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THE NUMBER OF GENERATORS OF THE FIRST KOSZUL HOMOLOGY OF AN ARTINIAN RING

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We study the conjecture that, if I, J are $\mu-$primary in a regular local ring (R, μ) with $\dim(R) = n$, then $\frac{I\cap J}{IJ} \cong \mathrm{Tor}_1(R/I, R/J)$ needs at least n generators, and a related conjecture about the number of generators of the first Koszul homology module of an Artinian local ring (A, m). In this manuscript, we focus our attention on the complete intersection defect of the Artinian ring and its quotient by the Koszul elements. We prove that the number of generators of the first Koszul homology module of $x_1, \ldots, x_n \in m$ on an Artinian local ring (A, m) is at least $n + \mathrm{cid}(A) - \mathrm{cid}(\frac{A}{(x_1,\ldots,x_n)A})$, where $\mathrm{cid}A$ denotes the complete intersection defect of the Artinian local ring A.

Key Words: Artinian ring; Koszul homology.

2010 Mathematics Subject Classification: 13D03.

1. INTRODUCTION

We study the conjecture that, if (R, m) is a regular local ring, $I + J$ is m-primary, and $R/I, R/J$ are Cohen–Macaulay, then $\mathrm{Tor}_1(R/I, R/J)$ needs at least $\mathrm{ht}(I) + \mathrm{ht}(J) - n$ generators. The conjecture follows in the equal-characteristic case from the conjecture that the number of generators of $H_1(x_1, \ldots, x_n; A)$ is at least n, where (A, m) is an Artinian local ring and $x_1, \ldots, x_n \in m$.

The question for Artinian rings studied here was raised by M. Hochster in a talk at a conference in honor of Joseph Lipman held at Purdue University in July 2004, but had earlier been discussed by Dutta, Huneke, as well as Hochster and, somewhat later, by Aberbach. A version of Hochster's talk [3] is available at http://www.math.lsa.umich.edu/~hochster/Lip.text.pdf.

The conjecture involving the least number of generators of $\mathrm{Tor}_1^R(R/I, R/J)$ is mentioned there in the case where I, J are primary to the maximal ideal of the regular ring R. The fact that Conjecture 10 below implies Conjecture 9 in equicharacteristic has been known for a long time, but appears to be unpublished.

First, we recall several concepts in commutative algebra.

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Definition 1 (Complete Intersection Defect). If A is a local ring and M is a finitely generated A-module, let $v(M)$ denote the least number of generators of M. If $A = T/I$, where T is regular local, then the complete intersection defect of A, denoted by $\mathrm{cid}(A)$, is defined as $v(I) - \mathrm{ht}(I)$.

Remark 2. This notion is well defined: see [1], Section 3, p. 33.

Remark 3. Notice that, if A is a complete intersection, namely a regular local ring modulo a regular sequence, then $\mathrm{cid}(A) = 0$. Also, $\mathrm{cid}(A) \geq 0$ for any local ring A, since for any ideal I, $\mathrm{ht}(I) \leq v(I)$ by Krull's height theorem.

We use $T_\bullet$ for the total complex of a double or multiple complex.

Definition 4 (Koszul Complex and Koszul Homology). Given a ring A and $x_1, \ldots x_n \in A$, for an $A-$module M, the Koszul complex of M with respect to $x_1, \ldots, x_n$, denoted by $K_\bullet(x_1, \ldots, x_n; M)$, is defined as follows. If $n = 1$, $K_\bullet(x_1; M)$ is the complex $0 \to M \xrightarrow{x_1} M \to 0$, where the left-hand copy of M is in degree 1 and the right-hand copy in degree 0. Recursively, for $n > 1$,

$$K_\bullet(x_1, \ldots x_n; M) = T_\bullet(K_\bullet(x_1, \ldots x_{n-1}; A) \otimes_A K_\bullet(X_n; M)).$$

Said differently,

$$K_\bullet(x_1, \ldots x_n; M) = T_\bullet(K_\bullet(x_1; A) \otimes_A \cdots \otimes_A K_\bullet(x_n; A) \otimes_A M)$$

The ith homology of the complex is the ith Koszul homology, denoted as $H_i(x_1, \ldots, x_n; M)$ or $H_i(\underline{x}; M)$.

We give several properties of the Koszul complex.

Proposition 5. *An A-linear map $f: M \longrightarrow N$ induces, in a covariantly functorial way, a map of Koszul complexes $K_\bullet(\underline{x}; M) \longrightarrow K_\bullet(\underline{x}; N)$, and, hence, a map of Koszul homology $H_\bullet(\underline{x}; M) \longrightarrow H_\bullet(\underline{x}; N)$. If $M = N$ and the map is multiplication by $a \in A$, the induced map on Koszul complexes and on their homology is also given by multiplication by the element a.*

This is obvious from the definition of Koszul homology: see Section 1.6 in [2].

Corollary 6. *$I = \mathrm{Ann}_A M$ kills all the Koszul homology modules $H_i(\underline{x}; M)$.*

Proposition 7. *$(x_1, \ldots, x_n)A$ kills every $H_i(x_1, \ldots, x_n; M)$*

A proof is provided in Section 1.6 in [2].

We can characterize the Koszul complex $K(\underline{x}; A)$ from the point of view of exterior algebra. Let $K_1(\underline{x}; A) = Au_1 \oplus Au_2 \oplus \cdots \oplus Au_n$. Then $K_t(\underline{x}; A) \cong \wedge^t K_1(\underline{x}; A)$.

Note that the formula for d_i comes from the fact that it is derivation on the exterior algebra, namely for $j_1 < j_2 < \cdots j_t$,

$$d(u_{j_1} \wedge u_{j_2} \cdots \wedge u_{j_i}) = \sum_{t=1}^{i} (-1)^{t-1} x_{j_t} u_{j_1} \wedge u_{j_2} \wedge \cdots \wedge u_{j_{t-1}} \wedge u_{j_{t+1}} \cdots \wedge u_{j_t}$$

Remark 8. The d_i are uniquely determined by d_1, where $d_1(u_j) = x_j$. Given an invertible linear transformation $L : A^n \to A^n$ such that $L(x_i) = y_i$, we have an induced isomorphism $\wedge A^n \to \wedge A^n$, which gives an isomorphism $K_\bullet(\underline{x}; A) \cong K_\bullet(\underline{y}; A)$. Therefore, $H_i(\underline{x}; A) \cong H_i(\underline{y}; A)$ and $H_i(\underline{x}; M) \cong H_i(\underline{y}; M)$.

The following conjectures motivate the study of the minimal number of generators of $H_1(\underline{x}; A)$ when A is an Artinian local ring.

Conjecture 9. *Let (R, m) be a regular local ring with Krull dimension n. Suppose $I + J$ is an $m-$primary ideal in R and R/I, R/J are Cohen–Macaulay. Then* $\mathrm{Tor}_1(R/I, R/J)$ *needs at least $ht(I) + ht(J) - n$ generators. In particular, if I, J are two $m-$primary ideals, then* $\mathrm{Tor}_1(R/I, R/J)$ *needs at least n generators.*

Conjecture 10. *If (A, m) is Artin local, $x_1, \ldots, x_n \in m$, then $H_1(\underline{x}; A)$ has at least n generators as an A-module.*

The fact that the second conjecture implies the first is one of the reasons we study the minimal number of generators of the first Koszul homology module.

Discussion 11. We note that Conjecture 10 reduces to the case where the Koszul elements $x_1, \ldots, x_n$ are minimal generators of the ideal they generate. If not, by applying an invertible matrix N (see Remark 8) to $x_1, \ldots, x_n$, we may assume $x_n = 0$. In this case, $K_\bullet(x_1, \ldots, x_{n-1})$ is the homology of the mapping cone of the 0 map from $K_\bullet(x_1, \ldots, x_n; M)$ to itself, which implies $K_1(x_1, \ldots, x_{n-1}, 0; M) \cong K_0(x_1, \ldots, x_{n-1}; M) \oplus K_1(x_1, \ldots, x_{n-1}; M)$. Applying this to $M = A$ yields the result.

Proposition 12. *Conjecture* 10 *implies conjecture* 9 *in the equi-characteristic case.*

The proof uses Serre's method of reduction to the diagonal: see [4], pp. V-4 through V-6.

Remark 13. With R, I, J as in the proposition above, if R contains a field, we may replace R by $\hat{R}$, and then $\hat{R} \cong k[[X_1, \ldots, X_n]]$, the formal power series ring, and $m = (X_1, \ldots, X_n)$. By Serre's result on reduction to the diagonal,

$$\mathrm{Tor}_i^R\left(\frac{R}{I}, \frac{R}{J}\right) \cong \mathrm{Tor}_i^S(S/(I^e + J'^e), R).$$

Here, $R := k[[X_1, \ldots, X_n]]$, $R' := k[[Y_1, \ldots, Y_n]]$, and in the second Tor, R is viewed as the $S-$module $S/(X_1 - Y_1, \ldots, X_n - Y_n)$. Then $R \cong R'$ in an obvious way, where $X_i \mapsto Y_i$. Let J' be the image of J in R'. $S := R\widehat{\otimes_k}R' \cong k[[X_1, \ldots X_n, Y_1, \ldots, Y_n]]$, $I \subset R$, $J' \subset R'$, and I^e and J'^e are the extensions of I and J' to S.

We give the proof of Conjecture 9 from Conjecture 10.

Lemma 14. *If x is a regular element on M, then $H_i(x, y_1, \ldots, y_n; M) \cong H_i(y_1, \ldots, y_n; \frac{M}{xM})$.*

A proof can be found in Section 1.6 in [2].

Proof of Conjecture 9 from Conjecture 10. Recall that $v(M)$ denotes the least number of generators of a module M. First, enlarge k to an infinite field by replacing (R, m) by $R[t]_{mR[t]}$. The hypothesis and conclusion are not affected. We have

$$\mathrm{Tor}_1^R\left(\frac{R}{I}, \frac{R}{J}\right) \cong \mathrm{Tor}_1^S\left(\frac{R}{I}\hat{\otimes}_k \frac{R'}{J'}; \frac{S}{(X_1 - Y_1, \ldots, X_n - Y_n)}\right).$$

Let $(B, \mu) = \frac{R}{I}\hat{\otimes}_k \frac{R'}{J'}$. Then by the results of [4], B is a Cohen–Macaulay ring of dimension $\delta = \dim(\frac{R}{I}) + \dim(\frac{R'}{J'}) = n - \mathrm{ht}(I) + n - \mathrm{ht}(J)$. The images of the $X_i - Y_i$ generate a μ-primary ideal in B, since

$$\frac{R}{I}\hat{\otimes}_k \frac{R'}{J'} \otimes \frac{S}{(X_1 - Y_1, \ldots, X_n - Y_n)} \cong \frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I + J},$$

and $I + J$ is $m-$primary. Let $Z_i = X_i - Y_i$. Because $Z_1, \ldots, Z_n$ is a regular sequence in S, the Koszul complex $K_\bullet(\underline{Z}; S)$ gives a resolution of $S/(\underline{Z})$. Tensoring this resolution with B gives $\mathrm{Tor}_i^S(B, S/(\underline{Z}))$, but it also gives the $H_i^S(\underline{Z}; B)$

Now that k is infinite, we can choose δ k-linear combinations of the $X_i - Y_i$ that form a regular sequence on B, say $z_1, \ldots, z_\delta$, and let $z_1, \ldots, z_n$ extend this to elements that span the same k-vector space as the $X_i - Y_i$:

$$\begin{aligned} H_1(X_1 - Y_1, \ldots, X_n - Y_n; B) = H_1(\underline{Z}; B) &\cong H_1(z_1, \ldots, z_n; B) \\ &\cong H_1\left(z_{\delta+1}, \ldots, z_n; \frac{B}{(z_1, \ldots, z_\delta)}\right), \end{aligned}$$

where the last congruence is a result of iteration of Lemma 13. But $\frac{B}{(z_1, \ldots, z_\delta)}$ is Artinian local. Therefore, by Conjecture 10, it will need $n - \delta = n - (n - \mathrm{ht}(I) + n - \mathrm{ht}(J)) = \mathrm{ht}(I) + \mathrm{ht}(J) - n$ generators. $\square$

We need the following well-known fact.

Lemma 15. *Given a finite complex $A_\bullet : 0 \to A_k \xrightarrow{d_k} \cdots \xrightarrow{d_1} A_0 \to 0$ such that each A_i has finite length, the alternating sum of the lengths of homologies equals the Euler characteristic of the complex, i.e., $\sum_{i=0}^k (-1)^i l(H_i(A_\bullet)) = \chi(A_\bullet) = \sum_{i=0}^k (-1)^i l(A_i)$.*

In the case of a Koszul complex, the Euler characteristic is zero.

Theorem 16. *Suppose (A, m) is local, $x_1, \ldots, x_n \in m$, M is finitely generated, and $\frac{M}{(x_1, \ldots, x_n)M}$ has finite length. If $\dim(M) < n$, then $\chi(x_1, , .x_n; M) := \sum (-1)^i l(H_i(\underline{x}; M)) = 0$. If $\dim(M) = n$, $\chi(\underline{x}; M)$ is the multiplicity of M on $(x_1, \ldots, x_n)R$.*

A proof can be found in [4], Théorème 1, p. V-12.

For Conjecture 10, the cases when $n = 1$ or 2 are known (S. Dutta and M. Hochster, personal communication). We include the argument for the convenience of the reader.

Proof of Conjecture 10 When $n = 1$ *or* 2. When $n = 1$, $H_1(x; A)$ is just $\mathrm{Ann}_A x$, since $x \in m$ and A is Artinian, so m consists of zero divisors, thus, $\mathrm{Ann}_A x \neq 0$, so it has to be generated by at least 1 element as an A-module.

When $n = 2$, suppose $\nu(H_1(\underline{x}; A)) < 2$, in which case it is cyclic. Since A is Artinian, every finite module has finite length, and the Euler characteristic of the Koszul complex is zero. Thus, by Theorem 15, $l(H_1(x_1, x_2; A)) - l(\frac{A}{(x_1,x_2)}) - l(\mathrm{Ann}_A(x_1, x_2)) = 0$. We know $\mathrm{Ann}_A(x_1, x_2) \neq 0$, so $\nu(H_1) \neq 0$, and if $H_1(\underline{x}; A)$ is cyclic, then, since it is killed by (x_1, x_2) by Proposition 7, it is a quotient module of $A/(x_1, x_2)$. Therefore, $l(H_1(\underline{x}; A)) \leq l(A/(x_1, x_2))$. But, as we noted earlier, $l(\mathrm{Ann}_A(x_1, x_2)) > 0$, so the alternating sum $l(H_1(x_1, x_2; A)) - l(\frac{A}{(x_1,x_2)}) - l(\mathrm{Ann}_A(x_1, x_2)) < 0$, a contradiction. Thus, we have $\nu(H_1(x_1, x_2; A)) \geq 2$. $\square$

2. MAIN RESULT

Theorem 17 (Main Theorem). *If* $(A, \mathfrak{m}, k)$ *is an Artinian local ring, and* $x_1, \ldots, x_n \in \mathfrak{m}_A$, *then* $\nu(H_1(\underline{x}; A)) \geq n + \mathrm{cid}(A) - \mathrm{cid}(\frac{A}{(\underline{x})})$.

By the Cohen Structure theory, there is a coefficient ring, which may be a Discrete Valuation Ring (V, t, k) or a cofficient field k (when A is equicharacteristic). In the latter case, let V denote the field as well, with $t = 0$. Fix a coefficient ring $V/(t^h) \subset A$. Suppose $y_1, \ldots, y_m$ is a minimal set of generators of the maximal ideal of $\frac{A}{(\underline{x})A}$. Define $\theta : V[[Y_1, \ldots, Y_m]] \to A/(\underline{x})A$. Then there exists a map $\tilde{\theta} : V[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]] \twoheadrightarrow A$ such that $X_i \mapsto x_i$ and the image of Y_i in $A/(\underline{x}A)$ is y_i.

Let $m_1 = m$ when A is equicharacteristic (namely, $t = 0 \in V$), and $m_1 = m + 1$ otherwise. In either case, $m_1 = \dim V[[Y_1, \ldots, Y_m]]$.

Let $\mathfrak{a}$ denote $\ker(\tilde{\theta})$ and I denote $\ker(\theta)$. Then we have a map $\pi : V[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]] \to V[[Y_1, \ldots, Y_m]]$, where $X_i \mapsto 0$, $Y_i \mapsto y_i$.

We have the following commutative diagram with exact rows, where ϕ is the restriction of π. We have that $\pi(\mathfrak{a}) \subseteq I$ because the right-hand square commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "\phi"] & V[[X_1, ..., X_n, Y_1, ..., Y_m]] \arrow[r, two heads, "\tilde{\theta}"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & V[[Y_1, ..., Y_m]] \arrow[r, two heads, "\theta"] & A/(\underline{x})A \arrow[r] & 0 .
\end{tikzcd}$$

We claim ϕ is surjective because for any $F \in I$, if we abuse notation and write F as its image under the obvious injective map from $V[[\underline{Y}]]$ to $T = V[[\underline{X}, \underline{Y}]]$, we have $\theta(F) = 0 \Rightarrow \tilde{\theta}(F) \in (x_1, \ldots, x_n)A \Rightarrow \tilde{\theta}(F) = \sum_{i=1}^{n} x_i a_i$ for certain $a_i \in A$. Lift the a_i to $G_i \in T$. Define $\widetilde{F}_j = F_j - \sum X_i G_i \in \mathfrak{a}$. Then $\phi(\widetilde{F}_j) = \pi(\widetilde{F}_j) = F_j$ as required.

Before we prove the theorem, we shall prove several lemmas. We want to find polynomials $F_1, \ldots, F_{m_1}$ that form part of a minimal set of generators of

$I = \ker(\theta)$ and are also a system of parameters in $V[[Y_1, \ldots, Y_m]]$. By a standard prime avoidance argument, we have the following lemma.

Lemma 18. *There exist elements $F_1, \ldots, F_{m_1}$ that form part of a minimal set of generators of I and are also a system of parameters in $V[[Y_1, \ldots, Y_m]]$.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $V[[Y_1, \ldots, Y_m]]$. Suppose $F_1, \ldots, F_i$ are part of a minimal set of generators of I that are also part of a system of parameters ($i = 0$ is allowed). Let $p_1, \ldots, p_k$ be the minimal primes of $(F_1, \ldots, F_i)$. Then

$$I \nsubseteq \big((F_1, \ldots, F_n) + \mathfrak{m}I\big) \cup (\cup_j p_j),$$

unless $i = m_1$. Pick $F_{i+1} \in I - \big((F_1, \ldots, F_n) + \mathfrak{m}I\big) \cup (\cup_j p_j)\big)$. Then $F_1, \ldots, F_{i+1}$ are part of a system of parameters of $V[[Y_1, \ldots, Y_m]]$ and part of a minimal set of generators of I. □

Since $\phi : \mathfrak{a} \twoheadrightarrow I$, the F_j have lifts in T that are in $\mathfrak{a}$, say $\widetilde{F}_j = F_j - \sum X_i G_{ji}$, where $G_{ji} \in T$. We want to show that $\widetilde{F}_j$ is also part of a minimal set of generators for $\mathfrak{a}$.

Lemma 19. *If $F_1, \ldots, F_{m_1}$ is part of a minimal set of generators for $I = \ker(\theta)$, then their lifts $\widetilde{F}_j \in \mathfrak{a}$ are also part of a minimal set of generators for $\mathfrak{a} = \ker(\tilde{\theta})$.*

Proof. Since $\widetilde{F}_j$ are already in $\mathfrak{a}$, it suffices to prove that their images are linearly independent over k in $\mathfrak{a}/\mathfrak{m}_T\mathfrak{a}$.

Consider $\mathfrak{a}/\mathfrak{m}_T\mathfrak{a} \hookrightarrow T/\mathfrak{m}_T\mathfrak{a} \twoheadrightarrow T/(\mathfrak{m}_T\mathfrak{a} + (\underline{X})) \cong k[[\underline{Y}]]/\mathrm{Im}(\mathfrak{m}_T\mathfrak{a}) \cong k[[\underline{Y}]]/(\underline{Y})I$, where the last isomorphism holds because $\mathrm{Im}(\mathfrak{m}_T) = (\underline{Y})$ and $\mathrm{Im}(\mathfrak{a}) = I$. The images γ_j of the $\widetilde{F}_j$ in $\mathfrak{a}/\mathfrak{m}_T\mathfrak{a}$ map to elements $\bar{\gamma}_j$ in $I/(\underline{Y})I$, which are the same as the images of the F_j. The $\bar{\gamma}_j$ are k-linearly independent, so the γ_j are k-linearly independent. □

Let $S = T/(\tilde{F}_1, \ldots, \tilde{F}_{m_1})$. A is a quotient of S. Consider the minimal free resolution of A over S:

$$\cdots \to S^{n_j} \to \cdots \to S^{n_2} \to S^{n_1} \to S \twoheadrightarrow A. \tag{1}$$

Lemma 20. *If $x_1, \ldots, x_n$ is a regular sequence in a ring R, then $K_\bullet(x_1, \ldots, x_n; R)$ gives a free resolution of $R/(x_1, \ldots, x_n)R$. Furthermore, if (R, m) is local and $(x_1, \ldots, x_n) \in m$, then this is a minimal free resolution.*

The proof is given in Section 1.6 in [2].

Proof of the Main Result. The X_i form a regular sequence on S (since $X_i, \widetilde{F}_j$ form a regular sequence on T.) Thus $H_1^A(\underline{x}; A) \cong H_1^S(K_\bullet(\underline{X}; S) \otimes A) \cong \mathrm{Tor}_1^S(\frac{S}{(\underline{X})S}; A)$.

Tensor the minimal resolution (1) of A with $\overline{S} = S/(\underline{X})S$. This yields the following complex:

$$\cdots \to \overline{S}^{n_j} \to \overline{S}^{n_2} \to \overline{S}^{n_1} \to \overline{S} \twoheadrightarrow \overline{A} = A/(\underline{x})A. \tag{2}$$

The matrices of the maps of this resolution have entries that are contained in $\mathfrak{m}_{\overline{S}}$, since the original resolution of A as an $S-$module was a minimal one over S, so that the maps of the original resolution were contained in $\mathfrak{m}_S$.

Suppose

$$\widetilde{F_1}, \ldots, \widetilde{F_{m_1}}, G_1, \ldots, G_r$$

is a minimal generating set of $\mathfrak{a}$ and $J = (\overline{G_1}, \ldots, \overline{G_r}) \subseteq S$, i.e., the ideal generated by the images of the G_k in S.

Now

$$A = T/\mathfrak{a} = T/(\widetilde{F_1}, \ldots, \widetilde{F_{m_1}}, G_1, \ldots, G_r) = S/J.$$

Then r is the minimal number of generators of $J = \ker(S \twoheadrightarrow A)$. Thus $r = n_1$.

Now $\mathfrak{a}$ is minimally generated by $m_1 + n_1$ elements. Hence $m_1 + n_1 = \nu(\mathfrak{a}) = \mathrm{cid}(A) + \dim(T) = \mathrm{cid}(A) + (m_1 + n)$. Thus $n_1 = \mathrm{cid}(A) + n$.

Recall $\overline{S}$ denote $\frac{S}{(\underline{x})}$, let $\overline{A}$ denote $\frac{A}{(\underline{x})}$, let W denote $\ker(\overline{S} \twoheadrightarrow \overline{A})$, let Z denote $\ker(\overline{S}^{n_1} \to \overline{S})$, and let B denote $\mathrm{Im}(\overline{S}^{n_2}) \subseteq m_{\overline{S}} \cdot \overline{S}^{n_1}$.

We claim that $\nu(W) = \mathrm{cid}(\overline{A})$. Note $\mathrm{cid}(\overline{A}) = \dim T - \nu(\mathfrak{a} + (\underline{X})) = m_1 + n - \nu(\mathfrak{a} + (\underline{X}))$. $X_1, \ldots, X_n, \widetilde{F_1}, \ldots, \widetilde{F_{m_1}}$ is part of a minimal set of generators for $\mathfrak{a} + (\underline{X})$ and can be extended to a minimal set of generators of $X_1, \ldots, X_n, \widetilde{F_1}, \ldots, \widetilde{F_{m_1}}, w_1, \ldots, w_s$, where $s = \nu(W)$. Thus $\nu(W) = \mathrm{cid}(\overline{A})$.

Let $H = Z/B$, which is the first homology module of (2), i.e., $\mathrm{Tor}_1^S(\overline{S}; A) = H_1^A(\underline{x}; A)$.

We have the following short exact sequences:

$$0 \to Z \to \overline{S}^{n_1} \to W \to 0 \tag{3}$$

and

$$0 \to B \to Z \to H \to 0.$$

Now tensor (3) with k to get the exact sequence

$$Z \otimes_{\overline{S}} k \to k^{n_1} \to W \otimes_{\overline{S}} k \to 0.$$

But since $B \subseteq m \cdot \overline{S}^{n_1}$, the images of $\frac{Z}{\mathfrak{m}_{\overline{S}} Z}$ and $\frac{H}{\mathfrak{m}_{\overline{S}} H}$ in $\frac{\overline{S}^{n_1}}{\mathfrak{m}_{\overline{S}} \overline{S}^{n_1}}$ are the same. Now, when we tensor (3) with k, we get the exact sequence

$$H \otimes_{\overline{S}} k \to k^{n_1} \to W \otimes_{\overline{S}} k \to 0.$$

Therefore, $n_1 = \nu(k^{n_1}) \le \nu(H) + \nu(W)$, and $\nu(H) \ge n_1 - \nu(W) = n + \mathrm{cid}(A) - \mathrm{cid}(\overline{A})$. □

Corollary 21. *When* $\mathrm{cid}(A) \ge \mathrm{cid}(\frac{A}{(x_1, \ldots, x_n)A})$, *we have* $\nu(H_1(x_1, \ldots, x_n; A)) \ge n$.

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REFERENCES

[1] Avramov, L. L. (1977). Homology of local flat extensions and complete intersection defects. *Mathematische Annalen* 228(1):27–37.

[2] Bruns, W., Herzog, J. (1993). *Cohen-Macaulay Rings*. Cambridge: Cambridge UP.

[3] Hochster, M. Thirteen open questions in commutative algebra. Available at http://www.math.lsa.umich.edu/~hochster/Lip.text.pdf

[4] Serre, J.-P. (1965). *Algèbre Locale · Multiplicités*. Seconde édition. Lecture Notes in Math. No. 11. Berlin: Springer-Verlag.