CSC301 HW3

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Question 1

(a)

Since $n \geq n-1 \geq n-2 \geq n-3 \geq \ldots$, so that $n \cdot n \geq n \cdot (n-1)$. We can then apply this inequality with more numbers which

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \ldots 1 \leq n \cdot n \ldots n$$

This inequality holds true because each element on the left side is smaller than elements on the right side. Simplifying the inequality,

$$n! \leq n^n$$

which shows that it is true.■

(b)

Takes the $\log_{n/2}$ for $(n/2)^{n/2}$, which equals

$$\log_{n/2}(n/2)^{n/2} = n/2 \log_{n/2}(n/2) = n/2$$

Takesthe $\log_{n/2}$ for n factorial. This equals

$$\log_{n/2}(n!) = \sum_{i=0}^{n-1} \log_{n/2}(n-i)$$

Given a log function $\log_a b$, as long as $b \geq a$, $\log_a b \geq 1$. Expanding $\sum_{i=0}^{n-1} \log_{n/2}(n-i)$:

$$\sum_{i=0}^{n-1} \log_{n/2}(n-i) = \log_{n/2}(n) + \log_{n/2}(n-1) + \ldots + \log_{n/2} 1$$

We can get that all elements before $\log_{n/2}(n/2-1)$ is larger or equal to 1, and there are total $n/2+1$ elements before $n/2-1$ in this summation. Therefore, we can obtain the following inequality:

$$\sum_{i=0}^{n-1} \log_{n/2}(n-i) = \log_{n/2}(n) + \log_{n/2}(n-1) + \ldots + \log_{n/2} 1 \geq n/2+1$$

Which is the same as,

$$\log_{n/2}(n!) \geq n/2+1 \geq n/2 = \log_{n/2}(n/2)^{n/2}$$

Exponentiates both sides,

$$n! \geq (n/2)^{n/2}$$

Just as the prompt.■

(c)

From question (a) and (b), we can get the inequality,

$$n^n \geq n! \geq (n/2)^{n/2}$$

Takes the log for all of them,

$$n \log n \geq \log(n!) \geq (n/2) \log(n/2)$$

Case 1: Big-Oh
Let $f(n) = \log(n!)$ and $c \cdot g(n) = c \cdot n \log n$. By definition, Since

$$\log(n!) \leq n \log n$$

We can let $c = 1$ and $N = 1$, and plug in the number into inequality,

$$f(n) = \log(n!) \leq n \log n = g(n)$$

for all $n \geq N$. Therefore,

$$\log(n!) = O(n \log n)$$

Case 2: Big-Omega
Since $\log(n!) \geq (n/2) \log(n/2)$, we can do some transformation on the right hand side,

$$\log(n!) \geq (n/2) \log n - (n/2) \log 2$$

When $n \geq 4$, $n/4 \log n \geq n/2$ and substitudes $n/2 \log 2$ with $n/4 \log n$, we can get:

$$\log(n!) \geq (n/2) \log n - n/4 \log n = n/4 \log n \text{ when } n \geq 4$$

By definition, let $f(n) = \log(n!)$, and $c \cdot g(n) = c \cdot n/4 \log n$. We can assume that for $c = 4$ and $N = 4$, the inequality

$$\log(n!) \geq n \log n$$

holds.
So for all $n \geq N$, then

$$\log(n!) = \Omega(n \log n)$$

Overall, if $\log(n!) = O(n \log n)$, and $\log(n!) = \Omega(n \log n)$, then

$$\log(n!) = \Theta(n \log n)$$

■

Question 2

Question 3