MATH3871/MATH5970
Bayesian Inference and Computation

Tutorial Problems 2 (Solutions)

1.

$$\begin{aligned}\pi(\theta \mid x) &\propto \theta^{\sum_i (x_i - 1)}(1-\theta)^n \frac{\theta^{p-1}(1-\theta)^{q-1}}{\mathrm{B}(p,q)} \\ &= \frac{\theta^{\sum(x_i-1)+p-1}(1-\theta)^{n+q-1}}{\mathrm{B}(p,q)}\end{aligned}$$

hence $\pi(\theta \mid x) \sim \mathsf{Beta}(n\bar{x}_n - n + p, n + q)$.

2.

$$\pi(\theta \mid x) \quad \propto \quad \exp(-n\theta)\theta^{n\bar{x}_n} \exp(-\theta) = \exp(-(n+1)\theta)\theta^{n\bar{x}_n}$$

hence $\pi(\theta \mid x) \sim \mathsf{Gamma}(n\bar{x}_n + 1, n + 1)$.

3. $x \mid \theta \sim \mathsf{Bin}(100, \theta)$, so

$$\begin{aligned}\pi(\theta \mid x) &\propto \theta^3(1-\theta)^{100-3}\theta(1-\theta)^{199} \\ &= \theta^4(1-\theta)^{296},\end{aligned}$$

and so $\pi(\theta \mid x) \sim \mathsf{Beta}(5, 297)$.

4. With $\mathbb{E}(\theta) = p/(p+q) = 4/102$ and $\mathbb{V}\mathrm{ar}(\theta) = \frac{pq}{(p+q)^2(p+q+1)} = 0.003658$, the posterior Beta($p$,$q$) distribution must have parameters $p = 4$ and $q = 98$.

   If we have a Beta(p', q') prior distribution then by looking at the powers of θ and $(1-\theta)$ we know that $p' + x = 4$ and $q' + n - x = 98$. Therefore we know that $p' = 1$ and $q' = 1$.

5. Our prior is a $\mathsf{N}(10, 0.25)$ distribution, and the likelihood is $\mathsf{N}(\theta, 1)$. Hence, under this conjugate prior we immediately have $\pi(\theta \mid x) \sim \mathsf{N}(\frac{10*4+12*31/3}{4+12}, \frac{1}{4+12}) = \mathsf{N}(10.25, 0.0625)$. Hence, $\mathbb{P}(\theta \mid x > 10) = 0.84$.