ON WRAPPED FUKAYA CATEGORY AND THE LOOP SPACE OF LAGRANGIANS IN A LIOUVILLE MANIFOLD

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ABSTRACT. We introduce an A_∞ map from the singular complex of based loop space of Lagrangian submanifolds with legendrian boundary of a Liouville Manifold $C_{n-*}(\Omega_L \mathcal{L}ag)$ to wrapped floer cohomology of Lagrangian submanifold $\mathcal{CW}^*(L,L)$. In the case of a cotangent bundle and a Lagrangian cofiber, the composition of our map with the map from $\mathcal{CW}^*(L,L) \to C_{n-*}(\Omega_q Q)$ as defined in [3] shows that this map is split surjective.

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1. INTRODUCTION

Consider a Liouville Manifold with contact boundary $(M, \partial M, \omega)$. By definition, there is a one-form θ such that $\omega = d\theta$ is the symplectic form on M; there is also a Liouville vector field Z such that $i_Z\omega = \theta$, from which we get Z strictly points outward along ∂M. Denote the contact form on ∂M by $\alpha = \theta|_{\partial M}$. There is a collar neighbourhood of ∂M which is parametrized by a map $\varphi : \partial M \times (0,1] \to M$ such that $\varphi|_{\partial M \times \{1\}} = Id$. Consider the completion:

$$\hat{M} = M \cup_{\partial M} ([1,\infty) \times \partial M). \tag{1.1}$$

Denote radial coordinate by r. $\hat{M}$ is canonically equipped with an extension $\hat{\theta}$ of θ, symplectic form $\hat{\omega}$ which extends ω and Liouville vector field $\hat{Z}$ that extends Z. On $\partial M \times (0,\infty)$:

$$\hat{\theta} = r(\theta|_{\partial M}),\ \hat{\omega} = d(\hat{\theta}),\ \hat{Z} = r\partial_r$$

Moreover, we impose the following condition on the almost complex structures J on $\hat{M}$:

$$dr \circ J = -\hat{\theta}, \tag{1.2}$$

This means we can decompose $T(\partial M \times (0,\infty))$ as follows:

$$T(\partial M \times (0,\infty)) = \mathbb{R}^2 \oplus \xi \tag{1.3}$$

where the fist factor is spanned by $r\partial_r$ and $(0, Y)$ with the standard complex structure: $r\partial_r \to (0, Y)$. Any such structure is invariant under the flow of $r\partial_r$.

In the above collar neighborhood of ∂M, we may let M^{in} denote the complement of $\partial M \times [\frac{1}{2}, 1]$, thus M^{in} is a compact Liouville manifold with contact boundary.

In order to understand the topological structure of the space of exact Lagrangian submanifolds with Legendrian boundary of a Liouville manifold, whose homotopic information can be obtained from its based loop space, we show an A_∞ morphism between the *Pontryagin category* of decorated Lagrangians to the wrapped Floer category of the Liouville Manifold.

Here Pontryagin category of a topological space X is a DG category whose objects are the points of X and morphisms between two points are given by cubical chains on space of paths in X between the points. The associative product structure is given by concatenation of paths. The exact definition of Pontryagin Category of a path connected space is in section 3.

Each component $\widetilde{\mathcal{L}ag_b}$ of $Obj(\mathcal{P}(\mathcal{L}ag))$ is the space of decorated Lagrangian $\mathcal{L} = (L, \alpha)$ such that $L \in \mathcal{L}ag_b$, where $\mathcal{L}ag_b$ is the space of Lagrangians whose second Stiefel-Whitney class $w_2(L)$ is equal to some fixed homology class $b \in H^2(M, \mathbb{Z}/2)$ and $\alpha \in H^1(L, \mathbb{Z}/2)$. In other words, it is the space of all possible Pin structures of a Lagrangian. Details can be found in section 3.

$$\begin{tikzcd} H^1(L;\mathbb{Z}/2) \arrow[r] & \widetilde{\mathcal{L}ag_b} \arrow[d, "\pi"] \\ & \mathcal{L}ag_b \end{tikzcd}$$

Thus the main theorem is as follows:

Theorem 1.1. *For any Liouville manifold $(M, \partial M, \omega)$, there exists an A_∞ functor from the Pontryagin category of decorated Lagrangians to the wrapped Floer category of the ambient symplectic manifold:*

$$\mathcal{P}(\widetilde{\mathcal{L}ag}) \xrightarrow{\mathcal{F}} \mathcal{CW}^*(M). \tag{1.4}$$

Remark 1.2. The degree 0 map of the A_∞ morphism is a chain map $\mathcal{F}^1$:

$$C_*(\Omega_{\mathcal{L}_0,\mathcal{L}_1}\widetilde{\mathcal{L}ag}) \xrightarrow{\mathcal{F}^1} \mathcal{CW}^{-*}(\mathcal{L}_0, \mathcal{L}_1). \tag{1.5}$$

when $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}$, this induces a map on homology from homology of based loop space of Lagrangians with Legendrian boundary to the wrapped Floer homology of a single Lagrangian.

$$H_*(\Omega_{\mathcal{L}}\widetilde{\mathcal{L}ag}) \xrightarrow{H(F)} \mathcal{HW}^{-*}(\mathcal{L}). \tag{1.6}$$

If $M = T^*Q$ and $\mathcal{L} = T^*_qQ$ this is an isomorphism as shown by the theorem below, but it is not necessarily isomorphism in the general case.

The above theorem implies that we can use Wrapped Floer homology to detect nontrivial families of Lagrangians. As an example for computation, let the Liouville Manifold M be a cotangent bundle T^*Q of a compact manifold Q. In this case, M. Abouzaid constructed an A_∞ map $f : \mathcal{CW}^*(\mathcal{L}, \mathcal{L}; H) \to C_{n-*}(\Omega_q Q)$ [3]. Here each cotangent fibre T^*_qQ is contractible, thus the map $\widetilde{\mathcal{L}ag_b} \to \mathcal{L}ag_b$ is an isomorphism. By taking the above A_∞ functor on the level of automorphisms of a single Lagrangian and composing with the chain level homorphism from f mentioned above, we obtain the following homotopy commutative diagram:

Theorem 1.3. *With $L = T_q^*Q$, the map $(f \circ \mathcal{F} \circ \eta)_*$ in the diagram below equals the identity map on $H_{n-*}(\Omega_q Q)$.*

$$\begin{tikzcd} C_{n-*}(\Omega_q Q) \arrow[r, "\eta"] \arrow[d, "id"] \arrow[dr, dotted] & C_{n-*}(\Omega_L \mathcal{L}ag) \arrow[d, "\mathcal{F}"] \\ C_{n-*}(\Omega_q Q) & \mathcal{CW}^*(\mathcal{L}, \mathcal{L}; H) \arrow[l, "f"] \end{tikzcd}$$

where the map η sends every loop in the base to the corresponding loop of cotangent fibers in the space of Lagrangian embeddings.

Remark 1.4. There are quite a few previous results on the case of cotangent bundles: C.Viterbo proposed the Floer homology of the cotangent bundle of a closed manifold is isomorphic to the cohomology of the loop space of the zero section using generating functions in [18] and [19]. A. Abbondandolo and M. Schwarz proved that there is a chain complex isomorphism from the Morse chain complex of action functional on the space of smooth loops on a closed manifold to the Hamiltonain Floer homology of its cotangent bundle In [1] and later in [4] that this map also preserves the ring structure in that the Chas-Sullivan loop product corresponds to the pair-of-pants product under this map. In [3] M. Abouzaid constructed the inverse map as a special case of an A_∞ functor from the wrapped Fukaya category of a Liouville manifold M to the category of modules on the differential graded algebra of chains over the based loop space of an exact relatively Pin Lagrangian submanifold.

Remark 1.5. Recently in a series of Papers[8, 9, 10], Ganatra, Pardon and Shende used the theory of microlocal sheaves to study the symplectic structure of Liouville manifolds and it is very likely that one can also use their machinery to achieve the results of this paper.

This paper first reviews the theory of Wrapped Fukaya Category and Pontryagin Category, then construct the moduli spaces that give rise to the functor, ending with an appendix that gives a C^0 estimate of the moduli spaces with moving Lagrangian boundary condition inside a Liouville manifold. The main techniques of the proof require an understanding of the stratified spaces of the compactification of the moduli space of pseudo-holomorphic curves with moving Lagrangian boundary condition and an adapted Hopf maximum principle for C^0 estimates of such moduli spaces.

2. Wrapped Fukaya Category

2.1. **Preview of Floer theory.** In [4] Abouzaid and Seidel constructed the wrapped Floer cohomology on Liouville domains in analogue with the Lagrangian Floer cohomology of a closed symplectic manifold. They defined $HW^*(L; H) \cong \varinjlim_w HF^*(L; wH)$ from a direct system of continuation maps from Lagrangian Floer cohomology with Hamiltonians of increasing slope at infinity.

Later in [3], Abouzaid used Hamiltonians that are quadratic at infinite ends to define Wrapped Floer Homology, we are going to follow this definition. From now on we work on open Liouville manifold (M, ω) with Liouville vector field Z and infinite conic ends. thus we have

$$i_Z\omega = \theta \tag{2.1}$$

where θ is the 1-form that agrees with $r\partial_r$ along the cylindrical ends of M. We use ψ^ρ to denote the image of the Liouville flow for time $\log(\rho)$ outward along the cylindrical end. Namely,

$$\psi^\rho(r,x) = (\rho \cdot r, x).$$

Let $\mathcal{H}(M) \subseteq C^\infty(M,\mathbb{R})$ denote the set of smooth functions H such that

$$H(r,x) = r^2 \tag{2.2}$$

away from some compact set of M. let $\mathcal{H}_Q(M)$ denote those functions that in addition vanish on Q. We fix such a function for the purpose of defining Floer homology, and let X_H denote the Hamilotnian flow of H defined by the equation

$$i_{X_H}\omega = -dH \tag{2.3}$$

We denote by $\mathcal{L}ag$ the space of exact Lagrangians with vanishing Maslov index,equipped with Legendrian boundary, namely Lagrangian submanifolds of M such that $L \cap \partial M = \Lambda$ is a Legendrian submanifold of the contact manifold ∂M. Thus by exactness, for any $L \in \mathcal{L}ag$,

$$\theta|_L = df \tag{2.4}$$

for some $f \in C^\infty(L,\mathbb{R})$.

For each pair of Lagrangians $L_0, L_1 \in \mathcal{L}ag$, we define $\mathcal{X}(L_0, L_1)$ to be the set of time-1 flow lines of X_H that start from L_0 and stops at L_1, namely a map $x : [0,1] \to M$ such that

$$\begin{cases} x(0) \in L_0 \\ x(1) \in L_1 \\ \frac{dx}{dt} = X_H \end{cases}$$

In order to get well-defined moduli spaces, we can assume without loss of generality that

(2.5) all time-1 Hamiltonian chords of H with boundaries on L_0 and L_1 are non-degenerate.

It is convenient to remember that the Hamiltonian chords above correspond bijectively to the intersection points of L_1 with the time-1 flow $\phi_H^1(L_0)$ of L_0 under the Hamiltonian flow of H. In particular, the non-degeneracy of the chords is the same as transversal intersection of L_1 and $\phi_H^1(L_0)$. Condition 2.5 therefore is true after we do a generic Hamiltonian perturbation on the Lagrangians that intersect transversally with ∂M^{in}.

Since Lagrangian Floer Homology is invariant under Hamiltonian perturbation, we are OK. We briefly introduce the definition of Lagrangian branes for the sake of proper definition of Lagrangian Floer homology: Let (M,ω,J) be symplectic with $2c_1(M) = 0$, therefore it is equipped with a non-vanishing quadratic volume form η_M^2, namely a global section of the line bundle $(\Omega^{n,0}M)^{\otimes 2} = (\wedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$. We define the squared phase map of TM as follows:

Definition 2.1. *Given a symplectic vector bundle $\varphi : E \to B$ of rank r such that $2c_1(E) = 0$, we have a global nonvanishing complex volume form $\eta_E^2 \in H^0(M, (\wedge_{\mathbb{C}}^r E)^{\otimes 2})$. For a Lagrangian* *sub bundle* $\phi : F \subset E$*, consider the following squared phase map of* F*:*

$$\alpha_E : Gr(E) \times B \to S^1,$$
$$\alpha_E(F, x) := \frac{\eta_x^2(v_1 \wedge v_2 \wedge \cdots v_r)}{|\eta_x^2(v_1 \wedge v_2 \wedge \cdots v_r)|}$$

Where $Gr(E)$ *is the set of lagragian sub bundles of* E*,* $v_1, \cdots, v_r$ *is a set of basis of* E_x*. Note this function* α_E *is well defined with respect to choices of the basis.*

We can define the associated grading $\alpha^{\#}$ on F to be:

$$\alpha^{\#} : B \to \mathbb{R}$$
$$\text{such that } exp(2\pi i \alpha^{\#}(x)) := \alpha_E(F, x)$$

Now we further assume that for a fixed $b \in H^2(B; \mathbb{Z}/2)$, the second Stiefel-Whitney class $w_2(\phi) \in H^2(B; \mathbb{Z}/2)$ is the same as b, namely $b = w_2(\phi)$.Now we fix a triangulation of B, We can choose an oriented vector bundle over the three skeleton $B_{[3]}, \psi : V \to B_{[3]}$ such that $w_2(\psi) = b|_{B_{[3]}}$. Then our assumption on $w_2(\phi)$ gives

$$w_2(\phi|_{B_{[3]}} \oplus \psi) = 0 \tag{2.6}$$

From standard literature([17] (section 11i), [12]), we know that second Stiefel-Whitney class is precisely the obstruction to the existence of Pin structures:

Definition 2.2. *A relative Pin structure on a Lagrangian sub bundle* F *is the choice of of a Pin structure on the vector bundle* $F|_{B_{[3]}} \oplus V$*.*

Definition 2.3 (brane structure)**.** *Given a Symplectic vector bundle* $E \to B$ *and* $b \in H^2(B; \mathbb{Z}/2)$*, consider a Lagrangin sub bundle* $F \to B$ *such that* $2\mu(F) = 0$*,* $w_2(F) + b = 0$*, then a brane structure with respect to* b *consists of a pair* $(\alpha^{\#}, P^{\#})$*, where* $\alpha^{\#}$ *satisfies the equation* (**??**) *and* $P^{\#}$ *is a relative Pin structure on* F*.*

Remark 2.4. Note that the obstruction to the existence of a grading is the Maslov class $\mu(F) \in H^1(B; \mathbb{Z})$ represented by the map $\alpha_E(F, -) : B \to S^1$; if $b = 0$, then we have a Pin-structure on F in the sense of ([17]), whose obstruction is the second Stiefel-Whitney class $w_2(E) \in H^2(B, \mathbb{Z}/2)$. Here we take the direct sum with V so as to "kill" the obstruction given by possible non zero b. If both obstructions vanish, then the set of brane structures on F is a torsor of $H^0(B; \mathbb{Z}) \times H^1(B, \mathbb{Z}/2)$.

Now in our situation, the base B of the fibration is an exact Lagrangian sub manifold $L \in M$ with Legendrian boundary, the subbundle is just $TL \subset TM|_L$ whose Maslov class $\mu(TL)$ and second Stiefel Whitney class $w_2(TL)$ vanish.

Therefore, to each Hamiltonian chord $x \in \mathcal{X}(L_0, L_1)$, we can assign a grading $\mu(x)$ from the gradings on L_0, L_1 for given transverse brane structures on L_0, L_1. Here we define $\mu(x) := [\alpha_1^{\#} - \alpha_0^{\#}] + 1$, where$[\cdot]$ is the floor function that lies in $\mathbb{Z}$. We also define the orientation space $o_x = Hom(P_0^{\#}, P_1^{\#}) \otimes TL_1^{\otimes \mu(x)}$.Further examine For details we refer to section (11) of [17].

Note that the Hamiltonian chords $x \in \mathcal{X}(L_0, L_1)$ are critical points of an action functional $\mathcal{A}$ on the space $\Lambda(L_0, L_1)$ of smooth path between L_0, L_1. It is defined as

$$\mathcal{A}(x) = -\int_0^1 x^*\theta + \int H(x(t))dt + f_{L_1}(x(1)) - f_{L_0}(x(0))$$

Note that equation 2.3implies for any chord that intersects some slice $\partial M^{in} \times \{r\}$ is contained inside this slice and has action

$$\mathcal{A}(x) = -r^2 + f_1(x(1)) - f_0(x(0)) \tag{2.7}$$

From the above observation we realize that $\mathcal{A}$ is a proper map from $\mathcal{X}(L_0, L_1)$ to $\mathbb{R}$. Since there are only finitely many non-degenerate chords in any compact subset of M, and the above equation implies that the action of a sequence of chords that escapes any compact subset must go to $-\infty$.

2.2. **Moduli spaces.** Given a pair $x_0, x_1 \in \mathcal{X}(L_0, L_1)$, we define $\tilde{\mathcal{M}}(x_0; x_1)$ to be the moduli space of maps

$$u : \mathbb{R} \times [0,1] \to M$$

satifying the fololowing conditions

$$\begin{cases} u(\mathbb{R} \times \{1\}) \subseteq L_1, \\ u(\mathbb{R} \times \{0\}) \subseteq L_0, \\ \lim\limits_{s \to -\infty} u(s, \cdot) = x_0(\cdot), \\ \lim\limits_{s \to \infty} u(s, \cdot) = x_1(\cdot). \end{cases}$$

such that u solves the Cauchy-Riemann equation

$$(du - X_H \otimes dt)^{0,1} = 0 \tag{2.8}$$

which can also be written explicitly with the almost complex structure J:

$$\partial_s u + J_t(\partial_t u - X_H dt) = 0$$

We have the following:

Proposition 2.5. *The moduli space $\tilde{\mathcal{M}}(x_0; x_1)$ is regular for a generic choice of almost complex structures J_t and has dimension $\mu(x_0) - \mu(x_1)$.*

Since equation (2.8) is invariant under translation of the variable s, there is an action of $\mathbb{R}$ on $\tilde{\mathcal{M}}(x_0; x_1)$. After taking the quotient of free $\mathbb{R}$-action, we get $\mathcal{M}(x_0; x_1)$; and let it be empty if the action is not free.

We may add broken strips to this moduli space and get a manifold with boundary $\overline{\mathcal{M}}(x_0; x_1)$ whose strata are disjoint unions over all possible sequences of product of moduli spaces linking x_0 and x_1 of the following form:

$$\mathcal{M}(x_0; y_1) \times \mathcal{M}(y_1; y_2) \times \cdots \times \mathcal{M}(y_k; y_{k+1}) \times \mathcal{M}(y_{k+1}, x_1). \tag{2.9}$$

Since the Lagrangians L_0, L_1 have infinite ends, we need to show that $\overline{\mathcal{M}}(x_0; x_1)$ is compact.We have the following lemma:

Lemma 2.6. *Any solution of 1.3 that converges to $x_0(t)$ at $-\infty$ and $x_1(t)$ at ∞, then*

$$\mathcal{A}(x_0) \geq \mathcal{A}(x_1) \tag{2.10}$$

Furthermore, all such solutions lie within some compact subset of M that is dependent only on x_0 and x_1.

Remark 2.7. This is the Integrated Maximum Principle as shown in Lemma 7.2 of [4]. We give a brief discussion of the proof here.

Proof. The inequality (2.10) is due to the energy estimate from the positivity of H as a quadratic function. To prove cmopactness, consider any hypersurface $\partial M^{in} \times \{r\}$ such that x_0, x_1 from infinity. If a solution of (2.8) escapes this compact subset, we may find a compact subset Σ of $\mathbb{R} \times I$ mapping to $\partial M^{in} \times \{r\}$ with boundary conditions on $\{r\} \times \partial M^{in}$ and a pair of Lagrangians on which θ vanishes.(Why?) Stokes' theorem tells us that the topological energy of the map restricted to Σ

$$\begin{aligned}
0 < E^t(u|_\Sigma) &= \int_\Sigma u^*\omega - dH \otimes dt \\
&= \int_{\partial\Sigma} u^*\theta - H \otimes dt \\
&= \int_{\partial\Sigma} u^*\theta(du - X_H \otimes dt) \\
&= \int_{\partial_r\Sigma} u^*(\theta \circ J) \circ (du - X_H \otimes dt) \circ j
\end{aligned}$$

Where $\partial_r\Sigma$ is the preimage of $\partial M^{in} \times \{r\}$. Note the restriction of X_H to $\partial M^{in} \times \{r\}$ is a multiple of the Reeb flow (proof beforehand), thus $\theta(JX_H)$ vanishes, so we get

$$0 < \int_{\partial_r\Sigma} \theta \circ J \circ du \circ j \tag{2.11}$$

Meanwhile, if ξ is a tangent vector compatible with orientation of $\partial_r\Sigma$, then $j\xi$ points inward, thus $du(j\xi)$ pints towards the infinite direction. Since J is of contact type as is known in (1.2), we have $\theta(J \circ du \circ j\xi) \leq 0$. Contradiction. □

The above lemma together with the observation that $\mathcal{A}$ is a proper map at the end of section (2.1) implies that the image of all possible $\mathcal{M}(x_0; x_1)$ for a fixed $x_1 \in \mathcal{X}(L_0; L_1)$ are contained in a certain compact subset of M. Therefore we may apply Gromov's compactness and gluing to conclude:

Corollary 2.8. *For each chord $x_1(t)$, the moduli space $\overline{\mathcal{M}}(x_0; x_1)$ is empty for all but finitely many choices of x_0, and is a compact manifold with boundary of dimension $\mu(x_0) - \mu(x_1) - 1$ whenever J is a generic family of almost complex structure. Moreover, the boundary is covered by the closure of the imgages of the natural inclusions.*

$$\mathcal{M}(x_0; y) \times \mathcal{M}(y; x_1) \to \overline{\mathcal{M}}(x_0; x_1)$$

From now on, we fix the almost complex structure J for which the above corollary holds.

2.3. **Wrapped Floer complex.** We define the graded vector space underlying the Floer chain complex to be the direct sum

$$\mathcal{CW}^*(L_0; L_1) := \bigoplus_{x \in \mathcal{X}(L_0; L_1)} |o_x| \tag{2.12}$$

Where o_x is the determinant line bundle of the linearization of operator (2.8). We may simply assuming that it is just the vector space freely generated by the sets of chords unless one wants to worry about signs.

The differential is a count of the solutions of rigid moduli spaces of equation (2.8), namely those with $\mu(x_0) = \mu(x_1) + 1$. This defines an isomorphism:

$$o_{x_0} \cong o_{x_1}$$

Thus an orientation of o_{x_0} is induced from one on o_{x_1}, and the map on the orientation lines is denoted μ_u. We define the map:

$$\mu_1 : \mathcal{CW}^i(L_0; L_1) \to \mathcal{CW}^{i+1}(L_0, L_1) \tag{2.13}$$

$$[x_1] \mapsto (-1)^i \sum_u \mu_u([x_1]) \tag{2.14}$$

The proof that this count is a well defined differential of chain complexes is standard. Moreover, Corollary (2.8) shows that each chord can be the input of only finitely many solutions to (2.8), thus we know that on the right hand side of (2.13) there are only finitely many summands.

Remark 2.9. The graded vector space underlying $\mathcal{CW}^*(L_0; L_1)$ depends on L_0, L_1, H, ω, meanwhile the differential depends on J. When we want to distinguish them as in the next lemma, we write $\mathcal{CW}^*(L_0, L_1; \omega, H, J)$.

Lemma 2.10. *If $\psi : M \to M$ satisfies $\psi^*(\omega) = \rho\omega$ for some non-zero constant ρ, then we have a canonical isomorphism*

$$\mathcal{CW}(\psi) : \mathcal{CW}^*(L_0, L_1; \omega, H, J) \cong \mathcal{CW}^*(\psi(L_0), \psi(L_1), \frac{H \circ \psi}{\rho}, \psi^*(J)). \tag{2.15}$$

This was shown as lemma 3.4 in [3], I rewrite here for the convenience of the reader.

Proof. For a diffeomorphism ψ, we have an isomorphism of chain complexes

$$\mathcal{CW}^*(L_0, L_1; \omega, H, J) \cong \mathcal{CW}^*(\psi(L_0), \psi(L_1), \psi^*\omega, \psi^*H, \psi^*(J))$$

by composing each chord from L_0 to L_1 with ψ and getting a chord from $\psi(L_0)$ to $\psi(L_1)$, and each solution to equation (2.8) to obtain an equation with ψ^*J. Since $\psi^*\omega = \rho\omega$, we know that the Hamiltonian flow of $H \circ \psi$ with respect to $\psi^*\omega$ agrees with the flow of $\frac{H \circ \psi}{\rho}$ with respect to ω. Since only the Hamiltonian flow appears in equation (2.8), we obtain an identification

$$\mathcal{CW}^*(\psi(L_0), \psi(L_1); \psi^*\omega, H \circ \psi, \psi^*J) \cong \mathcal{CW}^*(\psi(L_0), \psi(L_1), \omega, \frac{H \circ \psi}{\rho}, \psi^*J) \tag{2.16}$$

So from now on, we omit the Floer data and define

$$\mathcal{CW}^*(\psi(L_0), \psi(L_1)) \equiv \mathcal{CW}^*(\psi(L_0), \psi(L_1); \omega; \frac{H \circ \psi}{\rho}, \psi^*J)$$

□

2.4. **product structure.** We would like to study the product operation on wrapped Floer complexes,

$$\mathcal{CW}^*(L_1; L_2; H) \otimes \mathcal{CW}^*(L_0; L_1; H) \to \mathcal{CW}^*(L_0; L_2; H) \tag{2.17}$$

However, the natually defined map would take product in $\mathcal{CW}^*(L_0; L_2; 2H)$. And the usual continuation map from $\mathcal{CW}^*(L_0; L_2; 2H)$ to $\mathcal{CW}^*(L_0; L_2; H)$ fails to be well defined.Thus we appeal to the rescaling trick in [2]. We briefly sketch here the main idea.

First, we define a map

$$\mu_2^{\psi^2} : \mathcal{CW}^*(L_1; L_2; H) \otimes \mathcal{CW}^*(L_0; L_1; H) \to \mathcal{CW}^*(\psi^2(L_0); \psi^2(L_2)). \tag{2.18}$$

Where ψ^2 is the time-log(2) Liouville flow .Then compose with the inverse of equation (2.16) to get

$$\mu_2 : \mathcal{CW}^*(L_1; L_2; H) \otimes \mathcal{CW}^*(L_0; L_1; H) \to \mathcal{CW}^*(L_0; L_2 H). \tag{2.19}$$

The map (2.18) counts solutions to the following version of Cauchy-Riemann equation:

$$(du - X_S \otimes \alpha_s)^{0,1} = 0 \tag{2.20}$$

Where S is the surface obtained from removing 3 points ξ^0, ξ^1, ξ^2 from the boundary of $\mathbb{D}^2$. In the above equation, α_S is a closed 1-form on S while X_S is the Hamiltonian vector field of H_S on M, which depends on S. We require that H_S come from a map

$$H_S : M \to \mathcal{H}(M) \tag{2.21}$$

and restrict to H near ξ^1, ξ^2 and to $\frac{H}{4} \circ \psi^2$ near ξ^0. Note that $\frac{H}{4} \circ \psi^2 \in \mathcal{H}(M)$ since $H(y, r) = r^2$ for y large enough,namely near enough the cylindrical end.

To specify the remaining data required for equation (2.20), we can pick strip like ends for S, namely $\Sigma_\pm$ for the positive(respectively, negative) half strips in Σ and embeddings

$$\epsilon^0 : \Sigma_- \to S \tag{2.22}$$

$$\epsilon^1 : \Sigma_+ \to S \tag{2.23}$$

$$\epsilon^2 : \Sigma_+ \to S \tag{2.24}$$

which map $\partial\Sigma_\pm$ to ∂S and converge to the respective marked points ξ^k for $k = 1, 2$.

For the closed 1-form α_S, it should vanish on $\partial\S$ and satisfy

$$\epsilon^{0^*}(\alpha_S) = 2dt \tag{2.25}$$

$$\epsilon^{k^*}(\alpha_S) = dt, k = 1, 2 \tag{2.26}$$

In addtiion, we choose a family of almost complex structures $J_S : S \to \mathcal{J}(M)$ parametrized by the Riemann surface S whose compositions with ϵ^k agreews with J if $k = 1, 2$ and $(\psi^2)^* J$ if $k = 0$.

The Lagrangian boundary conditions near ξ^1 should be given by (L_0, L_1),similarly near xi^2 given by (L_1, L_2) and near xi^0 given by $(\psi(L_0), \psi(L_2))$. those along the two segments of ∂S that converge to xi^0, we cannot have a constant Lagrangian condition, instead a moving boundary Lagrangian condition ([17]). In order to pick a simple moving Lagrangian condition we fix a map ρ_s from ∂D^2 to $[1, 2]$ such that

$$\rho_S(z) \equiv 1 \text{ if } z \text{ is near } \xi^1 \text{ or } \xi^2 \text{ and } \rho_S(z) \equiv 2 \text{ if } z \text{ is near } \xi^0. \tag{2.27}$$

Definition 2.11. *the moduli space $\mathcal{M}(x_0; x_1, x_2)$ is the space of solutions to equation (2.20) with boundary conditions*

$$\begin{cases} u(z) \in \psi^{\rho_S(z)}(L_1) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_2 \\ u(z) \in \psi^{\rho_S(z)}(L_2) & \text{if } z \in \partial S \text{ lies between } \xi_2 \text{ and } \xi_0 \\ u(z) \in \psi^{\rho_S(z)}(L_0) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_0 \end{cases} \tag{2.28}$$

and such that the image of u converges to x_1 and x_2 at the corresponding incoming strip-like ends, and to $\psi^2(x_0)$ at the outgoing strip-like end.

So from our construction, we have ensured that the pullback of equation (2.20) by the strip-like ends ξ^k is given by

$$(du - X \otimes dt)^{0,1} = 0 \text{ with respect to } J \text{ if } k = 1, 2$$
$$(du - 2X_{\frac{H}{4}\circ\psi^2} \otimes dt)^{0,1} = 0 \text{ with respect to } (\psi^2)^* J \text{ if } k = 0$$

with Lagrangian boundary conditions (L_0, L_1) near ξ^1, (L_1, L_2) near ξ^2, and $(\psi^2(L_0), \psi^2(L_2))$ near ξ^0. The inputs we get the same original Hamiltonian equation (2.8), while for the output, we get the same equation with Hamiltonian $\frac{H}{2} \circ \psi^2$. We conclude that the Gromov bordification of the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is obtained by adding the strata

$$\coprod_{y \in \mathcal{X}(L_0, L_1)} \mathcal{M}_2(x_0; y, x_2) \times \overline{\mathcal{M}}(y; x_1) \tag{2.29}$$

$$\coprod_{y \in \mathcal{X}(L_1, L_2)} \mathcal{M}_2(x_0; x_1, y) \times \overline{\mathcal{M}}(y; x_2) \tag{2.30}$$

$$\coprod_{y \in \mathcal{X}(L_0, L_2)} \overline{\mathcal{M}}(x_0; y) \times \mathcal{M}_2(y; x_1, x_2) \tag{2.31}$$

The above three cases correspond to the breaking of a holomorphic strip at each infinite end of the the punctured disc. Note since we are working over exact Lagrangians, so there is no bubble.

Lemma 2.12. *For a fixed pair (x_0, x_1), the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is empty for all but finitely many x_0, and for a generic family of almost complex structures J_S and Hamiltonians H_S, $\overline{\mathcal{M}}_2(x_0; x_1, x_2)$ is a compact manifold of dimension $\mu(x_0) - \mu(x_1) - \mu(x_2)$, the boundary of which is the union of the codimention 1 strata listed in equations (2.29)-(2.31)*

Standard Sard-Smale arguments give transversality as shown in Chapter 3 of ([16]), compactness of moduli spaces is given by similar arguments of lemma 2.8, and is explicitly explained in Lemma 3.2 of [2]. Whenever the dimension of the moduli space $\mathcal{M}_2(x_0 x_1, x_2)$ is 0, it consists of finitely many elements, each of which defines an isomorphism

$$o_{x_1} \otimes o_{x_2} \to o_{\psi^2(x_0)} \tag{2.32}$$

Also denote this by the notation μ_u for the map induced on orientation lines we cna thus define a product on the wrapped Floer complex as follows

$$\begin{aligned} \mu_2^{\psi^2} &: \mathcal{CW}^*(L_2, L_1) \otimes \mathcal{CW}^*(L_1, L_0) \to \mathcal{CW}^*(\psi^2(L_2), \psi^*(L_0) \\ &\mu_2^{\psi^2}([x_1], [x_2]) = \sum_{\substack{|x_0| = |x_1| + |x_2| \\ u \in \mathcal{M}_2(x_0; x_1, x_2)}} (-1)^{\mu(x_1)} \mu_u([x_2], [x_1]). \end{aligned} \tag{2.33}$$

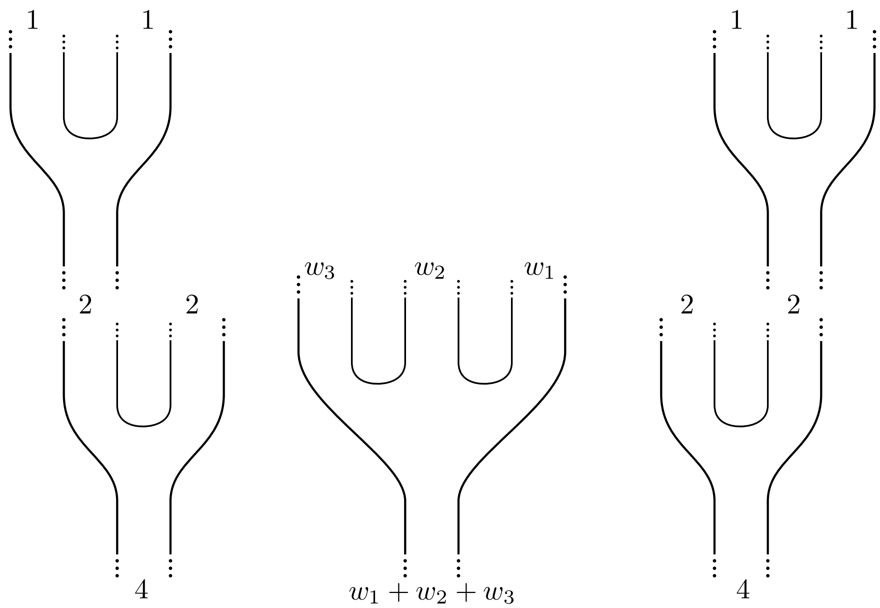

FIGURE 1.

2.5. A_∞ **structure.** We want to equip each infinite ends with some Floer data that is easy to define A_∞ structure and TQFT structure on the Wrapped Fukaya category. First, denote $\mathcal{R}_d$ by the moduli space of disks with d positive punctures $\{\xi^k\}_{k=1}^d$ on its boundary, ordered clockwise and a negative puncture ξ^0. Denote $\overline{\mathcal{R}}_d$ by the Deligne-Mumford compactification of this moduli space; and as in (9) of [17], fix a *universal and consistent* choice of strip like ends. Namely, for each surface S and eadch puncture, we have a map

$$\epsilon^k : Z_\pm \to S$$

whose source is Z_- if it is the negative puncture, and Z_+ otherwise, and that such a choice varies smoothly in the interior of $\overline{\mathcal{R}}_d$. Each boundary stratum σ of $\overline{\mathcal{R}}_d$ has the following form:

$$\sigma = \mathcal{R}_{d_1} \times \mathcal{R}_{d_2} \times \cdots \times \mathcal{R}_{d_j}$$

a neighborhood of σ looks like gluing of strip -like ends chosen on these lower dimensional moduli spaces:

$$\sigma \times [1, \infty)^{j-1} \to \mathcal{R}_d$$

Let us assign to each end ξ^i of a punctured disc a weight w_i, which is a positive real number, with the convention that any $\bar{\partial}$ operator on such a disc must pull back under ϵ^i to the Cauchy Riemann equation (2.8) up to applying ψ^{w_i}. In the product case where there are two incoming ends and one outgoing end, the inputs have weights 1 while the output has weight 2.

If we have three inputs and one output, the moduli space of all such possible configurations $\overline{\mathcal{R}}_3$ is actually an interval whose two ends are represents by nodal discs obtained by gluing the two possible different ways as shown in Figure **??**, two discs with two incoming ends of weight 1 and the other with weight 2. Now let us recall from the previous section (2.4) where we defined the product (2.19) to be the composition of (2.18) with rescaling. Instead of compositing with rescaling first, let us pull back by ψ^2 of al the data used to define $\mathcal{M}(x_0; x_1, x_2)$, and thus obtain a new $\bar{\partial}$ operator where the weights are now 2 at the incoming ends and 4 at the outgoing end.Then we might try to compose $\mathcal{CW}^*(\psi^2(L_3), \psi^2(L_2))$

with $\mathcal{CW}^*(\psi^2(L_2), \psi^2(L_0)$ after applying 2.18 with L_0, L_1, L_2. thus we may try to define the product with three inputs directly:

$$\mu_3 : \mathcal{CW}^*(L_3, L_2) \otimes \mathcal{CW}^*(L_2, L_1) \otimes \mathcal{CW}^*(L_1, L_0) \to \mathcal{CW}^*(L_3, L_0) \tag{2.34}$$

However, we immediately encounter the problem of ordering, namely when we multiply the first two terms then with the third, it is different from the result of multiplying the last two terms first on the chain level on the right hand side.Namely μ_2 is not associative at the chain level.

Yet these two give the same element on the cohomology level by looking at the following diagram as shown in [3].

(2.35)

$$\begin{tikzcd}
CW_b^*(L_2, L_3) \otimes CW_b^*(L_1, L_2) \otimes CW_b^*(L_0, L_1) \arrow[r, "\mu_2^{\psi^2} \otimes CW(\psi^2)"] \arrow[d, "CW(\psi^2) \otimes \mu_2^{\psi^2}"] & CW_b^*(\psi^2(L_1), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_1)) \arrow[d, "\mu_2^{\psi^4}"] \\
CW_b^*(\psi^2(L_2), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_2)) \arrow[r, "\mu_2^{\psi^4}"] & CW_b^*(\psi^4(L_0), \psi^4(L_3)). \arrow[d, "\cong"] \\
 & CW_b^*(L_0, L_3)
\end{tikzcd}$$

In order to show the existence of a homotopy of maps that links the former two orders of taking product, we need a moduli space of maps whose sources are disks with 4 punctures on its boundary and arbitrary conformal structure, take its compactification $\overline{\mathcal{R}}_3$ as in 1. Now whatever equation we define on $\overline{\mathcal{R}}_3$ has to restrict to equation (2.20) on each component of the boundary.In particular, the Cauchy-Riemann equation imposed on a disc whose conformal equivalence class is close to the boundary of $\overline{\mathcal{R}}_3$ is obtained by gluing the $\bar{\partial}$ operators associated to equation (2.20) at the node.This only works if the restriction of these $\bar{\partial}$ operators restrict to the two strip-like ends at the node agree, which is not true yet, since even the Lagrangian boundary conditions do not yet agree. However, after applying the time log-2 Liouville flow ψ^2, they agree. Thus we can glue two solutions to equation (2.20) after applying ψ^2 to one of them.

We also notice that the weights on the inputs of the three punctured disc have to be allowed to change with its modulus, as they are given by $(1, 1, 2)$ at one of the endpoints of $\bar{\partial}_3$ and $(2, 1, 1)$at the other endpoint. We now define auxilary data of families of Hamiltonians and almost complex structures on M, and of 1-forms on elements of $\bar{\partial}_3$ which would then help define an operation

$$\mathcal{CW}^*(L_3, L_2) \otimes \mathcal{CW}^*(L_2, L_1) \otimes \mathcal{CW}^*(L_1, L_0) \to \mathcal{CW}^*(L_3, L_0) \tag{2.36}$$

if we were given the homotopy in diagram (2.35). We note that this would only work with product with three inputs; actually we need arbitrarily many inputs, thus we proceed to the following generaral construction originally from ([2]).

Definition 2.13. *A* <u>*Floer datum*</u> D_S *on a stable disc* $S \in \overline{\mathcal{R}}_d$ *consists of the following choices:*

(1) Weights: A positive integer $w_{k,S}$ *assigned to the kth end such that*

$$w_{0,S} = \sum_{1 \le k \le d} w_{k,S}$$

(2) *Moving conditions: A map $\rho_S : \partial\bar{S} \to [1, +\infty)$ which agrees with w_k near the kth end.*
(3) *Basic 1-form: A closed 1-form α_S whose restriction to the boundary vanishes and whose pullback under ϵ^k agrees with $w_{k,S}dt$.*
(4) *Hamiltonian perturbations: A map $H_S : S \to \mathcal{H}(M)$ which agrees with $\frac{H \circ \psi^{w_{k,S}}}{w_{k,S}^2}$ near the kth end.*
(5) *Almost complex structure: A map $J_S : S \to \mathcal{J}(M)$ whose pullback under ϵ^k agrees with $(\psi^{w_{k,S}})^* J_t$.*

If we write X_S for the Hamiltonian flow of H_S, then these data allow us to write down a Cauchy-Riemann equation

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{2.37}$$

where the $(0,1)$ part is taken with respect to J_S.

The long list of definition (2.13) guarantees the following:

Lemma 2.14. *The pullback of equation (2.37) under ϵ^k is*

$$(du \circ \epsilon^k - X_{\frac{H \circ \psi^{w_{k,s}}}{w_{k,s}}} \otimes dt)^{0,1} = 0 \tag{2.38}$$

It agrees with equation (2.8) up to conformal rescaling with $\psi^{w_{k,s}}$.

Maybe at a later session:Note that the reason we have gone through such pain to define the Floer data is due to the failure of associativity for high products.

At the boundary of the moduli space $\mathcal{R}_d$, we require that Floer data be given by the choices be performed on smaller dimensional moduli spaces. When $d = 3$, we already observed that the choice of Floer data at the boundary can't be given precisely by the Floer data for $\mathcal{R}_2$ on each component, since these Floer data can't be glued. We shall consider the following notion of equivalence among Floer data which is weaker than equality:

Definition 2.15. *We say that a pair $(\rho_S^1, \alpha_S^1, H_S^1, J_S^1)$ and $(\rho_S^2, \alpha_S^2, H^2, J_S^2)$ of Floer data on a surface S are conformally equivalent if there exists a constant C such that*

$$\begin{aligned} \rho_S^2 &= C\rho_S^1, \\ \alpha_S^2 &= C\alpha_S^1, \\ J_S^2 &= \psi^{C*} J_S^1, \\ H_S^2 &= \frac{H_S^1 \circ \psi^C}{C^2} \end{aligned}$$

The idea of resacling by ψ^2 in equation (2.15) gives an identification of moduli spaces generalizes as follows:

Lemma 2.16. *Composition with ψ^C defines a bijective correspondence between solutions to equation (2.20) for conformally equivalent Floer data.*

We may now state the desired compatibility between Floer data at the boundary of the moduli spaces $\overline{\mathcal{R}}_d$:

Definition 2.17. *A universal and conformally consistent choice of Floer data $\mathbf{D}_\mu$ for the A_∞ structure, is a choice of Floer data for every element of $\overline{\mathcal{R}}_d$ and every integer $d \geq 2$, which varies smoothly over the interior of the moduli space, whose restriction to a boundary stratum* *is conformally equivalent to the product of Floer data coming from lower dimensional moduli spaces, and which near such a boundary stratum agrees to infinite order, in the coordinates* (2.9)*, with the Floer data obtained by gluing.*

Note that conformal consistency fixes, up to a constant depending on the modulus, the choice of Floer data on $\partial\overline{\mathcal{R}}_d$ once Floer data on each irreducible component has been chosen. For example, the choice of Floer data for $d = 2$ above determines the Floer data on the boundary of $\overline{\mathcal{R}}_3$ as used in the previous section, where we chose to use the original Floer data on the disc that does not contain the original end, and rescale it by ψ^2 on the other disc (see Figure 1). After introducing a perturbations of this Floer data which vanishes to infinite order at the boundary, we extend it from a neighbourhood of the boundary to the remainder of $\mathcal{R}_3$. One may proceed inductively to construct a universal datum $\mathbf{D}_\mu$ using the fact that the boundary of $\overline{\mathcal{R}}_d$ is covered by the images of codimension 1 inclusions

$$\overline{\mathcal{R}}_{d_1} \times \overline{\mathcal{R}}_{d-d_1+1} \to \partial\overline{\mathcal{R}}_d \tag{2.39}$$

and the contractibility of the space of Floer data on a given surface. The contractibility of this space also implies that we may extend any Floer data chosen on a given surface to universal data:

Lemma 2.18. *The restriction map from the space of universal and conformally consistent Floer data to the space of Floer data for a fixed surface S is surjective.*

Now we construct the A_∞ structure on the wrapped Fukaya Category which is given by higher products

$$\mathcal{CW}^*(L_d, L_{d-1}) \otimes \mathcal{CW}^*(L_{d-1}, L_{d-2}) \otimes \cdots \mathcal{CW}^*(L_1, L_0) \to \mathcal{CW}^*(L_d, L_0) \tag{2.40}$$

which cmoes from the count of solutions of equation ((2.20)).

Given a sequence of chords $\vec{x} = x_k \in \mathcal{X}(L_{k-1}, L_k)$ for $1 \leq k \leq d$ and $x_0 \in \mathcal{X}(L_0, L_d)$, we define $\mathcal{M}_d(x_0; \vec{x})$ to be the space of solutions to equation ((2.20)) whose source is an arbitrary element $S \in \mathcal{R}_d$ with marked points $(\xi^0, \cdots \xi^d)$ such that

$$\lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = \psi^{w_{k,S}} x_k \tag{2.41}$$

and with boundary conditions

$$u(z) \in \psi^{\rho_S(z)}(L_k) \text{ if } z \in \partial S \text{ lies between } \xi^k \text{ and } \xi^{k+1} \tag{2.42}$$

Note that lemma 2.14 shows that equation ((2.20)) restricts to equation (2.8) up to applying $\psi^{w_{k,S}}$.

If we don't take into account the strips breaking at the ends, the virtual codimension 1 strata of the Gromov bordification $\overline{\mathcal{M}}_d(x_0; \vec{x})$ lie over the codimension 1 strata of $\overline{\mathcal{R}}_d$. The consistency condition on $\mathbf{D}_\mu$ implies that whenever a disc breaks, each component is a solution to equation (2.20) for the Floer data $\mathbf{D}_\mu$ up to applying ψ^C for some constant C that depends on the modulus in $\overline{\mathcal{R}}_d$. Such rescaling identifies the solutions of rescaled equation with moduli space of the original equation, we conclude that for each integer k between, 0 and $d - d_2$ and chord $y \in \mathcal{X}(L_{k+1}, L_k)$, we get a natural inclusion:

$$\overline{\mathcal{M}}_{d_1}(x_0; \vec{x}^1) \times \overline{\mathcal{M}}_{d_2}(y; \vec{x}^2) \to \overline{\mathcal{M}}_d(x_0; \vec{x}) \tag{2.43}$$

where the sequences of inputs in the respective factors are given by $\vec{x}^2 = (x_{k+1}, \cdots x x_{k+d_2})$ and $\vec{x}^1 = (x_1, \cdots, x_k, y, x_{k+d_2+1}, \cdots, x_d)$.

Thus we apply the same arguments that prove Lemma (2.12), we have the following:

Lemma 2.19. *The moduli space $\overline{\mathcal{M}}_d(x_0;\vec{x}^2)$ are compact and are empty for all but finitely many x_0 once the inputs $\vec{x}$ are fixed. For a generic choice $\mathbf{D}_\mu$, they form manifolds of dimension*

$$|x_0| + d - 2 - \sum_{1\leq k\leq d} |x_k|$$

whose boundary is covered by the images of inclusions (2.43).

Whenever $|x_0| = x - d + \sum_{1\leq k\leq d}|x^k|$, there are therefore only finitely many elements of $\overline{\mathcal{M}}_d(x_0;\vec{x})$, every element $u \in \overline{\mathcal{M}}_d(x_0;\vec{x}) : S \to M$ induces an isomorphism

$$\mathrm{o}_{\psi^{w_{d,S}}x_d} \otimes \cdots \otimes \mathrm{o}_{\psi^{w_{1,S}}x_1} \to \mathrm{o}_{\psi^{w_{0,S}}x_0}. \tag{2.44}$$

Let us write μ_u for the induced map on orientation lines, omitting coposition iwth $\mathcal{CW}^*(\psi^{w_{k,S}})$ or its inverse from the notation, and define the dth higher product:

$$\mu_d : CW^*_b(L_{d-1}, L_d) \otimes \cdots \otimes CW^*_b(L_1, L_2) \otimes CW^*_b(L_0, L_1) \to CW^*_b(L_0, L_d) \tag{2.45}$$

Explicitly,

$$\mu_d([x_d], \ldots, [x_1]) = \sum_{\substack{|x_0| = 2-d+\sum_{1\leq k\leq d}|x_k| \\ u\in\mathcal{M}_d(x_0;\vec{x})}} (-1)^{\dagger}\mu_u([x_d], \ldots, [x_1]) \tag{2.46}$$

Where the sign is given by

$$\dagger = \sum_{k=1}^{d} k|x_k|. \tag{2.47}$$

This is just saying we count the elements of $\overline{\mathcal{M}}_d(x_0;\vec{x})$ with the right signs.

If we now consider the dimension 1 moduli spaces, Lemma 2.19 claims that their boundaries are the strata in equation (2.43) which are rigid and thus corresponds to the composition of operations μ_d. Taking into account of signs, we get the following proposition:

Proposition 2.20. *The operations μ_d define an A_∞ structure on the category $\mathcal{CW}^*(M)$. Explicitly, we have*

$$\sum_{\substack{d_1+d_2=d+1 \\ 0\leq k<d_1}} (-1)^{\maltese_1^k}\mu_{d_1}\left(x_d, \ldots, x_{k+d_2+1}, \mu_{d_2}(x_{k+d_2}, \ldots, x_{k+1}), x_k, \ldots, x_1\right) = 0 \tag{2.48}$$

where the sign is given by

$$\maltese_1^k = k + \sum_{1\leq j\leq k} |x_j|$$

3. the Pontryagin Category

Let $\mathcal{L}$ be any path connected topological space. Consider the topological category with objects the points of $\mathcal{L}$, and morphisms from L^0 to L^1 is given by the Moore path space

$$\Omega(L^0, L^1) \equiv \{\gamma : [0, R] \to \mathcal{L} | \gamma(0) = L^0, \gamma(1) = L^1\} \tag{3.1}$$

Where R is allowed to vary between 0 and ∞. The composition is given by concatenation:

$$\begin{aligned} \Omega(L^0, L') \times \Omega(L', L^1) &\to \Omega(L^0, L^1), \\ (\gamma_1, \gamma_2) \to \gamma_2 \cdot \gamma_1(l) &\equiv \begin{cases} \gamma_1(l) & \text{if } 0 \le l \le R_1 \\ \gamma_2(l - R_1) & \text{if } R_1 \le l \le R_1 + R_2. \end{cases} \end{aligned} \tag{3.2}$$

Where γ_i has domain $[0, R_i]$.

It is known that this formular defines an associative composition of paths. In order for this operation to induce the structure of a DGA on chains, we use cubical chains as in [3] instead of singular chains throughout this paper. From texts such as [13] we know that a map from a cube to a topological space is *degenerate* if it factors through the projection to a face. The underlying vector space for the cubical chain complex is:

$$C_i(X) = \frac{\mathbb{Z}[\mathrm{Map}([0,1]^i, X)]}{\mathbb{Z}[\text{degenerate maps}]}. \tag{3.3}$$

We write $\delta_{k,\epsilon}$ to denote the inclusion of the codimension -1 face where the kth coordinate is equal to$\epsilon \in \{0, 1\}$ constantly. The differential of $C_*(X)$ is defined as

$$\partial\sigma = \sum_{k=1}^{i} \sum_{\epsilon=0,1} \partial_{k,\epsilon}\sigma = \sum_{k=1}^{i} \sum_{\epsilon=0,1} (-1)^{k+\epsilon} \sigma \circ \partial_{k,\epsilon} \tag{3.4}$$

Note for cubical chains we can define products in a much easier way as the product of cubes is stil a cube, thus it is easy to to define the following map:

$$C_*(X) \times C_*(Y) \to C_*(X \times Y) \tag{3.5}$$

which may easily checked to be associative. Now we apply this to path space $\Omega_{\mathcal{L}}$, we obtain a DG category. Let

$$\mathcal{P}(\mathcal{L}) \tag{3.6}$$

denote the differential graded category with objects points of $\mathcal{L}$ and morphism spaces

$$\mathrm{Hom}_*(L^0, L^1) = C_*(\Omega(L^0, L^1)) \tag{3.7}$$

differentials and products are given by

$$\mu_1^P \sigma = \partial\sigma, \tag{3.8}$$

$$\mu_2^P(\sigma_2, \sigma_1) = (-1)^{|\sigma_1|} \sigma_2 \cdot \sigma_1. \tag{3.9}$$

$$\mu_d^P \cong 0 \text{for } d \ge 2 \tag{3.10}$$

And we have the following A_∞ relations:

$$\sum_{m,n} (-1)^{\maltese_n} \mu_{d-m+1}^P(\sigma_d, \cdots, \sigma_{n+m+1}, \mu_m^P(\sigma_{n+m}, \cdots \sigma_{n+1}), \sigma_n, \cdots, \sigma_1) = 0, \tag{3.11}$$

namely

$$\mu_1(\mu_2(\sigma_2, \sigma_1)) + \mu_2(\sigma_2, \mu_1(\sigma_1)) + (-1)^{|\sigma_1|+1} \mu_2(\mu_1(\sigma_2), \sigma_1) = 0.$$

Note the space $\mathcal{L}$ is path connected, thus every object is quasi-isomorphic to any other object. In particular, we may fully faithfully embed the A_∞ category with the single object L and morphism $C_{-*}(\Omega(L, L)$ to the Pontryagin category:

$$(L, C_{-*}(\Omega(L, L))) \to \mathcal{P}(\mathcal{L}). \tag{3.12}$$

Since the category $\mathcal{P}(\mathcal{L})$ is not necessarily triangulated, we enlarge it to its triangulated envolope (see section (3j) of [17]), which is the triangulated closure of the image of $\mathcal{P}(\mathcal{L})$ in its category of modules under the Yoneda embedding. We would like to study a practical model of twisted complexes(see (3l) of [17]). Here we follow the model of [3] or remark(3.26) of [17]. First enlarge $\mathcal{P}(\mathcal{L})$ by allowing shifts of all objects by any integer and define

$$\mathrm{Hom}_*(L^0[m_0], L^1(m_1)) \equiv \mathrm{Hom}_*(L^0, L_1)[m_1 - m_0]. \tag{3.13}$$

Differentials are given by the usual μ_1^P as defined above. As for μ_2^P, given a triple $(L^0[m_0], L^1[m_1], L^2[m_2])$, and morphisms $\sigma_i \in \mathrm{Hom}_* L^{i-1}, L^i$ we have

$$\mu_2^P(\sigma_2[m_2 - m_1], \sigma_1[m_1 - m_0]) = (-1)^{deg(\sigma_2)+1(m_1-m_0)}\mu_2^P(\sigma_2, \sigma_1)[m_2 - m_0] \tag{3.14}$$

Thus we define a twisted complex in the following sense:

Definition 3.1. *A twisted complex consists of the following data:*

(3.15) *A finite collection of objects $\{L^i\}_{i=1}^r$ and integers m_i, together with a collection of morphisms $\{\delta_{i,j}\}_{i<j}$ of degree 1 in $\mathrm{Hom}_*(L^i[m_i], L^j[m_j])$, such that*

$$\mu_1^P\delta_{i,j} + \sum_k \mu_2^P(\delta_{k,j}, \delta_{i,k}) = 0$$

Writing D for the matrix $(\delta_{i,j})$ and $T = (\oplus L^i[m_i], D))$ for such a twisted complex, note that the above equation can be rewritten as

$$\mu_1(D) + \mu_2(D, D) = 0 \tag{3.16}$$

Given two such complexes $T^1 = (\oplus L_1^i[m_i^1], D^1)$ and $T^2 = (\oplus L_2^i[m_i^2], D^2)$, we define the space of morphisms between them as

$$\mathrm{Hom}_*(T^1, T^2) = \bigoplus_{i_1,i_2} C_{-*}(\Omega)_{L_1^{i_1}, L_2^{i_2}}[m_{i_1}^1 - m_{i_2}^2]. \tag{3.17}$$

The differential of an element $S = \{\sigma_{i_1,i_2}\}$ in this space is given by an elementary matrix:

$$\sigma_{i_1,i_2} \mapsto \mu_1(\sigma_{i_1,i_2}) + \sum_{k<i_1} \mu_2(\sigma_{i_1,i_2}, \delta_{k,i_1}^1) + \sum_{k>i_2} \mu_2(\delta_{i_2,k}^2, \sigma_{i_1,i_2}), \tag{3.18}$$

or in terms of matrix multiplication as

$$\mu_1^{Tw(\mathcal{P})}S = \mu_1^P(S) + \mu_2^P(S, D^1) + \mu_2^P(D^2, S) \tag{3.19}$$

Composition is also defined in terms of matrix multiplication

$$\mu_2^{Tw(\mathcal{P})}(S_2, S_1) = \mu_2^P(S_2, S_1). \tag{3.20}$$

Bondal and Kapranov ([7])proved the following result:

Lemma 3.2. *Twisted complexes form a triangulated A_∞ category.*

Now let $\mathcal{L}ag_b$ denote the space of Lagrangians L with a relatively Pin structure with respect to the background class $b \in H^2(M)$. The first condition amounts to the vanishing of the second Stiefel-Whitney class $w_2(L) + b$. We know that the isomorphism classes of Pin structures on TL is a free torsor of $H^1(L; \mathbb{Z}/2)$ by examining the long exact sequence

$$H^1(L; \mathbb{Z}/2) \to H^1(L; Pin(n)) \to H^1(L, O(n)) \to H^2(L; \mathbb{Z}/2) \to \cdots \tag{3.21}$$

that arises from the exact sequence of classifying spaces:

$$B\mathbb{Z}/2 \to BPin(n) \to BO(n) \to B^2\mathbb{Z}/2 \to \cdots \tag{3.22}$$

Thus for each $L \in \mathcal{L}ag_b$, a given Pin structure $P^{\#}$ on TL and $\beta \in H^1(L;\mathbb{Z}/2)$, we may twist $P^{\#}$ by β as follows: take a line bundle ξ on L whose first Stiefel-Whitney class is β and the associated double cover $S(\xi)$. The Pin structure $P^{\#}$ is a Pin(n) principal bundle $P^{\#} \to L$ with an isomorphism $P^{\#} \times_{Pin(n)} \mathbb{R}^n \cong TL$, then take the product $S(\xi) \times_{\mathbb{Z}/2} P^{\#}$ where $\mathbb{Z}/2$ acts on $S(\xi)$ by ± 1 and on $P^{\#}$ by $\pm e$ which is the kernel of the double cover $Pin(n) \to O(n)$. This gives a new Pin(n)-structure on TL.

Also, recall a brane structure on L (Definition 2.3) consists of a grading $\alpha^{\#} : L \to \mathbb{R}$ and Pin structure $P^{\#}$, which are free torsors of $H^0(L,\mathbb{Z})$ and $H^1(L;\mathbb{Z}/2)$. Yet in our situation we don't require the vanishing of $2c_1(L)$, further more, we don't require that $w_2(L) = 0$, yet all the Lagrangians we consider has to satisfy $w_2(TL) = i_L^* b$ for some background class $b \in H^2(M,\mathbb{Z}/2)$. Thus we consider the space $\widetilde{\mathcal{L}ag}$ of Lagragians equipped with a specific Pin structure $(L, P^{\#})$ with a fixed background class b (we may omit this in later section and just denote the space $\widetilde{\mathcal{L}ag}$ when it is clear from the context), there is a fibration:

$$\begin{tikzcd} H^1(L;\mathbb{Z}/2) \arrow[r] & \widetilde{\mathcal{L}ag_b} \arrow[d, "\pi"] \\ & \mathcal{L}ag_b \end{tikzcd}$$

Note that for Lagrangians of different first cohomology groups, this fibration gives different fibers, thus in practice we only look at a component of any given L such that the fibration π is surjective with the same fiber $H^1(L;\mathbb{Z}/2)$.

We can now substitute $\mathcal{L}$ earlier in the section by $\widetilde{\mathcal{L}ag_b}$ get $\mathcal{P}(\widetilde{\mathcal{L}ag_b})$, from which we would like to construct a functor to the Wrapped Fukaya category of the ambient Symplectic manifold as the next section shows.

4. Construction of the functor

We would like to prove theorem (1.1)which generalizes the functors in [1] and [3]to the based loop space of Lagrangian embeddings with Legendrian boundary:

On the level of objects,the functor sends every graded Lagrangian with Pin structure $\mathcal{L} = (L, P^{\#})$ to itself; on the level of homomorphisms, we have:

$$\begin{aligned} Mor(\mathcal{P}(\widetilde{\mathcal{L}ag_b})) &\xrightarrow{\mathcal{F}} \mathcal{CW}(M) \\ C_*(\mathcal{L}_1, \mathcal{L}_0) &\mapsto \mathcal{CW}^*(\mathcal{L}_1, \mathcal{L}_0; H_{0,1}, J_{0,1}) \end{aligned}$$

For a generic choice of Floer data $(H_{0,1}, J_{0,1})$ that makes the moduli space of solutions regular. Note here that we have suppressed the notation of $(L, , P^{\#})$ of Pin structures on the notation of the wrapped floer complex when it is clear.

4.1. **Chain map.** Before proving Theorem (1.1), let us consider the following set up:

Consider the upper half plane $\mathbb{H}$ with 2 marked points z_1, z_2 on the boundary, which is biholomorphic to $U = \mathbb{D}^2 \setminus \{1 \text{ point on } \partial D^2\}$, let ξ denote the punctured point. The automorphism of $\mathbb{H}$ that preserves the point at infinity is the intersection of $PSL(2;\mathbb{R})$ and

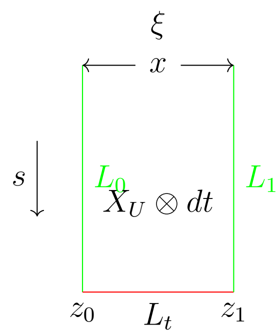

FIGURE 2. upper half plane

the affine transformations Aff $(\mathbb{R}^2)$, namely $z \mapsto az + b$ where $a \in \mathbb{R}_{>0}, b \in \mathbb{R}$. Thus after automorphism, we may assume $z_1 = (0,0), z_2 = (1,0)$.

We would like to consider a map:

$$H : U \to \mathcal{H}(M)$$

satisfying quadratic growth at infinity as in 6.4.Let X_U denote the Hamiltonian flow of H, consider the following Cauchy-Riemann equation:

$$(du - X_U \otimes dt)^{0,1} = 0 \tag{4.1}$$

We fix a positive strip-like end ϵ near ξ, and choose a map:

$$J_U : U \to \mathcal{J}(M)$$

which agress with J_t near ξ the infinite end $-\infty \times [0,1]$. Now given a σ representing an element in $C_{n-k}(\Omega_L \mathcal{L}ag)$, namely $\sigma : [0,1]^{n-k} \to \Omega_L \mathcal{L}ag$, thus $\sigma(\tau)$ is a path of lagrangians $[0,R] \to \mathcal{P}_{L_0,L_1}\mathcal{L}ag$ between L_0, L_1 for every $\tau \in I^{n-k}$, let $|\sigma(\tau)|$ denote the length R of this path; also fix $x \in \mathcal{X}(\mathcal{L}_0, \mathcal{L}_1)$.

Definition 4.1 (Moduli space of upper half plane)**.** *We define the moduli space of perturbed J-holomorphic maps with moving Lagrangian boundary $\mathcal{M}(x, \sigma, H, J)$ to be*

$$\left\{ \begin{array}{l} u : U \to \hat{M}, \\ (du - X_U \otimes dt)^{0,1} = 0, \\ u(z) \in L_0 \textit{ if } z \in \partial U \textit{ lies on the segement between } \xi \textit{ and } z_1 \ , \\ u(z) \in L_1 \textit{ if } z \in \partial U \textit{ lies on the segement between } \xi \textit{ and } z_2, \\ u(z) \in L_{\sigma(\tau)(\frac{t}{|\sigma(\tau)|})} \textit{ if } z = (t,0) \in \partial T \textit{ lies between } z_1 \textit{ and } z_2, \textit{ and } \tau \textit{ is some element in } I^{n-k} \\ lim_{s \to -\infty} u(s, \cdot) = x(\cdot), \textit{(asymptotic condition)} \end{array} \right\} \tag{4.2}$$

Actually, for generic almost complex structure $J \in \mathcal{J}$(the space of almost complex structures on $\hat{M}$), and Morse-Smale function H on $\hat{M}$, $\mathcal{M}(x, \sigma, H, J)$ is a smooth manifold of dimension $\mu(x) + |\sigma|$,where $|\sigma|$ denote the dimension of the cubical chain; notice from the standard transversality results in Floer theory we realize that the dimension of this moduli space is independent of H, J, thus we can omit it and write $\mathcal{M}(x, \sigma)$ when there is no confusion.

Fixing a choice of H, J, we would like to study the boundary of the Gromov compactification of $\mathcal{M}(x, \sigma, H, J)$. since U is topologically a 1-punctured disc with two boundary marked points, there is no non-trivial modulus, thus the only strata that we need to add to the compactification are the breakings of strip at ξ and the boundary chains of σ:

$$\coprod_{y \in \mathcal{X}(L_0, L_1)} \mathcal{M}(y; \sigma) \times \mathcal{M}(x; y) \tag{4.3}$$

$$\coprod_{\sigma' \subseteq \partial\sigma} \mathcal{M}(x; \sigma') \tag{4.4}$$

The standard compactness result implies that

Proposition 4.2. *The Gromov compactification $\bar{\mathcal{M}}(x; \sigma)$ is a compact manifold whose boundary is covered by the images of the strata (4.3)- (4.4).*

4.2. **The linear term.** Consider the following evaluation map:

$$ev : \overline{\mathcal{M}}(x; \sigma) \to \mathcal{X}(L_0, L_1) \tag{4.5}$$

which takes every punctured disc u to its infinite end x, a Hamiltonian chord between L_0 and L_1. If we consider the boundary strata (4.3), we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(y; \sigma) \times \mathcal{M}(x; y) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(x; \sigma) \arrow[d, "ev"] \\
\mathcal{M}(x; y) \arrow[r] & \mathcal{X}(L_0, L_1)
\end{tikzcd}$$

Where the first left vertical arrow is projection to the second factor and the top map is the inclusion of a boundary stratum.

Meanwhile, if we consider the boundary strata (4.4), we get the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(x; \sigma) \arrow[r] \arrow[dr] & \overline{\mathcal{M}}(x; \sigma) \arrow[d] \\
& \mathcal{X}(L_0, L_1)
\end{tikzcd}$$

We note that the following lemma implies theorem (1.1).

Lemma 4.3. *There exist fundamental cycles $[\overline{\mathcal{M}}(x; \sigma)] \in C_*(\overline{\mathcal{M}}(x; \sigma))$ such that the assignment*

$$\mathcal{F}^1(\sigma) = \oplus_x (-1)^\dagger ev_*([\overline{\mathcal{M}}(x; \sigma)]) \tag{4.6}$$

defines the chain map in 1.1.

Consider moduli spaces $\overline{\mathcal{M}}(x; \sigma)$ whose boundary only has codimension 1 strata, which must be products of closed manifolds. By taking the product of the fundamental chains of the factors in each boundary stratum, we obtain a chain in $\mathcal{CW}^*(L_0, L_1)$:

$$(-1)^{??} \sum_{x' \in \mathcal{X}(L_0,L_1)} [\mathcal{M}(x';\sigma)] \times [\mathcal{M}(x,x')] \cdot o_x \tag{4.7}$$

$$+ (-1)^{??} \sum_{\sigma' \subseteq \partial\sigma} [\mathcal{M}(x;\sigma')] \cdot o_x \tag{4.8}$$

We now define the fundamental chain

$$[\overline{\mathcal{M}}(x;\sigma)]$$

to be any chain whose boundary is the union of chains represented by equation (4.7),(4.8).

Proof of lemma 4.3. Consider $\sigma : I^k \to \Omega_{\mathcal{L}_0,\mathcal{L}_1}\widetilde{\mathcal{L}ag}$, fix $x \in \mathcal{CW}^{-k}(\mathcal{L}_1, \mathcal{L}_0; H, J)$ for some Hamiltonian H that is linear with respect to the radial coordinate p at cylindrical ends of M outside a compact set with slope c at infinity. Then $\mathcal{F}^1$ is the following map:

$$C_k(\Omega_{\mathcal{L}_1,\mathcal{L}_0}\widetilde{\mathcal{L}ag}) \to \mathcal{CW}^{-k}(\mathcal{L}_1, \mathcal{L}_0; H, J)$$
$$\sigma \mapsto \sum_{\dim \mathcal{M}(x,\sigma)=0} \#\mathcal{M}(x;\sigma) \cdot x$$

We would like to show this map preserves the differential structure, namely it is a chain map.

First look at $\mu^1_{\mathcal{CW}}$ on $\mathcal{CW}^*(\mathcal{L}_1, \mathcal{L}_0; H, J)$, which is given by $\mu^1(x) = \sum_{\dim \tilde{\mathcal{M}}(y,x)=1} \#\overline{\mathcal{M}}(y,x) \cdot y$. Here $\tilde{\mathcal{M}}(y,x)$ is the unparametrized moduli space of solutions of equation (2.8) with asymptotic ends at x, y. While $\mathcal{M}(y,x) = \tilde{\mathcal{M}}(y,x)/\mathbb{R}$ is rigid, thus a finite number of points.Hence so is $\overline{\mathcal{M}}(y,x)$.

$$\begin{aligned}\mu^1_{\mathcal{CW}} \circ \mathcal{F}^1(\sigma) - \mathcal{F}^1 \circ \mu^P_1(\sigma) =& \Big(\sum_{\substack{\dim \mathcal{M}(x,\sigma)=0. \\ \dim \mathcal{M}(y,x)=0}} \#\mathcal{M}(x,\sigma) \cdot \#\mathcal{M}(y,x) - \sum_{\dim \mathcal{M}(y,\mu^P_1\sigma)=0} \#\mathcal{M}(y,\mu^P_1\sigma)\Big) \cdot y \\ =& \partial\overline{\mathcal{M}}(y,\sigma) \cdot y\end{aligned}$$

Since $\dim \mathcal{M}(y,\sigma) = |y| + |\sigma| = |x| + 1 + |\sigma| = 1$, further more, this moduli space is a smooth and compact manifold with boundary for generic choice of Floer data, thus counting boundary with signs gives $\partial\mathcal{M}(y,\sigma) = \varnothing$. Thus $\mathcal{F}^1$ satisfies the first A_∞ relationship, namely

$$\mu^1_{\mathcal{CW}}(\mathcal{F}^1(\sigma)) = \mathcal{F}^1(\mu^P_1(\sigma)) \tag{4.9}$$

□

4.3. **Homotopy between compositions.** Having constructed the chain map $\mathcal{F}^1$, we would like to check it preserves the product structure, on the source it is given by the Chas-Sullivan product on the based loop space, namely the concatenation of paths; on the target it is the pair of pants product. As is the failure of pair of pants product, the preservation of product structure induced by $\mathcal{F}^1$ is only observed at the level of homology. At the level of chains, there is a homotopy between the $\mathcal{F}^1(\mu^P_2(\sigma_1, \sigma_2)))$ and $\mu^2_{\mathcal{CW}}(\mathcal{F}^1(\sigma_1), \mathcal{F}^1(\sigma_2)$;these two compositions are represented by the two outermost diagram in Figure (4.2). We shall therefore need to introduce a one-dimensional moduli space $\mathcal{Z}^{l_1,l_2}_4$ for each pair of positive l_1, l_2 representing the length of the path in $\mathcal{L}ag$, and whose boundary is represented by the broken curve and collapsing of two adjacent marked points as shown in Figure (4.2). As in the proof of the

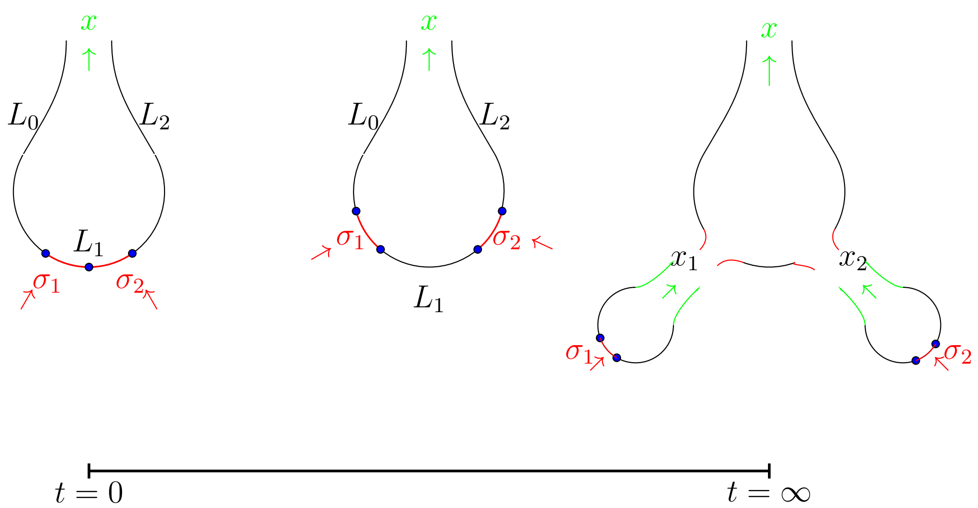

FIGURE 3. boundary strata of one-punctured discs with 4 boundary marked points

homotopy associativity of the product in Fukaya category, we shall define a family of Cauchy-Riemann equations on this abstract moduli space, interpolating between the equations on the two boundary figures, and the moduli space of solutions to this equation, with right boundary conditions, shall define the desired homotopy. We would like to describe the construction in the general case for all homotopies for an A_∞ functor.

4.4. **Abstract moduli spaces of upper half planes.** Given a tuple of positive values $l_1, \cdots, l_d$ write $\mathcal{Z}_{2d}^{l_1,l_2,\cdots,l_d}$ for the moduli space of homolorphic upper half planes with $2d$ consecutive boundary marked points $z_1, \cdots, z_{2d}$, all of which are incoming points, and that the distance $\tilde{l_i} = |z_{2i+1} - z_{2i}|$ satisfy $\frac{\tilde{l_i}}{\tilde{l_j}} = \frac{l_i}{l_j}$. This is bi-holomorphic to a disc with a puncture and $2d$ marked points on the boundary with fixed ratios. In addition, we fix an orientation on the moduli space $\mathcal{Z}_{2d}^{l_1,l_2,\cdots,l_d}$ using the conventions for Stasheff polyhedra ([17]) and the isomorphism:

$$\mathcal{Z}_{2d}^{l_1,l_2,\cdots,l_d} \cong \mathcal{R}_{d+1} \subseteq \mathcal{R}_{2d} \tag{4.10}$$

taking the odd number indexed incoming points as well as z_2 on the source to the first d incoming marked point on the target (actually we may take any z_{2k} $\forall k \in \{1, \cdots, d\}$ instead of z_2 as the $(d+1)$th marked point); and the outgoing marked point for $\mathcal{R}_{d+1}$ is the puncture at $\infty \in \bar{\mathbb{H}}$. we would like to quotient out the automorphism of $\mathbb{H}$ which fixes the ratio of different lengths $\frac{l_i}{l_j}$. This automorphism group $Aut_{2d}^{\vec{l}}$ turns out to be $PSL(2;\mathbb{R}) \cap \mathrm{Aff}(2;\mathbb{R})$, thus we have the following equivalence relation on $\mathbb{H}$:

$$z \simeq z' \iff z' = az + b \text{ for some } a \in \mathbb{R}_+, b \in \mathbb{R}, z \in \mathbb{H}$$

Therefore the dimension of the moduli space $\mathcal{Z}_{2d}^{l_1,\cdots,l_d}$ is:

$$2d - (d-1) - 2 = d - 1 \tag{4.11}$$

the first term $2d$ is the sum of $2d$ marked points on $\partial\mathbb{H}$; the term $(d-1)$ is due to the fact that as soon as we fix z_2, then the ratio $l_1 : \cdots : l_d$ would fix the remaining $d-1$ marked points $z_4, \cdots, z_{2d}$; the last term 2 is to mod out the automorphism group $Aut_{2d}^{\vec{l}}$.

Yet the compactification $\overline{\mathcal{Z}}_{2d}^{l_1,\cdots,l_d}$ of $\mathcal{Z}_{2d}^{l_1,\cdots,l_d}$ inside $\overline{\mathcal{R}}_{2d}$ is not quite the same as that of $\overline{\mathcal{R}}_{d+1}$ as we shall see below:

If two consecutive marked points z_{2k}, z_{2k+1} come together, as in the picture on the right hand side of figure (4.6), the topological type is determined by sequences $z_1, \cdots, z_{2k}, z_{2k+1}, \cdots, z_d$,thus we obtained a map:

$$\mathcal{Z}_{2d-1}^{l_1,\cdots,l_k+l_{k+1},\cdots,l_d} \to \overline{\mathcal{Z}}_{2d}^{l_1,\cdots,l_d} \tag{4.12}$$

where we regard the original two markings z_{2k}, z_{2k+1} as a single marked point z'_{2k}. From now on, if there is no confusion, we shall use $\mathcal{Z}_{2d}$ or $\mathcal{Z}_{2d}^{\vec{l}}$ to abbreviate for $\mathcal{Z}_{2d}^{l_1,\cdots,l_d}$. If two consecutive marked points indexed by z_{2k-1}, z_{2k} come close together, then by the property of *fixed ratio* of $z_{2i} - z_{2i-1}$ and $z_{2k} - z_{2k-1}$ $\forall i, k$, all two consecutive markings with such configurations have to come close; thus there are multiple bubbles with gluing parameters such that the ratio is still fixed. This phenomenon corresponds to the left hand side of Figure (4.6). Note the topological type is determined by sequences $\{1, \cdots, s_1\}, \cdots, \{s_1 + \cdots + s_{r-1} + 1, \cdots, d\}$, a partition of d into r sets of consecutive integers. We get a map:

$$\mathcal{Z}_{2s_1} \times \cdots \mathcal{Z}_{2s_r} \times \mathcal{R}_r \to \overline{\mathcal{Z}}_{2d+1} \tag{4.13}$$

Remark 4.4. Note the dimension of of the left hand side of 4.13 is

$$\sum_{i=1}^{r}(s_i - 1) + (r + 1 - 3) = (\sum s_i) - 2 = d - 2$$

which is 1 less than that of $\mathcal{Z}_{2d}^{\vec{l}}$. Here the second term $(r + 1 - 3)$ is the dimension of the moduli space of abstract discs $\mathcal{R}_r$ with r incoming marked points and a single outgoing marked point on the boundary.

We shall denote the unique disc representing $\mathcal{R}_r$ by level 1 disc, and call the other discs representing $\mathcal{Z}_{2s_i}$ level 2 discs.

4.5. **Floer data for half planes.** The following definition is a somewhat indirect extension of Definition (2.13) to the moduli space of upper half planes. Here we still have the outgoing end at z_0, but due to Gromov compactness conditions for moving boundary Lagrangian conditions, we don't have incoming ends on the marked points $z_1, \cdots, z_{2d}$ any more:

Definition 4.5. *A Floer datum D_U on a stable upper half plane $U \in \mathcal{Z}_{2d}^{\vec{l}}$ consists of the following choices on each component:*

(1) Weights: A positive real number $w_{k,U}$, where $k \in \{0, \cdots, r\}$ associated to each end for the 2nd type of of boundary strata (4.13) where the principle holormophic disc is broken into a level 1 disc with r incoming ends and a outgoing end and r level 2 discs each equipped with a single outgoing ends, namely $\mathcal{Z}_{2s_1} \times \cdots \mathcal{Z}_{2s_r} \times \mathcal{R}_r \subseteq \overline{\mathcal{Z}}_{2d+1}$ w_i satisfy $1 = w_0 \geq \sum_1^r w_{k,U}$.

For U inside the open strata or first type of boundary strata where there is no breaking of holomorphic discs (4.12), the weight is just 1.

(2) time shifting maps:For the curve breaking boundary (4.13), A map $\rho_U : \partial\bar{U} \to [1, +\infty)$ which agrees with $w_{k,U}$ near ξ^k where ξ^k denote the infinite ends and equals 1 for open strata and first type of boundary strata (4.12).

(3) Hamiltonian perturbation: A map $H_U : U \to \mathcal{H}(M)$ on each surface such that the restriction of H_U to a neighbourhood of each segment between z_{2i+1} and z_{2i+2} for

$1 \leq i \leq d$ takes value in $\mathcal{H}_Q(M)$(namely those functions that vanish near Q and are quadratic outside some compact neighborhood of Q.) and whose value near ξ^k for $0 \leq k \leq r$ is :

$$\frac{H \circ \psi^{w_{k,U}}}{w_{k,U}^2} \tag{4.14}$$

for curve breaking boundary (4.13) and equals H near ξ^0 for open strata and first type boundary strata (4.12).

(4) *Basic one-form: A subclosed one-form α_U whose restriction to the complement of a neighborhood of the intervals between z_{2i-1} and $z_{2i}(1 \leq i \leq d)$ vanishes; and whose pullback under ϵ^k for $0 \leq k \leq r$ agrees with $w_{k,U} d\tau$.*

(5) *Almost complex structure: A map $J_U : U \to \mathcal{J}(M)$ whose pullback under ϵ^k agrees with $\psi^{w_{k,U}} d\tau$.*

As before we write X_U for the Hamiltonian flow of H_U and consider the differential equation (4.1):

$$(du - X_U \otimes \alpha)^{0,1} = 0$$

with respect tot he U-dependent almost complex structure J_U. Note that the pullback of equation (4.1) under the ends ϵ^k agrees with

$$(du \circ \xi^k - X_{\frac{H \circ \psi^{w_{k,U}}}{w_{k,U}}} \otimes d\tau)^{0,1} = 0 \tag{4.15}$$

In order for the count of solutions to equation (4.1) to define the desired homotopies, we must choose its restriction to the boundary strata of the moduli spaces in a compatible way. In the case where $d = 2$, the moduli space is an interval, whose two endpoints may be indentified with the product $\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{R}_2$ and $\mathcal{Z}_3$ (see figure 4.2).

Our discusion in section (2) and (4.1)fixed the Floer data, respecitvely, on the unique elements of $\mathcal{R}_2$ and $\mathcal{Z}_2$. In the case of $\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{R}_2$ the equations on the outcoming ends of the level 2 discs and the incoming ends of the level 1 discs agree up to some rescaling. Thus we obain Floer data in a neighborhood of $\partial\overline{\mathcal{Z}}_4$, which may be extended to the interior of the moduli space. havign fixed this choice, we proceed inductively for the rest of the moduli spaces:

Definition 4.6. *A universal and conformally consistent choice of Floer data for the homomorphism $\mathcal{F}$, is a choice $\mathbf{D}_{\mathcal{F}}$ of such Floer data for every integer $d \geq 1$ and every element of $\mathcal{Z}_{2d}$ which varies smoothly over this compactified moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from either $\mathbf{D}_\mu$ or lower dimensional moduli spaces $\overline{\mathcal{Z}}_{d_i}$ and which near such a boundary agrees to infinite order with the Floer data obtained by gluing.*

The consistency condition implies that each irreducible component of a curve representing a point in the stratum (4.13) carries the restriction of the data $\mathbf{D}_\mu$ if it comes from the factor $\overline{\mathcal{R}}_r$, and the restriction of the datum $\mathbf{D}_{\mathcal{F}}$ if it comes from $\mathcal{Z}_{d_i}$, up to conformal equivalence.

4.6. **moduli space of upper half planes.** In order to extend $\mathcal{F}^1$ to an A_∞ functor $\{\mathcal{F}^d\}$, we need to construct corresponding moduli spaces. First of all, let us briefly review the moduli space constructed for $\mathcal{F}^1$: Consider the upper half plane $\mathbb{H}$ with 2 marked points z_1, z_2 on the boundary $\mathbb{R}$, with length $l = z_2 - z_1$. Note that this is biholomorphic to a unit disc with 1 puncture at i and 2 marked points on the boundary where $\infty \in \bar{\mathbb{H}}$ is identified with the puncture i. For $1 \leq i \leq d$, $\sigma_i \in C_{k_i}(\Omega_{\mathcal{L}_{i-1},\mathcal{L}_i}\widetilde{\mathcal{L}ag})$, namely $\sigma_i : I^{k_i} \to \Omega_{\mathcal{L}_{i-1},\mathcal{L}_i}\widetilde{\mathcal{L}ag}$ a map of based spaces, we would like to associate a family of moduli space $\tilde{\mathcal{M}}_3(\sigma; z)$ of upper half planes with boundary marked points with certain boundary conditions (the exact definition in (4.1)) over I^{k_i} such that $\forall \tau \in I^{k_i}$, the fiber is $\mathcal{M}_3(\sigma(\tau); x)$, the space of 1-punctured disc with two boundary markings ξ_1, ξ_2 such that the distance between $z_1 = r(\xi_1)$ and $z_2 = r(\xi_2)$ is $|\sigma(\tau)|$, where $r : \mathbb{D}^2 \to \bar{\mathbb{H}}$ is the Riemann mapping that sends i to ∞, 0 to i and $-i$ to 0. i.e. we have the following fibration:

$$\begin{tikzcd} \tilde{\mathcal{M}}_3(\sigma(\tau); x) \arrow[r] & \tilde{\mathcal{M}}_3(\sigma; x) \arrow[d, "\pi"] \\ & I^{k_i} \end{tikzcd}$$

Similarly, for any integer $d \geq 1$, we have $\mathbb{H}$ with $2d$ boundary markings, biholomorphic to the unit disc with 1 puncture and $2d$ markings, such that lengths $l_i = z_{2i} - z_{2i-1}$ denote the length of the moving lagrangian boundary condition, let $t_i = z_{2i+1} - z_{2i}$ represent the lenghths of the boundary component of $\mathbb{H}$ that lies in L_i, namely constant boundary condition; then $\forall\ \tau_i \in I^{k_i}$ we get the following fibration:

$$\begin{tikzcd} \tilde{\mathcal{M}}_{2d+1}(\sigma_1(\tau_1), \cdots, \sigma_d(\tau_d); x) \arrow[r] & \tilde{\mathcal{M}}_{2d+1}(\sigma_1, \cdots, \sigma_d) \arrow[d, "\pi"] \\ & I^{k_1+\cdots+k_d} \end{tikzcd}$$

Here define the above notification, namely moduli space of solutions of the equations satisfying floer data as in section 4.6 of [3]

Where $\mathcal{M}_{2d+1}(\sigma_1(\tau_1), \cdots, \sigma_d(\tau_d); x)$ is defined to be the moduli space of solutions to equation (4.1) whose source is an arbitrary element $U \in \mathcal{Z}_{2d}$, with boundary conditions:

$$\begin{cases} u(z) \in \psi^{\rho(z)}(L_0) & \text{if } z \in \partial U \text{ lies on the segment between } \xi^0 \text{ and } z_0 \\ u(z) \in \psi^{\rho(z)}(L_d) & \text{if } z \in \partial U \text{ lies on the segment between } \xi^0 \text{ and } z_{2d} \\ u(z) \in L_i & \text{if } z \in \partial U \text{ lies between } z_{2i-1} \text{ and } L_{2i} \text{ for } 1 \leq i \leq d-1. \\ u(z) \in L_{\sigma_i(s_i)(t)} & \text{if } z \in \partial U \text{ lies between } z_{2i} \text{ and } z_{2i+1} \text{ for } 1 \leq i \leq d \end{cases}$$

and with asymptotic conditions

$$\lim_{s \to +\infty} u(\epsilon^0(s,t)) = x(t)$$

Note the above fibers $\tilde{\mathcal{M}}_{2d+1}^{\sigma_1(\tau_1),\cdots,\sigma_d(\tau_d)} \in (\partial D^2)^{2d+1}$ lie in the space of unparametrized moduli spaces of upper half planes with markings, we would like to quotient out the automorphism of $\mathbb{H}$ which fixes the ratio of different lengths $\frac{l_i}{l_j}$. This automorphism group turns out to be $PSL(2;\mathbb{R}) \cap \mathrm{Aff}(2;\mathbb{R})$, thus we have the following equivalence relation on $\mathbb{H}$:

$$z \simeq z' \iff z' = az + b \text{ for some } a \in \mathbb{R}_+, b \in \mathbb{R}$$

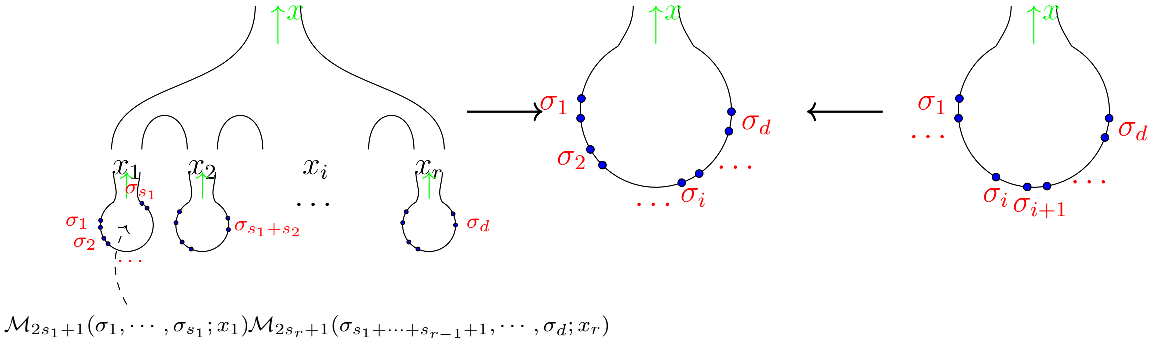

FIGURE 4. boundary strata of one-punctured discs with $2d$ boundary marked points

Writing $\vec{\sigma}(\tau)$ to denote $\sigma_1(\tau_1), \cdots, \sigma_d(\tau_d)$, we mod out the above automorphism group of each fiber to get the fibre-wise moduli space $\mathcal{R}_{2d+1}^{\vec{\sigma}(\tau)}$, which gives the following fibration:

$$\begin{array}{ccc} \mathcal{M}_{2d+1}(\vec{\sigma}(\tau); x) & \longrightarrow & \mathcal{M}_{2d+1}(\vec{\sigma}; x) \\ & & \downarrow \\ & & I^{\sum k_i} \end{array}$$

afterwards take fiberwise compactification to build a ~~smooth manifold with corners~~ stratified space [1] $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)$:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x) & \longrightarrow & \overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x) \\ & & \downarrow \\ & & I^{\sum_{i=1}^{d} k_i} \end{array}$$

prove that the boundary strata of the total space $\overline{\mathcal{M}}_{2d+1}$ consists of the union of the following pieces:

(1) $\overline{\mathcal{M}}_{2d}$, closure of strata representing upper half plane of $2d-1$ boundary marked points. Let $\sigma_{\mathfrak{i}}(\tau_{i,i+1})$ denote the concatenation of $\sigma_i(\tau_i), \sigma_{i+1}(\tau_{i+1})$, two paths in $\widetilde{\mathcal{L}ag}$. Then the fiber over such a strata would be $\overline{\mathcal{R}}_{2d}^{\sigma_1(\tau_1),\cdots,\sigma_{\mathfrak{i}}(\tau_{i,i+1}),\cdots,\sigma_d(\tau_d)}$. Note that $\overline{\mathcal{M}}_{2d}$ is a fibration over $I^{\sum d_i}$ with fiber $\overline{\mathcal{R}}_{2d}^{\sigma_1(\tau_1),\cdots,\sigma_{\mathfrak{i}}(\tau_{i,i+1}),\cdots,\sigma_d(\tau_d)}$. This corresponds to $t_i \to 0$.

(2)

$$\begin{aligned} \mathcal{M}_{d_1,\cdots,d_r}(\vec{\sigma}(\tau)) :=& \mathcal{M}_{2d_1+1}(\sigma_1(\tau_1), \cdots, \sigma_{d_1}(\tau_{d_1})) \\ & \times \cdots \times \mathcal{M}_{2d_r+1}(\sigma_{\sum_{i=1}^{r-1}+1}(\tau_{\sum_{i=1}^{r-1}+1}), \cdots, \sigma_d(\tau_d)) \times \mathcal{M}(y; x_1, \cdots, x_r). \end{aligned}$$

This corresponds to the multiple bubbling in equation (4.13) and left side of Figure (4.6).

last update here

[1]This is not necessarily a manifold with corners, as shown in dimension 3 case where there is a vertex that is adjacent to 4 edges, see example 6.3 on P14 of [14].

4.7. **proof of main theorem.** To prove $\mathcal{F}$ preserves A_∞ compositions, we extend $\mathcal{F}^1$ to a sequence of of multilinear maps of of all orders $d \geq 1$:

$$\mathcal{F}^d : C_*(\Omega_{\mathcal{L}_d,\mathcal{L}_{d-1}}\widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_*(\Omega_{\mathcal{L}_1,\mathcal{L}_0}\widetilde{\mathcal{L}ag}) \to \mathcal{CW}^*(\mathcal{L}_d, \mathcal{L}_0)[1-d]$$

To be more precise,

$$\mathcal{F}^d : C_{k_d}(\Omega_{\mathcal{L}_d,\mathcal{L}_{d-1}}\widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_{k_1}(\Omega_{\mathcal{L}_1,\mathcal{L}_0}\widetilde{\mathcal{L}ag}) \to \mathcal{CW}^{-|\vec{k}|-d+1}(\mathcal{L}_d, \mathcal{L}_0) \tag{4.16}$$

where $|\vec{k}| = \sum_{i=1}^d k_i$. $\{\mathcal{F}^d\}$ should satisfy the following equations:

$$\sum_r \sum_{s_1,\cdots,s_r} \mu^r_{\mathcal{CW}}(\mathcal{F}^{s_r}(\sigma_d, \cdots \sigma_{d-s_r+1}), \cdots, \mathcal{F}^{s_1}(\sigma_{s_1}, \cdots, \sigma_1)) \tag{4.17}$$

$$= \sum_{m,n} (-1)^{\maltese_n} \mathcal{F}^{d-m+1}(\sigma_d, \cdots, \sigma_{n+m+1}, \mu^P_m(\sigma_{n+m}, \sigma_n, \cdots, \sigma_{n+1}), \cdots, \sigma_1) \tag{4.18}$$

where the left hand side is the sum over all $r \geq 1$ and partitions $s_1 + \cdots + s_r = d$, and right hand sum is over all possible $1 \leq m \leq d, 0 \leq n \leq d-m$, and $\maltese_n$ is defined the same as in (2.20). Note in our definition of the Pontryagin category $\mathcal{P}(\widetilde{\mathcal{L}ag})$, the only nontrivial A_∞ operations are μ^P_1, μ^P_2. Thus the above equation (4.17)can be rewritten as

$$\sum_r \sum_{\vec{s}} \mu^r_{\mathcal{CW}}(\mathcal{F}^{s_r}(\cdots), \cdots, \mathcal{F}^{s_1}(\cdots)) \tag{4.19}$$

$$=(-1)^{\maltese_n} \mathcal{F}^d(\sigma_d, \cdots, \sigma_{n+2}, \mu^P_1(\sigma_{n+1}), \sigma_n, \cdots, \sigma_1) \tag{4.20}$$

$$+(-1)^{\maltese_n} \mathcal{F}^{d-1}(\sigma_d, \cdots, \sigma_{n+3}, \mu^P_2(\sigma_{n+2}, \sigma_{n+1}), \sigma_n, \cdots, \sigma_1) \tag{4.21}$$

Following the same strategy as the proof of Lemma (4.3), we may equip these moduli spaces with appropriate fundamental chains in the wrapped Fukaya category:

Lemma 4.7. *There exists a family of fundamental chains:*

$$[\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)] \in \mathcal{CW}^*(\overline{\mathcal{M}}_{2d}(\vec{\sigma}; x)) \tag{4.22}$$

such that

$$\partial[\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)] = \sum (-1)^{??}[\overline{\mathcal{M}}_{2d}(\sigma_d \cdots \sigma_{i\cup(i+1)}, \cdots, \sigma_1; x)]$$
$$+ \sum (-1)^{??}[\overline{\mathcal{M}}_{2d_r+1}(\sigma_d, \cdots, \sigma_{\sum_{i=1}^{r-1} d_i+1}; x_r)] \times \cdots \times [\overline{\mathcal{M}}_{2d_1+1}(\sigma_{d_1}, \cdots, \sigma_1; x_1)] \times [\overline{\mathcal{M}}_{r+1}(x_r, \cdots, x_1; x)]$$
$$+ \sum_{i=1}^d [\overline{\mathcal{M}}_{2d+1}(\sigma_d, \cdots, \partial\sigma_i, \cdots, \sigma_1; x)]$$

Definition 4.8. *We define $\mathcal{F}^d$ for higher homotopies as follows:*

$$\mathcal{F}^d : C_{k_d}(\Omega_{\mathcal{L}_d,\mathcal{L}_{d-1}}\widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_{k_1}(\Omega_{\mathcal{L}_1,\mathcal{L}_0}\widetilde{\mathcal{L}ag}) \to \mathcal{CW}^{-|\vec{k}|-d+1}(\mathcal{L}_d, \mathcal{L}_0) \tag{4.23}$$

$$\sigma_d \otimes \cdots \otimes \sigma_1 \mapsto \sum_{\dim(\mathcal{M}(\sigma_d,\cdots,\sigma_1;x))=0} \#\mathcal{M}(\vec{\sigma}; x) \cdot o_x. \tag{4.24}$$

proof of main theorem. From the description of the moduli spaces $\overline{\mathcal{Z}}_{2d}$ and its boundary strata from section (4.6) and the definition above; we realize that the this is quite similar to the proof of lemma (4.3). We just need to identify the boundaries of the moduli space (4.22) and that of the equations of the A_∞ relations:

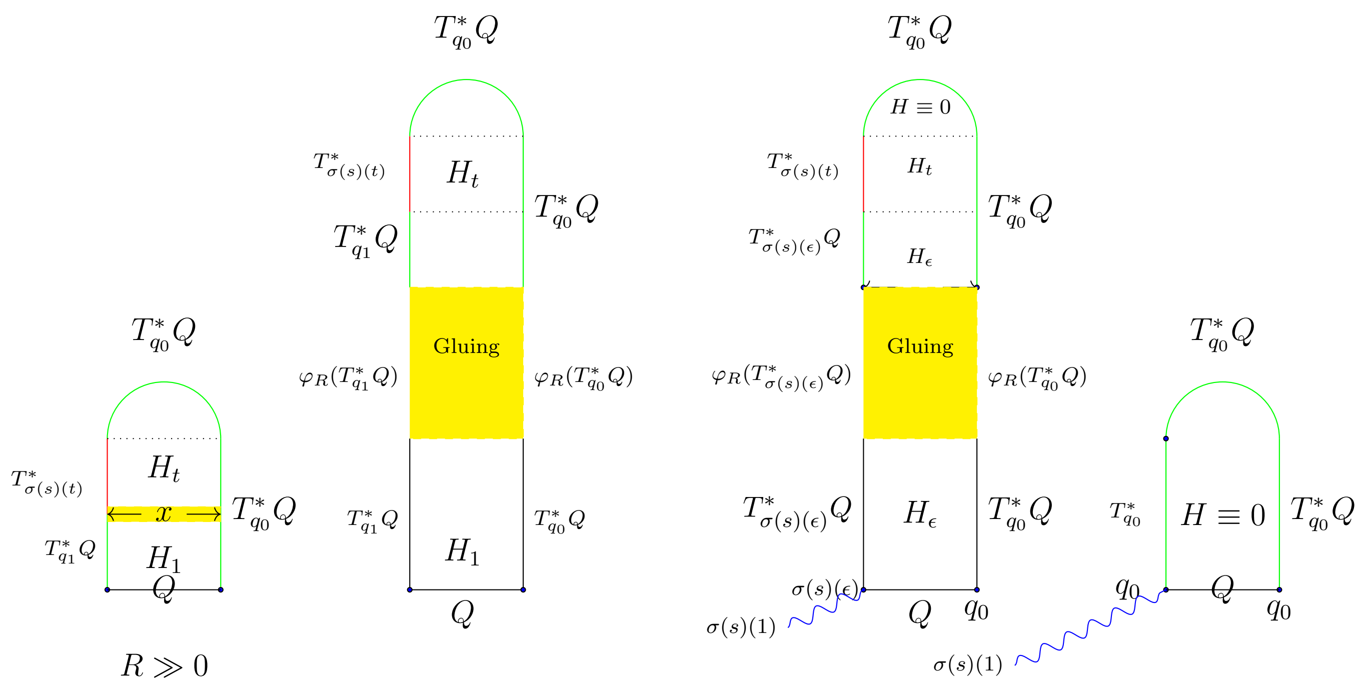

Figure 5. Homotopy between Id and $f \circ F \circ \eta$, where we glue two moduli spaces with moving Lagrangian boundaries for a path from $q_0 = \sigma(s)(0)$ to $\sigma(s)(\epsilon)$ for $0 \leq \epsilon \leq 1$ with gluing parameter $R \in [1, \infty)$. On the right end is the identity functor while on the left it is the composition $f \circ F \circ \eta$

(1) the Left hand side of equations (4.19) corresponds to the 2nd terms on the Right hand side of equation (4.22) (when $r \geq 2$) and the left hand side of equation (4.22) (when $r = 1$).
(2) the 2nd terms on the Right hand side of equation (4.19) corresponds to the Left hand side of equation (4.22).
(3) the first summands on the right hand side of equation (4.19) corresponds to the 3rd summands of equation (4.22).

□

5. Proof of Main Theorem

In this section we construct a chain homotopy equivalence between $Id_{C_{n-*}(\Omega_q Q)}$ and $f \circ F \circ \eta$ in the following commutative diagram:

$$\begin{tikzcd}
C_{n-*}(\Omega_q Q) \arrow[r, "\eta"] \arrow[dr, dotted] & C_{n-*}(\Omega_{T^*_q Q}\mathcal{L}ag) \arrow[d, "\mathcal{F}"] \\
 & \mathcal{CW}^*(L, L; H) \arrow[ul, bend left, "f"]
\end{tikzcd}$$

We consider the concatenation of paths on the zero section Q and evaluation of moduli spaces of pseudo-holomorphic discs with partial moving Lagrangian boundary condition defined below.

5.1. **Moduli spaces with partial moving Lagrangian boundary condition.** Given $\sigma : [0,1]^{n-k} \to \Omega(q_0, q_1)Q$, namely $[0,1]^{n-k} \times [0,1] \to Q$, a chain of paths on the compact manifold Q, we will define $\sigma_\epsilon : [0,1]^{n-k} \times [0,1] \to Q$ such that $\sigma_\epsilon(\tau)$ moves from q_0 to q_ϵ, where $q_\epsilon = \sigma(\tau)(\epsilon)$. We consider the moduli space of perturbed J-holomorphic maps with moving boundary $\mathcal{M}(\sigma_\epsilon, x, H, J)$ as defined in Definition 4.1. From [3] section 4 we also have a moduli space $\mathcal{H}(x; q_0, q_1)$ of J-holomorphic maps from upper half space with 2 punctures on the boundary to T^*Q such that the two punctures are mapped to $q_0, q_1 \in Q$ and boundary component of the upper half plane between the two punctures is mapped into Q. Evaluation map from the compactification $\bar{\mathcal{H}}(q_0, q_1, x) \to \Omega(q_0, q_1)$ gives the chain map $f^1 : \mathcal{CW}_b^k(T^*_{q_0}Q, T^*_{q_1}Q) \to C_{n-k}(\Omega_{q_0,q_1}Q)$.

Similar to the fibration in section (4.6), we have the following fibration:

$$\begin{array}{ccc} \mathcal{M}(\sigma_\epsilon(\tau); x) & \longrightarrow & \mathcal{M}(\sigma_\epsilon; x) \\ & & \downarrow \\ & & I^k \end{array}$$

Glue the two moduli spaces $\mathcal{H}(x, q_0, \sigma)$ with $\mathcal{M}(\sigma, x, H, J)$ with a gluing parameter $R \in [1, \infty)$. When $R \to \infty$, get a moduli space of J-holomorphic maps $u : D \backslash \{\xi^1, \xi^2\} \to T^*Q$ from a disc with two punctures on the boundary ξ^1, ξ^2, such that $\lim_{z \to \xi^1} u(z) = q_0, \lim_{z \to \xi^2} u(z) = q_1$, and that the boundary between ξ^1 and ξ^2 gets mapped to the zero section Q as in $\mathcal{H}(x; q_0, q_1)$. Perturb the gluing parameter R to finite value while $\varphi_R^*(T^*_{q_1}Q) \pitchfork$??? Afterwards for $\tau \in I^k$, Consider the Muduli space of partial Lagrangian moving boundary condition:

Definition 5.1.

The Floer data for $\mathcal{M}(\sigma_\epsilon, q_0)$ are similar to that of (4.1), namely weights, moving conditions, Hamiltonian perturbation, one-forms and almost complex structures. We are mostly concerned with the moving boundary conditions and Hamiltonian perturbation:

As is shown in the appendix, we use linear Hamiltonian flow to generate the moving Lagrangian cofibers. Given any $\tau \in I^{n-k}$, $\sigma(\tau)$ is a loop in Q: $I \to Q$. Consider the following function:

$$\psi : \mathbb{R} \to [0,1], \psi = \begin{cases} 1 & s \leq 0 \\ 1 - \exp(\frac{(1-s)^2}{1-(1-s)^2}) & s \in [0,1] \\ 0 & s \geq 1 \end{cases}$$

Notice $\psi|_I$ is a diffeomorphism from $[0,1]$ to itself and it is C^∞ at $s = 0$. Therefore, we may write σ as $\sigma' \circ \psi$ for some function $\sigma' : I^{n-k} \times \mathbb{R} \to Q$, namely $\sigma(\tau)(s) = \sigma'(\tau)(\psi(s))$ for any given $\tau \in I^{n-k}$.

Definition 5.2. *The boundary condition $\sigma_\epsilon : I^{n-k} \times \mathbb{R} \to [0,1]$ is given by:*

$$\sigma_\epsilon = \sigma' \circ \epsilon \cdot \psi$$

namely:

$$\sigma_\epsilon(\tau)(s) = \sigma'(\tau)(\epsilon \cdot \psi(s))$$

where ψ is defined as above.

The Hamiltonian function is as in the Appendix, namely quadratic in the distance to the zero section and large enough slope with respect to s as in (6.4), namely $H = -A \cdot |p|^2 \cdot f(s)$ for some smooth bump function that vanishes outside $[-1, 2]$ and is constant 1 inside $[0, 1]$.

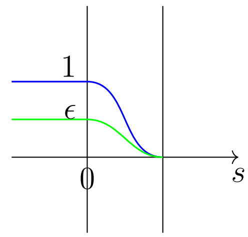

FIGURE 6. ψ and ψ_ϵ

Now for any given $\mathcal{M}(\sigma_\epsilon(\tau), q_0)$, take the evaluation map and get a path in $\Omega(q_0, \sigma_\epsilon(\tau))(Q)$, then intersect with the evaluation of moduli space of paths at the end point at time ϵ: $ev_\epsilon : \mathcal{P}(\sigma_\epsilon(\tau), q_1)$ to obtain a path from q_0 to q_1 inside Q. When we move ϵ from 0 to 1, we get the homotopy between $Id_{\mathcal{P}(Q)}$ and $f \circ \mathcal{F} \circ \eta$.

6. APPENDIX

We show the following C^0 estimate for moduli space of pseudo-holomorphic curves with moving Lagrangian boundary inside a Liouville manifold.

Theorem 6.1. *Given a Riemann surface Σ with punctures $\{\xi_1, \cdots \xi_k\}$ on the boundary, the moduli space of pseudo-holormorphic curves in a Liouville manifold $\widehat{M}$ from the domain Σ and moving Lagangian boundary conditions generated by the flow of a Hamiltonian with linear growth rate at infinity has a C^0 bound. Namely, there is a choice of Hamiltonian data such that all solutions of the inhomogeneous pseudo-holomorphic equation*

$$(du - X_H \otimes dt)^{0,1} = 0 \tag{6.1}$$

stay within a compact subset of $\hat{M}$.

In this section we denote by M a Liouville domain with contact boundary and $\widehat{M}$ its completion, namely $\widehat{M} = M \cup \partial M \times [0, \infty)$. Denote the contact form on $\partial M by \alpha$.

In order to show that our moduli space of $J-$holomorphic curves is compact, we need to prove Gromov compactness by restricting to a compact subset of $\widehat{M}$, which comes from a C^0 bound on $\rho = r \circ u$. Here u is the pseudo-holomorphic curve from $\mathbb{D}$ to $\widehat{M}$, where $\mathbb{D}$ is the unit disc. Denote this domain of u by Σ.

A C^0-bound is achieved if ρ is constraint within a compact neighborhood in $\widehat{M}$. Suppose the moving Lagrangian with Legendrian boundary $L_s = \Lambda_s \times (0, \infty)$ near ∂M, where Λ_s is a family of Legendrian submanifolds generated by a Hamiltonian H_0 parametrized by s on ∂M. In other words there exists a vector field X_{H_0} such that $i_{X_{H_0}} d\alpha = -dH_0$ and Λ_s is the time-s flow of X_{H_0}. A calculation similar to that of exercise 3.48 in [15] gives $X_{H_0} = H_0 \cdot Y + Z$ where Y is the Reeb vector field on ∂M and Z lies in the contract distribution $\kappa = \ker(\alpha)$ such that $i(Z)d\alpha|_\kappa = dH_0|_\kappa$. Now consider the linear Hamiltonian H^l on $Q \times (0, \infty)$, where $H^l(q, r) := r \cdot H_0(q)$. This gives rise to a vector field on the conical end $X_{H^l}(q, r) = (X(q), -r dH_0(Y)\partial_r)$.

For the sake of Hamiltonian that defines the wrapped Fukaya category, we consider a different Hamiltonian $H \in C^\infty(\hat{M}, \mathbb{R})$ parametrized by Σ such that:

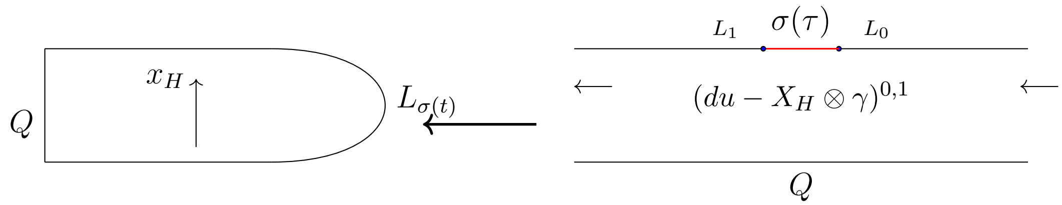

FIGURE 7. Maximum Principle for glued disc

$$H > 0 \text{ everywhere. Moreover, } H(r,q) = c(s) \cdot r^2 \text{ on the infinite cone for some smooth } c : \mathbb{R} \to \mathbb{R}_+. \tag{6.2}$$

Let X_H be the Hamiltonian vector field of H. On the infinite cone we have $X_H = 2c(s)r \cdot (0, Y)$, where Y is the Reeb vector field associated to $\theta|\partial M$ and the first coordinate corresponds to ∂_r component. By restricting, one sees that X_H satisfies $X|\partial M = c(1)Y$.

Suppose our domain is $\Sigma = \mathbb{R} \times [0,1]$ and on both infinite ends we have asymptote Hamiltonian chords x_0 and x_1 and that our equation satisfies the in-homogeneous Cauchy Riemann equation 2.8, which we write here for the reader's convenience:

$$\text{The map } u : \Sigma \to \hat{M} \text{ satisfies } (du - X_H \otimes dt)^{0,1} = 0, \text{ such that } u(s,1) \in K, u(s,0) \in L_0 \text{ when } s \gg 0,\ u(s,0) \in L_1 \text{ when } s \ll 0 \text{ and } u(s,t) \in L_s \text{ for } s \in [0,1]. \tag{6.3}$$

Namely we restrict the moving boundary Lagrangian condition to an interval on the lower boundary of Σ.

Given a perturbed J-holomorphic curve with asymptotic Hamiltonian chords at infinite strip like ends that satisfy (**??**), we want to give a C^0 estimate on $\rho = r \circ u$. Let us calculate the following:

$$\begin{aligned}
\partial_s \log(\rho) &= \partial_s(\log(r \circ u)) \\
&= \frac{dr}{r}(\partial_s u) = \frac{dr}{r} \circ (-J)(\partial_t u - X_H) \\
&= \frac{\theta}{r}(\partial_t u - 2c\rho X_\rho) = \alpha(\partial_t u) - 2c\rho\alpha(X_\rho) \\
&= \alpha(\partial_t u) - 2c(s)\rho
\end{aligned}$$

Where the third equality comes from the J-holomorphic curve equation, the fourth equality comes from the condition (**??**) and the last equality is due to the fact that $X_\rho = Y$. Similarly, we have:

$$\begin{aligned}
\partial_t \log(\rho) &= \frac{dr}{r} \cdot (-J) \cdot J \cdot (\partial_t u) = \alpha(J\partial_t u) \\
&= \alpha(-\partial_s u + JcX_\rho) = \alpha(-\partial_s u)
\end{aligned}$$

Suppose we have a bound on H_0: $H_0 \leq \mu := \max H_0$. Consider the following auxiliary function:

$$h_\mu(s,t) = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot (\frac{t^2}{2} - t)\mu & s \in (-1,0) \\ (\frac{t^2}{2} - t)\mu & s \in [0,1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot (\frac{t^2}{2} - t)\mu & s \in (1,2) \\ 0 & otherwise \end{cases}$$

Remark 6.2. The above construction is just to make $d^c h(\partial_s) \geq \mu$ on all of the lower boundary of Σ with $d^c h(\partial_s) = \mu$ when $s \in [0,1]$. While $d^c h(\partial_s) = 0$ on the upper boundary of Σ which maps onto a constant Lagrangian K. Further on, we apply the smoothing function so that $h_\mu(s,t)$ is close to 0 near infinite ends.

We want to show that Hopf maximum principle can be applied to the function $\log(\rho) - h_\mu$. Since h_μ is itself bounded above on Σ, thus an upper bound on $\log(\rho) - h_\mu$ would imply an upper bound on ρ. Note

$$\partial_t h_\mu = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot (t-1)\mu & s \in (-1,0) \\ (t-1)\mu & s \in [0,1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot (t-1)\mu & s \in (1,2) \\ 0 & otherwise \end{cases}$$

$$\partial_s h_\mu = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot \frac{-2s}{(1-s^2)^2}(\frac{t^2}{2} - t)\mu & s \in (-1,0) \\ 0 & s \in [0,1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot \frac{-2(s-2)}{(1-(s-2)^2)^2}(\frac{t^2}{2} - t)\mu & s \in (1,2) \\ 0 & otherwise \end{cases}$$

A picture of h_μ with respect to s can be found later in the picture of $f(s)$ in **??**.

$$\begin{aligned} d^c \log(\rho) &= d\log(\rho) \circ j \\ &= -\partial_s \log(\rho) dt + \partial_t \log(\rho) ds \\ &= (-\alpha(\partial_t u) + 2c(s)\rho)dt - \alpha(\partial_s u)ds \\ &= -u^*\alpha + 2c(s)\rho \cdot dt \end{aligned}$$

$$d^c h_\mu = -\partial_s h_\mu dt + \partial_t h_\mu ds$$

Let ξ denote $\partial_s u_0$, the vector field that is mapped to $\hat{M}$ from the lower boundary of Σ, note ξ is tangent to the Hamiltonian flow of L_s when $s \in [0,1]$. We may decompose ξ into three components: $\xi = a\partial_r + X_{H_0} + Z$ where X_{H_0} was the Hamiltonian vector field on Q generated by H_0 and $Z \in \kappa = \ker(\theta)$. Recall from earlier part of this section that $X_{H_0} = H_0 \cdot Y + Z'$ where $Z' \in \kappa$.

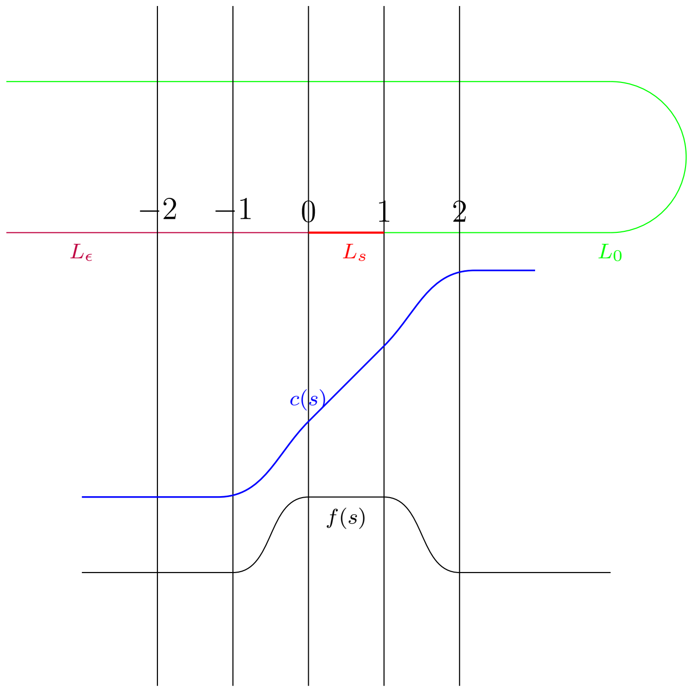

FIGURE 8. The blue curve in the middle illustrates the slope of coefficient of r^2 as we asked the Hamiltonian to be $H = c(s)\cdot r^2$; the bottom curve illustrates the bump function who vanishes outside a neighborhood of the moving Lagrangian.

$$\begin{aligned}(d^c\log\rho - d^ch_\mu)(\partial_s) &= -\alpha(\xi) + c(s)dt(\partial_s) - \partial_t h_\mu \\ &= -\alpha(a\cdot\partial_r + X_{H_0} + Z) + 0 - \partial_t h_\mu \\ &= -\alpha(H_0Y) - \partial_t h_\mu = -H_0 - \partial_t h_\mu = -H_0 + (1-t)\mu\cdot f(s)\end{aligned}$$

Note in the last equality $0 \le f(s) \le 1$, thus from the condition $H_0 \le \mu$, we get $(d^c\log\rho - d^ch_\mu)(\partial_s) \ge 0$ when $t = 0$, namely the lower part of $\partial\Sigma$. While $t = 1$, $(d^c\log\rho - d^ch_\mu)(-\partial_s) \equiv 0$ as $u(s,1) \in K$ and $K \cap \partial M \times [1,\infty) = \Lambda \times [1,\infty)$ for some Λ Legendrian on ∂M. Thus if $H_0 \le \mu$, then the maximum of $\log\rho - h_\mu$ can't occur on the boundary of Σ by Hopf Lemma, which claims that if the maximum occurs at the boundary for $\log\rho$, then the partial derivative on the outward normal direction would be strictly positive.

We also use the Laplacian to show that the maximum can't lie in the interior:

$$\begin{aligned}\Delta(\log\rho - h_\mu) &= -dd^c(\log\rho - h_\mu)(\partial_s,\partial_t) = u^*d\alpha - \partial_s[2c(s)\rho]ds\wedge dt - \Delta h_\mu \\ &= d\alpha(\partial_s u, \partial_t u) - 2c'(s)\rho - 2c(s)\partial_s\rho - \Delta h_\mu \\ &= d\alpha(\partial_s u, J\partial_s u + cX_\rho) - 2c'(s)\rho - 2c(s)\partial_s\rho - \Delta h_\mu \\ &= d\alpha(\partial_s u, J\partial_s u) - 2c'(s)\rho - 2c(s)\partial_s\rho - \Delta h_\mu\end{aligned}$$

Note we had the decomposition (1.3), thus if we let $\partial_s u = a \cdot Y + b\partial_r + Z$ where $Z \in \kappa$ as before. Then $J\partial_s u = -ar\partial_r + \frac{b}{r} \cdot Y + J \cdot Z$. Also $\omega = d(r \cdot \alpha) = dr \wedge \alpha + rd\alpha$. Therefore

$$\begin{aligned} d\alpha(\partial_s u, J\partial_s u) &= d\alpha(a \cdot Y + b\partial_r + Z, -ar\partial_r + \frac{b}{r} \cdot Y + J \cdot Z) \\ &= d\alpha(Z, J \cdot Z) = \frac{(\omega - dr \wedge \alpha)(Z, J \cdot Z)}{r} \\ &= \frac{\omega(Z, J \cdot Z)}{r} = \frac{|Z|_J^2}{r} \end{aligned}$$

From the above equation we get

$$\begin{aligned} \Delta(\log \rho - h_\mu) &= d\alpha(\partial_s u, J\partial_s u) - 2c'(s)\rho - 2c(s)\partial_s\rho\Delta h_\mu \\ &= \frac{|Z|_J^2}{r} - 2c'(s)\rho - 2c(s)\partial_s\rho\Delta - h_\mu \end{aligned}$$

We notice that in applying the Hopf maximum principle, we can ignore the term $\partial_s\rho$ at the maximum point. Therefore we only care about $\frac{|Z|_J^2}{r} - 2c'(s)\rho - \Delta h_\mu$. When $s \in [0,1]$, we have $\Delta h_\mu = \mu$, thus the above term becomes $\frac{|Z|^2}{r} - 2c'(s)\rho - \mu$. Thus as long as $2c'(s) + \mu < 0$, either $\rho \leq 1$ or the maximum of $\log \rho - h_\mu$ can't be in the interior, else $0 < \Delta(\log \rho - h_\mu) + 2c'(s) + \mu = |Z|^2 \geq 0$ a contradiction. When $s \in [-1, 0]$,

$$\Delta(\log \rho - h) = |Z|^2 - c'(s) - (1 + \frac{4s^2 - 2(1-s^2)^2 + 8s^2(s^2-1)}{(1-s^2)^4}(\frac{t^2}{2} - t))e^{\frac{-s^2}{1-s^2}}\mu$$

Some Calculation shows that $\frac{4s^2-2(1-s^2)^2+8s^2(s^2-1)}{(1-s^2)^4} \cdot e^{\frac{-s^2}{1-s^2}} \in [-200, 200]$ for $s \in [-1, 0]$; while $-\frac{1}{2} \leq (\frac{t^2}{2} - t)) \leq 0$. Similarly for $s \in [1, 2]$.

Thus if we impose the condition $c'(s) \leq -\frac{1}{2} \times 100|\max_Q H_0|$, then we get upper bound of ρ on all of Σ. We denote the constant on the right hand side of our inequality by $-A$.

Therefore, we can write the Quadratic Hamiltonian to be

$$H = -A \cdot sr^2 \cdot f(s) \tag{6.4}$$

. Here $f(s)$ is a bump function that vanishes outside $[-1, 2]$ and stays constant 1 between $1, 0$ as shown in figure () For example we can require $f(s)$ to be:

$$f(s) = \begin{cases} 0 & s \in (-\infty, -1] \\ \exp(-\frac{s^2}{1-s^2}) & s \in (-1, 0) \\ 1 & s \in [0, 1] \\ \exp(-\frac{(s-2)^2}{1-(s-2)^2}) & s \in [1, 2] \\ 0 & s \in [2, \infty] \end{cases} \tag{6.5}$$

Thus now $c'(s)$ would be $-A$ when $s \in [0, 1]$ and vanishes outside $[-1, 2]$. This is the same function in the definition of h_μ.

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