CSC HW3

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Question 1

(a)

By SVD, we can get that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$, which

$$\mathbf{A}^\top = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top)^\top = \mathbf{V}\mathbf{\Sigma}^\top\mathbf{U}^\top$$

Since with SVD, matrix $\mathbf{\Sigma}$ is a diagonal matrix, which means the transpose of it will still be itself. If the singular values for $\mathbf{A}$ are $\sigma_1, \sigma_2, \ldots, \sigma_n$, then the singular values for $\mathbf{A}^\top$ will not change since $\mathbf{\Sigma} = \mathbf{\Sigma}^\top$.

(b)

Using SVD,

$$\mathbf{A}^{-1} = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top)^{-1} = (\mathbf{\Sigma}\mathbf{V}^\top)^{-1}\mathbf{U}^{-1} = (\mathbf{V}^\top)^{-1}\mathbf{\Sigma}^{-1}\mathbf{U}^{-1}$$

Since $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices, their transpose equals their inverse. The only change is $\mathbf{\Sigma}^{-1}$. The diagonal matrix's inverse is just taking the reciprocals on the entries on main diagonal. So the singular values for $\mathbf{A}^{-1}$ will be $1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_n$

(c)

The matrix $\mathbf{A}$ with rank r can also be written as,

$$\mathbf{A} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

then

$$\alpha\mathbf{A} = \sum_{i=1}^{r} \alpha\sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

Because singular values are all scalars, when $\alpha > 0$, the singular values for $\alpha\mathbf{A}$ will be $\alpha\sigma_1, \alpha\sigma_2, \ldots, \alpha\sigma_n$

Questino 2

(a)

By definition, $\|\mathbf{A}\|_2 = \max \sigma = \sigma_1$ and $\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2}$. Since for all $i \le r$, $\sigma_i^2 \ge 0$, and $\sigma_1 = \sqrt{\sigma_1^2}$. Showing that $\sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2} \ge \sqrt{\sigma_1^2}$ which is the same as $\|\mathbf{A}\|_2 \le \|\mathbf{A}\|_F$.$\blacksquare$

(b)

In this case, $\sqrt{n}\|\mathbf{A}\|_2 = \sqrt{n}\sigma_1 = \sqrt{n\sigma_1^2}$. We know that $\text{rank}(\mathbf{A}) = r$, so $r \leq n$ and σ_1 is the largest singular value. We can get the inequality:

$$\sqrt{n\sigma_1^2} \geq \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2}$$

which is the same as $\sqrt{n}\|\mathbf{A}\|_2 \geq \|\mathbf{A}\|_F$.$\blacksquare$