Analysis of stick-slip motion using LuGre friction model

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1 LuGre model

Friction is a common phenomenon found in applications such as physical interaction between mechanical parts, aerodynamics or fluid dynamics. Due to the complexity of friction dynamics, its modeling is often neglected. But there are cases that require to take into consideration the friction model, because the differences between the mathematical model and reality become unacceptable. Thus, the control problem cannot achieve the desired performance, even using advanced control techniques.

In the specialized literature, several models that describe friction dynamics are proposed. Classical models combine static, viscous, Coulomb and Stribek friction. Another friction model, based on spring behavior was proposed by Dahl. Moreover, we can mention Karnopp, Bliman-Sorine and LuGre models.

The LuGre model was developed as a result of the collaboration between control groups Lund and Grenoble. It is an extension of the Dahl model that includes Stribek effect and can describe stick-slip motion.

1.1 Mathematical model

$$\begin{cases} \dot{z} = v - zh(v) = v - \sigma_0 \dfrac{|v|}{g(v)} z \\ F = \sigma_0 z + \sigma_1 \dot{z} + f(v), \end{cases} \qquad \text{where} \quad \begin{aligned} g(v) &= F_c + (F_s - F_c)e^{-|v/v_s|^\alpha} \\ f(v) &= \sigma_2 v \end{aligned}$$

Parameters:

- v - velocity of a contact area relative to another contact area
- z - internal friction (physical interpretation - "average bristle deflection")
- F - friction force estimated by the model
- σ_0 - microscale spring stiffness (see Tomlinson and Bristle models)
- σ_1 - microscale damping
- σ_2 - viscous friction coefficient
- F_s - stiction (static friction) coefficient
- F_c - Coulomb friction coefficient
- v_s - Stribek speed

We can write the expression of friction while in steady state F_{ss} (when $z = 0$ and $v = v_0$):

$$F_{ss}(v) = g(v)\ \mathrm{sgn}(v) + f(v).$$

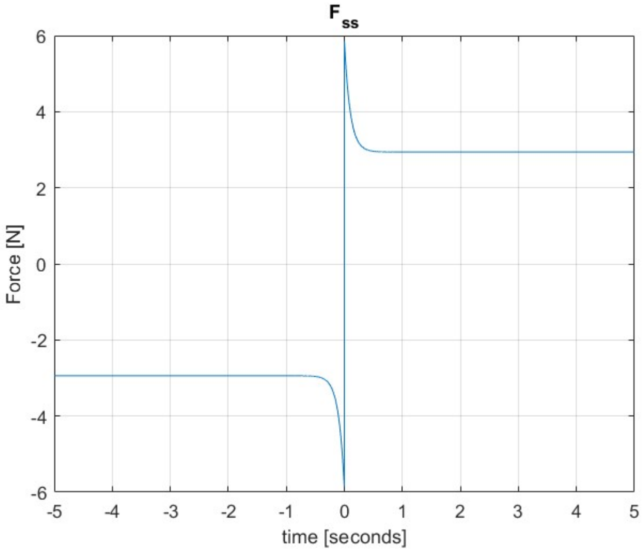

Figure 1: Friction force in steady state

2 Stick-slip motion

Stick-slip motion models the dynamics of a rigid body that slides on a surface (Fig.2). The main consequence of this displacement is the occurrence of friction at the contact area between the body and the static surface. As the name "stick-slip" suggests, we can see the two alternative phenomena, sliding and stopping, despite the traction force being constant. These phenomena cause highly nonlinear behavior in the system.

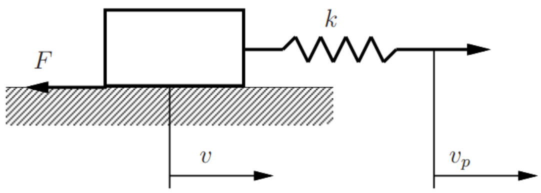

Figure 2: Stick-slip motion

In order to better express the two phenomena, we can classify friction forces into two categories: static and dynamic. Static friction is the force that keeps a body at rest, despite a traction force being applied. On the other hand, dynamic friction opposes the traction force while the body is moving.

2.1 Mathematical model

$$\begin{cases} \dot{l} = v_p - v \\ m\dot{v} = kl - F \\ \dot{z} = v - zh(v) \\ F = \sigma_0 z + \sigma_1 \dot{z} + f(v), \end{cases} \quad \text{where} \quad \begin{aligned} g(v) &= F_c + (F_s - F_c)e^{-|v/v_s|^{\alpha}} \\ f(v) &= \sigma_2 v \end{aligned}$$

We use the following states: l - spring elongation, v - velocity of the body relative to the surface and z - an internal state of the friction model. The input to the system is v_p – the velocity of the spring relative to the static surface and the output is state v.

Rewrite the model as

$$\begin{cases} \dot{x}_1 = u - x_3 \\ \dot{x}_2 = x_3 - x_2\sigma_0 \frac{|x_3|}{F_c+(F_s-F_c)\exp(-|x_3/v_s|^{\alpha})} \\ \dot{x}_3 = \frac{1}{m}\left(kx_1 - (\sigma_1+\sigma_2)x_3 - \sigma_0 x_2\left(1 - \frac{\sigma_1|x_3|}{F_c+(F_s-F_c)\exp(-|x_3/v_s|^{\alpha})}\right)\right). \end{cases}$$

2.2 Feedback linearization

Let us define a tracking control problem, that derives from the classic stabilization problem. Therefore, we introduce the error dynamics.

$$\begin{cases} \dot{e}_1 = \dot{r}_1 - \dot{x}_1 = \dot{r}_1 - u + x_3 \\ \dot{e}_2 = \dot{r}_2 - \dot{x}_2 = \dot{r}_2 - x_3 + x_2\sigma_0 \frac{|x_3|}{F_c+(F_s-F_c)\exp(-|x_3/v_s|^{\alpha})} \\ \dot{e}_3 = \dot{r}_3 - \dot{x}_3 = \dot{r}_3 - \frac{1}{m}\left(kx_1 - (\sigma_1+\sigma_2)x_3 - \sigma_0 x_2\left(1 - \frac{\sigma_1|x_3|}{F_c+(F_s-F_c)\exp(-|x_3/v_s|^{\alpha})}\right)\right) \end{cases}$$

For simplicity of calculations, let $A = \dfrac{|x_3|}{F_c + (F_s - F_c)\exp(-|x_3/v_s|^{\alpha})}$.

The output of the new system, corresponding to the physical system previously defined, is e_3, the velocity error. We can write $y = h(e)$, $\dot{e} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = f(e)$ and $g(e) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

We calculate the relative degree of the system by computing the derivatives of the output.

$y = e_3 = r_3 - x_3$

$$\dot{y} = L_f h(e) = \dot{e}_3 = \dot{r}_3 - \frac{1}{m}\left(kx_1 - (\sigma_1+\sigma_2)x_3 - \sigma_0 x_2\left(1 - \sigma_1 A\right)\right)$$

$$\ddot{y} = L_f^2 h(e) = \ddot{r}_3 - \frac{1}{m}\left(k(u - x_3) - (\sigma_1+\sigma_2)\dot{x}_3 - \sigma_0(x_3 - \sigma_0 x_2 A) + \sigma_0\sigma_1(Ax_3 - \sigma_0 x_2 A^2 + x_2\dot{A})\right)$$

As the signals $r_1(t), r_2(t), r_3(t)$ are known, their derivatives are also known. We note that the second order derivative of the output contains input u in its expression. Thus, the relative degree of the system is $\rho = 2$. We bring the system into the normal canonical form (see Khalil), using a diffeomorphism.

$$w = T(e) = \begin{bmatrix} \phi(e) \\ h(e) \\ L_f h(e) \end{bmatrix}$$

where $\phi(e)$ satisfies $\dfrac{\partial\phi}{\partial e}g(e) = 0$. The condition is rewritten as $\dfrac{\partial\phi}{\partial e_1} = 0$. Let $\phi(e) = e_2 + me_3$. We verify whether $T(e)$ is diffeomorphism. The dynamics of the system after the transformation becomes:

$$\begin{aligned}
\dot{w}_1 = &\quad \dot{e}_2 + m\dot{e}_3 = f_{w0}\left(w_1, \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}\right) \\
\dot{w}_2 = &\quad \dot{e}_3 = \dot{r}_3 - \tfrac{1}{m_1}\left(kx_1 - (\sigma_1+\sigma_2)x_3 - \sigma_0 x_2\left(1-A\right)\right) = w_3 \\
\dot{w}_3 = &\quad \ddot{r}_3 - \tfrac{1}{m}(k(u-x_3) - (\sigma_1+\sigma_2)\dot{x}_3 - \sigma_0(x_3 - \sigma_0 x_2 A) + \sigma_0\sigma_1(Ax_3 - \sigma_0 x_2 A^2 + x_2\dot{A}))
\end{aligned}$$

We calculate $\dot{A} = \dfrac{\partial A}{\partial x_3}\dot{x}_3$.

$$\dot{A} = \dot{x}_3 \frac{\operatorname{sgn}(x_3)(F_c + (F_s - F_c)\exp(-|x_3/v_s|^\alpha)) - |x_3|(F_s - F_c)\exp(-|x_3/v_s|^\alpha))\left(\frac{-1}{v_s^\alpha}\right)\alpha|x_3|^{\alpha-1}\operatorname{sgn}(x_3)}{\left(F_c + (F_s - F_c)\exp(-|x_3/v_s|^\alpha)\right)^2}$$

$$\dot{A} = \dot{x}_3 \frac{\operatorname{sgn}(x_3)(F_c + (F_s - F_c)\exp(-|x_3/v_s|^\alpha)(1 + |x_3|^\alpha \cdot \frac{\alpha}{v_s^\alpha}))}{\left(F_c + (F_s - F_c)\exp(-|x_3/v_s|^\alpha)\right)^2} = \frac{\operatorname{sgn}(x_3)(1 + |x_3|^\alpha \cdot \frac{\alpha}{v_s^\alpha})}{F_c + (F_s - F_c)\exp(-|x_3/v_s|^\alpha)}\dot{x}_3.$$

Rewrite $\dot{w}_3$.

$$\dot{w}_3 = \ddot{r}_3 - \frac{1}{m}(ku - (k+\sigma_0)x_3 - (\sigma_1+\sigma_2)\dot{x}_3 + A(\sigma_0^2 x_2 + \sigma_0\sigma_1 x_3) - \sigma_0^2\sigma_1 x_2 A^2 + \sigma_0\sigma_1 x_2\dot{A})$$

Consider the generic system:

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{\xi} = A\xi + B[\psi(x) + \gamma(x)u] \end{cases}$$

The control law $u = \gamma^{-1}(x)[-\psi(x) + v]$ brings the system to:

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{\xi} = A\xi + Bv \end{cases}$$

We return to $\dot{w}$.

$$\begin{aligned}
\dot{w}_1 = &\quad f_{w0}\left(w_1, \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}\right) \\
\dot{w}_2 = &\quad w_3 \\
\dot{w}_3 = &\quad \ddot{r}_3 - \tfrac{1}{m}(ku - (k+\sigma_0)x_3 - (\sigma_1+\sigma_2)\dot{x}_3 + A(\sigma_0^2 x_2 + \sigma_0\sigma_1 x_3) - \sigma_0^2\sigma_1 x_2 A^2 + \sigma_0\sigma_1 x_2\dot{A})
\end{aligned}$$

We have the correspondences $\eta = w_1$, $\xi = [w_2 \quad w_3]^T$, $f_0 = f_{w0}$, $A = A_w = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\gamma(w) = -\frac{k}{m} \implies \gamma^{-1}(w) = -\frac{m}{k}$ and

$$\psi(w) = \ddot{r}_3 - \frac{1}{m}(-(k+\sigma_0)x_3 - (\sigma_1+\sigma_2)\dot{x}_3 + A(\sigma_0^2 x_2 + \sigma_0\sigma_1 x_3) - \sigma_0^2\sigma_1 x_2 A^2 + \sigma_0\sigma_1 x_2 \dot{A})\Big|_{x=r-T^{-1}(w)}.$$

The subsystem $\begin{bmatrix} \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = A_w \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} + Bv$ can be easily controlled using state feedback, with v being the new control signal.

As the generic system $[\dot{\eta} \quad \dot{\xi}]^T$, defined above, is a cascade connection of two subsystems, the necessary and sufficient condition occurs (Lemma 9.2, Khalil) that the origin of the subsystem $\dot{w}_1 = f_{w0}(w_1, 0)$ must be asymptotically stable.

As $w_2(t) \equiv w_3(t) = \dot{e}_3(t) \equiv 0$ and $f_{w0}\left(w_1, \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}\right) = \dot{e}_2 + m\dot{e}_3$, we can modify the function $f_{w0} = m\dot{e}_3$ such that the dynamics cancel out and the system is at least stable. This change does not affect the selection conditions of $\phi(e)$, nor the diffeomorphism property of transform T.

2.3 Implementation and results

We used the following parameters for simulations:

$$m = 1 \text{ kg}, \; k = 2 \text{ N/m}, \; F_c = 2.94 \text{ N}, \; F_s = 5.88 \text{ N}, \; v_s = 0.1 \text{ m/s},$$
$$\alpha = 1, \; \sigma_0 = 2900, \; \sigma_1 = 107, \; \sigma_2 = 0.$$

In Fig. 3 we can see the open loop step response to the input $v_p = 2$ m/s. So if the spring is pulled with a constant velocity, a limit cycle occurs in the time evolution of the rigid body's velocity. The term "stick-slip" is justified by this alternative behavior.

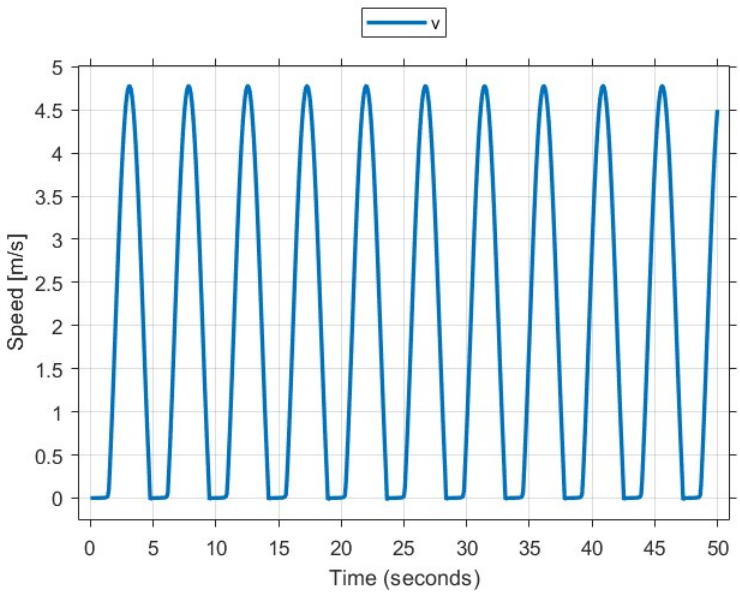

Figure 3: System's step response

Recall that the control problem is designed for tracking; specifically we want the output v to track the reference signal. For this reason, we defined the error dynamics and the control law was designed such that the error tends to zero.

The control law $u = \gamma^{-1}(x)[-\psi(x) + v]$ is well known in the context of feedback linearization, and the functions are defined in the previous section. The new control signal v is used to control the linear system, which results after mathematically eliminating the nonlinearity. In this sense, we consider a state feedback control w. We impose corresponding in the left half plane for the closed loop system.

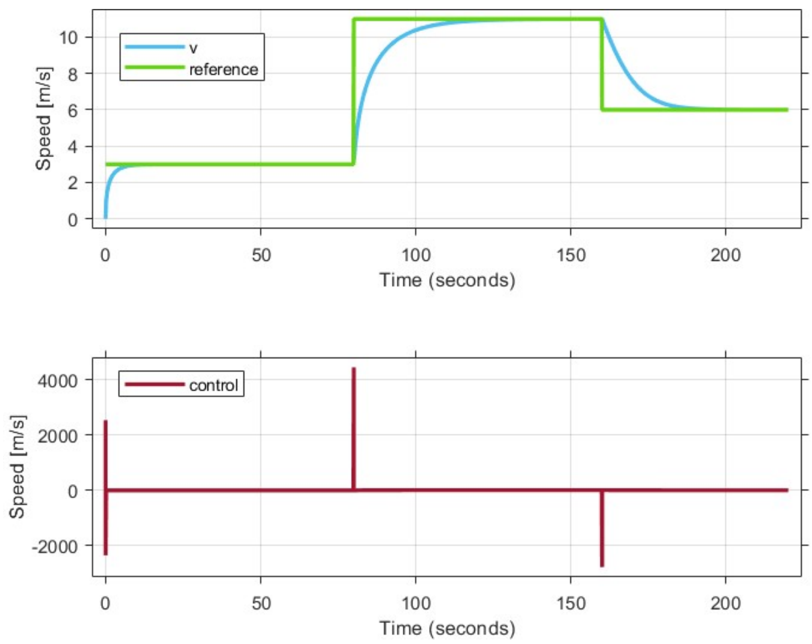

Figure 4: Tracking step signals

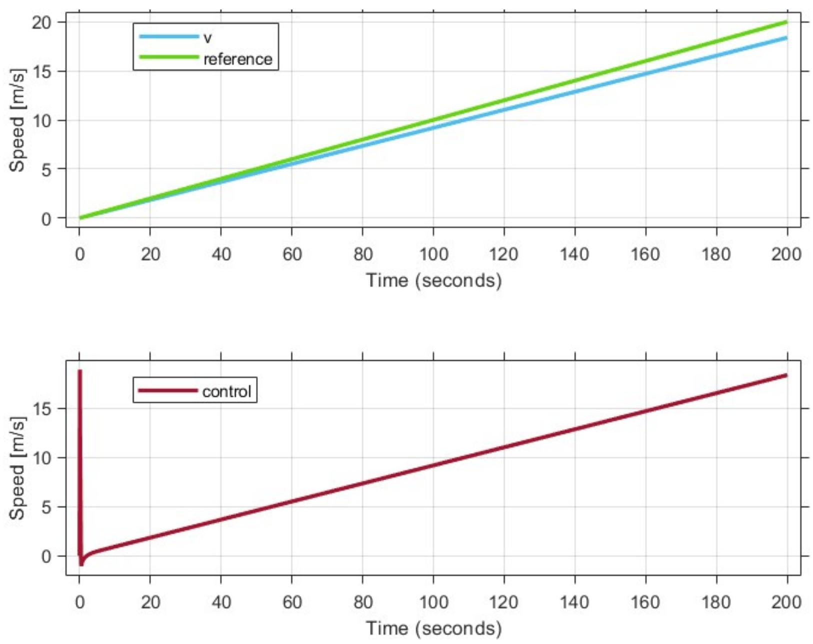

Figure 5: Tracking ramp signals

In Fig.4 we illustrate the closed loop system's response to a step signal. One can see that the steady-state position error is always zero as it was imposed and the settling time is variable depending on the values of the initial states. The control signal cannot be reproduced in practice. The imposed poles are $[-35 \quad -30]$. A pair of poles placed closer to the origin would reduce the control effort, but it would increase the settling time.

For ramp signals (Fig. 5 with slope value 0.1), a steady-state velocity error appears in this case at $t = 200$ seconds, the error is 1.6. This error can be decreased by imposing poles further from the origin. The control signal has normal values. As a note, tracking a slope value of 0.5 brings the system to numerical instability after 60 seconds.

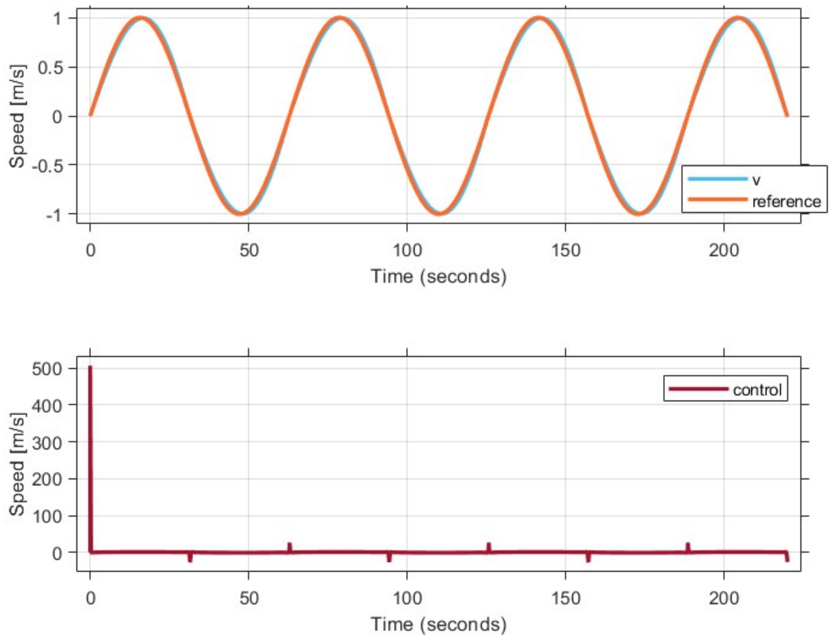

Figure 6: Tracking frequential signals

In Fig.6 we illustrate the system's response to the signal $r(t) = sin(2\pi \cdot 0.1 \cdot t)$. We notice the tracking performance is satisfactory, but it will get worse once we increase the frequency. Increasing the amplitude will affect the performance as well. Regarding the control signal it has great amplitude initially, but it later alternates in a favorable range.

To conclude, feedback linearization is clearly useful when dealing with nonlinear systems. However, the computational complexity becomes problematic in case of high order systems. Regarding tracking performance, feedback linearization combined with linear state feedback control achieves satisfactory performance in specific cases. However, when dealing with varying parameters, or they mismatch reality, other robust techniques are preferred, such as sliding mode control.

3 Bibliography

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