CSC355 PS1

Alex Zhang

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1 Problem 1

1.

Given $f(x) = (1 - x)\ln x$, $P_3(x)$ at $x_0 = 1$ will be

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f^2(x_0)}{2}(x - x_0)^2 + \frac{f^3(x_0)}{6}(x - x_0)^3 \\ &= 0 + 0/1(x - x_0) + -2/2(x - x_0)^2 + 3/6(x - x_0)^3 \\ &= -(x - 1)^2 + \frac{1}{2}(x - 1)^3 \end{aligned}$$

residual function $R_3(x)$ will be

$$R_3(x) = \frac{f^4(c(x))}{24}(x - 1)^4$$

2.

Plug in x = 0.5, $P_3(0.5)$ equals,

$$P_3(0.5) = -(0.5)^2 + \frac{1}{2}(-0.5)^3 = -0.3125$$

The residual function $R_3(0.5)$ will be $-\frac{0.0625}{24} \cdot \frac{2c(0.5)+6}{c(0.5)^4} = -0.0026 \cdot \frac{2c(0.5)+6}{c(0.5)^4}$. Because $c(x)$ is bound by 0.5 and 1, we can use the Extreme Value Theorem to find an absolute maximum value given that interval.

Taking the first derivative of $\frac{2c(0.5)+6}{c(0.5)^4}$ equals,

$$-\frac{6c(x) + 24}{c(x)^5}$$

We can see that when $c(x) = -4$, there is one extreme value, but it is not in the interval. We then calculate two end points.

$$\frac{2 \cdot 0.5 + 6}{0.0625} = 112$$
$$\frac{2 \cdot 1 + 6}{1} = 8$$

So the maximum value is 112, which means $|R_3(0.5)|$ is bounded by,

$$|R_3(0.5)| \leq \frac{0.0625}{24} \cdot 112 = 0.2917$$

which is the upper bound for $|f(0.5) - P_3(0.5)|$
The actual error is 0.0341, which I think there is a huge difference.

3.

Finding the bound of error $|f(x) - P_3(x)|$ for any $x \in [0.5, 1.5]$ is the same as finding bound for $|R_3(x)|$ in the same interval. We know that the choice of $c(x)$ in residual function is bounded by x and x_0, which in this case we can just assume $c(x)$ is in interval $[0.5, 1.5]$ given $x_0 = 1$.
Based on the previous problem, the interval does not reach -4, we can still calculate end points to get maximum value of that fourth derivative.

$$\frac{2 \cdot 0.5 + 6}{0.0625} = 112$$

$$\frac{2 \cdot 1.5 + 6}{1.5^4} = 1.7778$$

For the maximum value of $\frac{(x-1)^4}{24}$ for $x \in [0.5, 1.5]$, we can still use the extreme value theorem. Compared the end points and point where derivative is zero, we found out the maximum value of $\frac{(x-1)^4}{24}$ is $0.0625/24$.
The bound for the error $|f(x) - P_3(x)|$ for any $x \in [0.5, 1.5]$ is $\frac{0.0625}{24} * 112 = 0.2917$.

4.

The integral value of $\int_{0.5}^{1.5} P_3(x)dx$ will be,

$$\begin{aligned}
\int_{0.5}^{1.5} P_3(x)dx &= \int_{0.5}^{1.5} -(x-1)^2 + 1/2(x-1^3)dx \\
&= \int_{0.5}^{1.5} -x^2 + 2x - 1 + \frac{x^3}{2} - \frac{3x^2}{2} + \frac{3x}{2} - \frac{1}{2}dx \\
&= -\frac{x^3}{3} + x^2 - x + \frac{x^4}{8} - \frac{x^3}{2} + \frac{3x^2}{4} - \frac{x}{2}\bigg|_{0.5}^{1.5} \\
&= -\frac{1}{12}
\end{aligned}$$

5.

The integral $\int_{0.5}^{1.5} |R_3(x)|dx$ can be rewritten into

$$\int_{0.5}^{1.5} |R_3(x)|dx = \frac{1}{24}\int_{0.5}^{1.5} (x-1)^4 \frac{2c(x)+6}{c(x)^4}dx$$

We know that $\frac{2c(x)+6}{c(x)^4}$ in this interval is smaller than 112. This integral will be always smaller than

$$\frac{112}{24}\int_{0.5}^{1.5} (x-1)^4 dx$$

Calculating this integral, we get $112/24 * 0.0125 = 0.05833$, which is the upper bound for the absolute error in 4.
The actual error for that two expression is $|0.08802 - 0.83333| = 0.004687$. I think the upper bound for the absolute error is relatively large compared to the actual integral error.

Problem 2