

CSC301 HW2

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Question 1

Let algorithm computing x^y that makes at most $O(n)$ calls to integer multiplication called EXPO. The pseduocode is following below.

```
function z = EXPO(x,y)
    if y = 0 then return 1
    z = EXPO(x, math.floor(y/2))
    z = z*z
    if y is even then
        return z
    else
        return z*x
    end if
end function
```

Proof:

Base Case: $y = 0$

The algorithm then returns 1, which is true because $x^0 = 1$ for all x .

Induction: Assume $y = n - 1$ is true, we are going to prove that $y = n$ for $z = x^y$ is also true.

The recursion will get $z' = x^{\lfloor y/2 \rfloor}$.

Case 1: $y = n$ is even

The EXPO returns z . If y is even, then $\lfloor y/2 \rfloor = y/2$, which

$$z' = x^{y/2}$$

squaring both sides,

$$(z')^2 = x^y$$

So $z = (z')^2$ when y is even. Just as what EXPO will return.

Case 2: $y = n$ is odd

Then EXPO returns $z \cdot x$. In this case $\lfloor y/2 \rfloor = (y-1)/2$, which

$$z' = x^{y-1/2}$$

squaring both sides and multiplying by x ,

$$(z')^2 \cdot x = x^y$$

In this case $(z')^2 = z$. So $(z')^2 \cdot x = z \cdot x$. This matched the EXPO returning value. ■

1. Integer multiplication

Since we assume that x, y are all n -bits numbers, so in each recursion, y will chunk the last bit so there are total n times recursion calls. In each call, it will perform a integer multiplication on z it self, and sometimes possible calling multiplication for x and z . So overall, there are $O(n)$ integer multiplication calls for the whole recursion process.

2. Time Complexity

We have showed that there are total $O(n)$ calls to integer multiplication, and for multiplication itself, the time Complexity will be $O(n^2)$, which overall, the time complexity for EXPO will be $O(n^3)$

Question 2

(a)

If a number $1 \leq x < p$ for a given prime p is its own multiplicative inverse, then

$$x^2 \equiv 1 \pmod{p}$$

By definition, for some integer k :

$$\begin{aligned} x^2 - 1 &= pk \\ (x+1)(x-1) &= pk \end{aligned}$$

We have two case for $k = 0$ and $k \neq 0$.

Case 1: $k \neq 0$

Since $1 \leq x < p$ and x is relative prime to p , so either $x+1 = p$ or $x-1 = p$. However, the second case will not happen since $x-1 < p-1$. $x = p-1$ is one number that is its own multiplicative inverse.

Case 2: $k = 0$

This time either $x-1 = 0$ or $x+1 = 0$, but $x+1 \geq 2$ with the given condition so $x-1 = 0$ and $x = 1$.

There are two numbers 1 and $p-1$ which are their own inverse.

(b)

$(p-1)! = (p-1)(p-2)(p-3) \dots 2 \cdot 1$, so

$$(p-1)! \equiv -1 \pmod{p}$$

$$(p-1)(p-2)(p-3) \dots 2 \cdot 1 \equiv -1 \pmod{p}$$

$$-1 \cdot (p-2)(p-3) \dots 2 \cdot 1 \equiv -1 \pmod{p}$$

We need to show that $(p-2)(p-3) \dots 2 \equiv 1 \pmod{p}$.

Since for prime p , every number $1 \leq x < p$ is invertible modulo p , we have

$$ax \equiv 1 \pmod{p}$$

Taking $a \pmod{p}$, $a \in \{1, 2, \dots, p-2, p-1\}$ and since 1 and $p-1$ are their own inverse, $a \in \{2, \dots, p-2\}$. So for each x , there is going to be a multiplicative inverse $a \in \{2, \dots, p-2\}$. Therefore, we can pairing up every element in $\{2, \dots, p-2\}$ to make a multiplicative inverse respect to p .

$$-1 \cdot (p-2)(p-3) \dots 2 \cdot 1 \equiv -1 \pmod{p}$$

will become

$$-1 \cdot 1 \cdot 1 \dots 1 \cdot 1 \equiv -1 \pmod{p}$$

which is true. ■

(c)

Suppose N is a composite number, then there will be a factor $b > 1$ and $b \leq N-1$. This indicates that d divides $(N-1)!$, meaning that d does not divide $(N-1)! + 1$, same as

$$(N-1)! \not\equiv -1 \pmod{N}$$

So if N is not prime, then $(N-1)! \not\equiv -1 \pmod{N}$. ■

(d)

I think the problem is that Wilson's Theorem takes more time on doing the primality test. For example, the time complexity when N is a n -bits number will be $O(2^n n^2)$ because the the maximum times of integer multiplication will be $2^{n-1} - 1$ and each multiplication itself is $O(n^2)$. Whereas Fermat's Little Theorem only takes like $O(n^3)$ and maximum of $O(n^4)$ for doing n times of check.

Question 3

(a)

Based on the question, we know that p is prime, and for encryption integer e , and e is relative prime to $s = p - 1$. The following pseduocode is the method used for recovering x for $0 \leq x < p$.

```
function Decrypt(y,e,p)
    (d,a,b) = EXTEuclid(e,p-1)
    x = MODEXP(y,d,p)
    return x
end function
```

Proof:

Case 1: $x \not\equiv 0 \pmod{p}$

Since in this method, Extended Euclid's algorithm is used for getting integer d where $de \equiv 1 \pmod{s}$. This means $de - 1 = ks$ for some inetger k .

$$x^{de} = x^{1+ks} = x \cdot x^{(p-1)k}$$

By Fermat's Little Theorem, if $x \not\equiv 0 \pmod{p}$, then

$$x^{ks} \equiv x^{(p-1)k} \equiv 1^k \equiv 1 \pmod{p}$$

Therefore

$$x^{de} \equiv x \cdot x^{ks} \equiv 1 \cdot x \equiv x \pmod{p}$$

Case 2: $x \equiv 0 \pmod{p}$

Since for this RSA-like scheme, $0 \leq x < p$, then only possible number for x is 0. This case the method will also be true since

$$0^{de} \equiv 0 \pmod{p}$$

The correctness of this RSA-like scheme has benn proved. ■

Bit Complexity

- Alice generates one n-bits prime p
- Alice computes $s = p - 1$
- Alice chooses e relatively prime to s
- Eve knows e and p and computes s and $d \equiv e^{-1} \pmod{s}$
- Eve computes $x \equiv y^d \pmod{p}$

When p is a n-bits prime number, the time complexity for this method is $O(n^3)$ overall. In step four, Eve will use this method and use Extended Euclidean's Algorithm to get decryption integer d which takes $O(n^3)$. In step five, Eve then using the modular exponentiation to recover the original message x , which takes $O(n^3)$. Overall, this method will take polynomial time to recover x .

(b)

With given s and public key e , we can use the Extended Euclidean's Algorithm to get the multiplicative inverse of e where

$$de \equiv 1 \pmod{s}$$

The time complexity for Extended Euclidean's Algorithm is $O(n^3)$ and this is polynomial time.