Alexandra Yavnilovitch
Abigail Rechkin

Homework #12

We want to show that the total-elevation problem is NP-hard by reducing the subset-sum problem to it.

$$\text{SUBSET-SUM} \leq^{P}_{M} \text{TOTAL-ELEVATION}$$

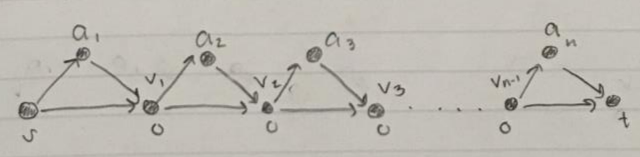
Subset-sum specifications has been detailed extensively in class; we will now describe the specifications of TOTAL-ELEVATION.

INPUT: a park map including all jogging checkpoints w/ their elevation and the roads connecting the checkpoints, an integer k which is the exact desired elevation change the jogger must run

OUTPUT: A route from the starting point s to the ending point that is of total elevation change k or nothing if that route doesn't exist.

ALGORITHM

Consider an instance of subset-sum φ which has an input of sum k and n integers $a_1, \ldots, a_n$. We will construct a mapping to instance φ' of TOTAL-ELEVATION like so:



The elevations of all of the checkpoints on the bottom of the triangles are 0 and the elevations of all checkpoints on the top are the values in the input list to subset sum. The desired total elevation change will be 2k because choosing to reach any non-zero checkpoint requires travelling double the elevation.

EXPLANATION

This mapping is done because it allows us to select which checkpoints to include in the route to achieve the desired elevation change. Doubling the K value ensures that we take into account travelling to and from a given check point. 0's are put as the elevations for the bottom checkpoints to ensure that the values from the input list to subset sum are accounted. For example, if we took the path from s to v_1 to a_2 to v_2 to a_3 to v_3 ... to t in TOTAL-ELEVATION, that would translate to summing $a_2 + a_3$ in our subset-sum problem. To reiterate, the only values that modify our total elevation change are the check points at the peaks of the triangles.

PROOF

Claim: There is a set of values in the input list of φ that sums to k if and only if there exists a route of total elevation change $2k$ in φ'.

[$\Rightarrow$] If there exists a set of x elements in φ which sum to k, then there exists x peaks that corresponds to the same values in φ! Thus, the jogger can visit each of these peaks and achieve a $2k$ elevation change resulting from traveling up to each peak and back down to 0. $(|0-a_1|+|a_1-0|=2a_1)$

[$\Leftarrow$] If there exists a route in φ' resulting in an elevation change of $2k$, we know that there are x peaks included in the path. These peaks map to x specific values in the input list of φ. These values must sum to k.

<u>TIME OF REDUCTION</u>: The time taken to construct our graph is $O(E+V)$ for TOTAL-ELEVATION. We have $3n$ edges and $2n+1$ vertices, so our time is $O(5n+1)$ which is linear. We conclude that G' can be constructed in poly-time.

Because SUBSET-SUM can be reduced to TOTAL-ELEVATION in poly-time and SUBSET-SUM is NP-Hard, TOTAL-ELEVATION is also NP-hard. :)