CSC352 HW4

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Question 1

Since $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$, let $\mathbf{q_1} = \frac{\mathbf{a_1}}{\|\mathbf{a_1}\|_2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$. In this case,

$$
\begin{aligned}
\mathbf{q_2} &= \mathbf{a_2} - (\mathbf{q_1}^\top \mathbf{a_2})\mathbf{q_1} \\
\mathbf{q_2} &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - (1/\sqrt{2} + 1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\
\mathbf{q_2} &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Normalizing $\mathbf{q_2}$,

$$
\mathbf{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

For $\mathbf{q_3}$,

$$
\begin{aligned}
\mathbf{q_3} &= \mathbf{a_3} - (\mathbf{q_1}^\top \mathbf{a_3})\mathbf{q_1} - (\mathbf{q_2}^\top \mathbf{a_3})\mathbf{q_2} \\
\mathbf{q_3} &= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - (3/\sqrt{2} + 1\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix} - (\mathbf{q_2}^\top \mathbf{a_3})\mathbf{q_2} \\
\mathbf{q_3} &= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - (3/\sqrt{2} + 1\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix} - (1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{q_3} &= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Normalizing $\mathbf{q_3}$,

$$\mathbf{q_3} = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

So $\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}$, And for $\mathbf{R}$, $\mathbf{r_{ij}} = \mathbf{q_i}^\top \mathbf{a_j}$, $\mathbf{r_{jj}} = \|\mathbf{a_j} - \sum_{i=1}^{j-1} \mathbf{r_{ij}} \mathbf{q_i}\|_2$,

$$\mathbf{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

The reduced $\mathbf{QR}$ decomposition will be

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

Question 2

(a)

For a matrix $\mathbf{X}$, and for a vector $\mathbf{v}$, their multiplication will be,

$$\mathbf{Xv} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} x_{11}v_1 + x_{12}v_2 + \cdots + x_{1n}v_n \\ x_{21}v_1 + x_{22}v_2 + \cdots + x_{2n}v_n \\ \vdots \\ x_{m1}v_1 + x_{m2}v_2 + \cdots + x_{mn}v_n \end{bmatrix}$$

There are n times of multiplication and $n-1$ addition in each row. The total number of flops in each row is $(2n-1)$. Since there are total m rows, the total number of flops will be

$$2mn - m$$

(b)

For a matrix $\mathbf{X}$ and a matrix $\mathbf{Y}$, assume their multiplication will be matrix $\mathbf{A}$.

$$\mathbf{XY} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1p} \\ y_{21} & y_{22} & \dots & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \dots & y_{np} \end{bmatrix} = A$$

Because of the matrix multiplication rule,

$$\mathbf{a}_{ij} = x_{i1}y_{1j} + x_{21}y_{2j} + \cdots + x_{in}y_{nj}$$

And since $\mathbf{A} \in \mathbb{R}^{m \times p}$, for matrix $\mathbf{A}$, there are total mp entries. for each entries, the number of flops will be n multiplication and $n-1$ addition, which is $2n-1$.
So the total number of flops for matrix times matrix will be

$$2nmp - mp$$

(c)

Given a matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, the product of its transpose and itself $\mathbf{X}^\top \mathbf{X}$ be a matrix $\mathbf{C}$, and each entry of $\mathbf{C}$ also follows that

$$\mathbf{c}_{ij} = x_{i1}x_{1j} + x_{21}x_{2j} + \cdots + x_{im}x_{mj}$$

However, $\mathbf{C}$ is a symmetric matrix since it equals $\mathbf{X}^\top \mathbf{X}$, which means we only need to calculate the upper right side and the main diagonal. There are total $(mn/2 + m/2)$ entries. For each entries, the number of flops will be $2m - 1$, so the total number of flops will be

$$m^2 n + m^2 - mn/2 - m/2$$

Question 3