find: $D(G \circ F)(1, 1)$, where $(1, 1) = (\vec{a})$

Apply the chain rule equation and get $= DG(1, e^3)DF(1, 1)$

$DF = \begin{matrix} 2xy & x^2 \\ 3e^{3x} & 0 \end{matrix}$ and $DG = \begin{matrix} 1 & 1 \\ y & x \\ 2\cos(2x - y) & -\cos(2x - y) \end{matrix}$

$DF(1, 1) = \begin{matrix} 2 & 1 \\ 3e^3 & 0 \end{matrix}$ and $DG(1, e^3) = \begin{matrix} 1 & 1 \\ e^3 & 1 \\ 2\cos(2 - e^3) & -\cos(2 - e^3) \end{matrix}$

Now, $D(G \circ F)(1, 1) = \begin{matrix} 2 + 3e^3 & 1 \\ 5e^3 & e^3 \\ 4\cos(2 - e^3) - 3e^3\cos(2 - e^3) & 2\cos(2 - e^3) \end{matrix}$

9.2 Polar Coordinate Examples

$x = r\cos\theta, y = r\sin\theta$

$DH(r, \theta) = DG(r\cos\theta, r\sin\theta)DF(r, \theta)$

$DH(r, \theta) = \frac{\partial G}{\partial x}\cos\theta + \frac{\partial G}{\partial y}\sin\theta - \frac{\partial G}{\partial x}r\sin\theta + \frac{\partial G}{\partial y}\cos\theta$

Example: Find DH

With a given $r, \theta, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$, we can find $DH(r, \theta)$ through the chain rule

Example: Find DG

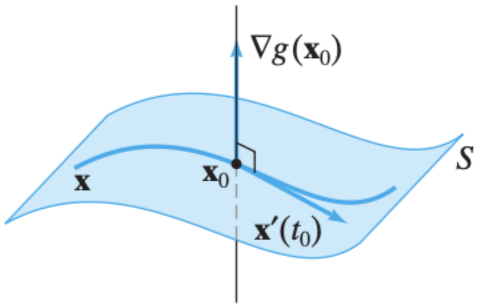
With a given $r, \theta, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial \theta}$, we can find DG with: $\left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right] = \left[\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right] \cdot DF^{-1}$

#Math

10 Applications of the Gradient

10.1 Gradients and level curves

If we have a level curve for the function $x^2 + y^2$, so $f(x, y) = c = x^2 + y^2$, then the gradient ∇F is always perpendicular to the tangent plane to the level curve



Thus, the equation of the tangent plane is given by $\nabla F \cdot (\vec{x} - \vec{a}) = 0, \forall \vec{x}$ on tangent plane, where $\vec{a}$ is the fixed reference vector

Example: Find equation of tangent plane given the function and the reference vector

$f(x, y) = x^2y + ye^x$ at $(0, 1, -1)$

Isolate and get the gradient: $f(x, y, z) = z - x^2y + ye^x$ $\nabla F = (-2xy + ye^x, -x^2 + e^x, 1)$ $\nabla F(0, 1, -1) = (1, 1, 1)$

$(1, 1, 1) \cdot (x - 0, y - 1, z + 1) = 0 \therefore x + y + z = 0$

10.2 Magnitude of ∇F

Consider the directional derivative $D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

In what direction does the function increase the most?

If θ is the angle between $\vec{v}$ and teh gradient vector $\nabla f(\vec{a})$, then we have:

$D_{\vec{v}}f(\vec{a}) = ||\nabla f(a)||||\vec{v}|| \cos\theta = ||\nabla f(\vec{a})|| \cos\theta$ because the magnitude of the unit vector $\vec{v} = 1$

Thus, the max ROC is at $\theta = 0, = ||\nabla f(\vec{a})||$

The min ROC is at $\theta = \pi, = -||\nabla f(\vec{a})||$ and is opposite to $\nabla f(\vec{a})$

10.2.1 Example

Given $f(x, y) = 3 \sin xy, \vec{a} = (1, \pi)$ find: 1. direction of max ROC, value of ROC at $f(\vec{a})$, and direction of tangent to the level curve at $\vec{a}$

1. Get gradient, plug in point, $\therefore$ max ROC is in the direction of gradient
2. Get magnitude of gradient at point, $\therefore$ this is the max ROC

3. ∇f is perpendicular to tangent line to the level curve at $(1, \pi)$. Find $\vec{v} \perp (-3\pi, -3)$

#Math

11 Conservative Vector Fields

A vector field is conservative if $\exists f : U \to \mathbf{R}$ such that $F = \nabla f$

The function f is called a potential function of F

Example: $F(x, y) = (2x, 2y)$

Thus, if $F = \nabla f$ and the potential function $f(x, y) = x^2 + y^2$, then $F(x, y)$ is conservative and f is the potential function

11.1 Test for conservative

Function $G(x, y, z)$ is conservative if

$$\begin{array}{ccc} (G_1)_y & = & (G_2)_x \\ \| & & \\ F_{xy} & = & F_{yx} \end{array} \qquad \begin{array}{ccc} (G_2)_z & = & (G_3)_y \\ \| & & \| \\ F_{yz} & = & F_{zy} \end{array} \qquad \begin{array}{ccc} (G_1)_z & = & (G_3)_x \\ \| & & \| \\ F_{xz} & = & F_{zy} \end{array}$$

11.2 Reconstruct a potential function given its gradient

Find $\nabla f = (f_x, f_y, f_z) = g = (g_1, g_2, g_3)$

1. Integrate g_1 wrt x

$f(x, y, z) = \int g_1 dx + h(y, z)$

2. Differentiate wrt y, set equal to g_2, solve for $h(y, z)$ by integrating wrt y and get a $k(z)$ term
3. Differentiate wrt z, set equal to g_3, solve for $k(z)$ up to constant C
4. Assemble final $f(x, y, z) + C$

#Math

12 Parametrization and Class

Definition of Path: a continuous function $f : I \to \mathbf{R^n}$ where $I \in \mathbf{R}$ is on the interval $[a, b]$

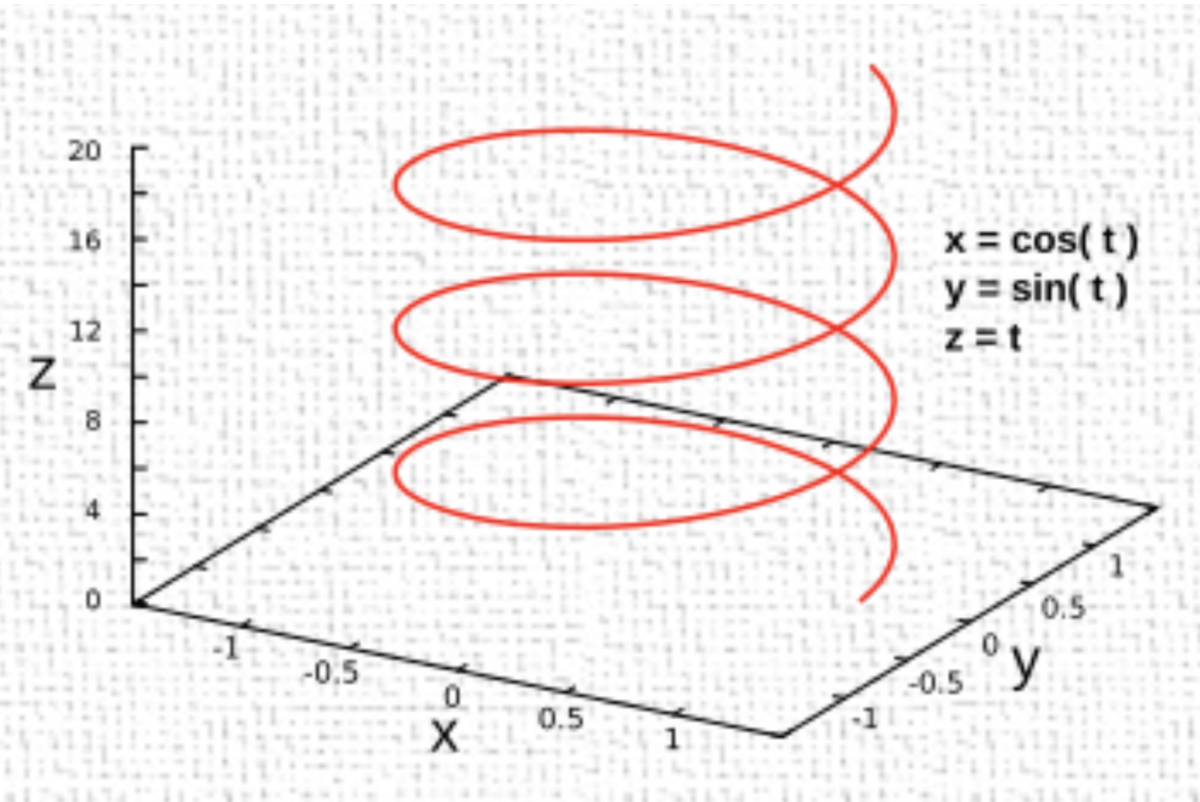
12.1 Parametrization

Ti parametrize a function means to express it in terms of one or more new variables, called parameters, instead of directly in terms of the original variables. Oftentimes, we introduce a variable t that "traces out" the function as it changes

[[Trigonometric Parametrization]]: Use cos and sin when parametrizing a circle, an ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$), or a super-ellipse ($x^{2/n} + y^{2/n} = 1$)

$f(a) =$ starting point of f, $f(b) =$ end point of f

The Im of the path, denoted by $f(I)$ is called the curve in $\mathbf{R^2}$ and f is a parametrization of C



Important result: Parametrization is not unique

$f(t) = (\cos t, \sin t)$ and $g(t) = (t, \sqrt{1 - t^2})$ have the same curve $\text{Im}(f) = \text{Im}(g)$

12.2 Class

Example: $y^2 = x^3$

Parametrized: $f(t) = (t, t^{3/2}) \to f'(t) = \left(1, \frac{3}{2} \cdot \sqrt{t}\right) \to f'' = \left(0, \frac{3}{4} \cdot \frac{1}{\sqrt{t}}\right)$, which is not defined at $t = 0$

$\therefore f$ is of class C^1 and not C^2

#Math

13 Arc Length, Divergence, and Curl

13.1 Arc Length

[Arc length] from a to b, with $f : I \to \mathbb{R}^m$, and c is a curve in f:

$L(f) = \int_a^b ||f'(t)||\, dx$

Method: get parametrization $f(t)$, get speed, then integrate w.r.t. bounds

13.2 Divergence of a vector field

If $Div(f) > 0$, consider the field as a source, flowing out If $Div(f) < 0$, consider the field as a sink, flows in

13.3 Curl of a vector field

$$Curl(F) = \nabla \times F = \begin{vmatrix} i & -j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$Curl(F) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\begin{array}{cc} y & 2 \\ z & 3 \\ x & 1 \\ y & 2 \end{array} \quad \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right.$$

#Math

14 Identities of Operations in $\mathbb{R}^3$

Scalar field: $f(x,y,z)$ Vector field: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

∇f inputs a scalar field and outputs a vector field

$\nabla \cdot F$ inputs a vector field $\vec{F}$ and outputs a scalar field

$\nabla \times F$ inputs a vector field $\vec{F}$ and outputs a vector field

14.1 Identities

The curl of a gradient, $\nabla \times (\nabla f) = \vec{0}$, gradient fields are irrotational

The divergence of a curl, $\nabla \cdot (\nabla \times \vec{F}) = 0$, curl fields have no net source

The divergence of a gradient, $\nabla \cdot (\nabla f)$ is the Laplacian, Δf, a scalar field

The curl of a divergence, $\nabla \times (\nabla \cdot \vec{F})$ is undefined, divergence can't input a scalar field

The gradient of a curl, $\nabla(\nabla \times \vec{F})$ is undefined, gradient can't input a vector field

The curl of a curl, $\nabla \times (\nabla \times (F)) = \nabla(Div(F)) - \nabla^2 F$, and is defined in $\mathbb{R}^3$

G is conservative if $\exists f : U \to \mathbb{R}$ such that $G = \nabla F$, where F is the potential function

The dot product of two vector fields, e.g. $F \cdot G$, is a scalar field defined by $\mathbb{R}^3 \to \mathbb{R}$

If $G : U(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$, so (G_1, G_2). If $Curl(G) = 0$, then G is conservative

If $G : U(\subseteq \mathbb{R}^3) \to \mathbb{R}^3$, if G is the curl of some vector field, then div=0

#Math

15 Special Domains and Conservative Functions

Let $U \subseteq \mathbb{R}^3$ be an open set

U is simply connected if:

1. U is connected (any two points can be connected by a path)
2. Every loop inside U can be shrunk continuously to a point inside U

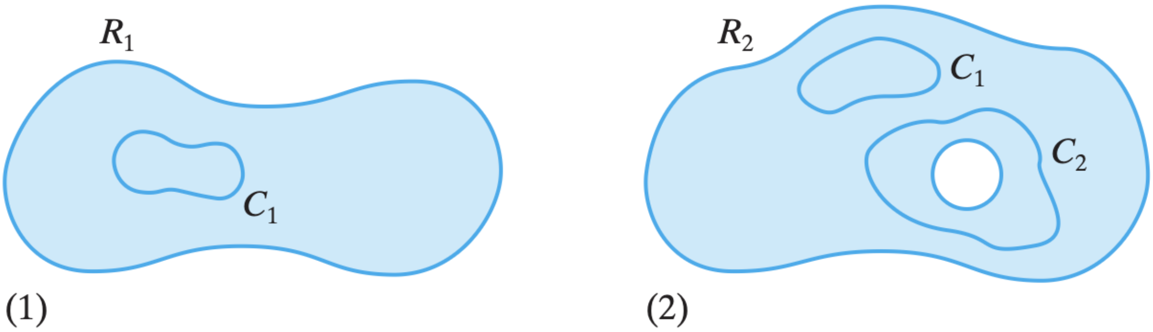


Figure 6.35 (1) The region $R_1 \subset \mathbf{R}^2$ is simply-connected: All points surrounded by any simple, closed curve in R_1 lie in R_1. (2) In contrast, R_2 is not simply-connected: Although the curve C_1 encloses points that lie in R_2, the curve C_2 surrounds a hole. Hence, C_2 cannot be continuously shrunk to a point while remaining in R_2.

If we let $U \subseteq \mathbb{R}^n$ be a simply connected open set, and $F : U \to \mathbb{R}^n$ be a vector field, then f is conservative if and only if $Curl(f) = 0$

Example:

Let $G(x, y, z) = (y^2, 2xy + z, y - \sin z)$, is G conservative? If so, find the potential function f such that $G = \nabla f$

Domain$(G) = \mathbb{R}^3$, simply connected, and $Curl(G) = (1 - 1, 0, 0 - 2y - 2y) = 0$, thus G is conservative

Let $(G_1, G_2, G_3) = (F_x, F_y, F_z)$

$F_x = y^2 \Rightarrow \int F_x dx = xy^2 + g(y, z)$

$F_y = 2xy + z \Rightarrow \frac{\partial F(x,y,z)}{\partial y} = 2xy + \frac{\partial g(y,z)}{\partial y} = 2xy + z \Rightarrow \frac{\partial g(y,z)}{\partial y} = z$

$g(y, z) = \int z dy = yz + h(z) \Rightarrow F(x, y, z) = xy^2 + yz + h(z)$

$F_z = y - \sin z \Rightarrow \frac{\partial F(x,y,z)}{\partial z} = y + \frac{dh(z)}{dz} = y - \sin z \Rightarrow \frac{dh(z)}{dz} = -\sin z$

$h(z) = \int -\sin z dz = \cos z + C$

$\therefore F(x, y, z) = xy^2 + yz + \cos z$

#Math

16 Riemann Sums

16.1 Single-variable Integration

Let $f[a, b] \to \mathbb{R}$ be a function

$\int_a^b f(x)\,dx$ represents the area under the curve

We partition $[a, b]$ into subintervals for **Riemann sums**

Area under $f \approx$ sum of area of rectangles, $A = f(\xi_i)\delta x_i, \Delta x_i = (a_i - a_{i-1}), \xi \in [a_{i-1}, a_i]$

A is integrable on $[a, b]$ if $\lim\limits_{\Delta x_i \to 0} \int \Sigma_{i=1}^n f(\xi_i)\Delta x_i$ exists

16.2 How to integrate functions of two variables

Let $f : [a, b] \times [c, d] \to \mathbb{R}$

$\Delta x_i = a_i - a_{i-1}, \Delta y_j = c_j - c_{j-1}$

V of partitions $= lbh = f(_\xi)\Delta x_i \Delta y_j$

$Vol(A) \approx \Sigma_{i=1}^n \Sigma_{j=1}^m f(\xi_i)\Delta x_i \Delta y_j$

f is integrable over $[a, b] \times [c, d]$ if $\lim\limits_{\Delta x_i \text{ and } \Delta y_j \to 0} \Sigma_{i=1}^n \Sigma_{j=1}^m f(\xi_i)\Delta x_i \Delta y_j$ exists, and is denoted by $\iint_{[a,b]\times[c,d]} f dA$

If f is continuous over $[a, b] \times [c, d]$, then it is integrable

Fubini's Theorem: let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous

Then, $\iint_{[a,b]\times[c,d]} = \int_c^d \int_a^b f(x, y)\,dx\,dy$ and can be reversed

#Math

17 Domains in Integration

When integrating over rectangle R such that $\iint_R f(x)dx, Domain(f) = R$

[[Type 1 Region]]:

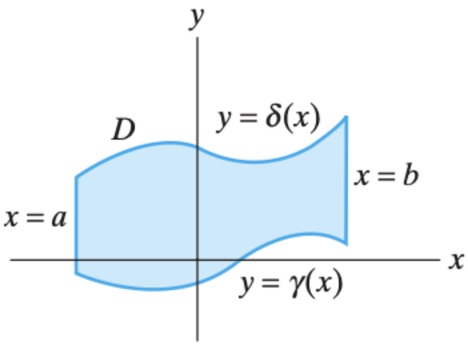
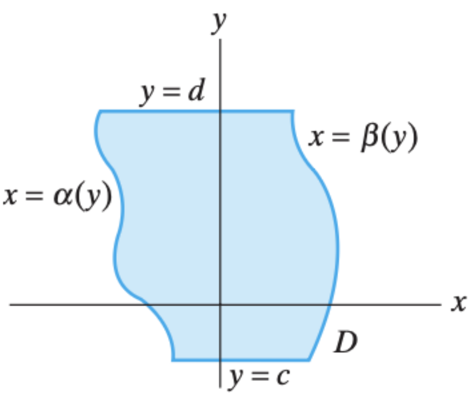


Figure 5.22 A type 1 elementary region.

$D = \{(x, y)|\gamma(x) \leq y \leq \delta(x), a \leq x \leq b\}$, where γ and δs are continuous on $[a, b]$

It is necessary to integrate wrt y first, because x is "uncertain"

[[Type 2 Region]]:



$D = \{(x, y)|\alpha(y) \leq x \leq \beta(y), c \leq y \leq d\}$

It is necessary to integrate wrt x first, because y is "uncertain"

18 Cheat Sheet

18.1 Delta-Epsilon

The condition $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ means our input point is inside the $\delta-$neighbourhood of (a, b)

The proof then shows that whenever the input point is that close to (a, b), the function value $f(x, y)$ lies in the $\varepsilon-$neighbourhood of the limit L: $|f(x, y) - L| < \varepsilon$

Proof: Given $\varepsilon > 0$ We want to find $\delta > 0$ such that if $0 < ||x-a|| < \delta$, then $|f(x)-L| < \varepsilon$

Start with $|f(x) - L|$ and manipulate it to relate it to $||x - a||$ For instance, show: $|f(x) - L| \leq c||x - a||$ for some $c > 0$

Choose $\delta = \frac{\varepsilon}{c}$ and show that $|f(x) - L| < c||x - a|| < c\delta = \varepsilon$

Therefore, $\lim\limits_{x \to a} f(x) = L$

18.2 Disproving a Multivariable Limit

1. Prove with direct substitution

If you get a determinate value (like 5, 0, or ∞) and the function is built from continuous functions, you're done

If you get an indeterminate form like 0/0, proceed with next steps.

2. Disprove with two-path test

For a limit approaching $(0, 0)$, common paths to test include: axis paths (along x, let $y = 0$, vice-versa), linear paths $y = mx$ and the limit d/n exist if it depends on m, parabolic paths

3. Disprove with polar coordinates $x = r\cos\theta, y = r\sin\theta$

18.3 Partial Derivative

Definition of partial derivatives at a point

$\frac{\partial F}{\partial x}(0, 0) = \lim\limits_{h \to 0} \frac{F(h,0)-F(0,0)}{h}$

18.4 Derivative Matrix

$D(G \circ F)(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$

$$Df = \begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \\ \nabla f_m \end{matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n} \\ \cdots \\ \frac{\partial f_m}{\partial x_1}, \frac{\partial f_m}{\partial x_2}, \ldots, \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Let $A = [a, b]$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae + bg & af + bh \end{bmatrix}$

Let A be of size $m \times m$ and B of size $p \times q$, then $C = A \times B$ has dimensions $m \times q$

18.5 Divergence and Curl

The divergence of F denoted by $\nabla \cdot F$ is $\mathbb{R}^3 \to \mathbb{R}$, measures the net rate of flow outward from a point, and is $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

The curl of F denoted by $\nabla \times F$ is $\mathbb{R}^3 \to \mathbb{R}^3$, measures the tendency to rotate or swirl around a point, and is $\nabla \times F =< \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} >$

The gradient of f denoted by ∇f is $\mathbb{R}^3 \to \mathbb{R}^3$ points in the direction of greatest increase of f, and its magnitude is the rate of increase, and is $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$

$\nabla \cdot (\nabla \times F) = 0$, or in words, the divergence of the curl of any vector field F is 0

18.6 Hyperbolic Functions

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Derivatives are the same as non-hyperbolic trig functions

$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$ $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

$\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1}(x) + C$ $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1}(x)$ $\int \frac{1}{1-x^2} dx = \tanh^{-1}(x) + C$

$\cosh^2(x) - \sinh^2(x) = 1$