CSC301 HW9

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Question 1

The pseudocode for PAR-MATMUL algorithm is below,

```
function PAR-MATMUL(A,B)
  if(dim(A,B) = 1)
    return A * B
  end if
  parallel for i = 1 to n do
    parallel for j = 1 to n do
      C_ij = DOT(A_i,B_j)
    end for
  end for
end function
```

```
fucntion d = DOT(x,y)
  if n = 1 then
    return x·y
  x = [x_L x_R], y = [y_L y_R]
  spawn d_L = DOT(x_L y_L)
        d_R = DOT(x_R y_R)
  sync
  return d_L + d_R
  end function
```

Costs Analysis Time

For DOT function, it's work recurrence is

$$T_1(n) = 2T_1(n/2) + O(1)$$

and span recurrence is

$$T_\infty(n) = T_\infty(n/2) + O(1)$$

Using master theorem, $T_1(n) = O(n)$, $T_\infty(n) = O(\log n)$.

For PAR-MATMUL function, it's work costs is

$$T_1(n) = O(DOT_{work}) \cdot O(n^2) = O(n^3)$$

It's span costs

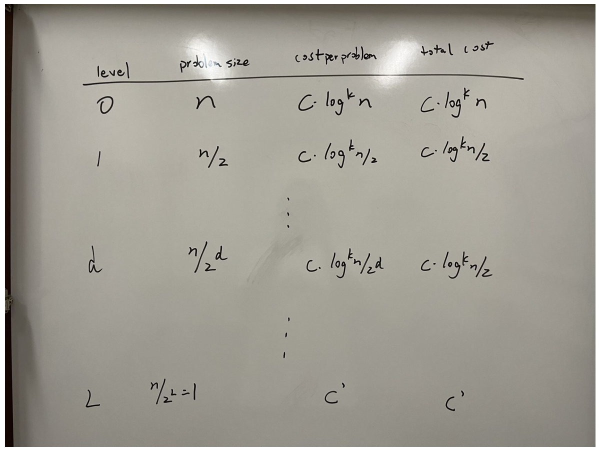
$$O(\log n) + O(\log n) + O(DOT_{span}) = O(3\log n)$$

The PAR-MATMUL function contains two parallel loops and a parallel dot product function, and it's span will still be $O(\log n)$.

Overall, This parallel algorithm for matrix multiplication will have $O(n^3)$ work and $O(\log n)$ span.

Question 2

Based on the recurrence, I can draw the following recurrence table:



From this table, there are total L level which $n/2^L \to L = \log_2 n$, which we can get the following total cost.

$$c \cdot \sum_{d=0}^{\log_2 n - 1} \log^k(n/2^d) + c'$$

doing transformation for each addition part,

$$c \cdot \sum_{d=0}^{\log_2 n - 1} (\log_2 n - \log_2 2^d)^k + c'$$

Case 1: Big-Oh
Since $\log_2 2^d = d$ and $d \geq 0$, $(\log_2 n - \log_2 2^d)^k$ will always smaller than $\log_2^k n$. This shows that

$$\begin{aligned}
c \cdot \sum_{d=0}^{\log_2 n - 1} (\log_2 n - \log_2 2^d)^k + c' &\leq c \cdot \sum_{d=0}^{\log_2 n - 1} (\log_2 n)^k + c' \\
c \cdot \sum_{d=0}^{\log_2 n - 1} \log^k(n/2^d) + c' &\leq c \cdot \sum_{d=0}^{\log_2 n - 1} (\log_2 n)^k + c' \\
&= c \cdot \log_2 n \cdot (\log_2 n)^k + c' \\
&= c \cdot \log_2^{k+1} n + c'
\end{aligned}$$

Because c and c' are both constant, let $g(n) = \log_2^{k+1} n$, $f(n) = \sum_{d=0}^{\log_2 n - 1} \log^k(n/2^d) + c'$, if $c, N > 0$

$$f(n) \leq c \cdot g(n)$$

for all $n \geq N$, then

$$f(n) = O(g(n)) = O(\log_2^{k+1} n)$$

Case 2: Big-Omega
Expanding the equation of total work we got from table, the summation part will be,

$$(\log_2 n)^k + (\log_2 n - 1)^k + (\log_2 n - 2)^k + \cdots + 2^k + 1^k$$

We can then transform these sums into

$$\sum_{d=1}^{\log_2 n} d^k$$

which

$$\sum_{d=\frac{\log_2 n}{2}}^{\log_2 n} d^k \leq \sum_{d=1}^{\log_2 n} d^k$$

Since on the left side of inequality, d goes from $\frac{\log_2 n}{2}$ to $\log_2 n$, let d in d^k be $\frac{\log_2 n}{2}$, the new inequality,

$$\sum_{d=\frac{\log_2 n}{2}}^{\log_2 n} (\frac{\log_2 n}{2})^k \leq \sum_{d=\frac{\log_2 n}{2}}^{\log_2 n} d^k$$

The left side's sum will be

$$(\frac{\log_2 n}{2} + 1) \cdot (\frac{\log_2 n}{2})^k = (\frac{\log_2^{k+1} n}{2^{k+1}}) + (\frac{\log_2^k n}{2^k}) \geq (\frac{\log_2^{k+1} n}{2^{k+1}})$$

Overall, we can put all inequalities together,

$$(\frac{\log_2^{k+1} n}{2^{k+1}}) \leq \sum_{d=\frac{\log_2 n}{2}}^{\log_2 n} (\frac{\log_2 n}{2})^k \leq \sum_{d=\frac{\log_2 n}{2}}^{\log_2 n} d^k \leq \sum_{d=1}^{\log_2 n} d^k = \sum_{d=0}^{\log_2 n-1} \log^k(n/2^d)$$

Since k is constant and $k \geq 0$, $\frac{1}{2^{k+1}}$ will also be constant and bigger than 0. Let $c = 1/2^{k+1}$,$N > 0$, $g(n) = \log_2^{k+1} n$, and $f(n) = \sum_{d=0}^{\log_2 n-1} \log^k(n/2^d)$, there is an inequality,

$$f(n) \geq c \cdot g(n)$$

for all $n \geq N$, then

$$f(n) = \Omega(g(n)) = \Omega(\log_2^{k+1} n)$$

Overall, since

$$\Omega(\log_2^{k+1} n) \leq f(n) \leq O(\log_2^{k+1} n)$$

then

$$f(n) = \Theta(\log_2^{k+1} n) = \sum_{d=0}^{\log_2 n-1} \log^k(n/2^d)$$

which also means

$$T(n) = \Theta(\log_2^{k+1} n)$$

■

Question 3

(a)

The following is the pseudocode for this algorithm,

```
function b = EXACT(S, a)
  if S = 0
    return 1
  if size(a) = 0
    return 0
  end if
  init 2D-array dp[a+1][S+1]
  dp(:,0) = 1
  dp(0,:) = 0
  for i = 1:n do
    for j = 1:S do
        dp[i][j] = dp[i-1][j]
      if j >= a[i-1] and dp[i][j-a[i-1]] != 0
        dp[i][j] = 1
      end if
    end for
  end for

  return dp[a][S];
  end function
```

Proof of Correctness

For the base case when integer $S = 0$, we can easily know that there exists a empty set which is the subset of the integers. For base case when size of nonnegative integers is 0, we can also know that there is no subset which can add up to exactly S. So both base cases are true.

For inductive process, the recurrence relation for $dp[i][j]$ will be,

$$dp[i][j] = \begin{cases} dp[i-1][j] & \text{if } dp[i-1][j] = 1 \\ dp[i-1][j-a[i-1]] & \text{if } dp[i][j-a[i-1]] \neq 0,\ a[i-1] <= j \end{cases}$$

Where 0 indicates, no subset exists in current situation, and 1 means there is a subset.

Case 1

In case $dp[i][j] = dp[i-1][j]$, if $dp[i-1][j] = 1$, this means up to a_{i-1}, a subset that satisfies the condition already exists. We can just pass that value to $dp[i][j]$.

Case 2

If $dp[i-1][j] = 0$, this means up to a_{i-1}, there is no subset. We then have to check if adding the current integer will create the subset. For condition $a[i-1] <= j$, we need to make sure the current capacity is bigger than the integer we are checking or we cannot add it anyway. The next step is for $dp[i-1][j-a[i-1]]$. This entry means before adding the current integer, whether the subset for capacity $j - a[i-1]$ exists. If it is true, we can just add the current integer and also increase the current capacity. Else, we will keep the number same as $dp[i-1][j]$. ■

Cost Analysis

In my pseudocode, the initialization takes $O(n + S)$, and there is a nested loop which takes $O(nS)$. Assume that comparison and assigning variables take constant time. The running time for my pseudocode will be $O(nS + n + S)$ where n is the size of nonnegative set and S will be the nonnegative number. Because nS takes the dominance, the running time will be,

$$O(nS)$$

The memory cost will also be $O(nS)$ because in my pseudocode, I create a table with size $n \times S$. So the space complexity will be

$$O(nS)$$

(b)

When checking the existence, we find that all entries in one column only depends on the left one column. Therefore, we can parallel each entry's calculations for every column. The pseudocode for parallel will be,

```
function b = EXACT(S, a)
  if S = 0
    return 1
  if size(a) = 0
    return 0
  end if
  init 2D-array dp[a+1][S+1]
  dp(:,0) = 1
  dp(0,:) = 0
  for i = 1:n do
    parallel for j = 1:S do
      dp[i][j] = dp[i-1][j]
      if j >= a[i-1]
        if dp[i-1][j-a[i-1]] != 0
          dp[i][j] = 1
        end if
      end if
    end for
  end for
  return dp[a][S];
  end function
```

It is similiar to the code in section (a), I just make the calculations for the inner loop become parallel.

Correctness

The justification of correctness is also similiar. In this case, everything in previous proof also holds true. For each thread, it will be responsible for each entry and check whether $dp[i-1][j] = 1$. If not each thread will also check conditions $j >= a[i-1]$ and $dp[i-1][j-a[i-1]]! = 0$. If these conditions are satisfied, $dp[i][j]$ will be labed as true.

Work and Span

The work for this algorithm will be $T_1(n) = O(nS)$. For the span, since each thread can spawn new threads, The costs of spawning will be $O(\log S)$. The overal span will be $T_\infty(n) = O(n \log S)$.

Parallelism

The parallelism $\overline{p}$ will be

$$\frac{T_1(n)}{T_\infty(n)} = \frac{nS}{n \log S} = \frac{S}{\log S}$$

which is representation of logarithmic.