CSC355 PS3

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Problem 1

Let $g(x) = \sum_{k=0}^{n} L_k(x)$. If we plug in all points from $x_0, \ldots, x_n$ into $g(x)$, we will get $g(x_k) = 1, \forall k = 0, \ldots, n$. Because $L_k(x_k)$, as a lagrange basis, will be 1 and be 0 for other k. We also know that because each $L_k(x)$ is a n-degree polynomial, their summation will be a polynomial with at most n degree, which is $g(x)$.
Instead of representing $g(x)$ in lagrange basis, we can try to represent $g(x)$ into monomial basis which is

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

We also have total n+1 function values at $g(x_k)$ for $k = 0, \ldots, n$. And we can create its Vandermonde matrix and a linear system.

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Since this Vandermonde matrix is a square matrix, it has an unique solution. Through observation, we can see the solution is when $a_0 = 1$, and everything else to be 0. This means our $g(x) = 1$. So $\sum_{k=0}^{n} L_k(x) = 1.$ ■

Problem 2

(a)

Coefficients for Newton basis with the given point are,

$$\begin{bmatrix} 0.0379 & 0.0856 & 0.1632 & 0.34133 & 0.029562 & -1.0447 & 1.9711 & -2.5477 & 2.5997 \end{bmatrix}$$

(b)

The evaluation values at these points are,

$$\begin{bmatrix} 0.043428 & 0.35741 & 0.47629 & 0.5 & 0.46928 & 0.34266 & 0.039373 \end{bmatrix}$$

(c)

The function values using polyfit and polyval is,

$$\begin{bmatrix} 0.043428 & 0.35741 & 0.47629 & 0.5 & 0.46928 & 0.34266 & 0.039373 \end{bmatrix}$$

The relative error between function value from polyval and my newton's evaluation method is around $1e-14$ for $x = \pm 1$. The error becomes smaller to $1e-16$ as we approaching to the middle point.

Problem 3

Based on the problem, we will have three cubic spline polynomials.

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$
$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$
$$S_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$$

There will be 12 equations for constraints and they are,

$$S_0(x_0) = f(x_0) \tag{1}$$
$$S_0(x_1) = f(x_1) \tag{2}$$
$$S_1(x_1) = f(x_1) \tag{3}$$
$$S_1(x_2) = f(x_2) \tag{4}$$
$$S_2(x_2) = f(x_2) \tag{5}$$
$$S_2(x_3) = f(x_3) \tag{6}$$
$$S_0'(x_1) = S_1'(x_1) \tag{7}$$
$$S_1'(x_2) = S_2'(x_2) \tag{8}$$
$$S_0''(x_1) = S_1''(x_1) \tag{9}$$
$$S_1''(x_2) = S_2''(x_2) \tag{10}$$
$$S_0''(x_0) = 0 \tag{11}$$
$$S_2''(x_3) = 0 \tag{12}$$

Substitute all with $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, and cubic polynomials.

$$a_0 = f(x_0) \tag{1}$$
$$a_0 + b_0h + c_0h^2 + d_0h^3 = f(x_1) \tag{2}$$
$$a_1 = f(x_1) \tag{3}$$
$$a_1 + b_1h + c_1h^2 + d_1h^3 = f(x_2) \tag{4}$$
$$a_2 = f(x_2) \tag{5}$$
$$a_2 + b_2h + c_2h^2 + d_2h^3 = f(x_3) \tag{6}$$
$$b_0 + 2c_0h + 3d_0h^2 = b_1 \tag{7}$$
$$b_1 + 2c_1h + 3d_1h^2 = b_2 \tag{8}$$
$$2c_0 + 6d_0h = 2c_1 \tag{9}$$
$$2c_1 + 6d_1h = 2c_2 \tag{10}$$
$$2c_0 = 0 \tag{11}$$
$$2c_2 + 6d_2h = 0 \tag{12}$$

With simplification,

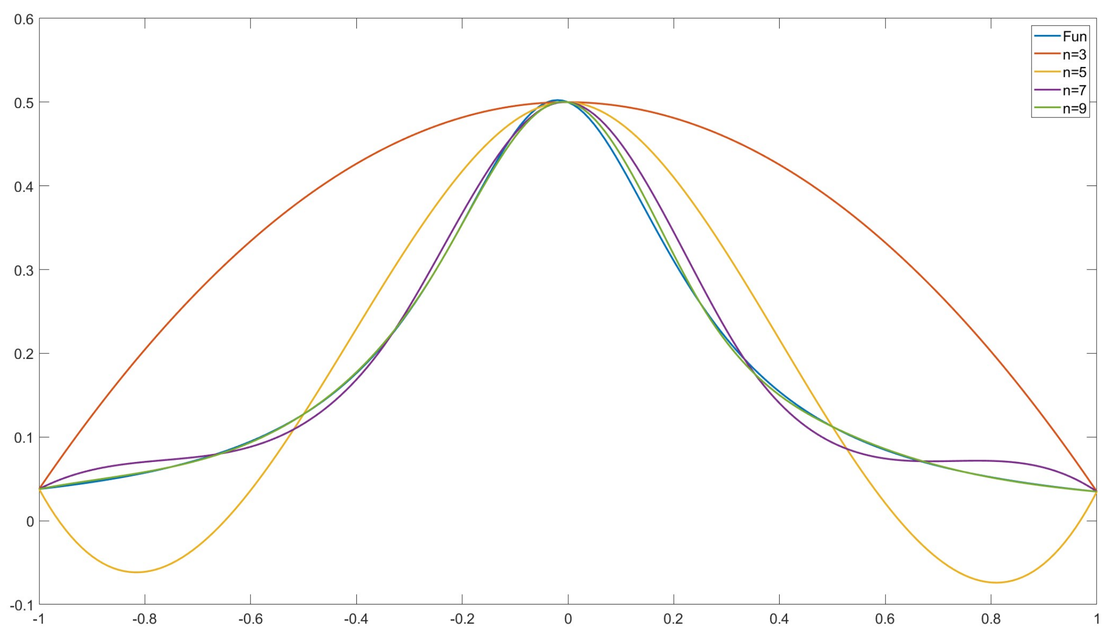
$$a_0 = f(x_0)$$
$$a_1 = f(x_1)$$
$$a_2 = f(x_2)$$
$$b_1 = b_0 + 3d_0h^3 \tag{1}$$
$$b_2 = b_1 + 6d_0h^3 + 3d_1h^3 \tag{2}$$
$$c_0 = 0$$
$$c_1 = 3d_0h$$
$$c_2 = 3d_0h + 3d_1h$$
$$d_0h + d_1h + d_2h = 0 \tag{3}$$
$$b_0h + d_0h^3 = f(x_1) - f(x_0) \tag{4}$$
$$b_1h + 3d_0h^3 + d_1h^3 = f(x_2) - f(x_1) \tag{5}$$
$$b_2h + 3d_0h^3 + 3d_1h^3 + d_2h^3 = f(x_3) - f(x_2) \tag{6}$$

The 12 equations above will become the linear system for this cubic spline

Problem 4

MATLAB's Spline Approximation

The result plot from -1 to 1 is,



The coefficients of the polynomial for n=7 is, (left most is the coefficient for $(x - x_i)^3$)

$$\begin{bmatrix} 1.5973 & -1.1383 & 0.32581 & 0.037925 \\ 1.5973 & 0.45903 & 0.099386 & 0.079211 \\ -7.1174 & 2.0564 & 0.93785 & 0.2225 \\ 7.4753 & -5.061 & -0.063694 & 0.5 \\ -2.0279 & 2.4143 & -0.94594 & 0.1933 \\ -2.0279 & 0.38636 & -0.01239 & 0.07113 \end{bmatrix}$$

Single Polynomial

the coefficients of polynomial of all cases are, (left most is the highest degree).
For 7 points,

$$\begin{bmatrix} -4.6497 & -0.1183 & 7.6023 & 0.1790 & -3.4162 & -0.0622 & 0.5000 \end{bmatrix}$$

For 9 points,

$$\begin{bmatrix} 15.4153 & 0.6104 & -30.1368 & -1.1445 & 18.8478 & 0.6474 & -4.5900 & -0.1148 & 0.5000 \end{bmatrix}$$

For 11 points,

$$\begin{bmatrix} -50.8766 & -2.7658 & 116.0274 & 6.1430 & -92.6163 & -4.6473 & 32.3396 & 1.4295 & -5.3378 & -0.1610 \\ 0.5000 \end{bmatrix}$$

For 13 points,

$$\begin{bmatrix} 167.1157 & 11.6309 & -435.9938 & -29.7878 & 422.5907 & 27.9182 & -193.5345 & -11.9514 & 45.1281 & 2.3835 \\ -5.7699 & -0.1951 & 0.5000 \end{bmatrix}$$

For 15 points,

$$\begin{bmatrix} -546.6420 & -46.6521 & 1606.5310 & 135.2212 & -1830.1796 & -150.5143 & 1039.7656 & 82.0073 & -319.4911 & -23.1852 \\ 55.5576 & 3.3391 & -6.0050 & -0.2176 & 0.5000 \end{bmatrix}$$

Cubic Spline

The coefficients for different polynomials are represented in matrix form, which each row means coefficients for a certain interval. They all follows $f(x) = a(x - x_i)^3 + b(x - x_i)^2 + c(x - x_i) + d$ expression.
For 7 points,

$$\begin{bmatrix} 1.5973 & -1.1383 & 0.32581 & 0.037925 \\ 1.5973 & 0.45903 & 0.099386 & 0.079211 \\ -7.1174 & 2.0564 & 0.93785 & 0.2225 \\ 7.4753 & -5.061 & -0.063694 & 0.5 \\ -2.0279 & 2.4143 & -0.94594 & 0.1933 \\ -2.0279 & 0.38636 & -0.01239 & 0.07113 \end{bmatrix}$$

For 9 points,

$$\begin{bmatrix} 0.68011 & -0.21836 & 0.11787 & 0.037925 \\ 0.68011 & 0.29172 & 0.13621 & 0.064371 \\ 1.2475 & 0.8018 & 0.40959 & 0.12728 \\ -10.814 & 1.7374 & 1.0444 & 0.29928 \\ 11.964 & -6.3732 & -0.11456 & 0.5 \\ -2.9205 & 2.6 & -1.0579 & 0.25997 \\ -0.21786 & 0.40963 & -0.30546 & 0.11238 \\ -0.21786 & 0.24623 & -0.14149 & 0.058209 \end{bmatrix}$$

For 11 points,

$$\begin{bmatrix} 0.26668 & 0.066301 & 0.072992 & 0.037925 \\ 0.26668 & 0.22631 & 0.13151 & 0.057309 \\ 1.9124 & 0.38632 & 0.25404 & 0.094797 \\ -1.3183 & 1.5337 & 0.63805 & 0.17636 \\ -12.894 & 0.74274 & 1.0933 & 0.35477 \\ 15.189 & -6.9934 & -0.15678 & 0.5 \\ -1.8644 & 2.1203 & -1.1314 & 0.31042 \\ -1.0143 & 1.0016 & -0.50702 & 0.15404 \\ -0.33462 & 0.39302 & -0.22809 & 0.084587 \\ -0.33462 & 0.19225 & -0.11103 & 0.052014 \end{bmatrix}$$

For 13 points,

$$\begin{bmatrix} 0.36753 & 0.009349 & 0.079905 & 0.037925 \\ 0.36753 & 0.19311 & 0.11365 & 0.053204 \\ 0.61092 & 0.37688 & 0.20865 & 0.079211 \\ 2.6071 & 0.68234 & 0.38518 & 0.12728 \\ -4.8436 & 1.9859 & 0.82989 & 0.2225 \\ -13.568 & -0.43592 & 1.0882 & 0.39356 \\ 17.191 & -7.2202 & -0.1878 & 0.5 \\ 0.21864 & 1.3755 & -1.1619 & 0.34773 \\ -1.7209 & 1.4848 & -0.68519 & 0.1933 \\ -0.64355 & 0.62437 & -0.33366 & 0.11238 \\ -0.27561 & 0.3026 & -0.17917 & 0.07113 \\ -0.27561 & 0.16479 & -0.10127 & 0.048398 \end{bmatrix}$$

For 15 points,

$$\begin{bmatrix} 0.26209 & 0.060999 & 0.074146 & 0.037925 \\ 0.26209 & 0.17332 & 0.10762 & 0.050526 \\ 0.69417 & 0.28565 & 0.17319 & 0.070202 \\ 1.0302 & 0.58315 & 0.2973 & 0.1028 \\ 2.5123 & 1.0247 & 0.52699 & 0.16017 \\ -8.5664 & 2.1014 & 0.97357 & 0.26369 \\ -13.23 & -1.57 & 1.0495 & 0.42068 \\ 18.187 & -7.2398 & -0.20905 & 0.5 \\ 2.8546 & 0.55463 & -1.1641 & 0.37541 \\ -2.0699 & 1.778 & -0.83084 & 0.22875 \\ -0.96593 & 0.89094 & -0.44956 & 0.14031 \\ -0.54388 & 0.47697 & -0.25414 & 0.091457 \\ -0.22287 & 0.24388 & -0.15116 & 0.0633 \\ -0.22287 & 0.14837 & -0.095128 & 0.046033 \end{bmatrix}$$

The 10 approximations to the integral are,

```
    >> integral_approx
Single interpolating polynomial:
-------------------------------------
Points       Integral value
    7            0.434959
    9            0.294270
   11            0.449111
   13            0.256646
   15            0.508593
```

```
Piecewise cubic polynomial:
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Points        Integral value
    7             0.374510
    9             0.365390
   11             0.364625
   13             0.364522
   15             0.364483
```

I wil trust the last value from cubic spline. Because I used MATLAB integral function to also compute function's integral. It seems that 0.36448 is really close to the value I got from MATLAB integral.
I believe about 4 decimal digits are correct. First is I try to test my integral value with more points. It seems that the value will converge to 0.364471, so the first 4 digits are correct.
Second idea is I tried to represent each cubic spline polynomials into lagrange basis. Then I can get an error for each cubic polynomial which is $\frac{f^{(4)}(c(x))}{24}(x-x_i)^4$. Because we are taking the integral, the error should be bounded by

$$\frac{\max f^{(4)}(c(x))}{24 \cdot 5}(x-x_i)^5\Big|_{x_i}^{x_{i+1}}$$

Because the lower bound is x_i, the evaluation will be 0, which means our integral value will be

$$\frac{\max f^{(4)}(c(x))}{120}h^5$$

where h is the space between each point(we have equispaced points so h will be the same). Evaluating fourth derivative through MATLAB syms, we get the final error to be 0.00098172. I guess this suggest we should be confident with 3 decimal points but consider this is an approximation, 4 decimal digits may work.