

# CSC HW3

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## Question 1

(a)

By SVD, we can get that  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ , which

$$\mathbf{A}^\top = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top)^\top = \mathbf{V}\mathbf{\Sigma}^\top\mathbf{U}^\top$$

Since with SVD, matrix  $\mathbf{\Sigma}$  is a diagonal matrix, which means the transpose of it will still be itself. If the singular values for  $\mathbf{A}$  are  $\sigma_1, \sigma_2, \dots, \sigma_n$ , then the singular values for  $\mathbf{A}^\top$  will not change since  $\mathbf{\Sigma} = \mathbf{\Sigma}^\top$ .

(b)

Using SVD,

$$\mathbf{A}^{-1} = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top)^{-1} = (\mathbf{\Sigma}\mathbf{V}^\top)^{-1}\mathbf{U}^{-1} = (\mathbf{V}^\top)^{-1}\mathbf{\Sigma}^{-1}\mathbf{U}^{-1}$$

Since  $\mathbf{U}$  and  $\mathbf{V}$  are orthogonal matrices, their transpose equals their inverse. The only change is  $\mathbf{\Sigma}^{-1}$ . The diagonal matrix's inverse is just taking the reciprocals on the entries on main diagonal. So the singular values for  $\mathbf{A}^{-1}$  will be  $1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_n$

(c)

The matrix  $\mathbf{A}$  with rank  $r$  can also be written as,

$$\mathbf{A} = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

then

$$\alpha \mathbf{A} = \sum_{i=1}^r \alpha \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

Because singular values are all scalars, when  $\alpha > 0$ , the singular values for  $\alpha \mathbf{A}$  will be  $\alpha \sigma_1, \alpha \sigma_2, \dots, \alpha \sigma_n$

## Question 2

(a)

By definition,  $\|\mathbf{A}\|_2 = \max \sigma = \sigma_1$  and  $\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2}$ . Since for all  $i \leq r$ ,  $\sigma_i^2 \geq 0$ , and  $\sigma_1 = \sqrt{\sigma_1^2}$ . Showing that  $\sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2} \geq \sqrt{\sigma_1^2}$  which is the same as  $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$ . ■

(b)

In this case,  $\sqrt{n}\|\mathbf{A}\|_2 = \sqrt{n}\sigma_1 = \sqrt{n\sigma_1^2}$ . We know that  $\text{rank}(\mathbf{A}) = r$ , so  $r \leq n$  and  $\sigma_1$  is the largest singular value. We can get the inequality:

$$\sqrt{n\sigma_1^2} \geq \sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2}$$

which is the same as  $\sqrt{n}\|\mathbf{A}\|_2 \geq \|\mathbf{A}\|_F$ . ■