Markov Chain Monte Carlo in Cosmology

Searching for Primordial Magnetic Fields signatures in the CMB

Alex Zucca, October 26th 2020

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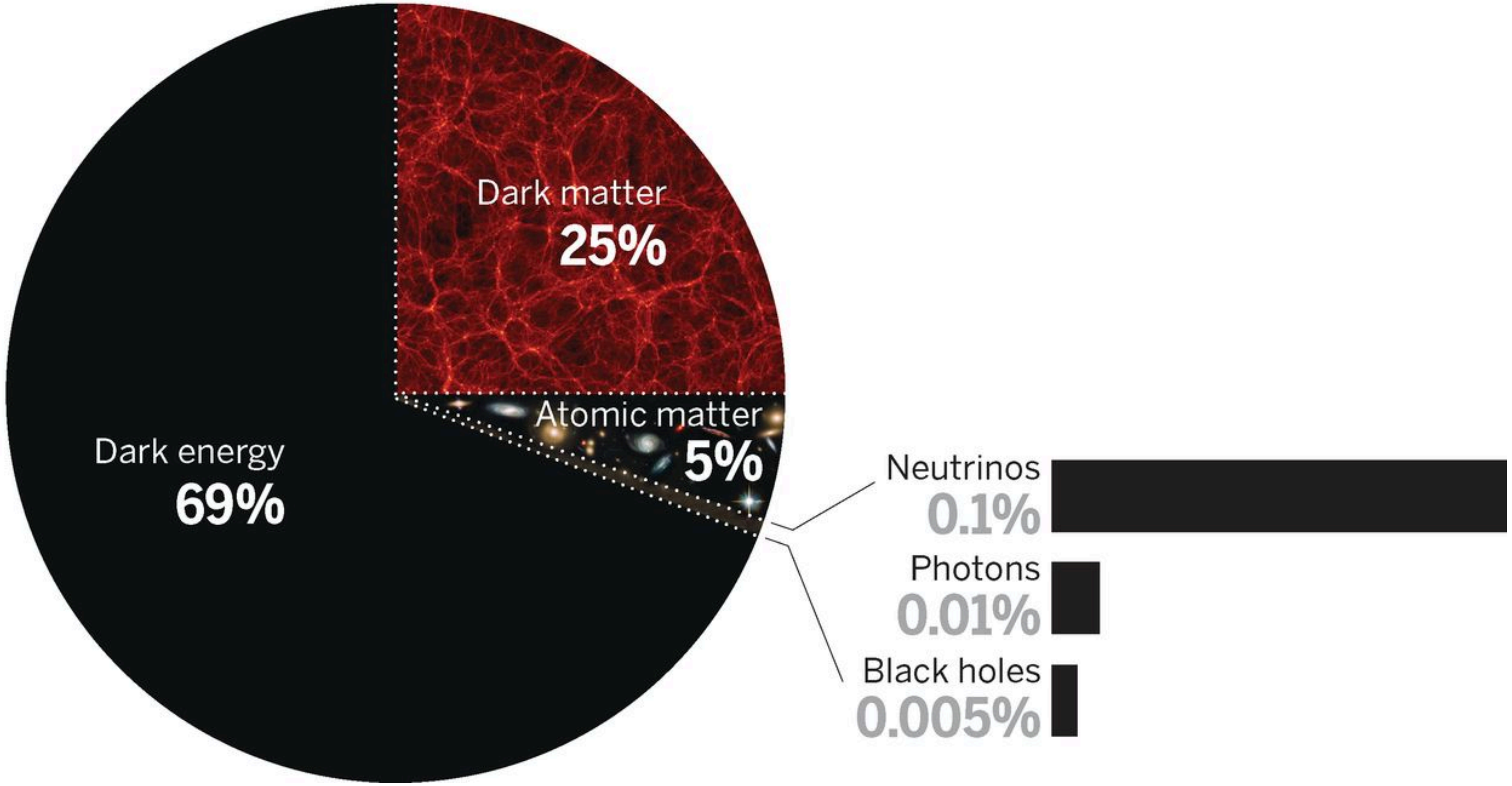
1. Theory

1.1 Intro to Cosmology

Homogeneous Background of the ΛCDM model



Friedmann Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_\kappa^2(r,R)d\Omega\right]$$

Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{m_0^2}T_{\mu\nu} - g_{\mu\nu}\Lambda$$

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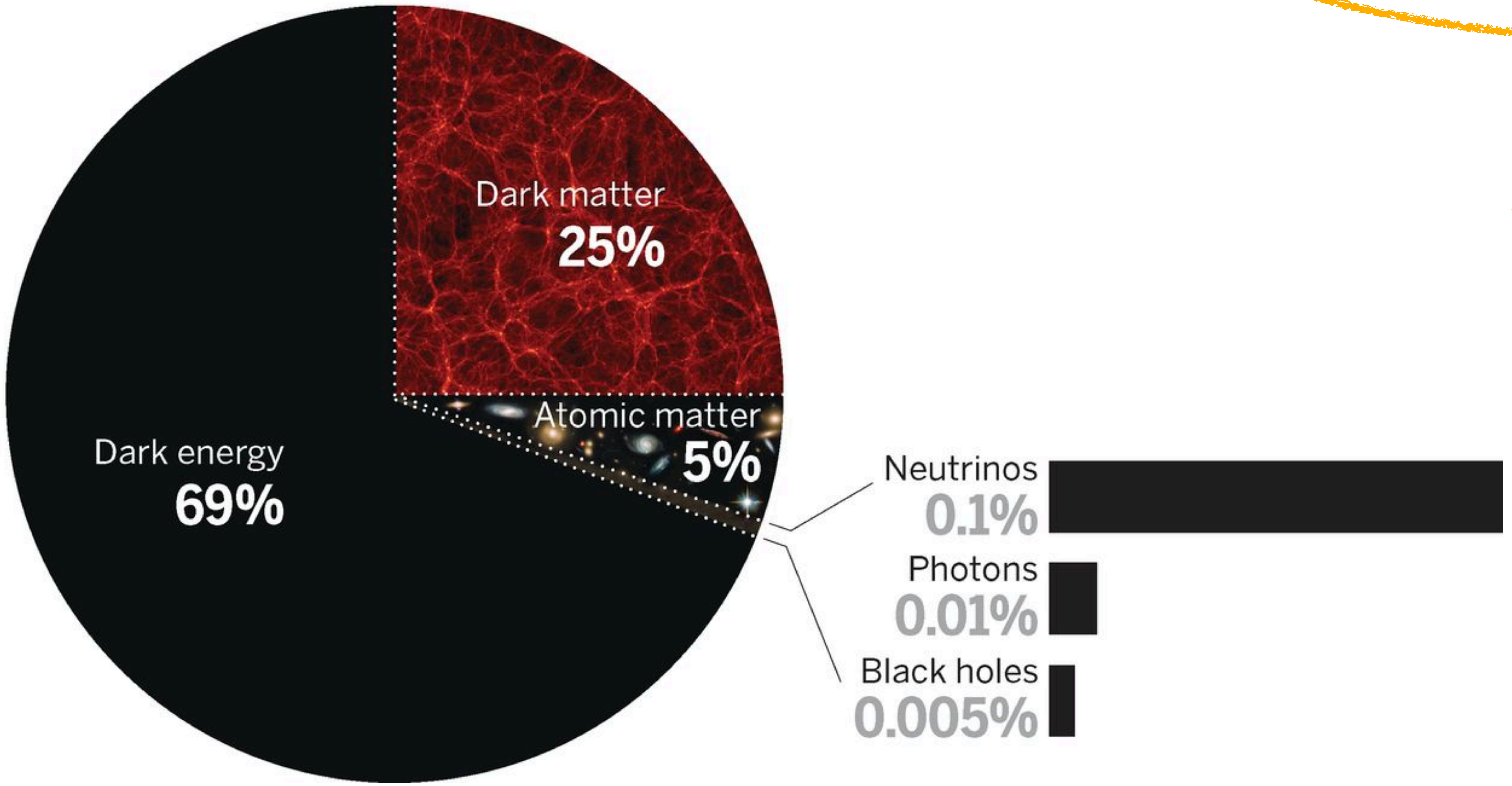

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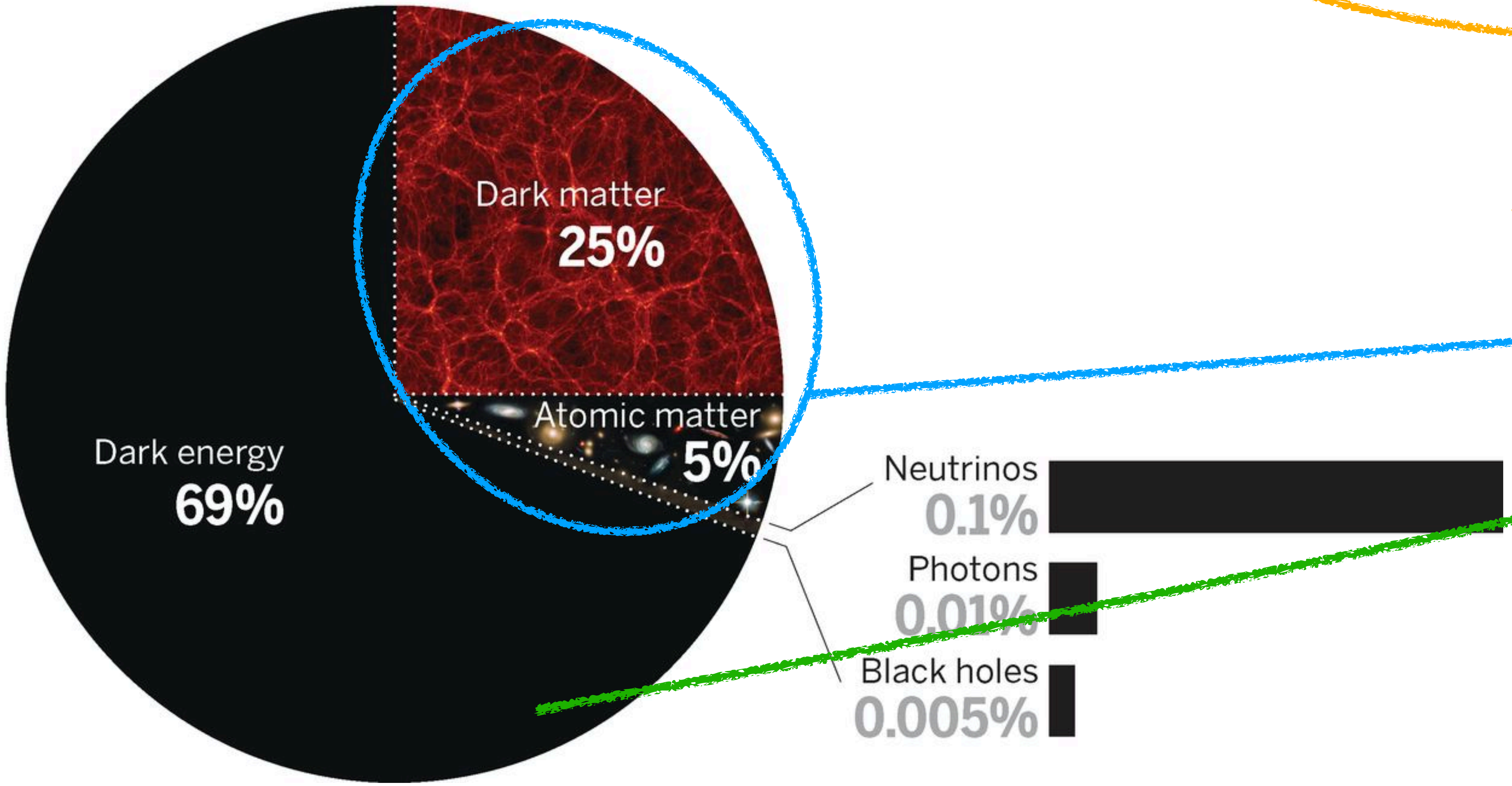

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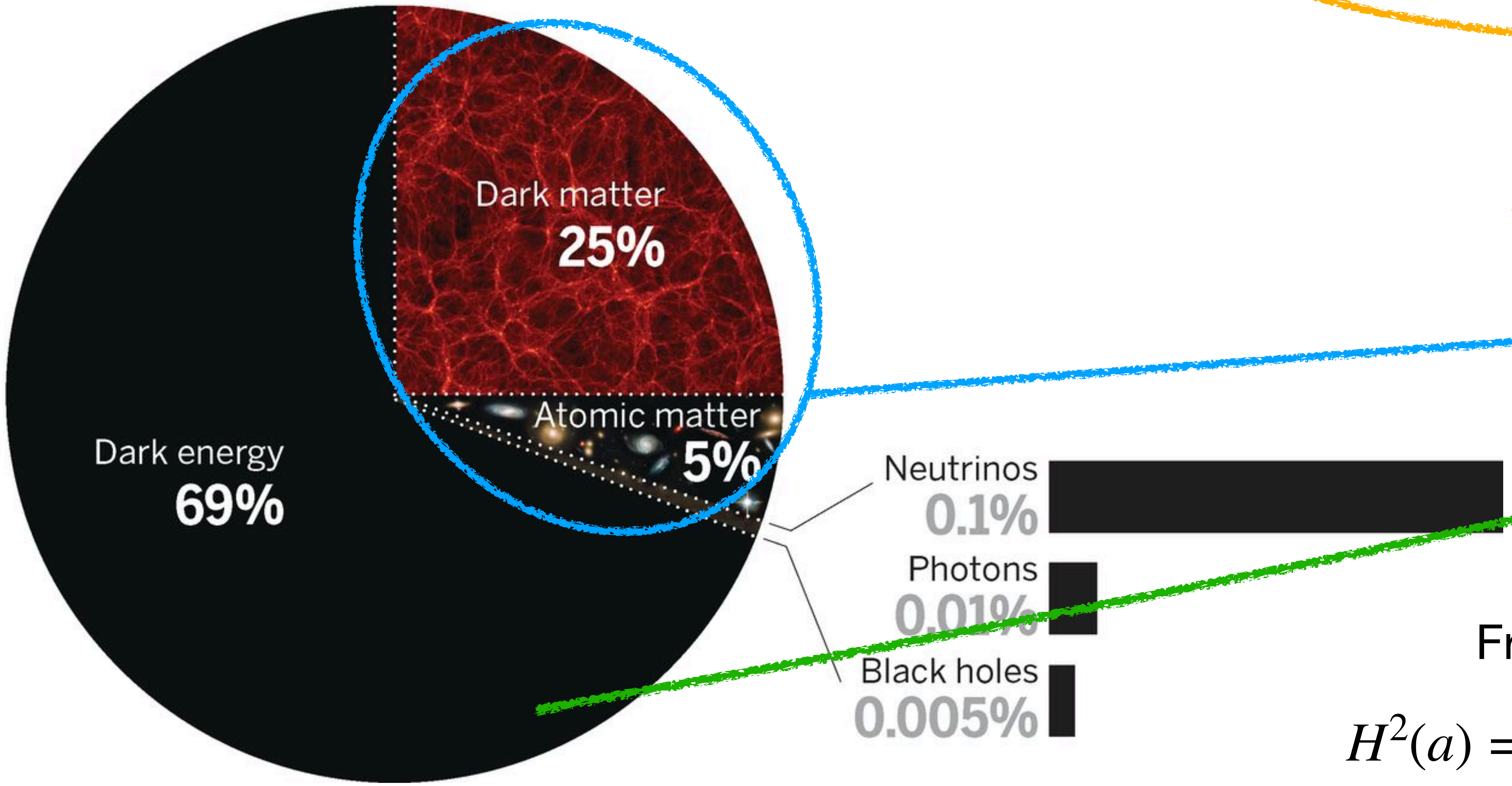

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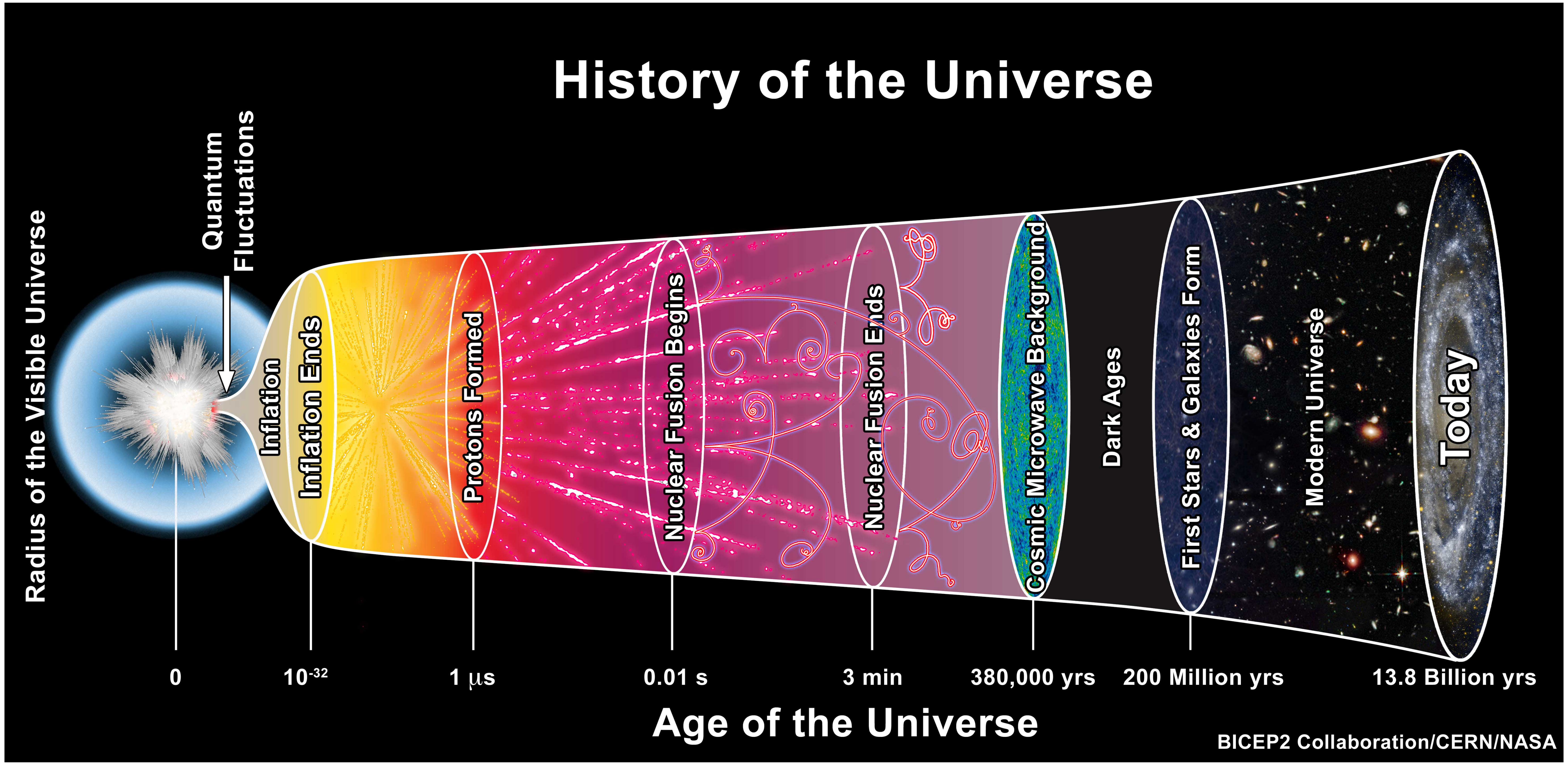
Friedmann Robertson-Walker metric

$$H^2(a) = H_0^2\left[\Omega_m a^{-3} + \Omega_\kappa a^{-2} + \Omega_r a^{-4} + \Omega_\Lambda\right]$$

$$H \equiv \frac{1}{a}\frac{da}{dt}$$



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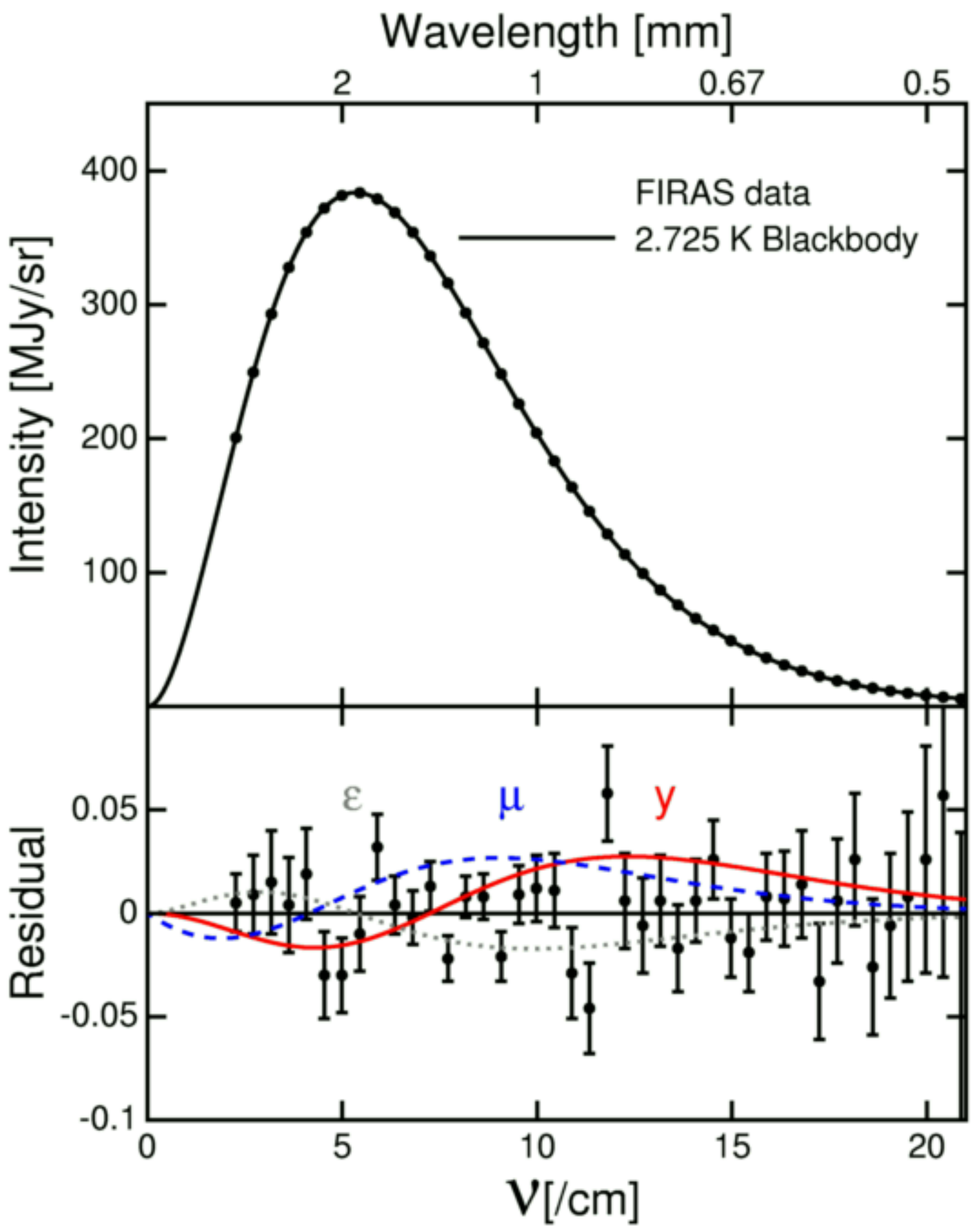
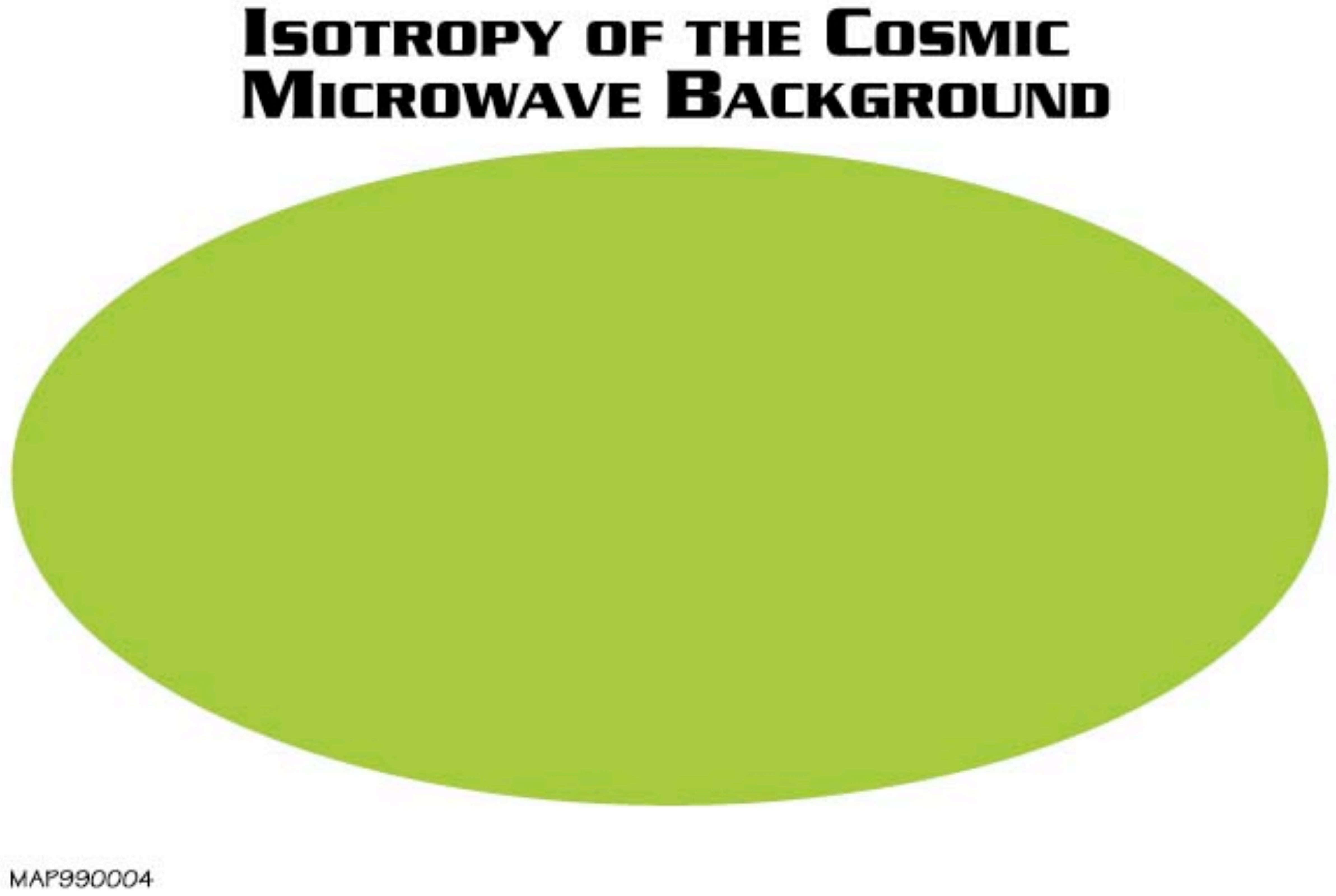
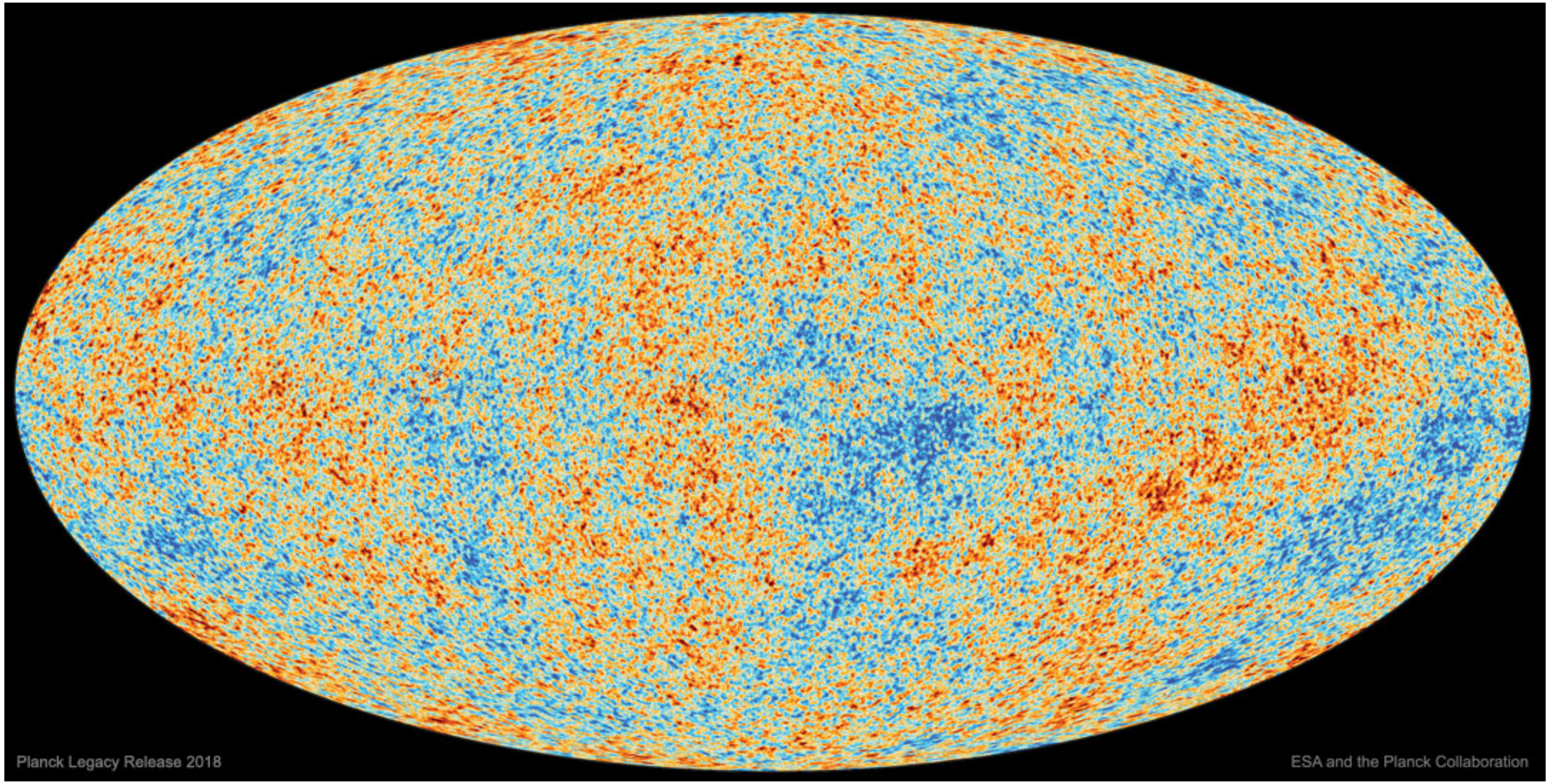
1.2 The Cosmic Microwave Background (CMB)





CMB anisotropies

map of $\Delta T_{\mathrm{CMB}}(\theta, \varphi)$



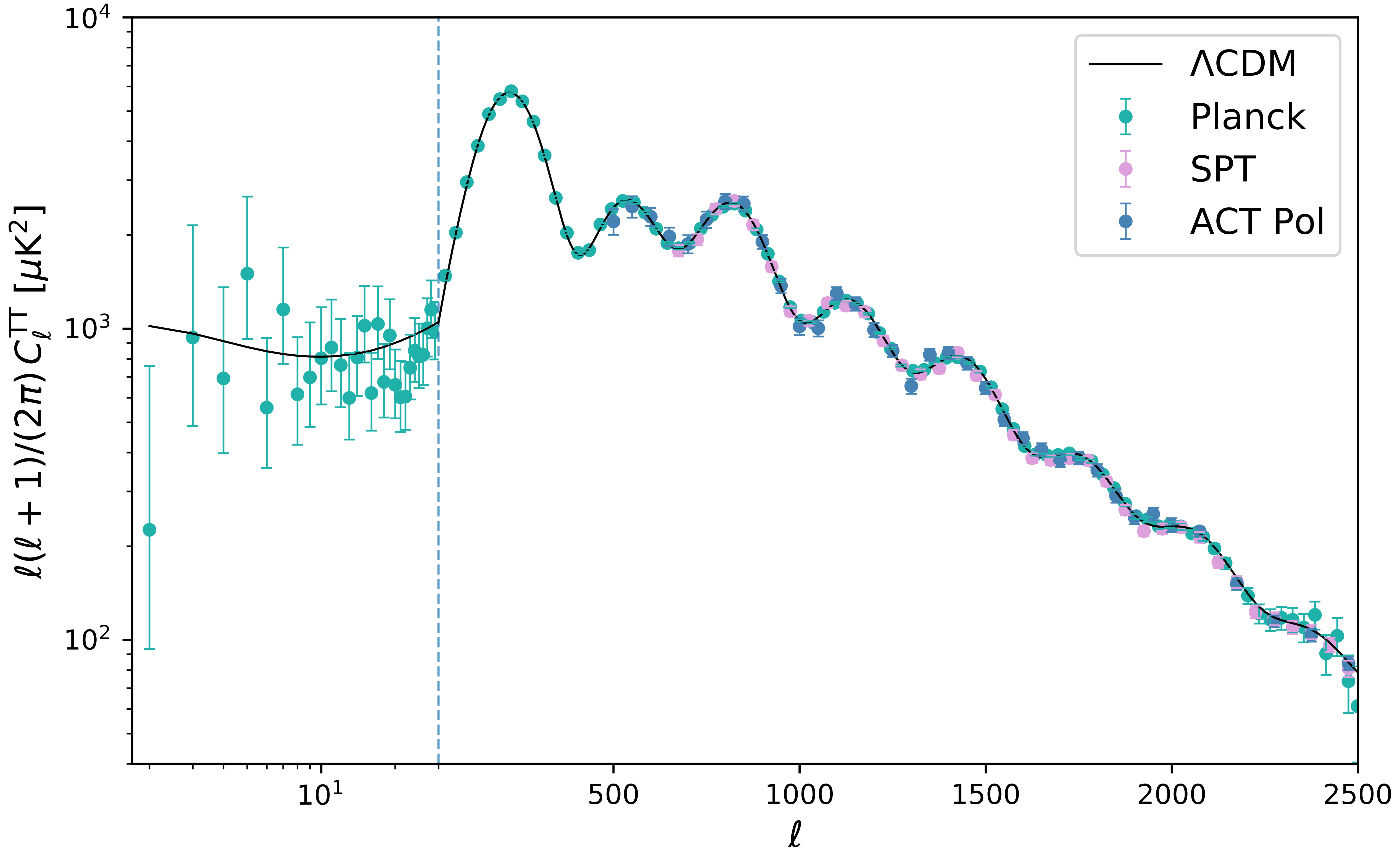
https://wiki.cosmos.esa.int/planck-legacy-archive/index.php/Frequency_maps

How CMB maps are used to test cosmological models

$$\Theta(\hat{\mathbf{n}}) \equiv \frac{\delta T_{\rm CMB}}{\bar{T}_{\rm CMB}} = \sum_{\ell,m} a_{\ell,m} Y_\ell^m(\hat{\mathbf{n}})$$

we cannot predict $T_{\rm CMB}$ at a given position in the sky, but we can predict correlations between different positions in the sky

$$\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{TT} \mathscr{P}_\ell(\theta)$$



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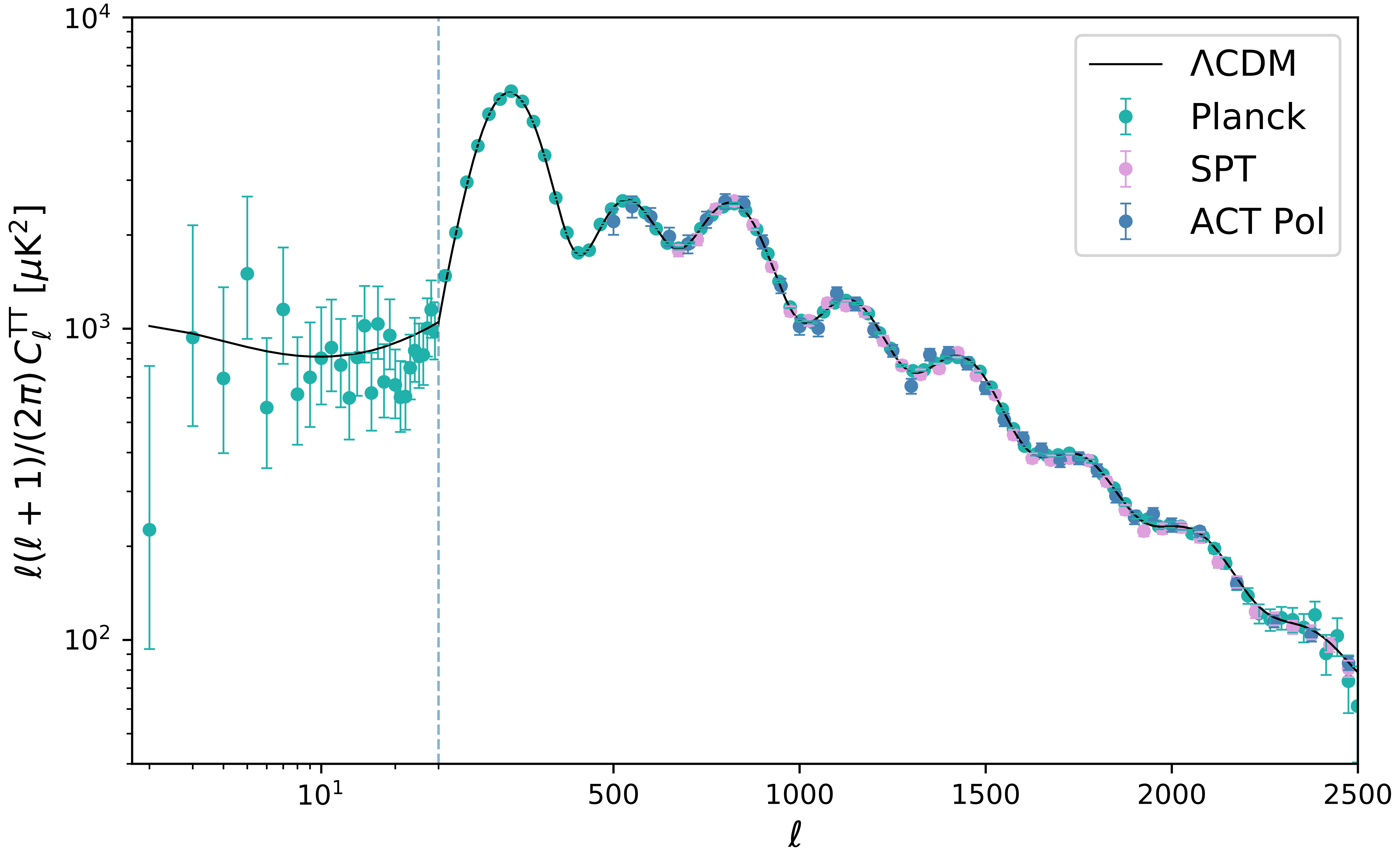
Boltzmann codes: $C_\ell^{\rm TT,\,th.}$

e.g.
CAMB:
CLASS
PICO



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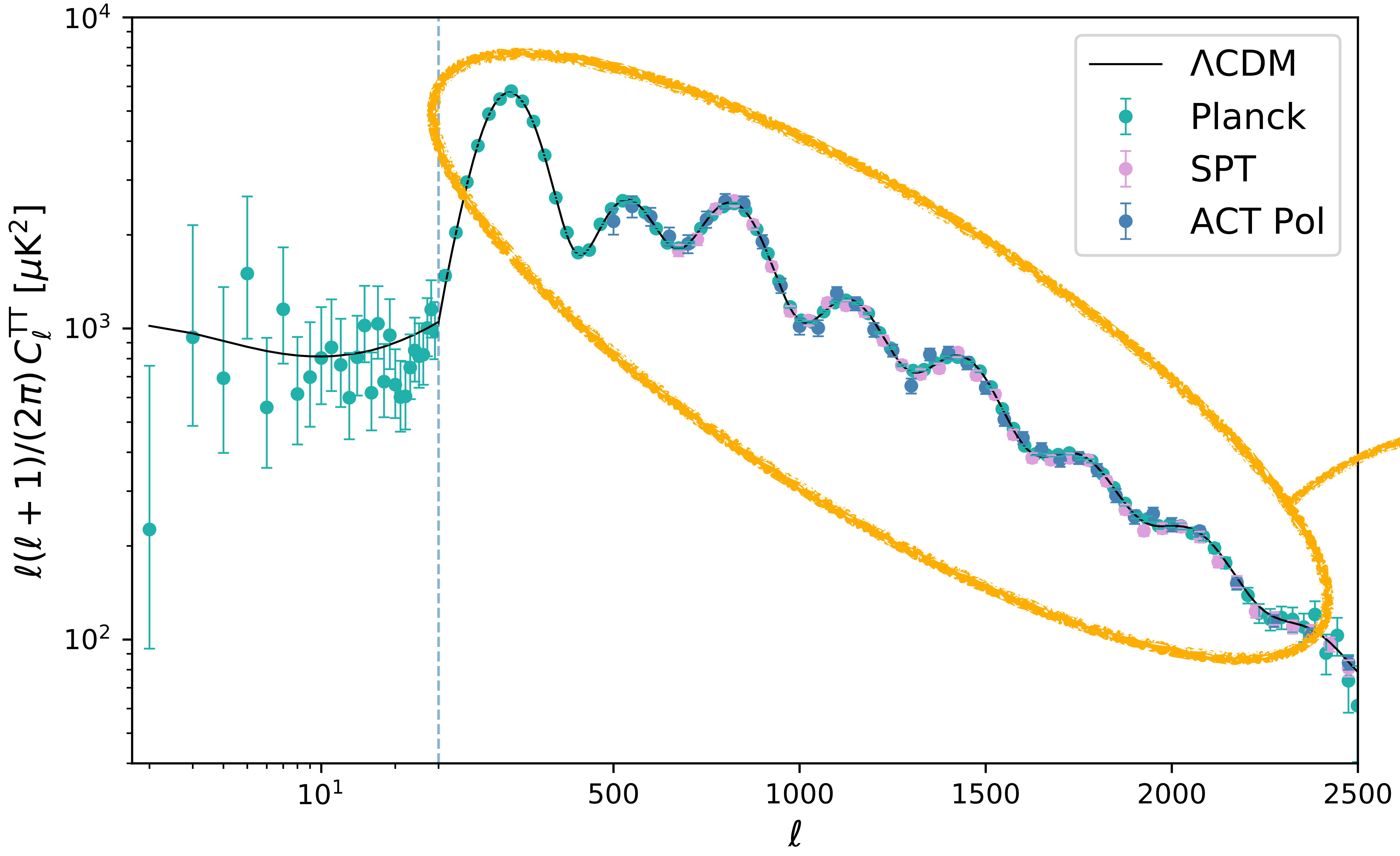
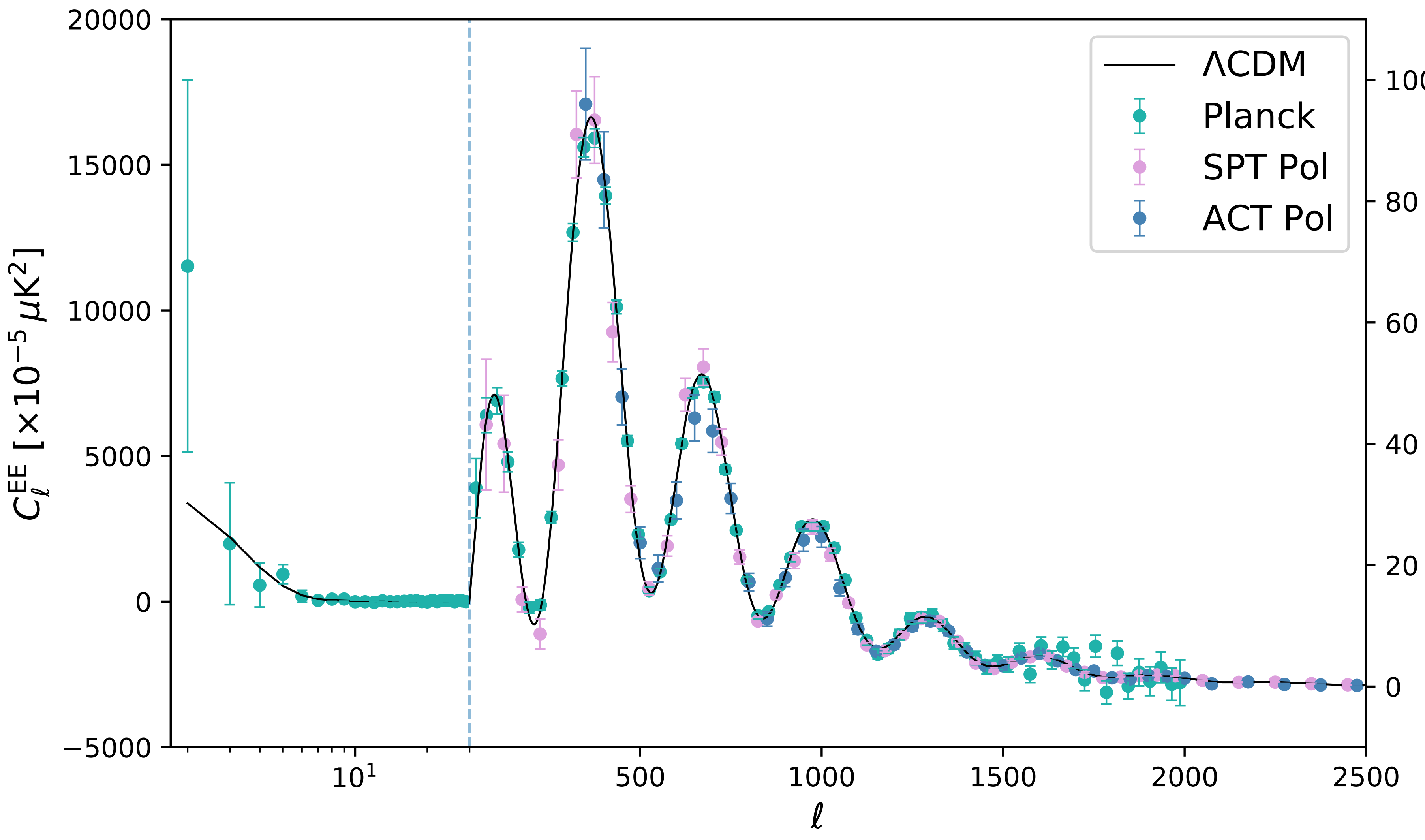
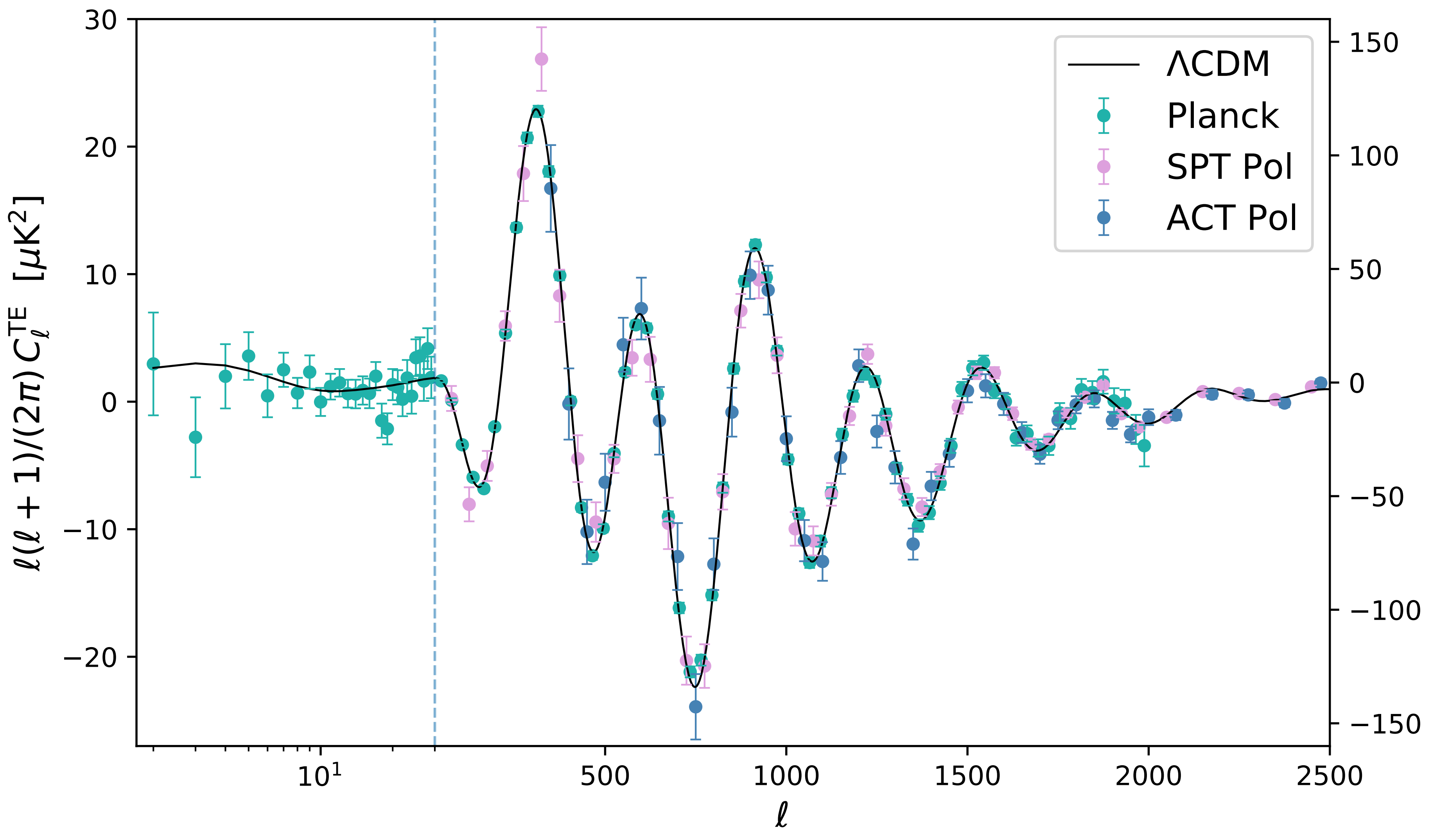
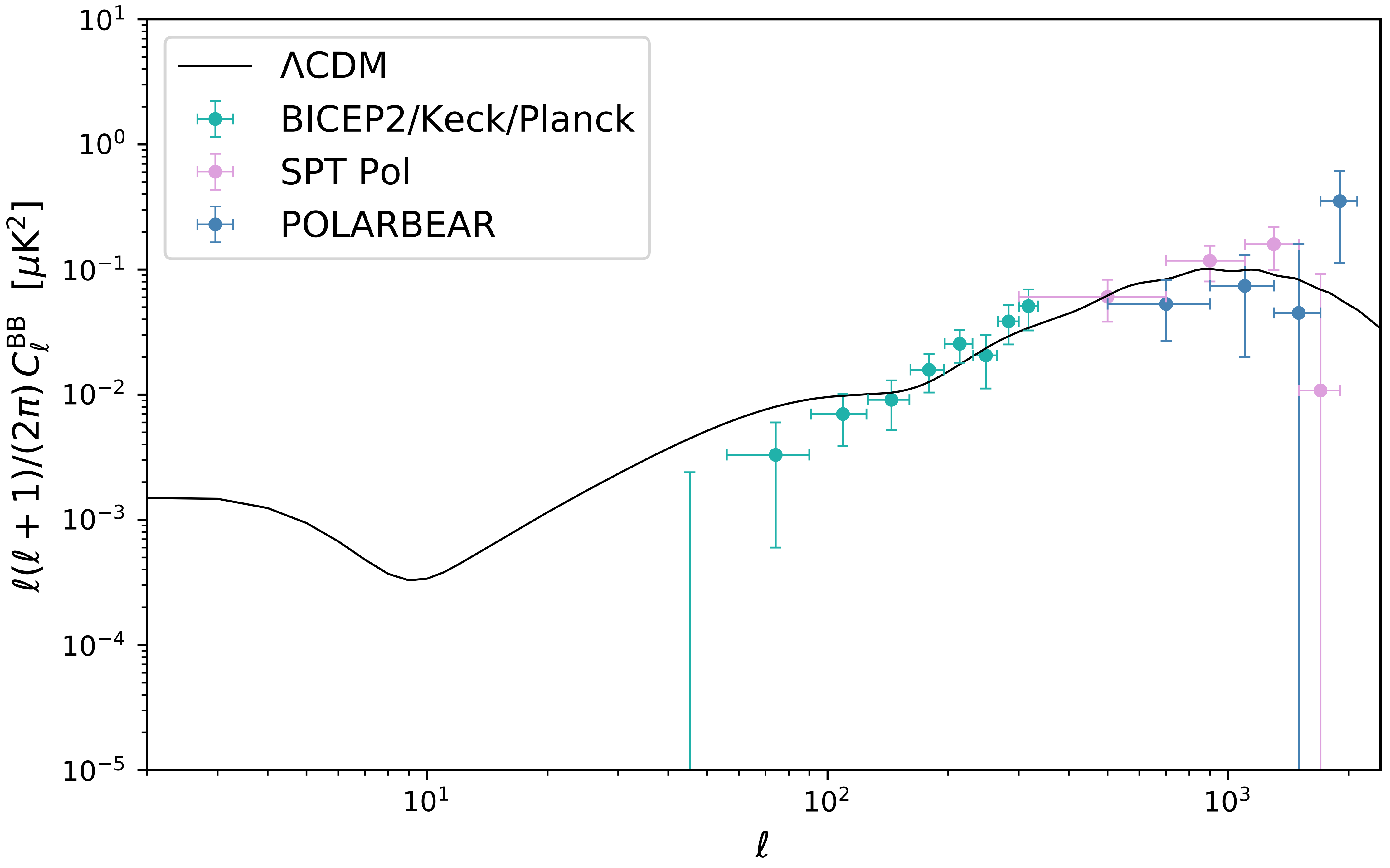


acoustic oscillations

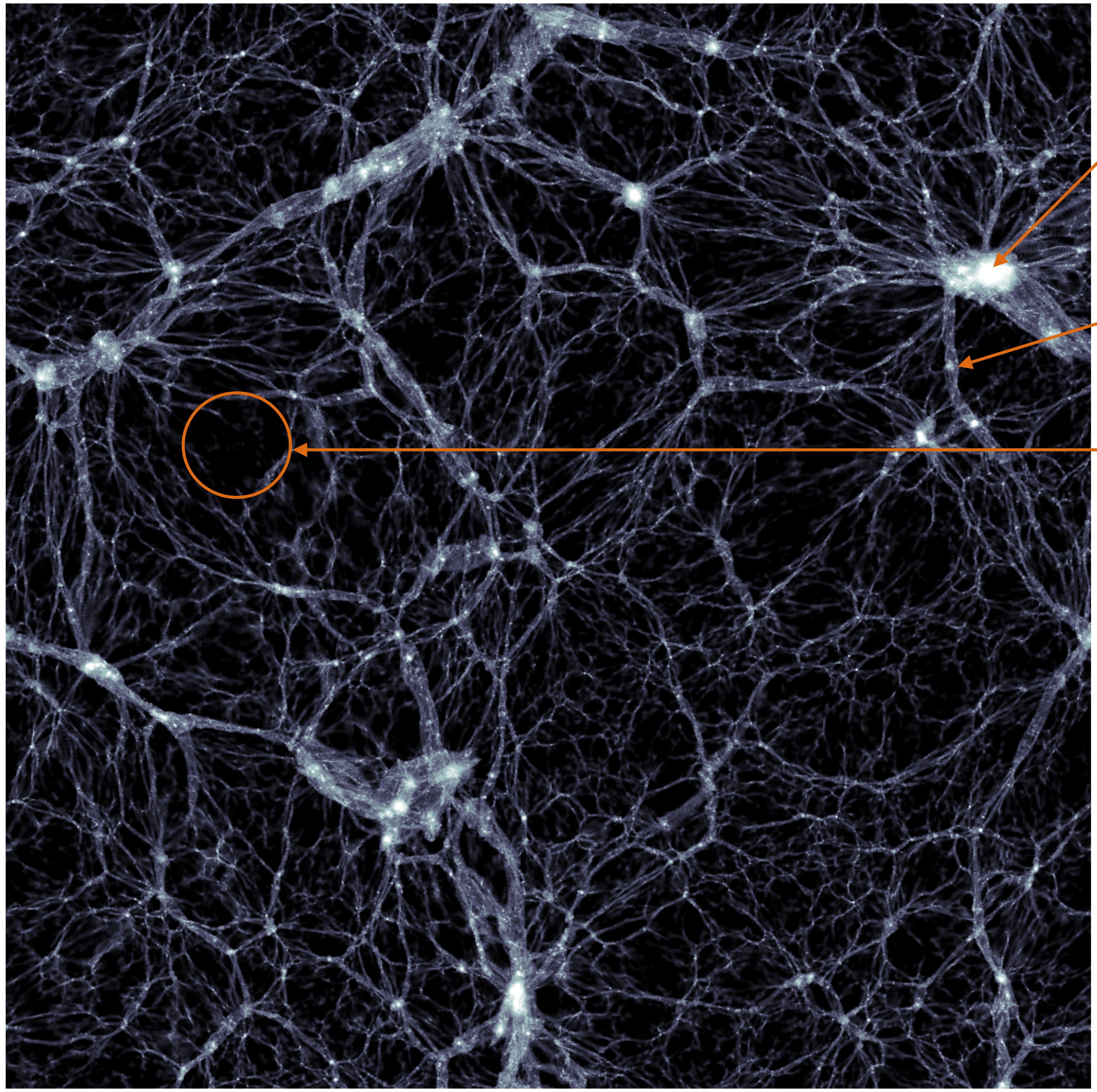
CMB polarization







1.3 Magnetic Fields in the Universe



Galaxies: $B \approx \text{few}\, \mu G$

Galaxy Clusters: $B \approx \text{few}\, \mu G$

Galaxy Filaments: $B \approx 10\, nG$

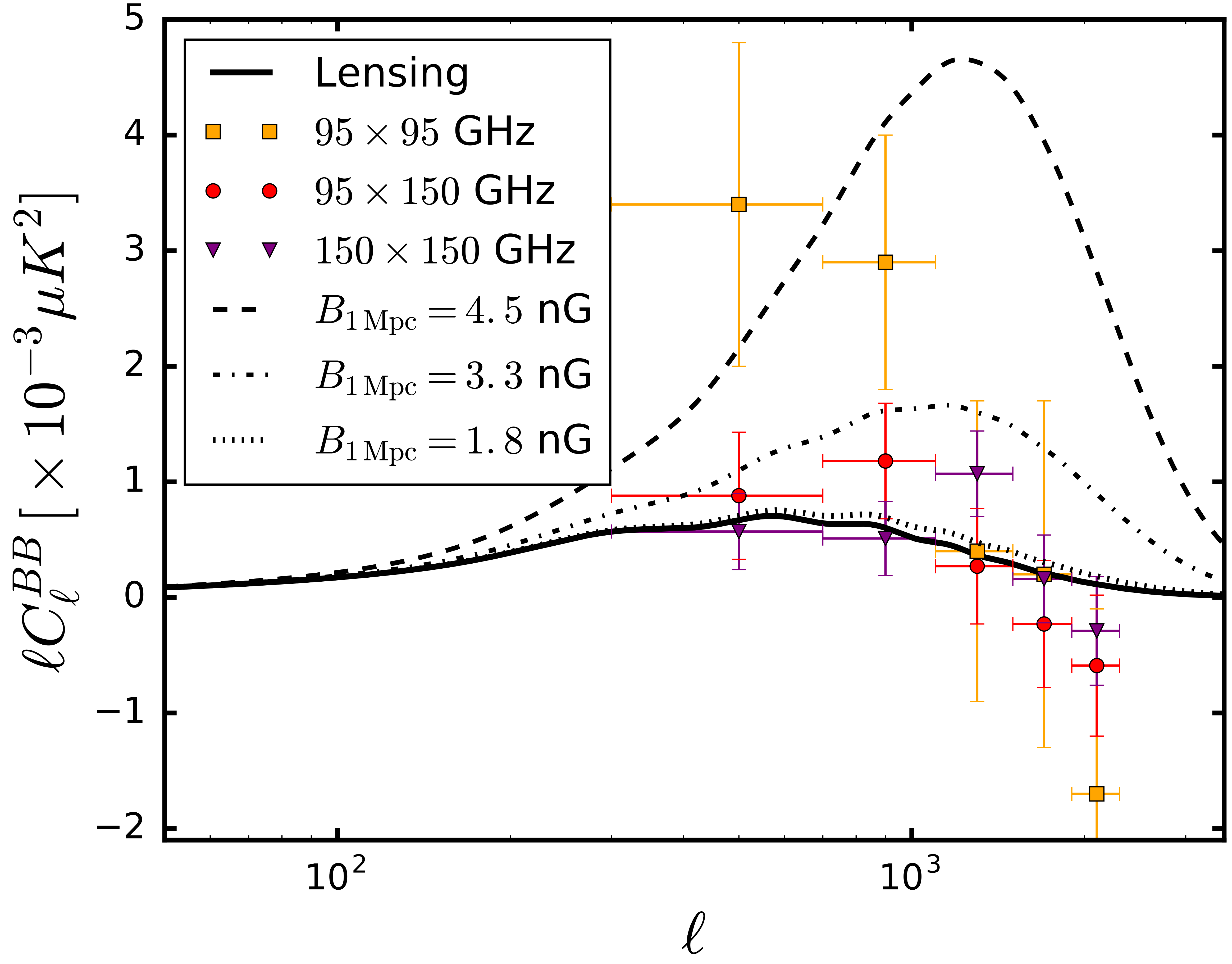
Voids: $B \gtrsim 10^{-16} G$ (?)

A.Neronov, I.Vovk, Science (2010)

Origin unknown

- Astrophysical?
- **Primordial?**

PMF can be detected in the CMB B-mode signal at small scales



1.4 Markov Chain Monte Carlo

Data Analysis

Likelihood function: $\mathscr{L}(\{d_i\} \mid \{w_a\}) \equiv P(\{d_i\} \mid \{w_a\})$ probability of the data $\{d_i\}$, given a theory $\{w_a\}$

We are interested in the probability of the theory $\{w_a\}$, given the data $\{d_i\}$, called **POSTERIOR** probability.

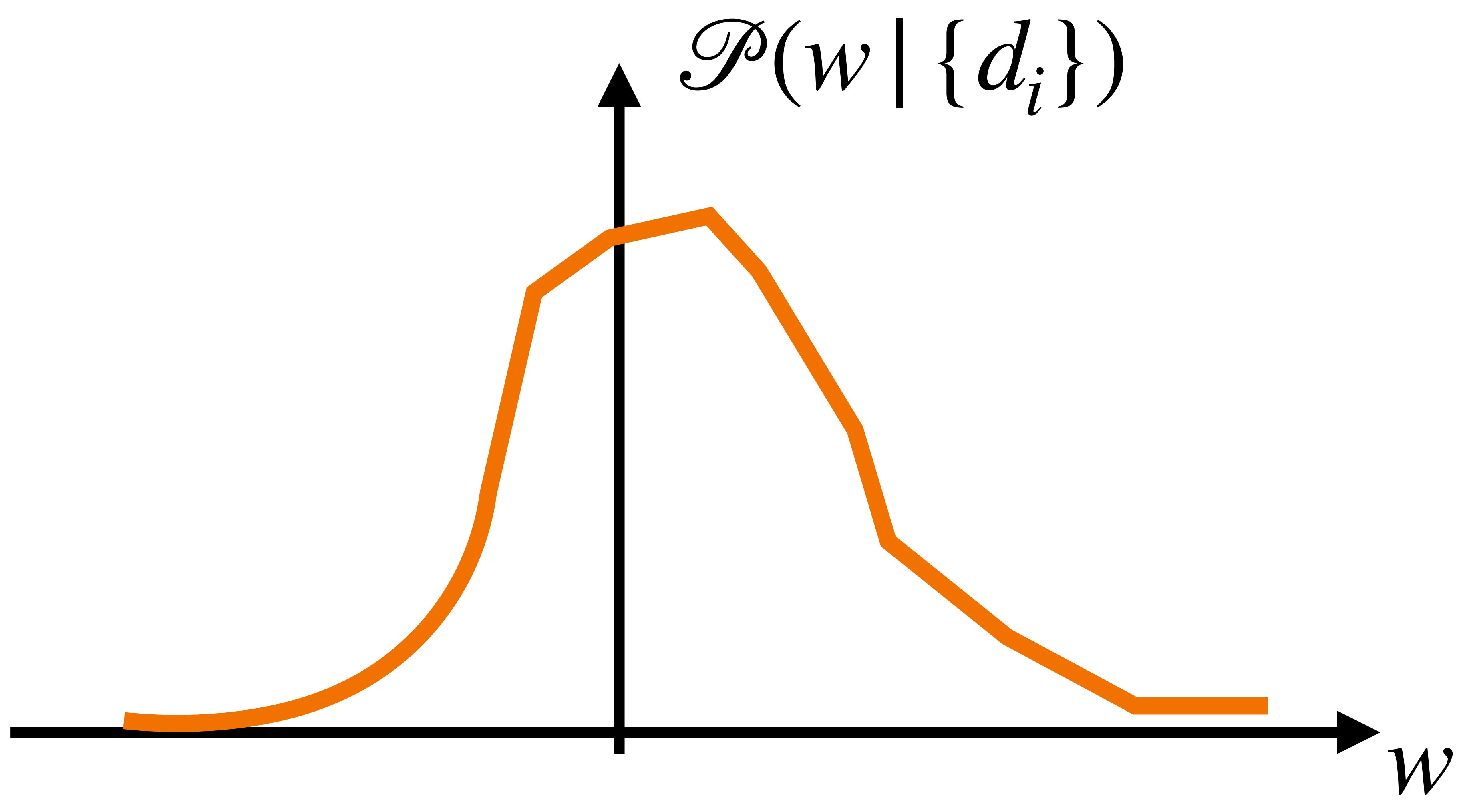
$$\mathscr{P}(\{w_a\} \mid \{d_i\}) = \frac{\mathscr{L}(\{d_i\} \mid \{w_a\}) \times P(\{w_a\})}{P(\{d_i\})}$$

PRIOR

GOAL 1: best $\{w_a\}$ where $\mathscr{P}(\{w_a\} \mid \{d_i\})$ peaks.

GOAL 2: marginalize over "nuisance" parameters: $\{w_a\} = \{w_a^{\text{phys.}}, w_b^{\text{nuis.}}\}$

$$\mathscr{P}(\{w_a^{\text{phys.}}\}) = \int \prod_b dw_b^{\text{nuis.}} \mathscr{P}(\{w_a^{\text{phys.}}\}, \{w_b^{\text{nuis.}}\})$$



$$\mathbb{E}[w] = \int dw w \mathscr{P}(w\,|\,\{d_i\}) \approx \frac{\text{width}}{M} \sum_{k=1}^{M} w_k \mathscr{P}(w_k)$$

$\mathcal{O}(M)$ likelihood computations

If we have n parameters, we require $\mathcal{O}(M^n)$ likelihood computations.

if $t_{\text{like}} \approx 1s$, and $n = 30$, and we want to have a grid
of 100 points in each dimension, we require $(10^2)^{30} \approx 3 \times 10^{52}$yrs!!!

IDEA: generate a sequence of points in parameter space $\{\mathbf{w}_i\}$ (approximately) distributed as $\mathscr{P}(\mathbf{w}\,|\,\mathbf{d})$, then

$$\langle f \rangle \approx \frac{1}{M} \sum_{i=1}^{M} f(\mathbf{w}_i)$$

The sequence $\{\mathbf{w}_i\}$ is generated through a Markov Chain

Markov Chain

Random walk: $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$, where the points are asymptotically distributed as a target probability $\mathscr{P}(\mathbf{w})$.

It is defined by a transition probability $T(\mathbf{w}_i \to \mathbf{w}_j)$. If T depends only on $\mathbf{w}_i$ and not on the previous history - how we got to $\mathbf{w}_i$ - the chain is called a **Markov Chain**

It is sufficient to satisfy to **detailed balance** condition: $\mathscr{P}(\mathbf{w}_i)T(\mathbf{w}_i \to \mathbf{w}_j) = \mathscr{P}(\mathbf{w}_j)T(\mathbf{w}_j \to \mathbf{w}_i)$

Metropolis-Hastings algorithm

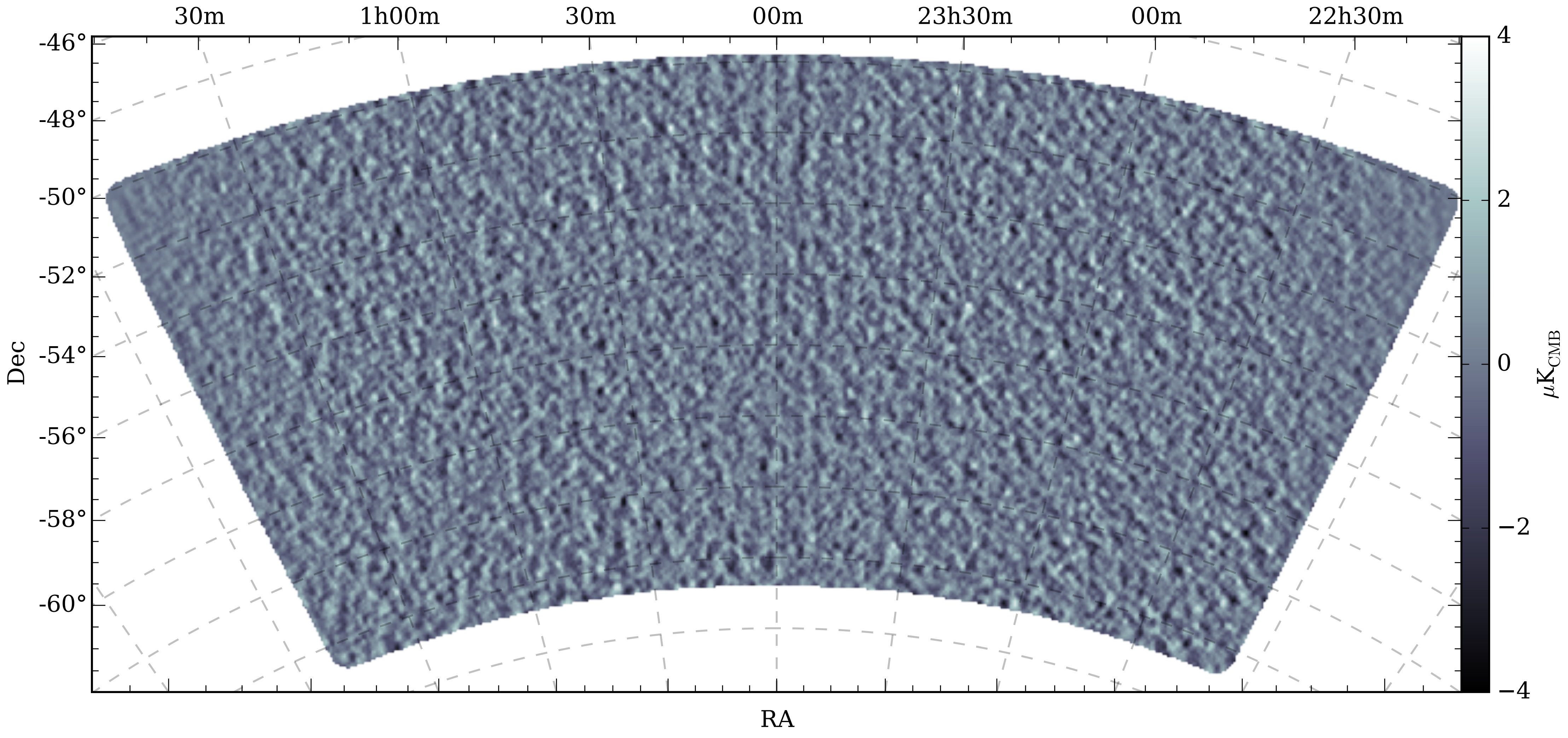
Propose a new point, $\mathbf{w}_{\text{trial}} = \mathbf{w}_i + \delta\mathbf{w}_i$ and accept the move with probability $p = \min\left[1, \dfrac{\mathscr{P}(\mathbf{w}_{\text{trial}})}{\mathscr{P}(\mathbf{w}_i)}\right]$

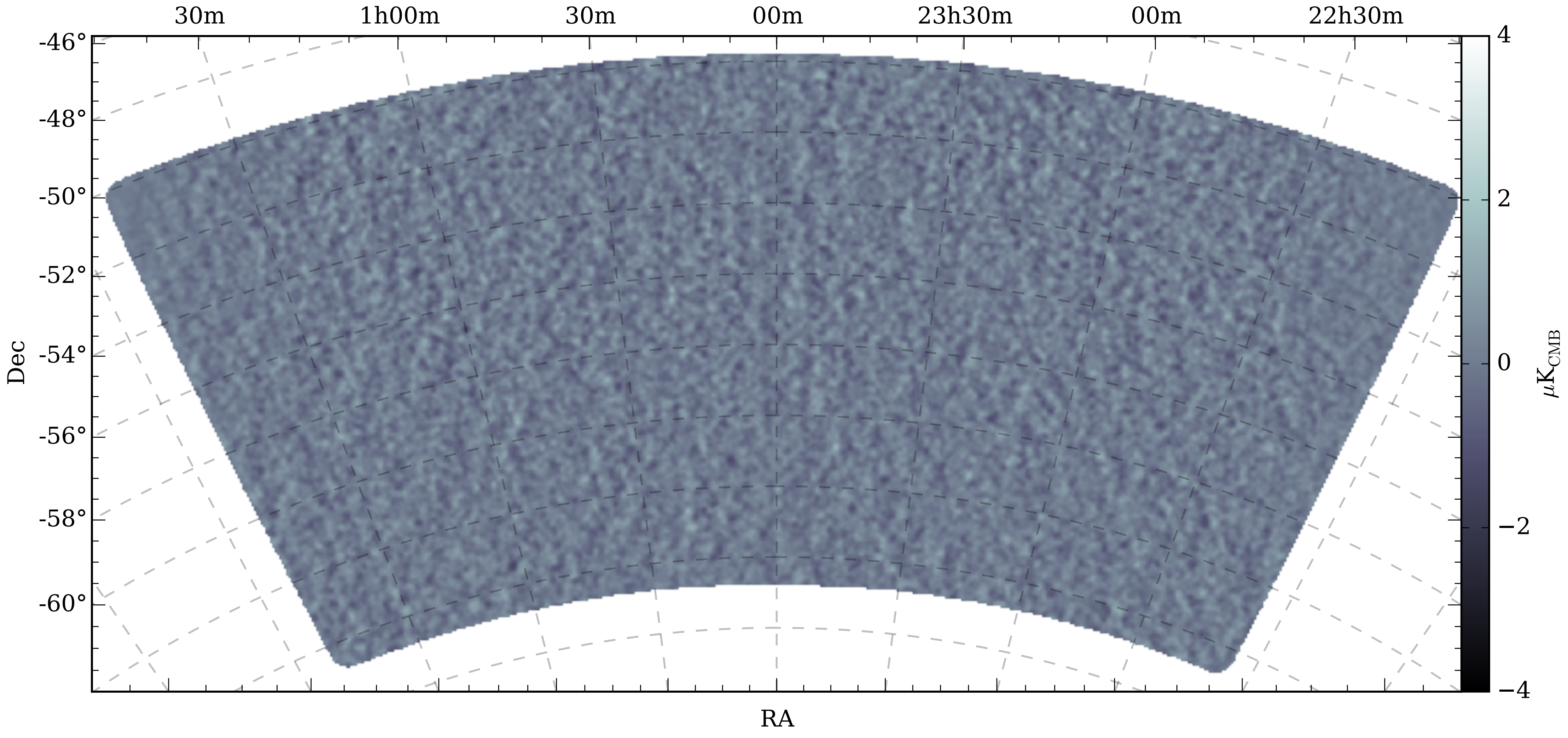
If accepted, $\mathbf{w}_{i+1} = \mathbf{w}_{\text{trial}}$, else $\mathbf{w}_{i+1} = \mathbf{w}_i$

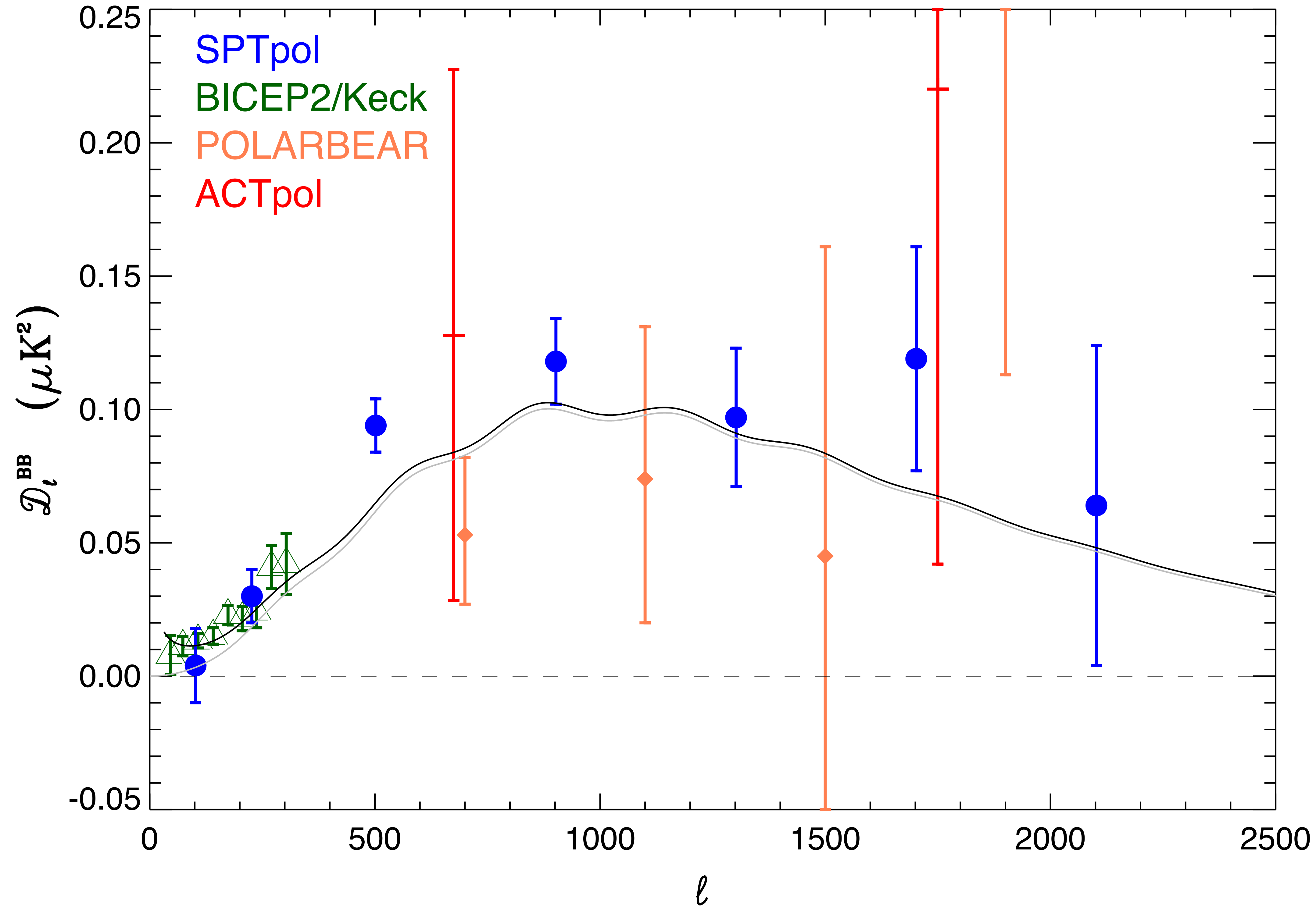
2. Application

Code is here: https://github.com/alexzucca90/py_spt_emcee

We'll use the MCMC Python package `emcee` to fit the **South Pole Telescope** (SPT) CMB B-mode data to the standard Cosmological predictions + PMF







Phys. Rev. D 101, 122003 (2020)

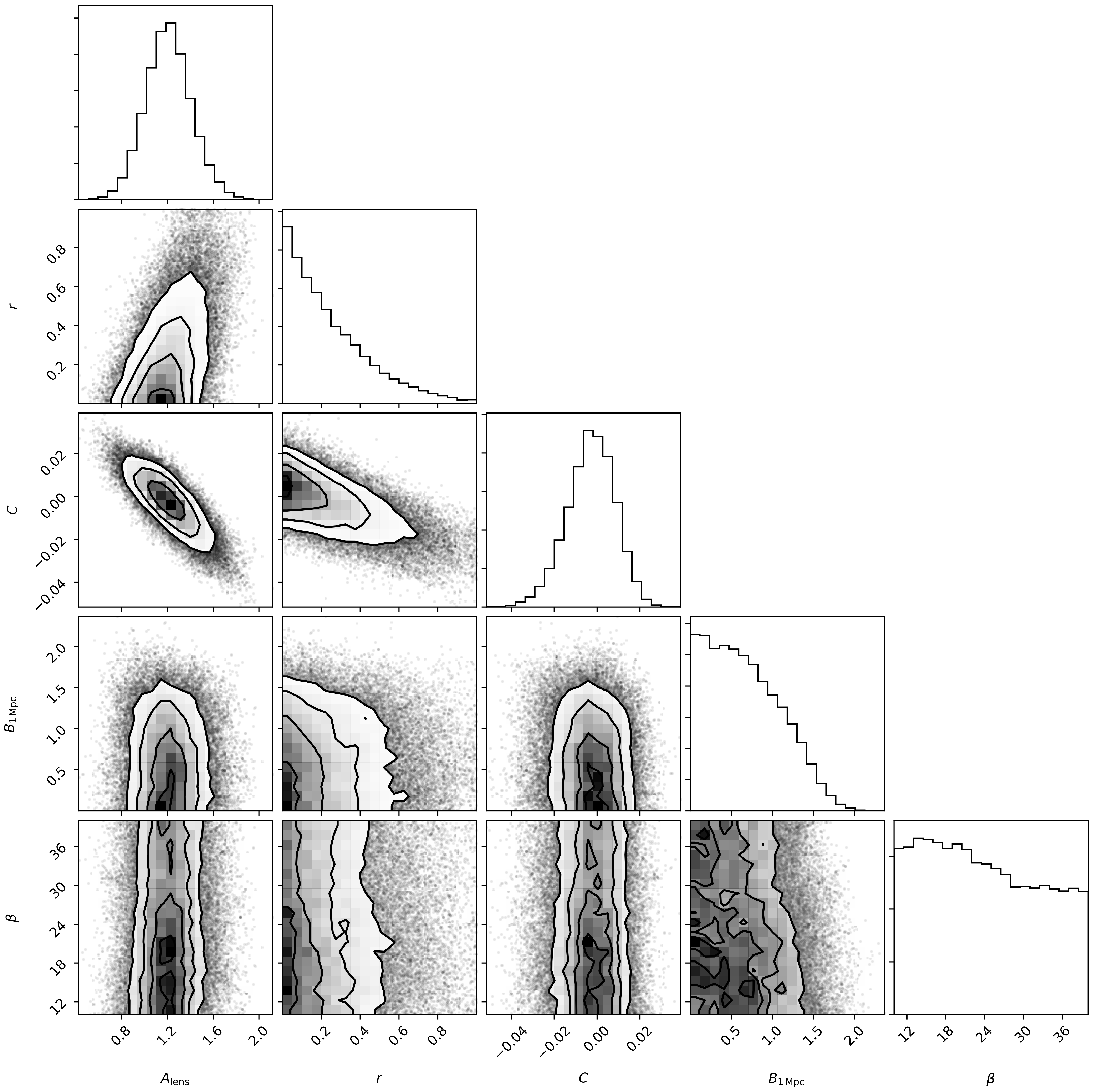
emcee performs the MCMC for us, we just need to give it the likelihood

SPT likelihood: $-\log \mathscr{L}_{\rm SPT} = \frac{1}{2}\left[\Delta \mathscr{D}_{\ell'} C^{-1}_{\ell'\ell} \Delta \mathscr{D}_{\ell} + \log |C|\right]$

$$\mathscr{D}_\ell = A_{\rm lens}\mathscr{D}_\ell^{\rm lens.} + r\mathscr{D}_\ell^{\rm tens.} + C + A_{\rm PMF}\left[\mathscr{D}_\ell^{\rm PMF,V} + A_\beta \mathscr{D}_\ell^{\rm PMF,T}\right] + P_L^{\nu_1\times\nu_2}\mathscr{D}_\ell^{\rm Poiss.} + A_{\rm dust}S_{150}^{\nu_1\times\nu_2}\mathscr{D}_\ell^{\rm dust,150}$$

$$\Delta \mathscr{D}_{\ell_{\rm bin}} = f_{\rm beam}\mathscr{D}_{\ell_{\rm bin}} - \mathscr{D}_{\ell_{\rm bin}}^{\rm data}$$

$$\mathscr{D}_{\ell_{\rm bin}} = W_{\ell_{\rm bin}\ell}^{\nu_1\times\nu_2}\mathscr{D}_\ell / A_{\rm cal}^{\nu_1\times\nu_2}$$



r
C
$B_{1\,\mathrm{Mpc}}$
β
A_{lens}
r
C
$B_{1\,\mathrm{Mpc}}$
β

Thank you!