CSC352 HW5

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Question 1

(a)

Since $\mathbf{x} \in \mathbb{R}^m$, let $\mathbf{q_1} = \frac{\mathbf{x_1}}{\|\mathbf{x_1}\|_2}$. Because $\mathbf{x}$ is a vector, it only has one column, the matrix $\mathbf{Q}$ is just $\mathbf{q_1}$. For $\mathbf{R}$, since there is only one columne, $\mathbf{R} = \mathbf{r}_{11} = \|\mathbf{x_1}\|_2$. The QR decomposition will be

$$\mathbf{x} = \mathbf{QR} = \frac{\mathbf{x_1}}{\|\mathbf{x_1}\|_2} \cdot \|\mathbf{x_1}\|_2$$

(b)

Given and orthogonal matrix $\mathbf{J} \in \mathbb{R}^{m \times n}$. Let $\mathbf{q}_1 = \frac{\mathbf{j}_1}{\|\mathbf{j}_1\|_2}$, and $\mathbf{q}_2$ is,

$$\mathbf{q}_2 = \frac{\mathbf{j}_2 - (\mathbf{q}_1^\top \mathbf{j}_2)\mathbf{q}_1}{\|\mathbf{j}_2 - (\mathbf{q}_1^\top \mathbf{j}_2)\mathbf{q}_1\|_2}$$

Since $\mathbf{J}$ is an orthogonal matrix, $\mathbf{q}_1 \cdot \mathbf{j}_2 = 0$, indicating

$$\mathbf{q}_2 = \frac{\mathbf{j}_2}{\|\mathbf{j}_2\|_2}$$

This case can be generalized into any column $\mathbf{q}_i$ for matrix $\mathbf{Q}$. For matrix $\mathbf{R}$, $\mathbf{r}_{ij} = \mathbf{q}_i^\top \mathbf{j}_j$. In this case, since $\mathbf{Q}$ is orthonormal matrix to $\mathbf{J}$, $\mathbf{r}_{ij} = 0\ \forall i,\ j \leq m,\ n$. And for diagonal entries

$$\mathbf{r}_{jj} = \|\mathbf{j}_j - \sum_{i=1}^{i-1} \mathbf{r}_{ij}\mathbf{q}_i\|_2$$

Where $\sum_{i=1}^{i-1} \mathbf{r}_{ij}\mathbf{q}_i\|_2 = 0$ because $\mathbf{r}_{ij} = 0$. Therefore, $\mathbf{r}_{jj} = \|\mathbf{j}_j\|_2$.

Overall, after doing QR decomposition on an orthogonal matrix, we get $\mathbf{Q}$ is an orthonormal matrix and each column is the normal vector from $\mathbf{J}$. $\mathbf{R}$ is a diagonal matrix with each entry represents the 2-norm of the corresponding column vector in $\mathbf{J}$.

(c)

Given an upper triangular matrix $\mathbf{T} \in \mathbb{R}^{m\times n}$ Let $\mathbf{q_1} = \frac{\mathbf{t_1}}{\|\mathbf{t_1}\|_2}$, which is just $\mathbf{e_1}$ in this case. For $\mathbf{q_2}$, it should be

$$\mathbf{q}_2 = \frac{\mathbf{t}_2 - (\mathbf{q}_1^\top \mathbf{t}_2)\mathbf{q}_1}{\|\mathbf{t}_2 - (\mathbf{q}_1^\top \mathbf{t}_2)\mathbf{q}_1\|_2}$$

$$\mathbf{q}_2 = \frac{\begin{bmatrix} 0 \\ \mathbf{t_2 2} \\ \vdots \\ 0 \end{bmatrix}}{\| \begin{bmatrix} 0 \\ \mathbf{t_2 2} \\ \vdots \\ 0 \end{bmatrix} \|_2}$$

which is just $\mathbf{e_2}$. We can then general all column $\mathbf{q_i}$ in matrix $\mathbf{Q}$ as $\mathbf{e_i}$. Therefore, $\mathbf{Q}$ is just an identity matrix with dimention $m \times m$.

Based on the informaiton that $\mathbf{Q}$ is an identity matrix, we can then just get $\mathbf{R}$ which $\mathbf{R}$ is just the original matrix since a matrix times an identity matrix will still be itself.

$$\mathbf{T} = \mathbf{QR} = \mathbf{IT}$$

Question 2

Let $\mathbf{u_1} = \mathbf{a_1} + \|\mathbf{a_1}\|_2\mathbf{e_1} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -2 \\ 1 \end{bmatrix}$.

new $\mathbf{a_1} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 1 \end{bmatrix} - 2\frac{45}{90}\begin{bmatrix} 9 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

new $\mathbf{a_2} = \begin{bmatrix} -3 \\ -14 \\ 14 \\ -7 \end{bmatrix} - 2\frac{-90}{90}\begin{bmatrix} 9 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 10 \\ -5 \end{bmatrix}$, which is $\begin{bmatrix} -10 \\ 10 \\ -5 \end{bmatrix}$

new $\mathbf{a_3} = \begin{bmatrix} 4 \\ -3 \\ 0 \\ 15 \end{bmatrix} - 2\frac{45}{90} \begin{bmatrix} 9 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 2 \\ 14 \end{bmatrix}$, which is $\begin{bmatrix} -5 \\ 2 \\ 14 \end{bmatrix}$.

Similar, $\mathbf{u_2} = \mathbf{a_2} - \|\mathbf{a_2}\|_2 \mathbf{e_1} = \begin{bmatrix} -10 \\ 10 \\ -5 \end{bmatrix} - 15 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -25 \\ 10 \\ -5 \end{bmatrix}$

new $\mathbf{a_2} = \begin{bmatrix} -10 \\ 10 \\ -5 \end{bmatrix} - 2\frac{375}{750} \cdot \begin{bmatrix} -25 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$.

new $\mathbf{a_3} = \begin{bmatrix} -5 \\ 2 \\ 14 \end{bmatrix} - 2\frac{75}{750} \cdot \begin{bmatrix} -25 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix}$, which is $\begin{bmatrix} 0 \\ 15 \end{bmatrix}$.

$\mathbf{u_3} = \mathbf{a_3} + \|\mathbf{a_3}\|_2 \mathbf{e_1} = \begin{bmatrix} 0 \\ 15 \end{bmatrix} + 15 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$

new $\mathbf{a_3} = \begin{bmatrix} 0 \\ 15 \end{bmatrix} - 2\frac{225}{450} \cdot \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \end{bmatrix}$, therefore, we can get $\mathbf{R}$

$$\begin{bmatrix} -5 & 15 & -5 \\ 0 & 15 & 0 \\ 0 & 0 & -15 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 3

Given $\mathbf{A}^\dagger = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top$,

(a)

The left side can be transformed into:

$$\begin{aligned} \mathbf{A}\mathbf{A}^\dagger\mathbf{A} &= \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{A} \\ &= \mathbf{A}\mathbf{A}^{-1}(\mathbf{A}^\top)^{-1}\mathbf{A}^\top\mathbf{A} \end{aligned}$$

Since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and $(\mathbf{A}^\top)^{-1}\mathbf{A}^\top = \mathbf{I}$,

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}\mathbf{I}\mathbf{A} = \mathbf{A}$$

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(b)

The left side,

$$\begin{aligned}\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger &= \mathbf{A}^\dagger\mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top \\ &= \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^{-1}(\mathbf{A}^\top)^{-1}\mathbf{A}^\top\end{aligned}$$

Since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and $(\mathbf{A}^\top)^{-1}\mathbf{A}^\top = \mathbf{I}$.

$$\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$$

■

(c)

Left side is,

$$\begin{aligned}(\mathbf{A}\mathbf{A}^\dagger)^\top &= ((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top)^\top\mathbf{A}^\top \\ &= \mathbf{A}((\mathbf{A}^\top\mathbf{A})^{-1})^\top\mathbf{A}^\top \\ &= \mathbf{A}((\mathbf{A}^\top\mathbf{A})^\top)^{-1}\mathbf{A}^\top \\ &= \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top \\ &= \mathbf{A}\mathbf{A}^\dagger\end{aligned}$$

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