

Principal Components Analysis Notes

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1 Background

1.1 History

Principal Components Analysis (PCA) goes all the way back to Harold Hotelling in the 1933 when the cutting edge topics in statistics at the time were the multivariate methods we know today. It was first introduced as a technique for obtaining a smaller set of orthogonal linear projections of a single collection of correlated variables where the projections are ordered by decreasing variance. Note that it is also called the *Karhunen-Loève transform* in communications theory and *empirical orthogonal functions* in atmospheric science.

1.2 Eigendecomposition

Let \mathbf{A} be a square n by n matrix with n linearly independent eigenvectors e_i for $i = 1, \dots, n$. Then \mathbf{A} can be factored as

$$\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$$

where \mathbf{E} is the square n by n matrix whose i 'th column is the eigenvector e_i of \mathbf{A} and $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues i.e $\mathbf{\Lambda}_{i,i} = \lambda_i$. Note that this is a special case of the *spectral theorem*.

1.3 Singular Value Decomposition

Let $T \in \mathcal{L}_{V \rightarrow W}$ where V and W are two finite dimensional inner product spaces both over \mathbb{C} . Let $T^* \in \mathcal{L}_{W \rightarrow V}$ be the *adjoint* of T and recall that a map is called *self-adjoint* if it equals its adjoint. From this, one can deduce that the eigenvalues of T^*T are non-negative.

Define the *singular values* of T to be the non-negative square roots of the eigenvalues of T^*T in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T . Some important properties of

2 References

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