Phys 607: Project-2

Aleyna Akyüz and John Batarekh

I. Introduction

In this project, we will develop a new metamaterial that is energy-sensitive to photons that pass through it. When given an initial current, our metamaterial acts as RLC circuit that is governed by the second-order ODE

$$\ddot{I} + 2\alpha\dot{I} + \omega_0^2 I = 0 \tag{1}$$

where $\alpha = R/2L$ and $\omega_0^2 = \frac{1}{LC}$. We find that these values are $R = 400n\Omega$, $L = 10\mu H$, and $C = 1fF$, such that $\omega_0^2 =$ 10GHz.

We observe that when light passes through our metamaterial, the photon frequency couples to the oscillator and changes the ODE as follows:

$$\ddot{I} + 2\alpha\dot{I} + (\epsilon_c\omega^2 + \omega_0^2)I = 0 \tag{2}$$

where ω^2 is the photon's frequency and ϵ_c is the coupling efficiency. We find that the photon coupling efficiency is related to the position that the photon passes through relative to the edges of our metamaterial sample and the polarization angle such that

$$\epsilon_c = \frac{x^2 + y^2 + a^2}{2a^2}(\cos(\theta) - \sin(\theta)) \tag{3}$$

Where x and y are positions when the sample is centered at $(0, 0)$, a is the side length of the metamaterial sample, and θ is the polarization angle. (I now realize that this will imply that we are able to measure both the frequency and position of the photon, but we just have to go with it)

We want to simulate the average energy fluctuation which we define as the difference between the expected average energy and the average simulated energy generated by a microwave black-body source. We will do this simulation using the Monte-Carlo method. The specifics of which will be discussed in the next section.

II. Method

A. Rejection Sampling

In our Monte-Carlo simulation, we will be randomizing frequencies created by a black body source with and effective temperature of about $.1K$ such that that the frequency of the generated light is in the microwave regime. To do this we will use Planck's Law of spectral radiance

$$B(f, T) = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{k_B T}} - 1} \tag{4}$$

Since this equation does not have a simple integral, it is not worth attempting the inverse CDF randomization method. Instead, we will use the rejection sampling method. We did this by calculating the frequency that maximizes the spectral radiance function

$$f_{max} = \frac{3.157 \cdot k_B T}{h} \tag{5}$$

Then generate N random spectral radiances between 0 and $B(f_{max}, T)$, where N is an integer number of photons we want to create. Next, we generate N random frequencies between $.001 * f_{max}$ and $5 * fmax$. Lastly, we compare the value of our randomized spectral radiance and the value of $B(f_{rand}, T)$, and if the randomized spectral radiance is less than the value of spectral radiance calculated at random frequency, then we keep the value of the random frequency. The one assumption we created with this process is the bounds of the random frequency. Technically, all values from 0 to ∞ are possible, however it is representative to use the bounds described earlier and also makes the time complexity much lower.

B. Inverse CDF

In our Monte-Carlo simulation, we want to assume that our light source is reasonably close to the detector. This implies that the light will be more likely to be close to the center of the detector since the intensity is proportional to $\frac{1}{r^2}$ such that

$$\frac{1}{r^2} = \frac{1}{x^2 + y^2 + d^2} \tag{6}$$

where x is the distance in the horizontal direction, y is the distance in the vertical direction, and d is the shortest distance between the source and the detector. As it turns out, this function is not integrable, in fact I believe it is less integrable than the spectral radiance equation. However, a good approximation is a 2-dimensional Gaussian. It is by no means perfect but we can see from the following graphs that the $1/r^2$ and the Gaussian have similar shapes. We also plot out the percent error between the two plots and we see that if we choose $\sigma = \sqrt{1.185}a$ we can minimize the percent error to about 1%.

To calculate the inverse CDF of the 2D Gaussian, we first needed to normalize the function.

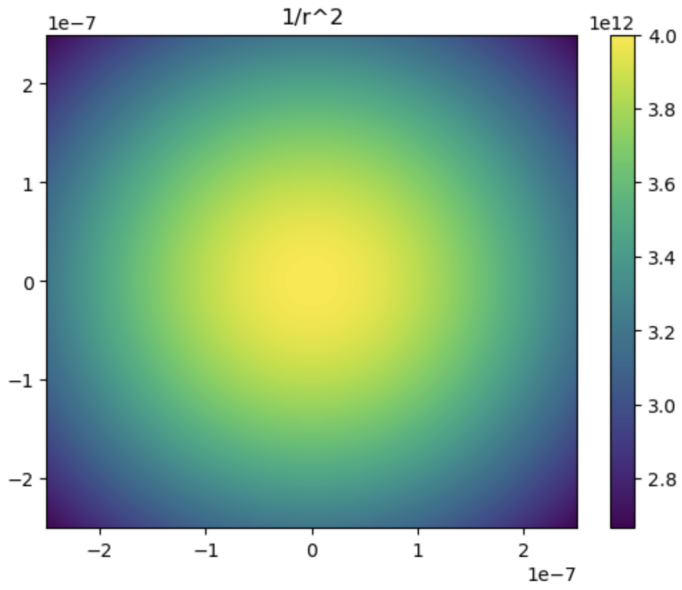
$$1 = \frac{1}{N}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} e^{-\frac{(x^2+y^2)}{\sigma^2}} dxdy \tag{7}$$

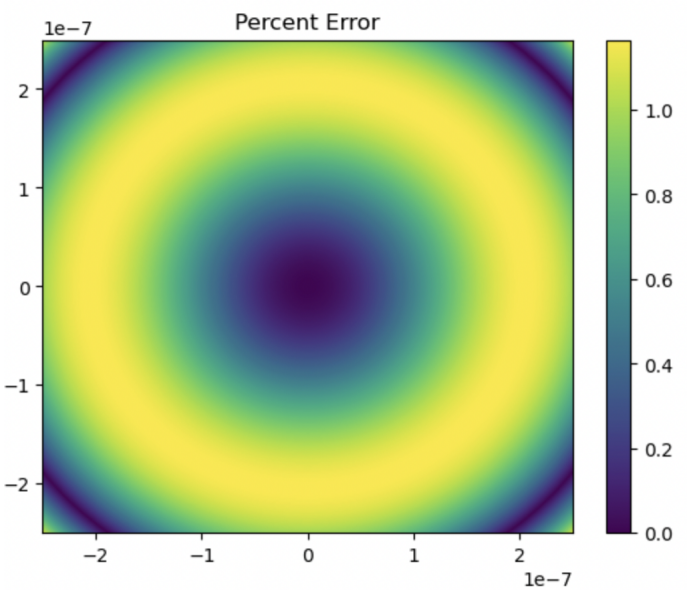
Thus,

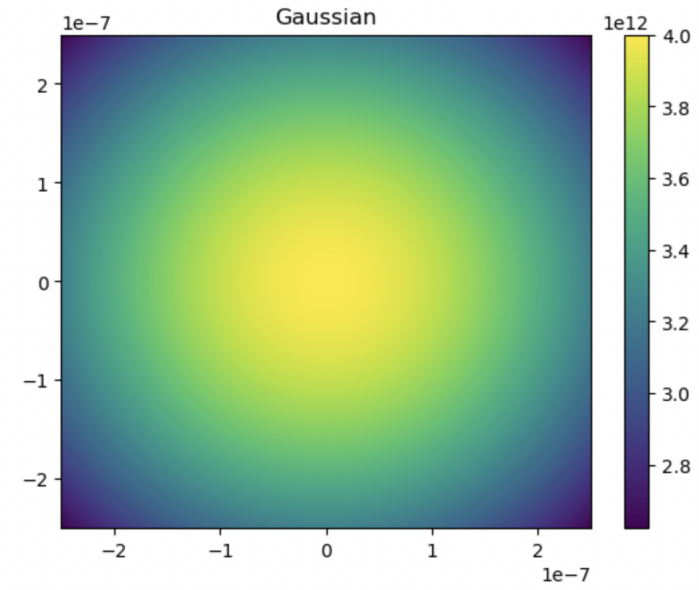
$$N = \pi\sigma^2 erf^2(\frac{a}{2\sigma}) \tag{8}$$

Next, we integrate the normalized function and then set it equal to a uniformly distributed random number and invert the function with respect to one variable while holding the other variable at its maximum, in this case, $a/2$. Since this function is symmetric it ends up being

$$x = y = \sigma erf^{-1}(erf(\frac{a}{2\sigma})(2u - 1)) \tag{9}$$







Where u is a random number. Note that u for x cannot be the same as u for y because then x and y would be correlated and not random.

C. Scipy Solve IVP

We used the initial value problem solver of Scipy library scipy.integrate.solveivp with fifth-order Runge Kuta method RK45.

D. Individual Iteration

In this section we will discuss clearly how each iteration in our simulation works. Imagine a time domain. First, we break up the time domain into several shorter time domains. Then we progressively iterate through each one. Starting at the first sub-domain, we generate a number of photons depending on the intensity of our light (usually 1 photon per division), and then we calculate how much of their frequency is absorbed by the materials using the previously discussed coupling efficiency and randomization methods. Next, we calculate how the current changes on the sub-domain using the frequency perturbation and the initial conditions. We then repeat this for every sub-domain using the last solution of the previous sub-domain as the initial conditions for the current sub-domain.

III. The Conclusion

We observed that even though the final values of the current are quite close, they are quite different in time as we can see in figure11.

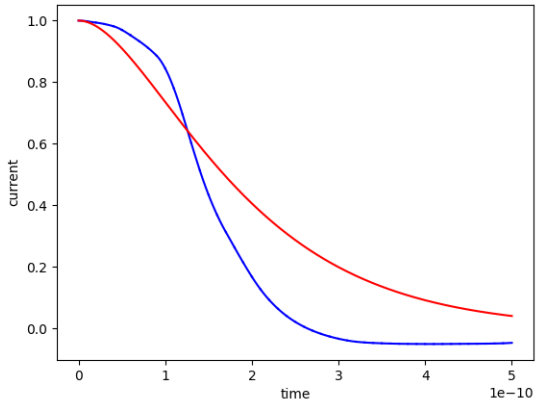


Fig. 1. Solutions for ODEs. The red line describes the solution before the photon interacts and the blue line describes the photon interacts.

Then we calculated the average power for both RLC circuits as following

$$P_{avg} = I_{RMS}^2 * R \tag{10}$$

where I_{RMS} is the root mean square of the current. Then, we calculated the average energy difference over 10 iterations and found that the standard deviation is $8.85 \times 10{-}6$, the median is 2.34×10^{-5} and the mean is 2.16×10^{-5}. Originally, we expected the difference in average energy to be 0 since the average frequency coupled would be 0. This may be incorrect because the frequency perturbations at the beginning of the simulation are more influential than those toward the end which is why there is a preference based on the frequency shift at the beginning.