MIE377 Financial Optimization Models

Project 2

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Contents

1 Introduction

The purpose of this project is to compare five different investment strategies by applying them to build the optimal portfolios on the same set of stocks over a period of three years. We will analyze in details on how the portfolios are constructed and also how they performed, to find out the optimal investment strategy. The three strategies are the following:

The purpose of this project is to compare five different investment strategies by applying them to build the optimal portfolios on the same set of stocks over a period of three years. We will analyze in details on how the portfolios are constructed and also how they performed, to find out the optimal investment strategy. The five strategies are as following:

1. The nominal MVO model, which minimizes portfolio variances while ensuring that a target return is met. We are allowed to short-sell the assets.
2. The Robust MVO model, which acknowledges the fact that the parameters used in the model are noisy, and incorporate their uncertainties as deterministic variability into the model.
3. The Resampling MVO model, an "alternate" Robust model that draws the braces against the uncertainty in parameters by taking the average of hundreds optimal portfolio, each formed using slightly different parameters due to the uncertainty in them.
4. The Most-diverse MVO, which chooses a few assets to mimic the market performance. This is done by having one asset as representative of a cluster of assets that are similar to each other.
5. The Conditional Value-at-Risk (CVaR) optimization, which minimizes the expected loss for each asset using Monte Carlo simulations.

Each model mentioned above will be explained in details in section 2.

2 Methodology

2.1 Dataset information

For the purpose of this project, there are 20 stocks in the market to be considered, all constituents of the S&P 500. In other words, our investment universe in this project consists of 20 stocks (In this project, we use $n = 20$ to denote the total number of assets). The list of stocks and their number of stocks outstanding can be found in Table 1.

Table 1: List of assets and their number of shares outstanding

Company Tickers	F	CAT	DIS	MCD	KO	PEP	WMT	C	WFC	JPM
Number of Shares(billions)	3.640	0.986	1.505	1.091	4.544	1.543	4.345	3.326	5.157	4.525
Company Tickers	AAPL	IBM	PFE	JNJ	XOM	MRO	ED	T	VZ	NEM
Number of Shares(billions)	6.938	0.858	6.456	3.080	3.682	0.770	0.332	5.468	3.899	0.578

2.2 Calibration and Investment Periods

The dataset consists of weekly adjusted closing prices corresponding to these 20 stocks from 20-Dec-2011 to 31-Dec-2015 (Quandl.com, 2017). Within the four years of data, the first year(20-Dec-2011 to 31-Dec-2012) will be used to calibrate the regression model and to estimate the optimization parameters. After which we will construct three portfolios based on the optimization models described, with the investment horizon ranges from the start of 2013 to the end of 2015, for a total of six investment periods. The portfolios must be rebalanced every six months at the start of every January and July. The calibration period should immediately precede the start of the investment period. Once an investment period is over, we will re-calibrate our parameters using the most recent one-year window available.

2.3 Factor Models

The Fama-French three-factor model (Fama and French, 1993) is used to explain our observed rates of return, the equation for which is shown below:

$$r_i - R_f = \alpha_i + \beta_i m(f_m - R_f) + \beta_i sSMB + \beta_i vHML + \epsilon_i \tag{1}$$

where r_i is the asset return, R_f is the risk-free rate of return, α_i is the intercept from regression, β_{im} is the market factor loading, $(f_m - R_f)$ is the excess market return factor, β_{is} is the size factor loading, SMB is the size factor, β_{iv} is the value factor loading, HML is the value factor, and ϵ_i is the residual from regression. The factor data have been provided as part of the project (French, 2016), including the weekly risk-free rate from 20-Dec-2011 to 31-Dec-2015, which is used to calculate excess market return.

The parameters (f_m, R_f), SMB, HML are all given in the factor data, while α_i, β_{im}, β_{is}, and ϵ_i can be calculated using ordinary least-squares regression (OLS), with the procedure as follows:

$$\epsilon_i = r_i - B * \begin{bmatrix} \alpha \\ \beta_{im} \\ \beta_{is} \\ \beta_{iv} \end{bmatrix} \tag{2}$$

This is the closed form solution for OLS, where B is the 4 by 1 matrix consists of

$$B = [1 \quad (f_m - R_f) \quad SMB \quad HML] \tag{3}$$

After the coefficients have been calculated, we can find the vector of residual:

$$\epsilon_i = r_A - B \begin{bmatrix} \alpha_i \\ \beta_{im} \\ \beta_{is} \end{bmatrix} \tag{4}$$

Then, we could calculate the asset expected return and asset covariance matrix that can be put into our investment optimization models:

$$\mu = \alpha + V^T \overline{f}$$
$$Q = V^T F V + D$$

where
$\mu \in \mathbb{R}^{20\times1}$ is the expected return of each of the assets,
$\overline{f} \in \mathbb{R}^{3\times1}$ is the vector geometric mean of the factor returns
$Q \in \mathbb{R}^{20\times20}$ is the covariance matrix ,
$V \in \mathbb{R}^{3\times20}$ is the matrix of factor loadings ($\beta' s$),
$\bar{f} \in \mathbb{R}^{3}$ is the is the vector of expected factor returns,
$F \in \mathbb{R}^{3\times3}$ is the diagonal matrix of factor variances (i.e., the factor covariance matrix),
$D \in \mathbb{R}^{20\times20}$ is the (diagonal) matrix of residual variance $\sigma^2_{\epsilon i}$

2.4 Sharpe Ratios

The Sharpe ratio measures the performance of an asset or portfolio through comparing the ratio between return and risk of the asset/portfolio. It is the ratio between the excess rate of return per unit of risk. The Sharpe ratio can also help explain whether a portfolio's excess returns are due to smart investment decisions or a result of too much risk. A negative Sharpe ratio indicates that a risk-less asset would perform better than the security being analyzed.
Sharpe ratio can be calculated in two ways:

1. **Ex Ante**: Use estimated parameters, which gives us an idea of what our future expectation is

$$SR_p = \frac{\mu^T x - R_f}{\sqrt{x^T Q x}}$$

2. **Ex Post**: Use realized values. Here, the historical portfolio values are used and the ratio of

realized return to volatility incurred during a time series is calculated,

$$SR_p = \frac{\mu_p - R_f}{\sigma_p}$$

This is useful for comparison of risk-adjusted performances among multiple portfolios.

For this project, Sharpe Ratios are calculated at the end of each re-balancing period. For both Sharpe Ratios, the risk-free rate R_f for each week is given as part of the data used for this project.

2.5 Transaction Costs

For the purpose of this project, we must record our transaction costs every time we rebalance the portfolio. The cost of buying or selling an asset is 0.5% of the traded volume. This means that if a stock is currently quoted at \$25 and the investor buys 10 additional shares, then the transaction fee will be equal to 0.0052510 = \$1.25. Please note that, for the sake of simplicity, we are allowed to buy, sell, and hold fractions of stocks (e.g., we are allowed to hold 32.22 shares of a stock). There is no cost associated with the construction of the starting portfolios. In other words, we only need to start measuring the transaction costs during our first portfolio rebalance.

2.6 The MVO Model

The typical MVO problem seeks to minimize variance while achieving a target return, based on estimated asset returns μ and covariance matrix Q. It is shown below:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & x^\top Q\, x \\ \text{s.t.} \quad & \mu^\top x \geq R \\ & 1^\top x = 1 \end{aligned}$$

Where x is a n by 1 column vector where element x_i represents the weight to be invested in stock i to construct the optimal portfolio, where$n$ is the number of stocks in the portfolio, which in this case equals 20. Q is the n by n covariance matrix of excess return of each asset, μ is the n by 1 column vector expected return for each stock. Both Q and μ are estimated from the factor models described in Section 2.3. $R = 0.2\%$ is the weekly target return, which is taken to be the average estimated expected return of all assets in the portfolio 0.2%. Note that we are allowed to short-sell the assets.

2.7 Robust MVO

The robust MVO model seeks to minimize variance while maximizing return, while taking uncertainty into account. The Nominal MVO model uses estimated expected returns, μ, which has uncertainties that are not incorporated into the model. Without consideration of uncertainties, a small estimation error in μ can have a significant impact on the optimal portfolio weights generated by the MVO model.

To solve this problem, the robust MVO model incorporates the uncertainty in μ by formulating a robust counterpart. It is modeled as follows:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \lambda x^\top Q\, x - \mu^\top x \\ \text{s.t.} \quad & 1^\top x = 1 \end{aligned} \tag{5}$$

Where the value λ is set to be 50. In addition, we also have an ellipsoidal uncertainty set around the expected return:

$$\mu_{true} \in \mathcal{U}(\mu) = \{\mu_{true} \in R^n : \ (\mu^{true} - \mu)^\top \Theta^{-1} (\mu^{true} - \mu) \leq \epsilon_2^2\} \tag{6}$$

where μ_{true} is the vector of true expected returns, ϵ_2^2 is the maximum distance from the estimated expected returns where we believe the true expected returns will lie, which is related to the confidence level around the estimate.

θ in the uncertianty set is calculated as :

$$\theta = \frac{diag(Q)}{n} \tag{7}$$

Where Q represents the covariance matrix of assets

ϵ is found using the inverse cumulative distribution function of the chi-squared distribution with N degrees of freedom, with the input being $1 - \alpha$. for this project, the confidence interval is set to be 90% ($\alpha = 10\%$).

$$\epsilon = X_n(1 - \alpha) \tag{8}$$

This changes the formulation of the problem to:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & x^\top x - \epsilon y - \lambda x^\top Q\, x \\ \text{s.t.} \mu^t opx - epsilon & \qquad\qquad y^2 = x \top \theta x \\ & 1^\top x = 1 \\ & y \geq 0 \end{aligned} \tag{9}$$

Note that we are allowed to short-sell the assets.

2.8 Resampling MVO

The Resampling MVO is an alternative to the Robust MVO technique. Instead of using the original estimate of μ and incorporating a robust counterpart into the nominal MVO model, the resampling model finds the average portfolio weights from different sets of estimates. The following algorithm is used to compute a portfolio under the resampling MVO model:

1. Estimate the asset expected returns μ and covariance matrix Q. In this project, these parameters are found from the Fama-French Factor model described in Section 2.3.

2. Collect a sample of T observations by darwing randomly generated values from the asset return distribution, which is assumed to be normal, $r_t \sim N(\mu, Q)$ for $t = 1, ..., T$, with $r_t \in \mathbb{R}^{20}$. For this project, we choose $T = 100$.

3. Use the T randomly generated observations to estimate a new set of expected returns μ' and covariance matrix Q'. Note that, these estimations are calculated from the generated data directly, without using the factor model.

4. Find optimal weights x' using the nominal MVO model and the newly estimated parameters μ' and Q', and store these weights.

5. Repeat steps 2 to 4 60 times, and calculate the final portfolio by taking the average of all x's.

2.9 Most-diverse MVO

The Most-diverse MVO is similar to the nominal MVO problem, except that the assets available to construct the optimal portfolio is no longer 20 assets, but instead 12 assets that are determined to be representative of all 20 assets. The motivation for Most-diverse MVO is similar from cardinality-constrained MVO, which is to reduce the number of assets used to form a portfolio thus reducing transaction cost and making it easier to manage. The representative assets are found using the following steps:

First the 20 assets are categorized into 12 clusters, each cluster contains assets that are similar to each other, all assets in the same cluster has high correlation with each other, with correlation between asset i and asset j being:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \tag{10}$$

Next each cluster will have a representative that represents all assets in the cluster, the representative is chosen to be the one with highest correlation with all other assets within the cluster, and can be as a mixed-integer linear programming problem:

$$
\begin{aligned}
\max_{\boldsymbol{x}} \sum_{i=1}^{n} \sum_{j=1}^{n} & \rho_{ij} z_{ij} \\
\text{s.t.} \qquad & \sum_{i=1}^{n} y_j = k \\
& \sum_{j=1}^{n} z_{ij} = 1 \ for \ i = 1, ...n \\
& z_{ij} \leq y_j \ for \ i = 1, ...n; \ j = 1 = 1, ...n \\
& z_{ij} \in \{0, 1\} \quad y_j \in \{0, 1\}
\end{aligned}
\tag{11}
$$

Where z_{ij} and y_j are decision variables:

$$
y_j = \begin{cases} 1 & if \ true \\ 0 & otherwise \end{cases} \tag{12}
$$

$$
z_{ij} = \begin{cases} 1 & if j \ is \ the \ most \ similar \ asset \ in \ the \ portfolio \\ 0 & otherwise \end{cases} \tag{13}
$$

After solving the LP problem to get z_{ij} we can get weights for each representative assets:

$$
x_j = \frac{\sum_{i=1}^{n} V_i z_{ij}}{\sum_{i=1}^{n} V_i} \tag{14}
$$

where V_i is the market capitalization of asset i and x_j is the optimal weight of our representative asset j.

Note that our target return should still be the average value of the expected returns from all 20 assets. We are allowed to short-sell the assets.

2.10 Conditional Value-at-Risk Optimization with Monte Carlo simulations

2.10.1 Monte Carlo Simulations

Monte Carlo simulations, also known as 'stochastic simulation', generates hypothetical observations (scenarios) from repeated random sampling. It allows us to solve problems where a deterministic solution is difficult to achieve by using a probabilistic analog, and the solution of problem is represented as a parameter of the hypothetical population. Sufficiently large number of scenarios will converge towards the 'true' deterministic solution. To model stock prices over time, geometric random walks are used to generate hypothetical observations. The following time series properties are considered when generating scenarios:

- **Drift**: The change of the average value of a random process, or the long-term trend of the time series.
- **Volatility**: The variation over time, or standard deviation of randomness.
- **Volatility to drift relationship**: Typically, volatility increases when drift decreases, and vice versa.

For this optimization model, we use geometric random walks to generate hypothetical stock returns. It is assumed that the returns follow a normal distribution, $r_t \sim N(\mu, \sigma)$, and that the returns follow a Gaussian process

$$r_t = \frac{S_t - S_{t-1}}{S_{t-1}} = \mu + \sigma\epsilon_{t-1}$$

where ϵ_0, ..., ϵ_{t-1} are independent normal random variables. Thus, for a given time step dt, we can compute the stock price in the next time period as

$$S_{t+1} = S_t exp[(\mu - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{t} * \epsilon]$$

dt must be in the same time frame as for estimates of μ and σ, thus, $dt = t/n = 26/1 = 26$ is a 26-week(or six months) time step in this project.

To simulate the price path of a single stock, the following steps are followed:

1. Estimate μ and σ from historical data. In this project, they are estimated with Fama-French factor model described in Section 2.3.
2. Select the number of time steps to take ....
3. Generate stock price at each time step t for asset i with the equation

   $$S^i_{t+1} = S^i_t exp[(\mu_i - \frac{1}{2}\sigma_i^2)dt + \sigma_i\sqrt{t} * \xi^i_t]$$

   Where we define a new vector

   $$\xi = L\epsilon$$

   to accommodate for correlated assets. To enforce the correlation, we find $LL^T = \rho$, where $\rho \in \mathbb{R}^{20\times20}$ is the correlation matrix, in MatLab by

   $$L = chol(\rho,' lower')$$

4. Repeat step 2 to generate 2,000 paths

2.10.2 Conditional Value-at-Risk Optimization

Value-at-Risk (VaR) is a risk metric which measures the loss at the tail of a Profit and Loss (PnL) Probability distribution. Conditional Value-at-Risk (CVaR) measures the expected loss exceeding VaR. Using portfolio variances as a risk measure and minimizing it has the drawback for simulating an investor averse to both upside and downside movements. By optimizing CVaR instead, we could minizmizing downside risk only, i.e. minimizing $CVaR_\beta(x)$, which is more realistic.

The $CVaR_\beta(x)$ is defined as,

$$CVaR_\beta(x) = \frac{1}{1-\beta}\int_{f(x,r)\geq VaR_\beta(x)} f(x,r)p(r)dr$$

Where x is the portfolio, r is the random vector of returns, $p(r)$ is the density of r and $f(x,r)$ is the loss of portfolio for a realization of r, and β is the confidence level.

It is difficult to set up an optimization model with this definition of $CVaR_\beta(x)$, because it is defined in terms of $VaR_\beta(x)$, which is not sub-additive and non-convex. Therefore, we change the definition of $CVaR_\beta(x)$ to the following:

$$F_\beta(x,\gamma) = \gamma + \frac{1}{1-\beta}\int_{f(x,r)\geq\gamma}(f(x,r)-\gamma)^+p(r)dr$$

Where $\gamma = VaR_\beta(x)$ is an auxiliary variable introduced as placeholder for VaR during the optimization process, and the function $a^+ = max\{a,0\}$.

In this setup, $F_\beta(x,\gamma)$ is a convex function of γ.

Next, to avoid using the density function $p(r)$, we use a scenario representation, r_s for $s = 1,...,S$, which is the realization of scenario s. Each scenario is equally likely, so we can approximate $F_\beta(x,\gamma)$ as

$$\tilde{F}_\beta(x,\gamma) = \gamma + \frac{1}{(1-\beta)S}\sum_{s=1}^{S}(f(x,r_s)-\gamma)^+$$

Then we deal with the non-linearity by:

$$(f(x,r_s)-\gamma)^+ = \begin{cases} z_s \geq 0,\ \ for\ s = 1,...,S \\ z_s \geq f(x,r_s)-\gamma,\ \ for\ s=1,...,S \end{cases}$$

For this project a confidence level of $\beta = 95\%$ was chosen. The final $CVaR_{95\%}(x)$ optimization model is setup as

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{\gamma}} \quad & \gamma + \frac{1}{(1-\beta)S}\sum_{s=1}^{S} z_s \\
\text{s.t.} \quad & z_s \geq 0, \qquad s = 1, ..., S \\
& z_s \geq f(x, r_s) - \gamma, \qquad s = 1, ..., S \\
& x \in \chi
\end{aligned}
\tag{15}
$$

where χ includes budget, target return and no-short-selling constrains to the portfolio weights x; $S = 2000$ represents the total number of scenarios scenario; z_s for $s = 1, \dots, S$ represents the loss in excess of VaR in scenario s; $f(x, r_s)$ is the loss function in scenario, where r_s is the realization of scenario s. In this project, we treat the loss as negative values of the portfolio return,

$$f(x, r_s) = -r_s^T x$$

3 Results

3.1 Portfolio Performance

The performance of all five investment strategies are summarized in Figure 1 and table 2 below.

Figure 1: Portfolio Return for All Five Models

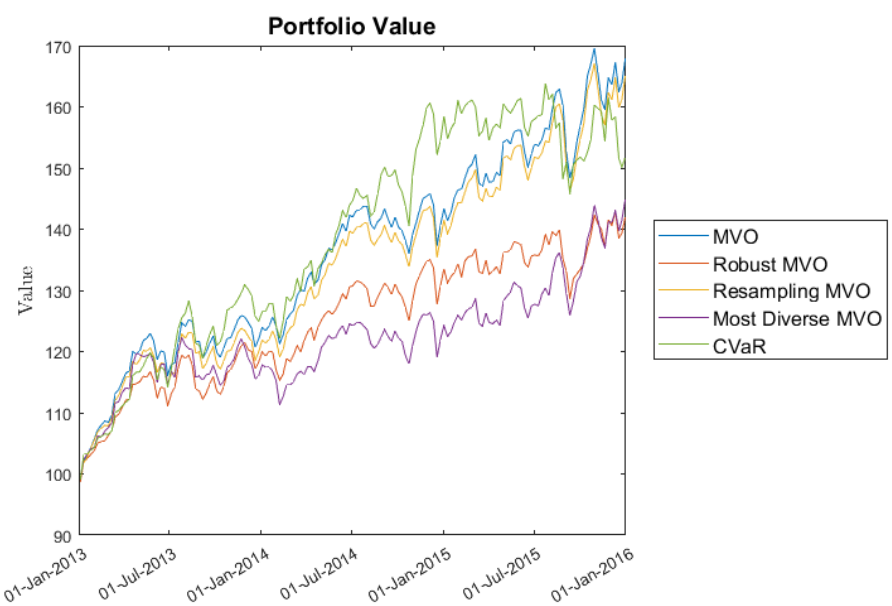


Table 2: Portfolio Performance Summary

	Weekly Mean Excess Return	Variance	Standard Deviation
MVO	0.336%	0.000234	0.0153
Robust MVO	0.259%	0.000214	0.0146
Resampling MVO	0.337%	0.000231	0.0152
Most Diverse MVO	0.257%	0.000226	0.0150
CVaR	0.265%	0.000332	0.0182

As shown in the table, portfolios from all five models yield monthly return of greater than 0.2 %, the average expected return of all 20 assets. Moreover, all five portfolios suffer huge drop in values in August 2015, due to A substantial concern of global market instability caused by Greece default on June 30 on an International Monetary Fund loan payment.

Among the five portfolios, the portfolio values for MVO and Resampling MVO are almost identical across the span of three years. This is reasonable since the parameters for Resampling are derived from 100 random observations per iteration for 60 iteration. Since the sample size is so large, the average values from these observations converge to the mean value of parameters, which are used in the nominal MVO model.

Note that one of the motivations for the Resampling MVO is that, even when the parameters are altered by a small amount, the portfolio weighting and resulting portfolio values may be changed by a lot. but from figure 1, after considering uncertainty in parameters, the portfolio value stays the same. Thus we can conclude that in our case, either the nominal MVO is robust against uncertainties in parameters, or the number of iterations used in Resampling is high enough, such that the effect of varying parameters is not cleared seen.

We can also see that the Robust MVO portfolio has much lower weekly return compare to MVO. One of the key reason for this is that Robust MVO model does not incorporate target return, unlike nominal MVO. Instead it minimizes variances while maximizes return, and the weighting of importance between the two is determined by λ in formulation 5. With $\lambda = 50$, the model emphasizes more than minimizing variance than getting a higher return, as evident in it having the lowest variance among all models (see table 2).

Most-diverse portfolio yields the lowest return during most of the rebalancing period, except in the last 3 months where the portfolio value catches up to Robust MVO. This is to be expected, since the most-diverse portfolio is just like the nominal MVO, except that it only has 12 assets to construct the portfolio, instead of 20 in nominal MVO. The 12 assets are chosen based on only their correlation with other assets, not on whether they are ideal to be invested in. Thus by using Most-diverse portfolio we make sure our portfolio is diversified thus reduces its variance, but we also have less assets to choose from, and as a result yield lower return compare to nominal MVO.

Lastly, the CVaR portfolio value significantly outperforms all other models in majority of rebalancing periods, except for the last three months, where CVaR's value has a steep fall in the last three months of 2015, resulting its ending value lower than nominal MVO and Resampling MVO. The reason for the CVaR model to outperform the rest is that, it is the only model that has an asymmetric setup: it only considers the downside risk, while the other models also consider, and minimizes upside risk (which are perferred by investors). By allowing upside risk, CVaR can generate high return by allowing assets to have higher than expected return. Note that, during August 2015 Greece default crisis, CVaR suffers the largest drop among all models, which leading it to under-perform in the end. By checking table 2 we can see that CVaR indeed gives the highest variance, meaning when the market is unstable, CVaR portfolio's value will also become highly unstable. This is due to the fact that CVaR measures the expected loss exceeding VaR, which measures the loss at the tail of a Profit and Loss probability distribution. When the market is unstable, the probability distribution would also be unstable, thereby causing CVaR to be unstable.

Note that the transaction cost during each rebalance has not been reflected in Figure 1, which can affect the net profit using each investment strategy. The affect of transaction cost will be analyzed in section 3.12.

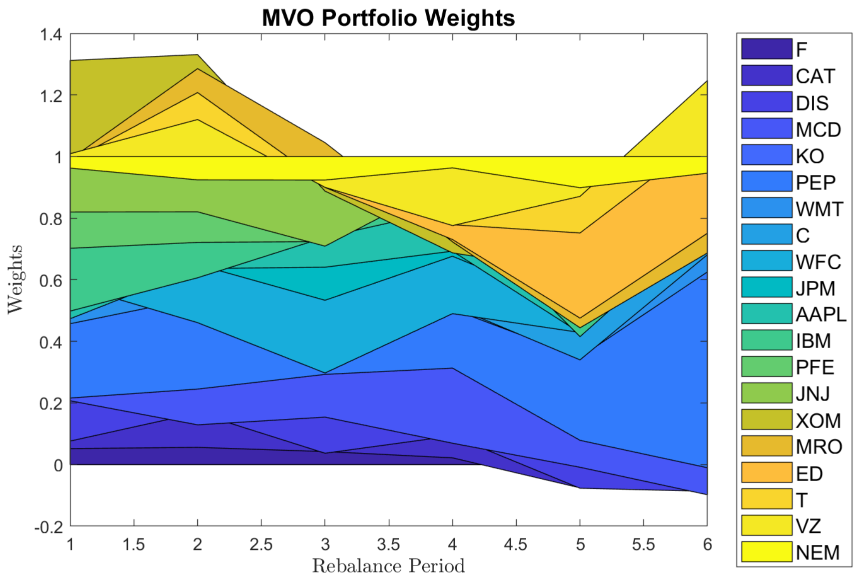
3.2 MVO Portfolio Weights

Here in Figure 2 we see the optimal weights of the 20 assets used to construct the optimal portfolio throughout the six investment periods using the MVO model. Note that short-selling is allowed in this model, thus the optimal weights for some assets would be less than zero, in which case we are shorting such assets.

In the figure we can spot the problem of over-concentration with this model, especially in the last period (period 5 to 6). During this time there are assets that have significant weights(PEP and ED); while some other assets that have close to zero weights(C, IBM and F). By checking out the results of in Matlab, we can see that in the final optimal portfolio at the end of sixth period, PEP has the largest weight of 63% budget, while stocks like F have no weights at all. This agrees with our discussion in class, that MVO can likely to result in over-concentrated portfolios.

Other than over-concentration, we can see that some assets have weightings changing significantly throughout the rebalance periods. This means that assets are constantly being traded to rebalance the optimal portfolio, which can incur a rather large transaction cost.

Figure 2: MVO Portfolio Weights

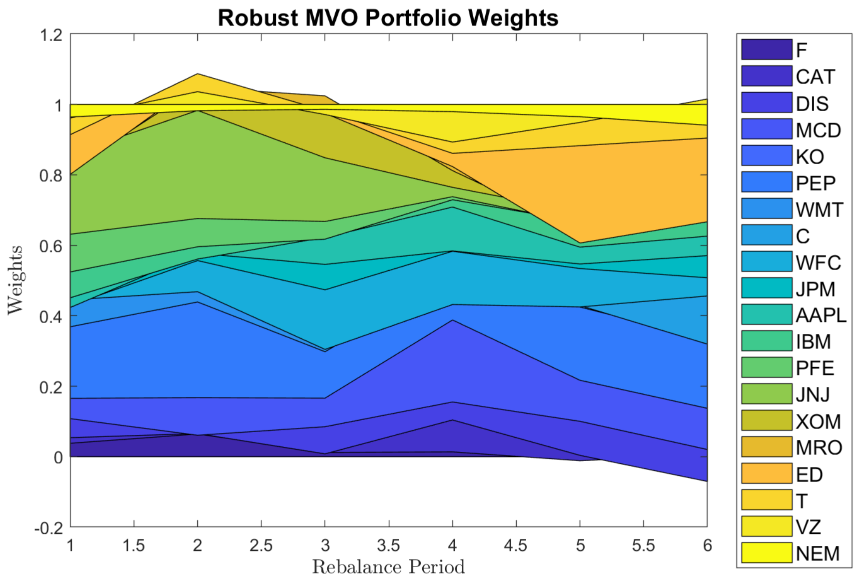


3.3 Robust MVO Portfolio Weights

Here in Figure 3 we see the optimal weights of the 20 assets generated from the robust MVO model. Comparing to the weights found from the nominal MVO model, it is clear that over-concentration of assets starting from period 5 (around July 2015) has improved significantly, as there are roughly 10 assets with significant weights toward the end of rebalance periods. The rapid increase in value for a few stocks after period 5 has caused MVO to place heavy weights on them. However, such increase might have large uncertainty to occur again. Because the robust MVO model takes uncertainty into account, it neutralizes the occasional increase in the few asset values, thereby reducing over-concentration.

In addition, the weightings for individual assets appeared to be a bit ”smoother” compared to the nominal MVO, this means less transaction costs than the nominal MVO. This is because robust MVO considers a 90% confidence interval of expected return for each asset, thus when a particular asset's return varies from one period to another,as long as the return is still within the confidence interval, the robust MVO will not ”overreact” by drastically changing its weightings, unlike in nominal MVO.

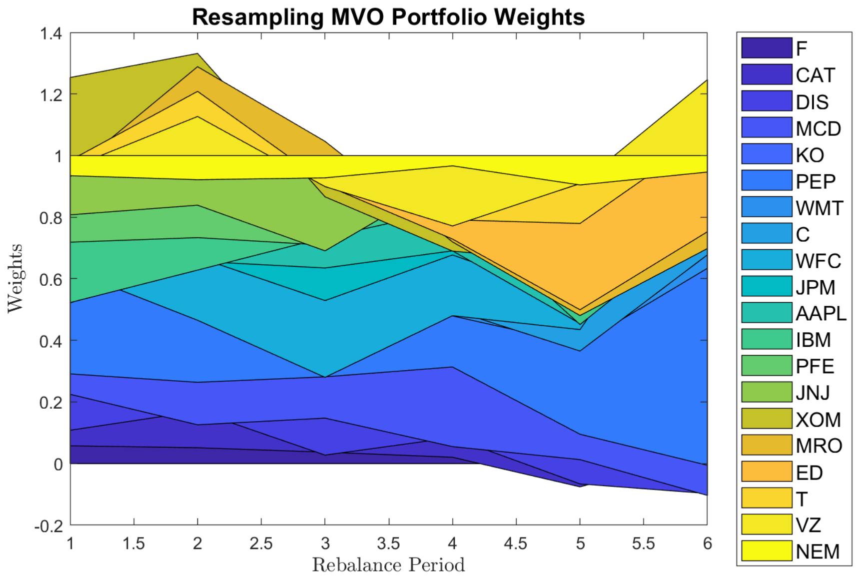
Figure 3: Robust MVO Portfolio Weights



3.4 Resampling MVO

Figure 4 displays the optimal weights of the 20 assets used to construct the optimal portfolio throughout the six investment periods using the Resampling MVO model. These weights are very similar to those produced by the nominal MVO model. This explains why the performance of portfolios created by these two models are very similar. As explained in Section 3.1, the large number of iteration would cause the results of the resample MVO model to converge to those of the nominal MVO. Whether the similar performances between the two model is due to nominal MVO in this case is robust against uncertainties or or due to large number of simulations, will be discussed in detail in section **??**

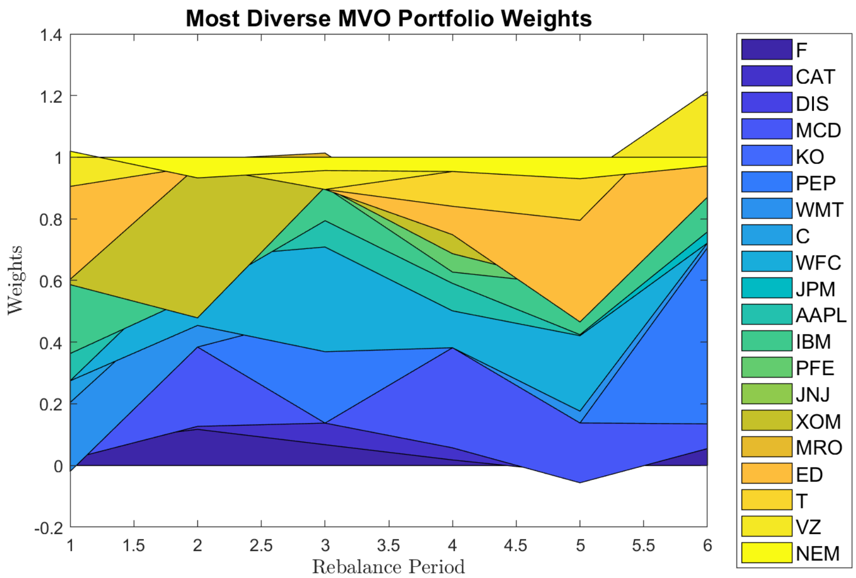
Figure 4: Resampling MVO Portfolio Weights



3.5 Most-diverse MVO

The optimal weights of 12 assets used to construct the optimal portfolio generated by the Most-diverse MVO model is shown in Figure 12. In each period only 12 assets are chosen as representative and being plugged into the nominal MVO model. Similar to nominal MVO, the most-diverse MVO suffers over-concentration problem with most weights being put in PEP and ED, especially during the last period. It is an interesting fact that the assets over-concentrated by the nominal MVO just so happened to be representative assets in Most-diverse MVO. Furthermore, the Most-diverse portfolio also does a large amount of rebalancing despite it only having 12 assets to choose from. This is partly due to the representative assets are changed after each rebalancing period. Thus a representative that has significant weights in the previous period may no longer be a representative in the current period, and all of its previous weights will need to be redistributed, thus causing higher transaction cost.

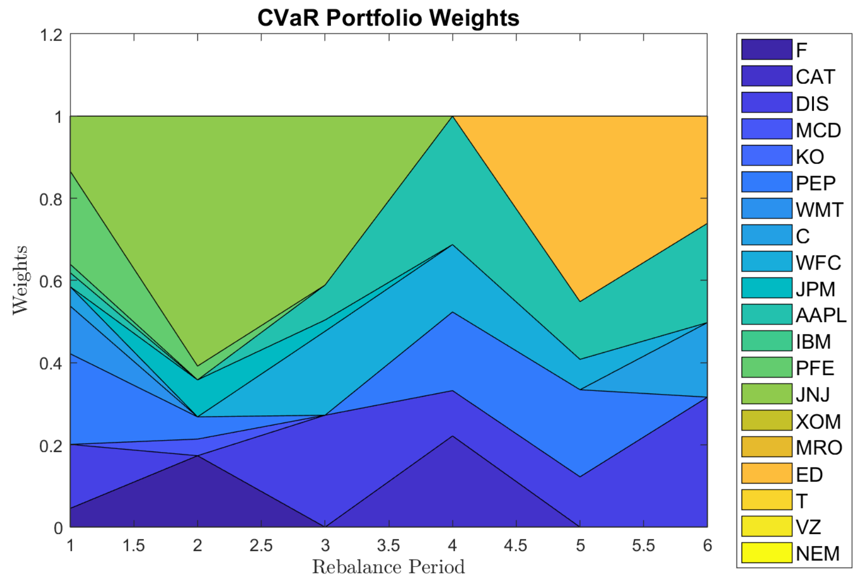
Figure 5: Most-Diverse MVO Portfolio Weights



3.6 CVaR weights

The weightings of optimal portfolio using CVaR is shown in figure 12. Compared to the previous model, CVaR suffers the most severe over-concentration problem. This is to be expected, since the formulation of CVaR model doesn't consider variances of assets, and instead finds the one asset that has the lowest downside risk for each of Monte Carlo simulation, denoted as a_i. Since there are 2000 simulations in total, it is likely for a_i to be different for each simulations. In the end, there are multiple a_i generated from simulations, which are used to construct the optimal portfolio. But the number of a_i are still much lower than the total number of assets avaliable in the market, resulting in over-concentration.

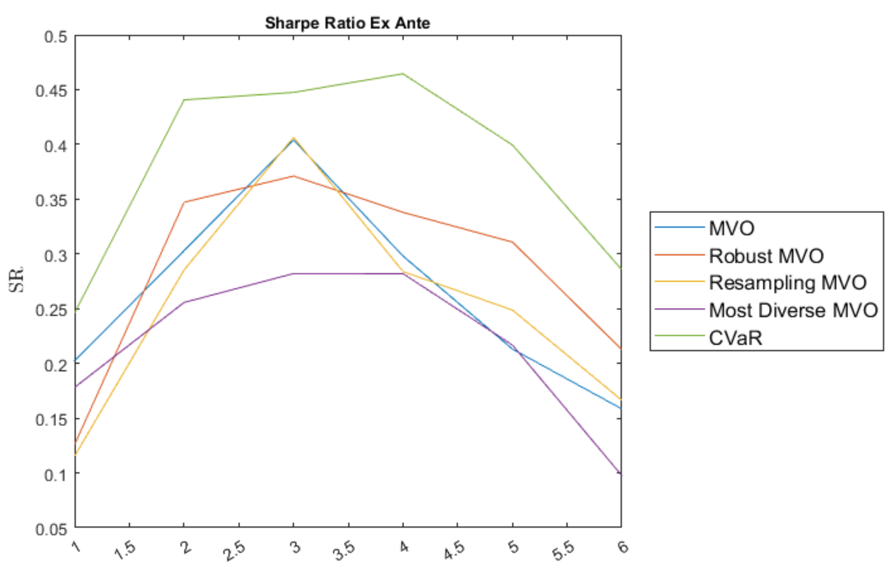
Figure 6: CVaR Portfolio Weights



3.7 Sharpe Ratios

3.7.1 Comparison of Sharpe Ratios among different models

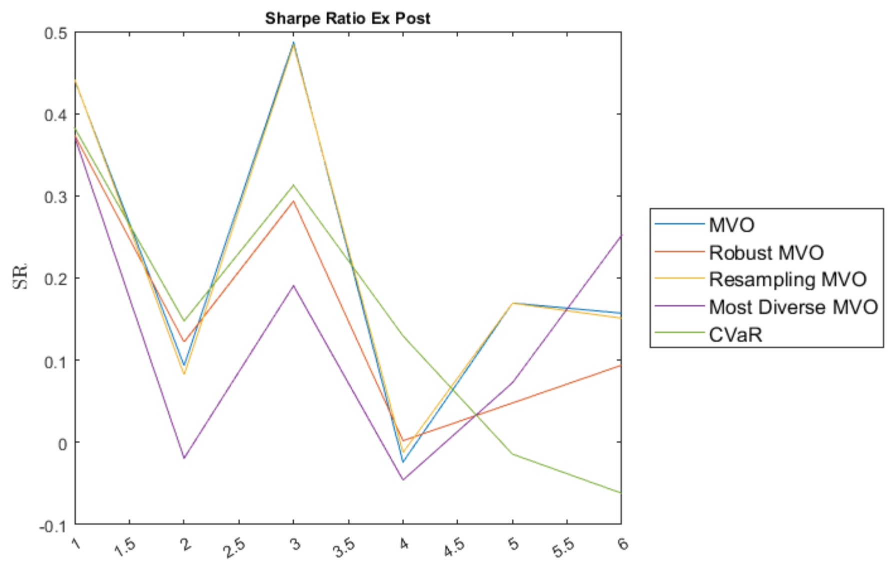
Figure 7: Sharpe Ratios Ex Ante for different models



The Ex Ante Sharpe Ratio (Figure 7) measures the future expectation excess rate of return per unit of risk. Overall, the CVaR model has significantly higher risk-adjusted performance compared to other models because known. The MVO and Resampling MVO have very similar risk-adjusted performance, as expected from figure 1. Most-Diverse MVO performs poorest by having the worst risk-adjusted performance, meaning that by expectation, although Most-diverse MVO successfully reduces risk by choosing assets that are relatively uncorrelated with each other, by reducing the number of assets to 12, its return suffers so much such that it is not worth it to implement Most-diverse MVO compare to other models.

Overall all five models have concave down graphs. Meaning that for the first few rebalance periods, the market is doing well and it is relatively stable(low risk) and/or all assets have higher expected return; while for the last few periods, the expected return dropped and/or have higher risks due to market instability.

Figure 8: Sharpe Ratios Ex Post for different models



The Ex Post Sharpe Ratio uses historical data to calculated realized excess returns over volatility. It can be used to compare the risk-adjusted performance of different portfolios. In Figure 8, the Ex Post Sharpe Ratio at the end of each rebalancing period for each optimization model is displayed.

At first glance, the Ex Post SR graph is not as smooth as the Ex Post SR graph, the obvious reason being that, in real life environment the asset performances are much less straight forward, as there are a lot more parameters that can affect a model but are not considered by our models, also there can be global economic events that happens that can greatly affect the stock market(such as the Greece default), that are impossible to be predicted by our models. Thus the model expectation are always "naive" compare to how stocks actually behaves. As a result, model gives a more straight forward prediction on asset performances.

Needless to say, MVO and Resampling MVO have very similar Ex Post values because their optimal weights after each reblancing are similar. Note that the Most-Diverse MVO has the lowest Sharpe Ex Post ratio among all models for most rebalancing periods. This is consistent with figure 7 and figure 1 where Most-Diverse is also ranked lowest in terms of portfolio value and Ex Ante ratio. Also at time step 2 and 4, Most-Diverse has negative Ex Post value, meaning that in the six months following these points in time, investment using Most-Diverse MVO yields lower return than the risk-free rate. In addition, at time step 4, almost all models have close to zero or negative Ex Post value, this agrees with 1, because if we compare the portfolio values between July 2014 and Jan 2015, all portfolios other than CVaR have very little return in these six months. Interestingly, the Greece default in August 2015 didn't lead to low Ex Post value, because after a slump in price the market quickly recovers from it and portfolio values rise back up.

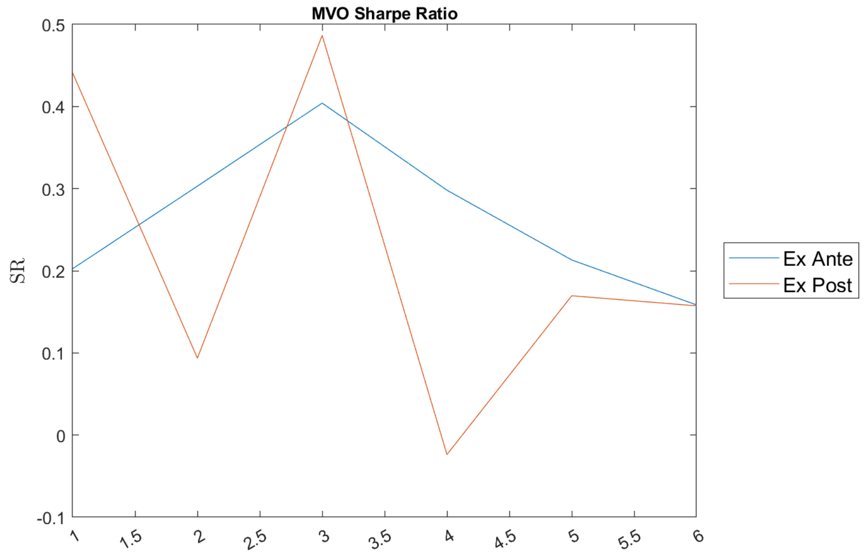
Lastly, one useful application of Ex Ante and Ex Post values is to use them to evaluate our investment decisions. For example, Ex Ante shows that CVaR always has the best risk-adjusted performance, implying we should always follow CVaR when investing; however, from Ex Post we see that most of the time CVaR does not give the highest risk-adjusted performance, thus we would be making wrong decision if we invest following CVaR.

3.7.2 Comparison of Ex Ante and Ex Post Sharpe Ratio for each model

To compare how the realizations of portfolio values compare to the forecasts, the Ex Ante Sharpe Ratio and the Ex Post Sharpe Ratio are plotted on the same graph for each optimization model (Figure $9 \sim 13$).

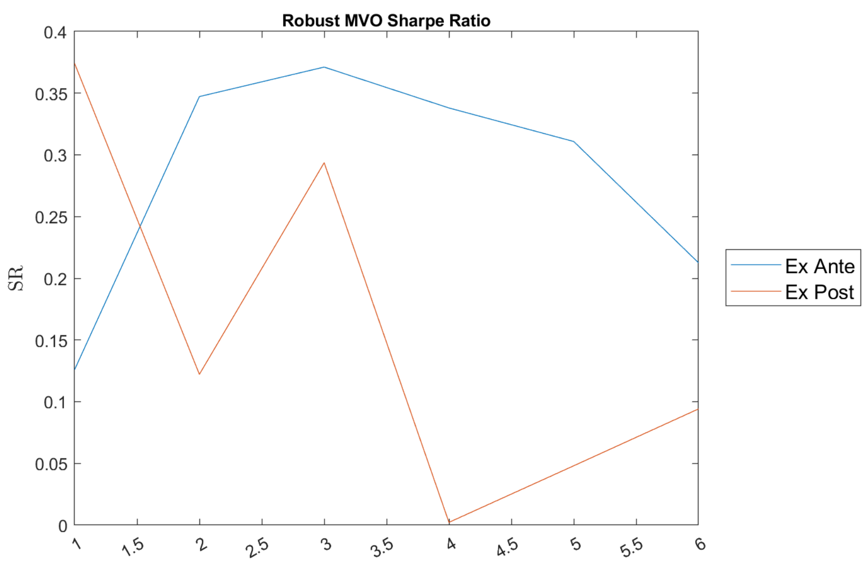
For the MVO model, the Ex Ante Sharpe Ratio generally exceeds the Ex Post portfolio values. This could be an indication that, in general, we have over estimated the portfolio returns. However, at period 6, the Ex Ante converges with Ex Post.

Figure 9: Sharpe Ratios for MVO Model



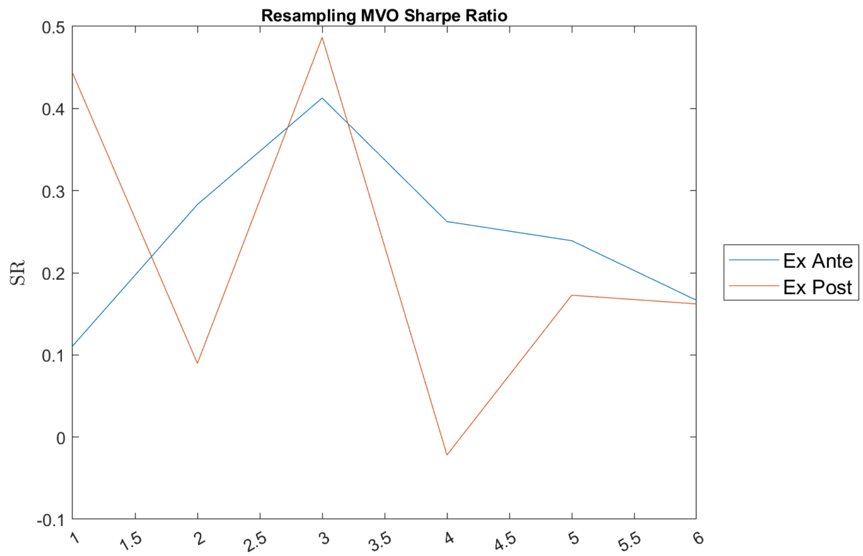
For the Robust MVO model, the Ex Ante Sharpe Ratio also exceeds the Ex Post portfolio values in most periods, which also could be an indication of over-estimation in portfolio returns.

Figure 10: Sharpe Ratios for Robust MVO model



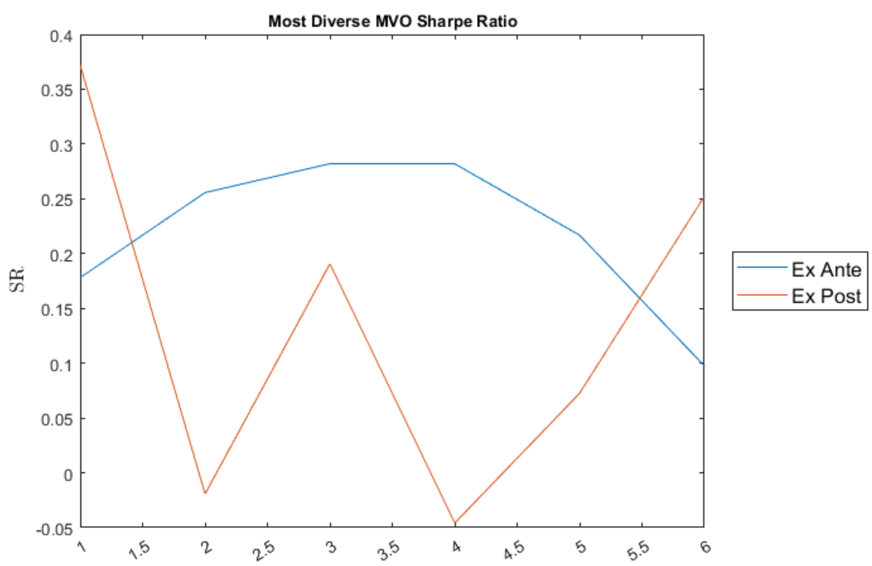
Very similar to the sharpe ratios for the MVO model, the Resampling MVO model also has higher Ex Ante in most periods, and its return is over estimated for most periods, except at period 6.

Figure 11: Sharpe Ratios for Resampling MVO Model



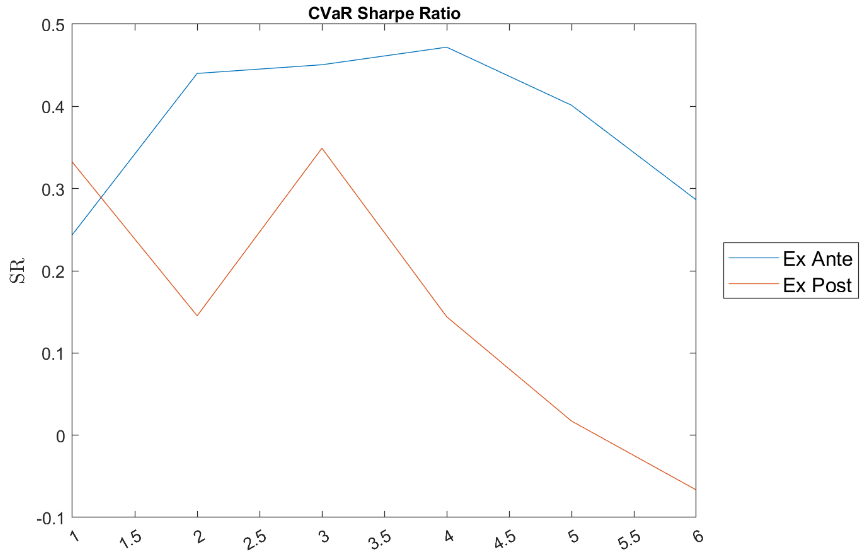
For the Diverse MVO model, the realization of portfolio exceeds the forecast at the end of period 6 and has an upward growing trend, while the forecast at period 6 has a downward growing trend. This could suggest that the realized return tends to exceed the forecast at period 6, and that the forecast has an underestimation.

Figure 12: Sharpe Ratios for Most Diverse MVO model



For the CVaR MVO model, the Ex Ante Sharpe Ratio exceeds the Ex Post portfolio values in most periods, which is an indication of over-estimation in portfolio returns.

Figure 13: Sharpe Ratios for CVaR model



Note that even though Ex Ante values appear to be higher than Ex Post values most of the time for all models, we can't really compare the two directly because Ex Ante is calculated using expected return computed using past year's data, while Ex Post is the realized return for the next rebalance period, which is in the span of six months. Since Ex Ante has one year of data points and Ex Post has six months of data points, the difference in sample size will result in different in standard deviation. Thus if we compare the two to determine whether our model is overestimating or underestimating portfolio performance without considering the difference in standard deviation, it would be meaningless.

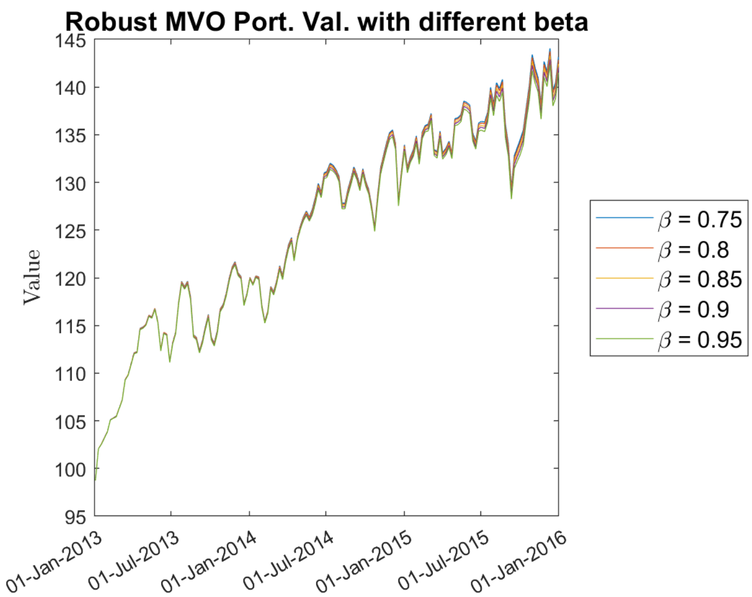
3.8 Effect of having different Confidence Levels

The effect of having different confidence levels (β) for the robust MVO model and CVaR model is studied by computing portfolios under different confidence levels and comparing the portfolio values.

3.8.1 Robust MVO

In Figure 14 , the portfolio computed by the Robust MVO model under different confidence levels are graphed. In general, the change of confidence interval does not affect the portfolio value significantly, for the data set given. However, one can still observe that with a higher confidence interval, the portfolio value tends to be higher. This could be a result of large magnitude of the weighting coefficients (i.e. λ). This makes the robust MVO model prioritize in minimizing the variance, rather than maximizing the return of the portfolio, which in turn makes the expected return's uncertainty set has less effect on the final result.

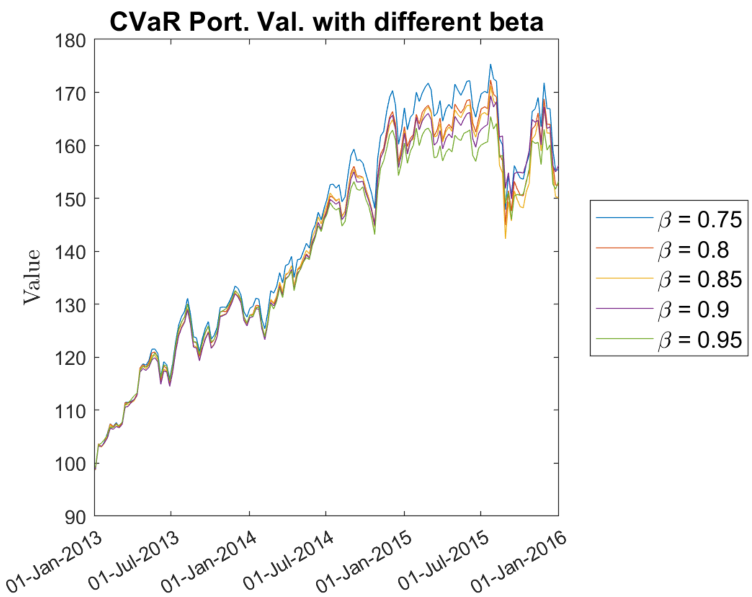
Figure 14: Robust MVO Portfolio Value with different confidence levels



3.8.2 CVaR

In Figure 15, the portfolio computed by the CVaR model under different confidence levels are graphed. Between July 2014 and July 2015, where the slope of the return is nearly horizontal, a lower confidence interval has higher portfolio returns. In other periods where the portfolio values grow or decline, all of them have nearly the same performance. This suggests that, when the financial market is stable, confidence intervals could have more impact on the portfolio returns under the CVaR model. Comparing to the Robust MVO model, where all portfolio values with different βs are similar in all periods, the CVaR model is more sensitive to the market growth.

Figure 15: CVaR Portfolio Value with different confidence levels



3.9 Effect of having different number of simulations

In this section, the effect of having different number of simulations, or randomly generated scenarios, is studied for the Resampling MVO model and the CVaR model.

3.9.1 Resampling MVO

It is obvious from Figure 16 that, when the scenario number is too low, the portfolio value does not converge to the portfolio value created by the nominal MVO. In addition, with each run, the portfolio value changes drastically when the scenario number is too low, as one can see in Figures 16 to 18, which are generated from different runs of the model with the exact same parameters. The reason for this phenomenon is that, when the sample size is small, the mean and variance of randomly generated samples will not converge to its true expected value. Thus we can conclude that, in order for Resampling MVO to be a valid model for portfolio and produce meaningful portfolio, we require a rather large number of scenarios with respect to the data size. Using the construction of our project, we would need at least 60 scenarios for Resampling Model to be valid. When there are sufficient number of scenarios, the portfolio value converges to the Nominal MVO portfolio value, as its return would converge to the return of the MVO model. Note that after increasing our number of scenarios from 60 to a higher number, our portfolio value is not significantly different and still has the same general shape, we can conclude that our original nominal MVO is relatively robust against uncertainties in parameters, which is the main cause for the resampling MVO portfolio and nominal MVO portfolio to have very similar performance.

Figure 16: Resampling Portfolio Value with different numbers of simulations Run 1

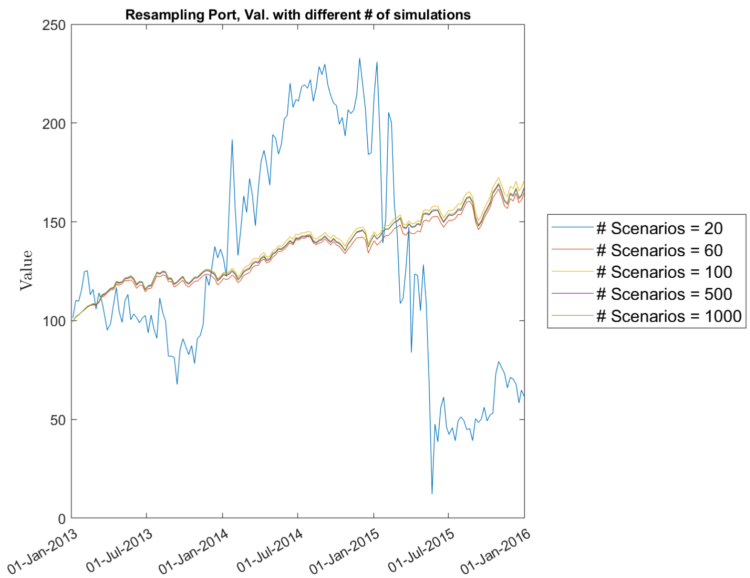


Figure 17: Resampling Portfolio Value with different numbers of simulations Run 2

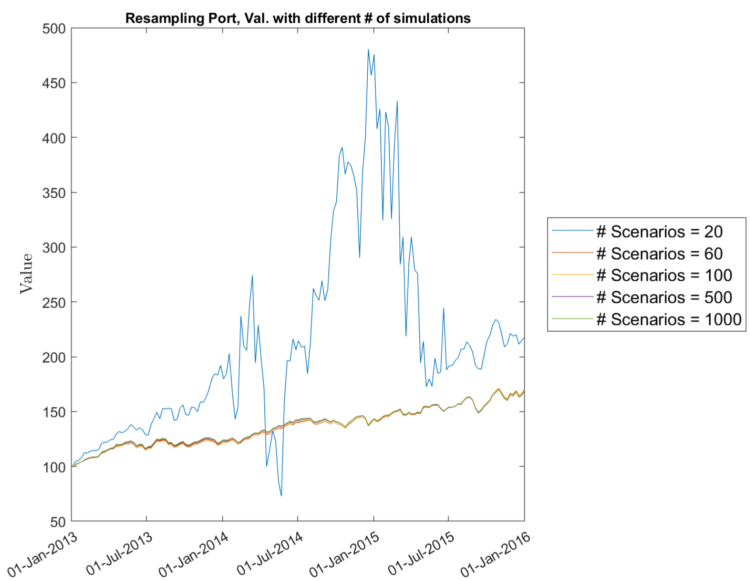
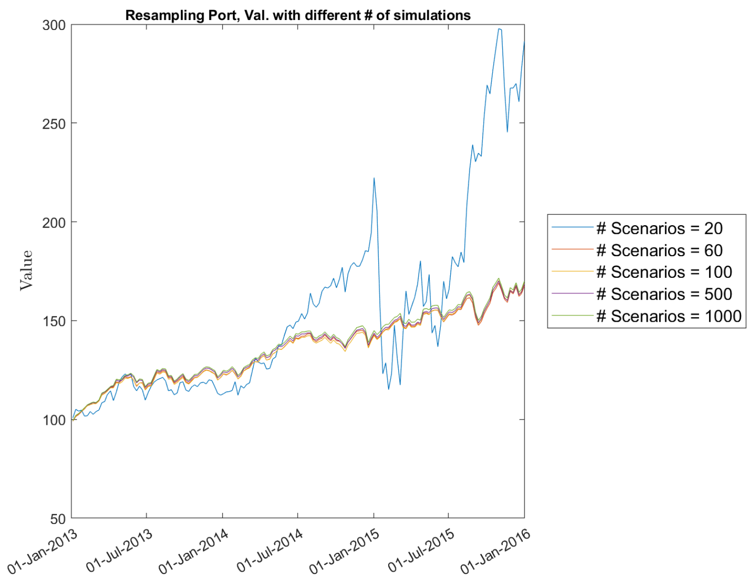


Figure 18: Resampling Portfolio Value with different numbers of simulations Run 3



3.9.2 CVaR

For this model, the portfolio value with different number of scenarios is very random: with the same number of scenarios, a portfolio can have very different values in different runs. In Figures 19 to 21, portfolios with the same numbers of scenarios are constructed three different times under the CVaR model. One can see that, the portfolio with the best performance happen to have different numbers of scenarios each time, and there is no clear trend of the change of portfolio value based on the change of scenario number. For example, in Figure 20, the portfolio that performs the best is the one with 500 different scenarios, while the portfolio with the most number of scenarios (2500) has a mediocre performance; however, in Figure 21, the portfolio with 500 scenarios performances significantly worse than the rest, while the portfolio with 2500 scenarios performs the best. However, it is noticed that, as the number of scenarios increases, the overvalue value of portfolio created would have less variances. For example,when we look at the periods starting from July 2014, when there are 500 scenarios, the portfolio values are around 160 in Figure 19, around 170 in Figure 20 and around 140 in Figure 21; on the other hand, when there are 2500 scenarios, the portfolio values in the same periods are around 160 in both Figure 19 and 20, and around 170 in 21. This is because, only sufficiently large number of scenarios will converge towards the true deterministic solution. Thus, it is better to increase the number of scenarios for the CVaR model, because it would make the final portfolio created have less uncertainty and converge towards the deterministic solution.

Figure 19: CVaR Portfolio Value with different numbers of simulations Run 1

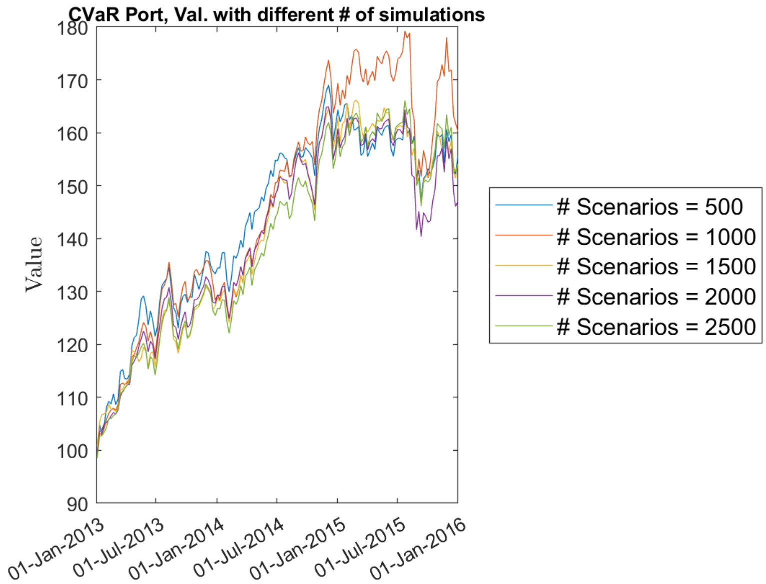


Figure 20: CVaR Portfolio Value with different numbers of simulations Run 2

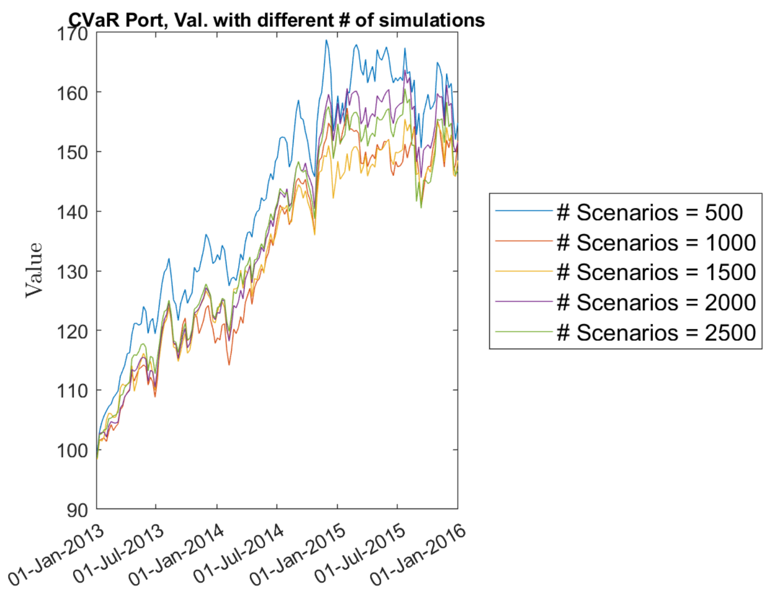
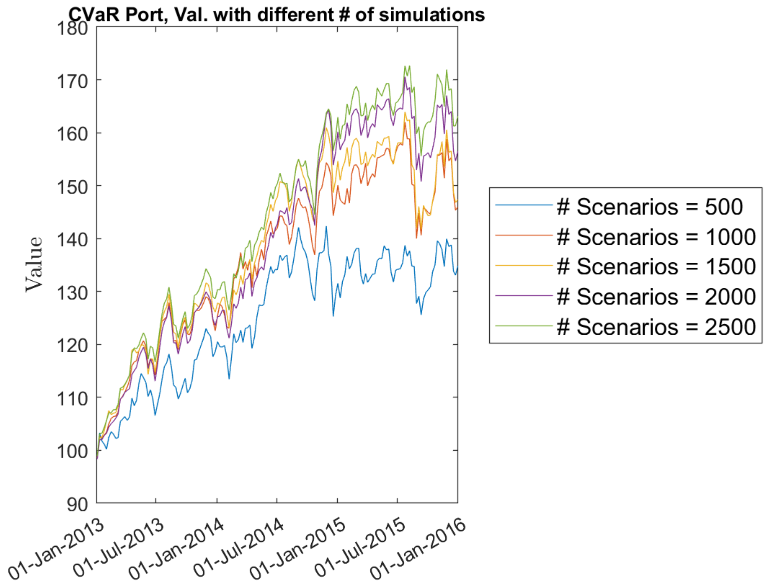


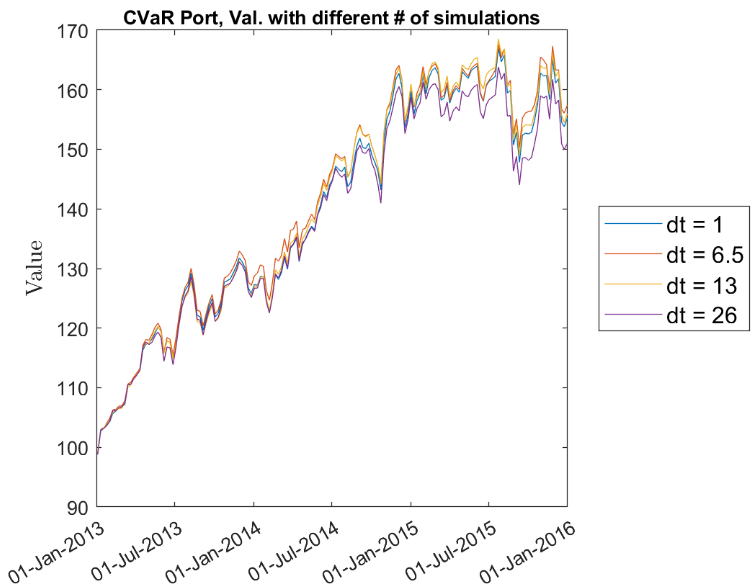
Figure 21: CVaR Portfolio Value with different numbers of simulations Run 3



3.10 CVaR with different step size (dt)

In this section, we studied the effect of taking different step sizes, dt, for the CVaR model. The resulting portfolio values are graphed in Figure 22. It can be observed that, when the number of scenarios is sufficiently large, changing the step size does not affect the final portfolio value too significantly, as they all converge to the true deterministic solution. However, it is important to note that, if the step size is small, the frequency of data generation is higher, there would be more uncertainty of our estimated returns; the ratio of standard deviation and expected return would increase as frequency increases. Therefore, for the return to converge, more scenarios should be generated for smaller step sizes.

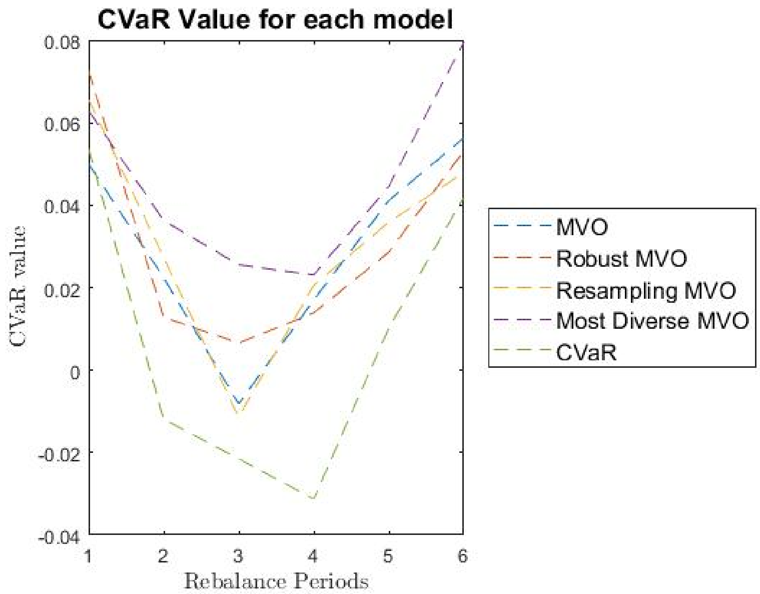
Figure 22: CVaR Portfolio Value with different step size



3.11 CVaR for different models

In this section, the Conditional Values at Risk (CVaR) are calculated for each of the portfolios across the investment periods, as presented in Figure 23. The 2,000 scenarios generated with Monte Carlo method that were used in the CVaR optimization model are applied here again. The lower the CVaR, there is less expected loss exceeding VaR with a probability of 5%. The performance of each model on the CVaR is consistent with the portfolio value performance discussed in Section 3.1, in which the CVaR optimization model performs the best, the Nominal MVO and Resampling MVO model has similar performances, and the Most Diverse MVO performs the worst. The CVaR optimization model is expected to outperform the rest with this measure, because the model minimizes CVaR, making it have the lowest exposure to risk. On the other hand, the Most Diverse MVO portfolio would have the largest expected loss exceeding VaR. This also makes sense because there are only 12 assets in the portfolio, which has larger exposure to risks.

Figure 23: CVaR Value for Optimal Portfolios from Different Models



3.12 Transaction Cost

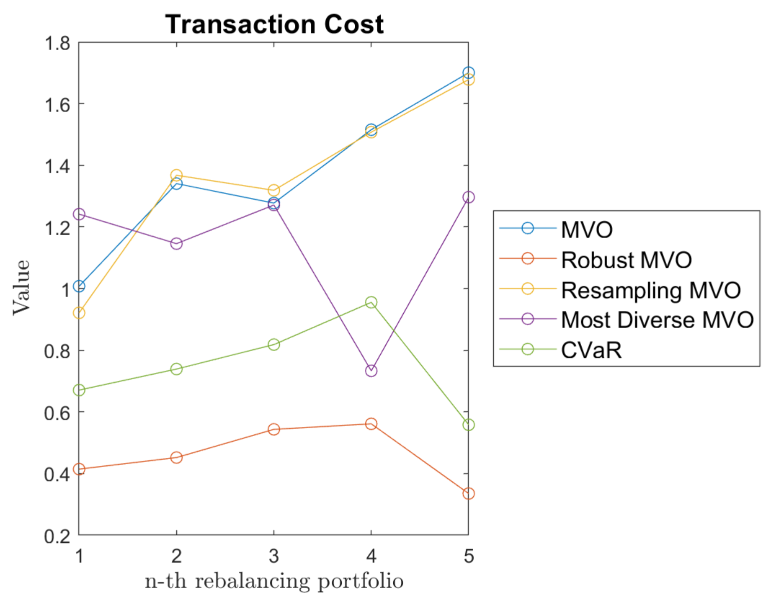
In figure 24 we plot the transaction cost incurred in each model every time when the portfolio rebalances. The time frame starts at 01-Jul-2013, as it is the beginning of the second investment period, which is the first time a portfolio rebalances and incurs transaction costs. The transaction costs are strongly related to the portfolio weights, because they incur when the weight changes.

We could see that transaction costs for CVaR and Robust MVO are both low. Since for cvar the portfolio weights are over-concentrated on few assets, the rebalancing is little as the over-concentrated assets remain at a high level. On the contrary, Robust MVO results in a diverse portfolio. However, Robust MVO's weighting plot is smoother, meaning that there are only few changes of assets' weights after each rebalance periods. Because the Robust MVO model has no target return, it only seeks to minimize variances. Thus, the portfolio it generates will put weights on assets with low variances, which are not expected to have much changes in values, hence weights, when the portfolio rebalances. Therefore, both models had very low transaction costs.

The transaction cost for Most Diverse MVO is generally lower than that for Nominal MVO and Resampling MVO. This is expected, because there are only 12 assets. However, it shifts up and down a lot as the portfolio weights concentrated on different assets in different periods. In our model we would choose 12 assets out of the 20 assets. Whenever the choice of representative assets changed during the test period, we would see a sudden rise in the transaction cost. If the representative assets choice remained the same, we would see a drop in the transaction cost. This is also the reason why transaction cost for Most Diverse MVO is higher than that for CVaR and Robust MVO, which both have 20 assets in the portfolio instead of 12.

Transaction cost for MVO and resampling cost are very close. This is expected since the optimal weights for the two portfolios are very similar. Large number of iteration would cause the results of the resampling MVO model to converge to the nominal MVO. In the graph of portfolio weights for these two models (Figure 2 and 4), there are sharp changes in weights, which would lead to high transaction costs.

Figure 24: Transaction Costs



4 Drawback and Improvement

4.1 Size of Data Set

Using higher frequency of measurement will increase the uncertainty in the estimated expected return, as the ratio between the standard deviation and mean is proportional to the square root of the number of measurements. On the other hand, using measurements with lower frequency will reduce the quality of estimates, which could cause the standard deviation of our estimated mean to be very large. To obtain estimates with higher quality and less uncertainty, we could collect more observations and obtain a larger data set. For example, if the data set is 10 times larger, the standard deviation of our estimated mean would reduce by a factor of $\sqrt{10}$. With a lower standard deviation, we would be more certainty about our asset's future returns.

4.2 Non-ideal Environment

For any multi-factor factor model, the ideal environment is as follows:

$$v(f_i, f_j) = 0, \forall i \neq j$$
$$cov(f_i, \epsilon_j) = 0, \forall i, j$$
$$cov(\epsilon_i, \epsilon_j) = 0, \forall i \neq j$$

Where fs are the factors and ϵs are the noises exist in the models. If the factor models is under a non-ideal environment, then the model performance may be lower than expected. However, we leaned in class that the FamaFrench model does not respect the ideal environment as it cannot be clearly said that the market factor, HML and SMB are uncorrelated. If we find the covariance matrix of the three factors in the project as seen in equation 16, we can see that the three factors are indeed correlated as their covariances are non-zero. Thus we can conclude that the ideal-environment conditions do not meet in this project, resulting lower returns. But we should also note that in real-life the models usually operate under non-ideal conditions, thus this project mimics the portfolio management in industry setting in some degree.

$$\begin{bmatrix} 2.688 & -0.00348 & -0.0584 \\ -0.00348 & 1.207 & -0.221 \\ -0.0584 & -0.221 & 0.815 \end{bmatrix} * 10^{-3} \tag{16}$$

4.3 Assumption of Normal Distribution

In our CVaR model, we generate random scenarios using Monte Carlo simulations, but one of the key assumptions used for Monte Carlo is that its arithmetic random walk follows a Gaussian process (i.e., its randomness will be governed by independent normal distributions). In addition, in Resampling MVO, the assets' return for each scenario is generated based on multi-variate normal distribution. However, this is not always valid. in other words, in real life the asset's performance is determined by a lot of different external factors, and may not necessary follow a normal distribution. As a result, the portfolio return predicted by these two models may be inaccurate due to this exact fact. In fact, in one of the article discussing Monte Carlo (David Blanchett), it is pointed out that the underlying distribution of assets have "fatter tail" compare to normal distributions, thus the normal distribution may not do a good job at predicting the asset's performance in extreme events (when asset performance is unexceptionally high/low). If we can find out what the true underlying distribution of assets is, and incorporate it into our models instead of normal distribution, we can potentially improve our portfolio returns.

4.4 Number of cluster used for Most-Diverse MVO

For the Most-Diverse MVO, we find the 12 most representative assets in the market to limit our number constituent. The number 12 is given by the project description, and by analyzing the results we found that Most-Diverse gives the poorest performance for both portfolio performance and sharpe ratio. This is potentially because we have limited our choose of assets, which can filter out some of the ideal assets that may have high expected return. This shows one of the drawback of Most-Diverse MVO: that when we are choosing the number of representative asset, we are constantly making a trade-off between easiness to manage portfolio and high return. In addition, we cannot tell whether a certain number of representative asset meets our need until we actually compute out the portfolio. Maybe after seeing the portfolio performance using 12 representative assets, we have decided that this is not the ideal number to use for us because it gives too low of a return, then we will need to change the number and run it again until we are satisfied with the return we see, which can be very time consuming.

To understand the effect of the number of representative assets on Most-Diverse portfolio's performance, we can produce a set of Most-Diverse portfolio using different number of representative assets while keeping other parameters the same. If we have close to 20 representative asset, the output portfolio will converge to nominal MVO; if we have very few assets, the portfolio will be very over-concentrated. By trial and error we can get insights the optimal number of representative to be used for a Most-Diverse model.

5 Conclusion

In this project, we analyzed the performance of the Nominal MVO, Robust MVO, Resampling MVO Most-Diverse MVO and the CVaR model. Although CVaR has the highest portfolio return most of the time, in the end Nominal MVO yields the highest weekly return while Most-Diverse MVO gives the lowest portfolio variance. CVaR gives significantly better Ex Ante Sharpe ratio, but with realized values Nominal MVO and Resampling have the highest Ex Post ratio. By varying the different parameters, we found that Resampling model can only be valid when the number of scenarios generated is 60 or above in our case; similarly, if the number of scenarios is too low, the portfolio value for CVaR will be fluctuating by a lot. CVaR for different models' optimal portfolios are also calculated, with most diverse having the highest CVaR and CVaR model having the lowest. Drawback and improvements of the models and this project as a whole are also discussed, such as the assumption of normal distribution, which may not be valid.

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