# HOMEWORK 11

ABIGAIL RECHKIN
ALEXANDRA YAVNILOVITCH

We will show that the given EXAM-QUESTIONS (EXAM-Q) problem is NP-hard by reducing np-hard problem MONO-SAT, which was covered in discussion, to our problem (EXAM-Q).

$$\text{MONO-SAT} \leq_m^p \text{EXAM-Q}$$

EXAM-Q Specification

Given
- An assignment of n TA's (each is assigned either exam, or makeup exam)
- A list of prefered problems for each TA $l_1, \ldots, l_n$
- A list of m available problems $p_1, \ldots, p_m$

Goal Check whether there exist disjoint lists of questions for the exam and the makeup exam, such that at least one question from each TA's list is included in the lists.

Let $\varphi$ be an instance for MONO-SAT with n clauses and m variables. We construct an EXAM-Q mapping $\varphi'$:
- each clause $c_i$ in $\varphi$ maps to $TA_i$. The corresponding list $l_i$ of preferences belongs to $TA_i$.
- The literals of each clause map to a corresponding problem p.
- A non-negated clause in $\varphi$ represents $TA_i$'s assignment to the EXAM. While, a negated clause signifies $TA_i$'s assignment to the MAKEUP EXAM in $\varphi'$.

EXAMPLE

exam exam makeup-exam makeup-exam

$\varphi$: $(p_0 \vee p_1) \wedge (p_1 \vee p_2) \wedge (\bar{p_0} \vee \bar{p_1}) \wedge (\bar{p_1} \vee \bar{p_2})$

TA1 TA2 TA3 TA4

$\varphi'$:

| EXAM | | MAKEUP | |
|---|---|---|---|
| TA1 | $p_0$ | TA3 | $p_0$ |
| | $p_1$ | | $p_1$ |
| TA2 | $p_1$ | TA4 | $p_1$ |
| | $p_2$ | | $p_2$ |

Explanation As previously stated, each clause represents a TA's preference list. The literals within the clause are ORed, which ensures that at least one course on the TA's list is selected. We chose to divide the expression by negated and non-negated clauses, in order to ensure that no question is included in both the EXAM and MAKEUP.

PROOF

CLAIM There is a satisfying assignment to $\varphi$ if and only if there is a satisfyin assignment to $\varphi'$.

$\Longrightarrow$ If there is an assignment which satisfies $\varphi$, then we know that for each clause there is at least one literal that makes it true. Thus, each clause represents a TA's preference list, from which at least one problem is satisfied, meaning $\varphi'$ is satisfied.

$\Longleftarrow$ If there is a satisfying assignment to $\varphi'$, then we know that each TA has at least one problem selected from his/her list. The preferences of each TA with the EXAM assignment are mapped to a nonnegated clause, while the preferenced of the MAKEUP TAs are mapped to negated clauses. Hence, at least one literal within each clause in $\varphi$ is true, which satisfies $\varphi$.

$\varphi'$ contains $n$ TAs (from $n$ clauses), and $m$ possible problems from the given literals. We can divide the expression into an exam an makeup assignments, by discerning the nonnegated and negated clauses, respectively. Finally, $\varphi'$ can be constructed in at most $O(n \cdot m)$ time, because each TA can list every problem on their list. This means that $\varphi'$ can be constructed in polynomial time.

Because MONO-SAT can be reduced to EXAM-Q in polynomial time, and MONO-SAT is NP-hard, EXAM-Q is also NP-hard.